Quadratic polynomial maps with Jacobian rank two

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Abstract

We classify all matrices $M \in \mathrm{Mat}_{m,n}(K[x])$ for which $\deg M = 1$ and $\operatorname{rk} M \le 2$. Furthermore, we classify all such matrices M such that $M = \mathcal{J}H$ for some polynomial map H. Among other things, we show that $\mathcal{J}H$ is similar to a matrix $\tilde{M}$ which has either only two nonzero columns or only three nonzero rows. In addition, we show that $\operatorname{trdeg}_K K(H) = \operatorname{rk} \mathcal{J}H$ for quadratic polynomial maps H over K such that $\frac{1}{2} \in K$ and $\operatorname{rk} \mathcal{J}H \le 2$.

Furthermore, we prove that nilpotent Jacobian matrices N for which $\deg N = 1$ and $\operatorname{rk} N \le 2$ are conjugation similar to a triangular matrix (with zeroes on the diagonal), regardless of the characteristic of K. This generalizes [dBY, Th. 3.4] (the case where K has characteristic zero) and [PC, Th. 1] (the case where $\frac{1}{2} \in K$ and $N(0) = 0$). In addition, we prove the same result for Jacobian matrices N for which $\deg N = 1$ and $N^2 = 0$. This generalizes [MO, §4] and [PC, Lem. 4] (the case where $\frac{1}{2} \in K$ and $N(0) = 0$).

Key words: quadratic polynomial map, Jacobian rank two, transcendence degree two, homogeneous, nilpotent, unipotent Keller map, linearly triangularizable, strongly nilpotent, similar, conjugation similar.

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1 Introduction

Throughout this paper, K is an arbitrary field. Let $H = (H_1, H_2, \ldots, H_m)$ be a polynomial map from K^n to K^m, i.e. $H_i \in K[x] := K[x_1, x_2, \ldots, x_n]$ for each

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i, where $x := (x_1, x_2, \ldots, x_n)$ is an n-tuple of indeterminates. The *degree* of H is defined by $\deg H := \max\{\deg H_1, \deg H_2, \ldots, \deg H_m\}$. We say that H *is homogeneous of degree* d if $y_1 H_1 + y_2 H_2 + \cdots + y_m H_m$ is homogeneous of degree $d+1$, where $y := (y_1, y_2, \ldots, y_m)$ is an m-tuple of indeterminates.

We write $\mathcal{J}H$ for the Jacobian matrix of H (with respect to x), i.e.

$$\mathcal{J}H = \begin{pmatrix} \frac{\partial}{\partial x_1} H_1 & \frac{\partial}{\partial x_2} H_1 & \cdots & \frac{\partial}{\partial x_n} H_1 \\ \frac{\partial}{\partial x_1} H_2 & \frac{\partial}{\partial x_2} H_2 & \cdots & \frac{\partial}{\partial x_n} H_2 \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \frac{\partial}{\partial x_1} H_m & \frac{\partial}{\partial x_2} H_m & \cdots & \frac{\partial}{\partial x_n} H_m \end{pmatrix}$$

Let R be a commutative ring with 1. We write $\mathrm{Mat}_{m,n}(R)$ for the set of matrices with m rows and n columns over R. So $\mathcal{J}H \in \mathrm{Mat}_{m,n}(K[x])$. We write $\mathrm{Mat}_n(R)$ for the ring of matrices with n rows and n columns over R. We define $\mathrm{GL}_n(R)$ as the group of invertible matrices over R, i.e. $\mathrm{GL}_n(R) := \{M \in \mathrm{Mat}_n(R) \mid \det M \text{ is a unit in } R\}$.

If R is a K-algebra, then we say that elements M and $\tilde{M}$ of $\mathrm{Mat}_{m,n}(R)$ are *similar (over K)* if there exists matrices $S \in \mathrm{GL}_m(K)$ and $T \in \mathrm{GL}_n(K)$ such that $\tilde{M} = SMT$. If $m = n$ and $S = T^{-1}$ in addition, then we say that M and $\tilde{M}$ are *conjugation similar (over K)*.

A matrix $M \in \mathrm{Mat}_n(R)$ is *upper (lower) triangular* if all entries below (above) the principal diagonal are zero, and *triangular (diagonal)* if M is either (both) upper or (and) lower triangular. The reader may verify the following.

Lemma 1.1. *Suppose that R is a K-algebra and $M \in \mathrm{Mat}_n(R)$.*

- **(i)** *If M is upper (lower) triangular, then M is conjugation similar to a lower (upper) triangular matrix $\tilde{M} \in \mathrm{Mat}_n(R)$*
- **(ii)** *If R is a reduced ring and M is both triangular and nilpotent, then the diagonal of M is totally zero.*

If $M \in \mathrm{Mat}_{m,n}(K[x])$, then we write $M(v)$ for the matrix

$$\begin{pmatrix} M_{11}(v) & M_{12}(v) & \cdots & M_{1n}(v) \\ M_{21}(v) & M_{22}(v) & \cdots & M_{2n}(v) \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ M_{m1}(v) & M_{m2}(v) & \cdots & M_{mn}(v) \end{pmatrix}$$

where $v \in K^n$. We say that $M \in \mathrm{Mat}_m(K[x])$ is *strongly nilpotent (over K)* if there exists an $r \geq 1$, such that

$$M(v^{(1)}) \cdot M(v^{(2)}) \cdot \cdots \cdot M(v^{(r)}) = 0$$

for all $v^{(1)}, v^{(2)}, \ldots, v^{(r)} \in K^n$. If K is infinite, then proposition 1.3 below gives a classification of strongly nilpotent matrices over $K[x]$. For the proof of proposition 1.3, we need the following lemma, which one can show by induction on r.

Lemma 1.2. *Suppose that $M \in \mathrm{Mat}_m(K[x])$ is of the form*

$$\begin{pmatrix} A & \emptyset \\ * & B \end{pmatrix} \qquad or \qquad \begin{pmatrix} A & * \\ \emptyset & B \end{pmatrix}$$

Let $\tilde{M} := M(v^{(1)}) \cdot M(v^{(2)}) \cdot \cdots \cdot M(v^{(r)})$, $\tilde{A} := A(v^{(1)}) \cdot A(v^{(2)}) \cdot \cdots \cdot A(v^{(r)})$ and $\tilde{B} := B(v^{(1)}) \cdot B(v^{(2)}) \cdot \cdots \cdot B(v^{(r)})$. Then $\tilde{M}$ is of the form.

$$\begin{pmatrix} \tilde{A} & \emptyset \\ * & \tilde{B} \end{pmatrix} \qquad or \qquad \begin{pmatrix} \tilde{A} & * \\ \emptyset & \tilde{B} \end{pmatrix} \tag{1.1}$$

respectively.

Proposition 1.3. *Suppose that L is infinite and an extension field of K. Let $M \in \mathrm{Mat}_m(K[x])$. Then M is strongly nilpotent over L, if and only if M is nilpotent and conjugation similar to a triangular matrix $\tilde{M} \in \mathrm{Mat}_m(K[x])$ over K.*

Proof. The 'if'-part is an straightforward exercise, so assume that M is strongly nilpotent over L. Then there are $v^{(2)}, v^{(3)}, \dots, v^{(r)} \in L^n$ such that for all $v^{(1)} \in L^n$,

$$M(v^{(1)}) \cdot M(v^{(2)}) \cdot \cdots \cdot M(v^{(r)}) = 0 \neq M(v^{(2)}) \cdot M(v^{(2)}) \cdot \cdots \cdot M(v^{(r)})$$

Since K is infinite, it follows that

$$M \cdot M(v^{(2)}) \cdot \cdots \cdot M(v^{(r)}) = 0 \neq M(v^{(2)}) \cdot M(v^{(2)}) \cdot \cdots \cdot M(v^{(r)})$$

so the columns of M are linearly dependent over L. Since L is a vector space over K, the columns of M are linearly dependent over K. Hence M is conjugation similar to a matrix $\tilde{M} \in \mathrm{Mat}_m(K[x])$, of which the last column is zero. From lemma 1.2, it follows that the leading principal minor matrix of size $(m-1) \times (m-1)$ of M strongly nilpotent. Hence it follows by induction on m that M is conjugation similar to a lower triangular matrix. □

The above proof has been extracted from that of [dB1, Th. 3.1], which is a more general result.

Corollary 1.4. *Suppose that $M \in \mathrm{Mat}_m(K[x])$ is of the form*

$$\begin{pmatrix} A & \emptyset \\ * & B \end{pmatrix} \qquad or \qquad \begin{pmatrix} A & * \\ \emptyset & B \end{pmatrix}$$

Then M is conjugation similar to a triangular matrix, if and only if A and B are conjugation similar to a triangular matrix.

Proof. From lemma 1.2, it follows that A and B are strongly nilpotent over an infinite extension field L of K if M is strongly nilpotent over L. Hence the 'only if'-part follows from proposition 1.3.

To prove the 'if'-part, suppose that A and B are strongly nilpotent over an infinite extension field L. Then there exists an integer r, such that $\tilde{A} = \tilde{B} = 0$ in (1.1), for every $v^{(1)}, v^{(2)}, \dots, v^{(r)} \in L^n$. It follows that

$$\big(M(v^{(1)}) \cdot M(v^{(2)}) \cdot \cdots \cdot M(v^{(r)})\big) \cdot \big(M(v^{(r+1)}) \cdot M(v^{(r+2)}) \cdot \cdots \cdot M(v^{(2r)})\big) = 0$$

for all $v^{(1)}, v^{(2)}, v^{(3)}, \dots, v^{2r-1}, v^{(2r)} \in K^n$. Hence the 'if'-part follows from proposition 1.3 as well. $\square$

Suppose that $M = \mathcal{J}H$ and $\tilde{M} = SMT$, where H is a polynomial map from K^n to K^m, $S \in \mathrm{GL}_m(K)$ and $T \in \mathrm{GL}_n(K)$. Let $\tilde{H} := SH(Tx)$. From the chain rule, it follows that

$$\mathcal{J}\tilde{H} = \mathcal{J}\big(SH(Tx)\big) = SM|_{x=Tx}T = \tilde{M}|_{x=Tx} \tag{1.2}$$

so $\tilde{M}$ itself is a Jacobian matrix up to an automorphism of $K[x]$. It follows that $\mathcal{J}\tilde{H}$ is (strongly) nilpotent or upper (lower) triangular, if and only if $\tilde{M}$ is (strongly) nilpotent or upper (lower) triangular respectively.

The *degree* of a matrix $M \in \mathrm{Mat}_{m,n}(K[x])$ is defined by

$$\deg M := \max\{\deg M_{11}, \deg M_{12}, \dots, \deg M_{1n}, \deg M_{21}, \deg M_{22}, \dots, \deg M_{mn}\}$$

§2 In section 2, we classify all matrices $M \in \mathrm{Mat}_{m,n}(K[x])$ for which $\deg M = 1$ and $\mathrm{rk}\, M \le 2$. Furthermore, we classify all such matrices M such that $M = \mathcal{J}H$ for some polynomial map H. Among other things, we show that $M = \mathcal{J}H$ is similar to a matrix $\tilde{M}$ which has either only two nonzero columns or only three nonzero rows.

In addition, we show that $\mathrm{trdeg}_K K(H) = \mathrm{rk}\, \mathcal{J}H$ for quadratic polynomial maps H over K such that $\frac{1}{2} \in K$ and $\mathrm{rk}\, \mathcal{J}H \le 2$. In general $\mathrm{trdeg}_K K(H) \le \mathrm{rk}\, \mathcal{J}H$ for a polynomial map H of any degree, with equality if K has characteristic zero. This is proved in [BMS, Lem. 9] and [BMS, Th. 8] respectively.

§3 In section 3, we prove that nilpotent Jacobian matrices N for which $\deg N = 1$ and $\mathrm{rk}\, N \le 2$ are conjugation similar to a triangular matrix (with zeroes on the diagonal), regardless of the characteristic of K. This generalizes [dBY, Th. 3.4] (the case where K has characteristic zero) and [PC, Th. 1] (the case where $\frac{1}{2} \in K$ and $N(0) = 0$). In [PC, Th. 1], which is the main result of [PC], the authors additionally assume that K is infinite, but one can derive the finite case from the infinite case by way of proposition 1.3 above.

At the end of section 3, we prove that nilpotent Jacobian matrices N for which $\deg N = 1$ and $N^2 = 0$ are conjugation similar to a triangular matrix (with zeroes on the diagonal), regardless of the characteristic of K. This generalizes [MO, §4] and [PC, Lm. 4] (the case where $\frac{1}{2} \in K$ and $N(0) = 0$).

We additionally show that $N(v^{(1)}) \cdot N(v^{(2)}) \cdot N(v^{(3)}) = 0$ if $\frac{1}{2} \in K$, where $v^{(1)}, v^{(2)}, v^{(3)}$ are as in the proof of proposition 1.3, using the fact that the proof of [PC, Lm. 4] shows that $N(v^{(1)}) \cdot N(v^{(2)}) = 0$ if $N(0) = 0$ in addition.

2 (Jacobian) matrices of degree one and rank at most two

A matrix of rank zero can only be the zero matrix, so we only need to distinguish rank one and rank two. Let us start with rank one.

Theorem 2.1. *Let M be a matrix whose entries are polynomials of degree at most 1 over K. If $\operatorname{rk} M = 1$, then M is similar to a matrix $\tilde{M}$ for which one of the following statements holds.*

(1) *Only the first column of $\tilde{M}$ is nonzero.*

(2) *Only the first row of $\tilde{M}$ is nonzero.*

If M is the Jacobian matrix of a (quadratic) polynomial map in addition, then the following assertion can be added to (1)*.*

(1) *The first column of $\tilde{M}$ is of the form $(*, \frac{1}{2}, 0, \ldots, 0)$.*

Proof. If the constant part $M(0)$ of M is zero, then we can substitute $x_i = x_i + 1$, where x_i is a variable which appears in M, to obtain $M(0) \neq 1$. So we may assume that $M(0) \neq 0$.

Since $\operatorname{rk} M(0) = 1$, we can choose $\tilde{M}$ such that

$$\tilde{M}(0) = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots \end{pmatrix}$$

Looking at the linear parts of the 2×2 minor determinants, we see that only the first row and the first column of $\tilde{M}$ may be nonzero.

Suppose that (1) does not hold. Then we may assume that the second entry of the first row of $\tilde{M}$ is nonzero. Suppose that (2) does not hold. Then we may assume that the first entry of the second row of $\tilde{M}$ is nonzero. This contradicts that the leading principal 2×2 minor determinant of $\tilde{M}$ is zero.

So we have proved the first part of this theorem. To prove the second part of this theorem, assume that $M = \mathcal{J}H$ for a polynomial map H. If we remove terms $x_1^{k_1} x_2^{k_2} \cdots$ of H for which $\frac{1}{k_i} \notin K$ for all i, then $M = \mathcal{J}H$ is preserved, and for every term t of H, there exists an i such that

$$\frac{\partial}{\partial x_i} t \neq 0$$

Since $\deg \mathcal{J}H \leq 1$, it follows that H becomes a polynomial map with terms of degree 1 and 2 only. So we may assume that $H(0) = 0$ and $\deg H \leq 2$.

Let $\tilde{H} := SH(Tx)$ and suppose that $\tilde{M}$ is as in (1). From (1.2), it follows that $\mathcal{J}\tilde{H}$ is an in (1) as well, i.e. only the first column of $\mathcal{J}\tilde{H}$ is nonzero.

If $\frac{1}{2} \in K$, then for all j, $\tilde{H}_j$ is linearly dependent over K on x_1^2 and x_1 only. If x_1 and x_1^2 are in turn linearly dependent over K on $\tilde{H}_1, \tilde{H}_2, \cdots$, then we can get the first column of $\mathcal{J}\tilde{H}$ and $\tilde{M}$ of the given form by way of row operations. Otherwise, we can get the first column of $\mathcal{J}\tilde{H}$ and $\tilde{M}$ of the form $(*, 0, 0, \cdots, 0)$ by way of row operations, so (2) is satisfied.

So assume that $\frac{1}{2} \notin K$. Then for all j, $\tilde{H}_j$ is dependent over K on $x_1, x_1^2, x_2^2, x_3^2, \ldots$. By row operations, we can obtain that the coefficients of x_1 of $\tilde{H}_j$ are zero for all $j \geq 2$. Hence only the first row of $\mathcal{J}\tilde{H}$ is nonzero and $\tilde{M}$ is as in (2). □

In [dB2, Th. 1.8], it is proved that over fields of characteristic zero, polynomial maps with an antisymmetric Jacobian matrix are linear. With essentially the same proof, one can draw the same conclusion if the characteristic of the field exceeds the degree of the polynomial map.

Lemma 2.2. *Let H be a polynomial map of degree at most d over K, such that $d! \neq 0$ in K. If $\mathcal{J}H$ is antisymmetric, then $\deg H \leq 1$.*

Proof. There is nothing to prove if $\frac{1}{2} \notin K$, so assume that $\frac{1}{2} \in K$. Suppose that $\mathcal{J}H$ is antisymmetric. Then

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}H_k = -\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}H_j = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}H_i = -\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}H_k$$

and hence $2\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}H_k = 0$, for all i, j, k. As $2d! \neq 0$ in K, it follows that $\deg H \leq 1$. □

Using lemma 2.2 above, we can proceed with rank two.

Theorem 2.3. *Let M be a matrix whose entries are polynomials of degree at most one over K. If $\operatorname{rk} M = 2$, then M is similar to a matrix $\tilde{M}$ for which one of the following statements holds.*

(1) *Only the first two columns of $\tilde{M}$ are nonzero.*

(2) *Only the first two rows of $\tilde{M}$ are nonzero.*

(3) *The first row and the first column of $\tilde{M}$ are nonzero, and $\tilde{M}$ is zero elsewhere.*

(4) *The leading principal 3×3 minor matrix of $\tilde{M}$ is anti-symmetric, with only zeroes on the diagonal, and $\tilde{M}$ is zero elsewhere. Furthermore, the three entries below the diagonal of this principal minor matrix are linearly independent over K.*

If M is the Jacobian matrix of a (quadratic) polynomial map in addition, then the following assertions can be added to (3) *and* (4) *respectively.*

(3) *The first column of $\tilde{M}$ is of the form $(*, *, \frac{1}{2}, 0, \ldots, 0)$.*

(4) *$\tilde{M}$ is symmetric, i.e. $\frac{1}{2} \notin K$.*

Proof. We first show that we may assume that

$$M(0) = \begin{pmatrix} 0 & -1 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots \end{pmatrix} \tag{2.1}$$

Notice that this is indeed the case if $\operatorname{rk} M(0) \geq 2$. Otherwise, we may assume that $\operatorname{rk} M(0) = 1$, because we can substitute $x_i = x_i + 1$ if $M(0) = 0$, just like in the proof of theorem 2.1. So we may assume that

$$M(0) = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots \end{pmatrix}$$

Suppose that an $\tilde{M}$ as in (3) cannot be obtained by interchanging the first and the second row of M. Then M has a nonzero entry outside the second row and the first column. Without loss of generality, we may assume that the second entry of the first row of M is nonzero.

Say that the coefficient of x_j of this entry of M is nonzero. Let C be the coefficient matrix of x_j of M. Then

$$x_j^{-1}\big(M - x_j C\big) + x_j^0 C$$

is the expansion of the matrix $x_j^{-1}M$, as a Laurant polynomial over x_j with matrix coefficients. Let $\tilde{C}$ be the matrix one gets by substituting $x_j = x_j^{-1}$ in $x_j^{-1}M$. Then

$$\tilde{C} = x_j^{+1}\big(M - x_j C\big) + x_j^0 C = x_j\big(M - M(0) - x_j C\big) + x_j M(0) + C$$

If $\operatorname{rk} C \geq 2$, then we can interchange C and $M(0)$ as coefficient matrices of M without affecting $\operatorname{rk} M = 2$, namely by replacing M by the result of substituting $x_i = x_j^{-1} x_i$ for all $i \neq j$ in $\tilde{C}$, to obtain $\operatorname{rk} M(0) \geq 2$ as above.

So assume that $\operatorname{rk} C = 1$. Since the second entry of the first row of C is nonzero, we may assume that the first row of C equals $(0\ 1\ 0\ 0\ \cdots\ 0)$. Since $\operatorname{rk} C = 1$, it follows that only the second column of C is nonzero, so we may assume that C is the transpose of $M(0)$. Now substitute $x_j = x_j - 1$ to obtain that $M(0)$ is as in (2.1).

So $M(0)$ is as in (2.1). Looking at the linear parts of the 3×3 minor determinants, we see that only the first two rows and the first two columns of M may be nonzero. Take $i \geq 3$ arbitrary.

Suppose that (2) does not hold. Then we may assume that the third row of M is nonzero. If the first entry of the third row is zero, then we can add the second column and the second row to the first column and the first row respectively, so we may assume that the first entry of the third row is nonzero.

Looking at the quadratic part of the leading principal 3×3 minor determinant, we see that the third column is dependent on the transpose of the third row. If $i > 3$, then we could interchange the third and the i-th column of M, so the i-th column of M is dependent on the transpose of the third row.

Suppose that (1) does not hold. Then we may assume that the third column of M is nonzero. Just as the i-th column of M is dependent on the transpose of the third row, we can deduce that the i-th row of M is dependent on the transpose of the third column. So the i-th row of M is dependent on the third row. In addition, the i-th column of M is dependent on the third column.

Since the third column is dependent on the transpose of the third row, the third entry of the first row of M is nonzero along with the first entry of the third row. Furthermore, we may assume that the leading principal 3×3 matrix of M is of the form

$$\begin{pmatrix} * & * & -a \\ * & c & -b \\ a & b & 0 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} * & * & b \\ * & c & \lambda b \\ a & \lambda a & 0 \end{pmatrix} \tag{2.2}$$

where a and b are linear forms over K and $\lambda \in K$.

We show that (3) is satisfied if M is of the form of the rightmost matrix of (2.2). Indeed, if M has this form, then we can subtract the first column and the first row λ times from the second column and the second row respectively, to obtain $\lambda = 0$, after which we can look at the leading principal 3×3 minor determinant to deduce that c has become zero.

So assume that M is of the form of the leftmost matrix of (2.2), but not of the form of the rightmost matrix of (2.2). Then a and b are independent linear forms. Looking at the leading principal 3×3 minor determinant, we see that $b \mid a^2 c$, so $b \mid c$ and $c = \mu b$ for some $\mu \in K$. Subtracting the third row μ times from the second row, we see that we may assume that $\mu = 0$. So we may assume that $c = 0$. More generally, we may assume that the diagonal of the leading principal 3×3 matrix of M is zero.

Now it is straightforward to check that the leading principal 3×3 matrix of M is antisymmetric. Furthermore, the three entries below its diagonal are linearly independent over K, because a, b and $f + 1$ are linearly independent over K for every linear form f. Since a and b are linearly independent, it follows that the i-th row of M is dependent over K on the third row. Since $-a$ and $-b$ are linearly independent, it follows that the i-th column of M is dependent over K on the third column. So we can make $\tilde{M}$ as in (4) from M by way of row and column operations.

So we have proved the first part of this theorem. To prove the second part of this theorem, assume that $M = \mathcal{J}H$ for a polynomial map H. Just as in the proof of theorem 2.1, we may assume that $H(0) = 0$ and $\deg H \le 2$. Let $\tilde{H} := SH(Tx)$. The case where $\tilde{M}$ is as in (3) follows in a similar manner as the case where $\tilde{M}$ is as in (1) in the proof of theorem 2.1.

Hence assume that $\tilde{M}$ is as in (4) and that $\frac{1}{2} \in K$. Then $\mathcal{J}\tilde{H}$ is as in (4) as well. Furthermore, $\tilde{H}_j \in K[x_1, x_2, x_3]$ for all j. From lemma 2.2, it follows that $\deg \tilde{H} = 1$. This contradicts that $\tilde{M}$ has three entries which are linearly independent over K. □

Corollary 2.4. *Let H be a quadratic polynomial map over K, such that $r := \operatorname{rk} \mathcal{J}H \le 2$. If $\frac{1}{2} \in K$, then $K[H] \subseteq K[f_1, \dots, f_r]$ for polynomials f_i. In particular,* $\operatorname{trdeg}_K K(H) = \operatorname{rk} \mathcal{J}H$.

Proof. From $\frac{1}{2} \in K$, it follows that for every term t of H,

$$\frac{\partial}{\partial x_i} t \ne 0 \Longleftrightarrow x_i \mid t$$

If $r = 0$, then H is constant. If $r = 1$, then it follows from theorem 2.1 that we may assume that either $K[H] = K[x_1]$ or $K[H] = K[H_1]$. So assume that $r = 2$. Then it follows from theorem 2.3 that we may assume that either $K[H] \subseteq K[x_1, x_2]$ or $K[H] = K[H_1, H_2]$ or $K[H] = K[H_1, x_1]$, because (4) of theorem 2.3 for $M = \mathcal{J}H$ requires $\frac{1}{2} \notin K$. □

Lemma 2.5. *Let H be a polynomial map over K and suppose that $\mathcal{J}H$ is symmetric. If for each i, the i-th entry of the diagonal of $\mathcal{J}H$ has no terms whose degrees with respect to x_i are equal to -2 in K, then there exists a polynomial $h \in K[x]$ such that $H = (\mathcal{J}h)^{\mathrm{t}}$ and $\mathcal{J}H = \mathcal{H}h$.*

Proof. Assume that the diagonal of $\mathcal{J}H$ is as indicated above. Then for each i, H_i has no terms whose degrees with respect to x_i are equal to -1 in K. From the proof of [vdE, Lem. 1.3.53], it follows that there exists a polynomial $h \in A[x]$ such that $H = (\mathcal{J}h)^{\mathrm{t}}$ (the α_i in that proof are nonzero). So $\mathcal{J}H = \mathcal{H}h$. □

Corollary 2.6. *Suppose that H is a polynomial map of degree at most 2 in dimension three over K, such that $\mathcal{J}H$ is antisymmetric with only zeroes on the diagonal. Suppose that $\mathcal{J}H$ is not constant. Then there exist a $\lambda \in K^*$ and $c_1, c_2, c_3 \in K$, such that*

$$\mathcal{J}H = \mathcal{H}\big(\lambda(x_1 + c_1)(x_2 + c_2)(x_3 + c_3)\big)$$

Furthermore, $\frac{1}{2} \notin K$ and $\operatorname{rk} \mathcal{J}H = 2 < 3 = \operatorname{trdeg}_K K(H)$.

Proof. From lemma 2.2, it follows that $\frac{1}{2} \notin K$ and that $\mathcal{J}H$ is symmetric. From lemma 2.5, it follows that $\tilde{M}|_{x=Tx} = \mathcal{J}\tilde{H} = \mathcal{H}h$ for some polynomial h.

Since $\deg H \le 2$, it follows that terms of degree greater than 3 of h cannot affect $\mathcal{H}h$. Since $\frac{1}{2} \notin K$, it follows that terms of degree at most 3 of h which are divisible by x_i^2 for some i cannot affect $\mathcal{H}h$. So we can remove terms of h of degree greater than 3 and terms of h which are divisible by x_i^2 for some i. Furthermore, we can remove terms of h of degree at most 1. After these removals, h will be of the form

$$h = \lambda x_1 x_2 x_3 + \tilde{c}_1 x_2 x_3 + \tilde{c}_2 x_3 x_1 + \tilde{c}_3 x_1 x_2$$

In particular $\deg h \leq 3$. Suppose that $\mathcal{J}H$ is not constant. Then $\deg h = 3$, so $\lambda \neq 0$. Hence $\mathcal{J}H$ is of the given form, with $c_i = \lambda^{-1}\tilde{c}_i$ for each i. Furthermore, $\operatorname{rk} \mathcal{J}H = 2$.

Suppose that $\operatorname{trdeg}_K K(H) \leq 2$. Then there exists a polynomial f such that $f(H) = 0$. If $\bar{f}$ is the leading homogeneous part of f and $\bar{H}$ is the leading homogeneous part of H, then $\bar{f}(\bar{H}) = 0$, so $\operatorname{trdeg}_K K(\bar{H}) \leq 2$. From [dB3, Th. 2.5], it follows that there exists an $S \in \mathrm{GL}_3(K)$ such that

$$S\bar{H} = (p, q, 0) \qquad \text{or} \qquad S\bar{H} = (p^2, pq, q^2)$$

for homogeneous polynomials p, q of the same degree. In the first case, the rows of $\mathcal{J}\bar{H}$ are dependent over K. In the second case, $\deg(p, q) = 1$ and the columns of $\mathcal{J}\bar{H}$ are dependent over K. This is however not the case for

$$\mathcal{J}\bar{H} = \lambda \begin{pmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix}$$

so $\operatorname{trdeg}_K K(H) = 3$. □

Corollary 2.7. *Let M be a matrix whose entries are polynomials of degree at most one over K. Suppose that $\operatorname{rk} M \leq 2$ and that M is the Jacobian matrix of a polynomial map H. Then there does not exists a polynomial map H such that $\operatorname{trdeg}_K K(H) = \operatorname{rk} \mathcal{J}H$ and $\mathcal{J}H = M$, if and only if M is as in (4) of theorem 2.3. In that case, $\frac{1}{2} \notin K$ and $\operatorname{trdeg}_K K(H) = 3 > 2 = \operatorname{rk} \mathcal{J}H$ for every polynomial map H such that $\mathcal{J}H = M$.*

Proof. The 'if'-part follows from corollary 2.6. The last claim follows from corollary 2.6 as well. The 'only if'-part follows from theorem 2.3 and the proof of corollary 2.4. □

3 Nilpotent Jacobian matrices of degree one and rank at most two

Before we prove the main result of this section, which is theorem 3.2 below, we formulate a lemma about nilpotent matrices N of degree 1 and size 2×2 or 3×3, such that $P^{-1}N(0)P$ has Jordan normal form for a permutation matrix P.

Lemma 3.1. *Let K be a field and N be a nilpotent matrix whose entries are polynomials of degree 1. Then the following holds.*

- **(i)** *If $N(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then N is conjugation similar to a triangular matrix.*
- **(ii)** *If $N(0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then N is lower triangular.*

(iii) *If* $N(0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$*, then* N *is conjugation similar to a triangular matrix.*

(iv) *If* $N(0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ *and* N *is* not *upper triangular, then* N *is similar to an antisymmetric matrix* $\tilde{N}$ *with only zeroes on the diagonal, such that the constant part of* $\tilde{N}$ *is equal to the right hand side of* (2.1) *and the last row of* $\tilde{N}$ *is* $(0\ b\ 0)$ *for some nonzero linear form* b.

Proof.

(i) Using the trace condition, we see that

$$N = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

for linear forms a, b. As polynomial rings have unique factorization, $b = \lambda a$ for some $\lambda \in K$. Using the deteminant condition, we see that $c = \lambda^{-1} a$, so the entries of $a^{-1}N$ are contained in K. Hence $a^{-1}N$ is strongly nilpotent over an infinite extension field of K. From proposition 1.3, it follows that $a^{-1}N$ is conjugation similar to a triangular matrix, and so is N.

(ii) Using the determinant condition,

$$N = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$$

(because the linear part is zero). Hence N is lower triangular.

(iii) Using the principal 2×2 minors condition, we see that

$$N = \begin{pmatrix} * & * & * \\ b & * & * \\ 0 & c & * \end{pmatrix}$$

for linear forms b, c (because the linear part is zero). On account of the determinant condition, $bc = 0$ (because bc is the quadratic part of $\det N$). If $b = 0$, then the leading principal 1×1 minor and the trailing principal 2×2 minor of N are nilpotent, hence conjugation similar to a triangular matrix on account of (i). If $c = 0$, then the leading principal 2×2 minor and the trailing principal 1×1 minor of N are nilpotent, hence conjugation similar to a triangular matrix on account of (i). From corollary 1.4, it follows that N is conjugation similar to a triangular matrix in both cases.

(iv) Using all principal minors conditions

$$N = \begin{pmatrix} -a & * & * \\ b & a+c & * \\ 0 & -b & -c \end{pmatrix}$$

for linear forms a, b, c (because the linear parts are zero). If $b = 0$, then N is upper triangular, so assume that $b \neq 0$. On account of the determinant condition, $a = c$ (quadratic part) and $b \mid a(a+c)c$ (cubic part). So $b \mid a$ if $\frac{1}{2} \in K$. If $\frac{1}{2} \notin K$, then $b \mid a^2$ on account of the principal 2×2 minors condition, so $b \mid a$ in any case.

Since $b \mid a$ and $b \mid c$, we may assume that $a = c = 0$, because we can replace N by $T^{-1}NT$, where

$$T := \begin{pmatrix} 1 & -\frac{a}{b} & 0 \\ 0 & 1 & -\frac{c}{b} \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad T^{-1} = \begin{pmatrix} 1 & \frac{a}{b} & \frac{ac}{b^2} \\ 0 & 1 & \frac{c}{b} \\ 0 & 0 & 1 \end{pmatrix}$$

On account of the determinant condition, the upper right corner of N is zero. On account of the principal 2×2 minors condition,

$$N = \begin{pmatrix} 0 & f+1 & 0 \\ b & 0 & f+1 \\ 0 & -b & 0 \end{pmatrix}$$

for a linear form f. Hence

$$N \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -f-1 & 0 \\ f+1 & 0 & -b \\ 0 & b & 0 \end{pmatrix}$$

is antisymmetric with only zeroes on the diagonal, its constant part is equal to the right hand side of (2.1), and its last row is $(0\ b\ 0)$ for some nonzero linear form b. □

Theorem 3.2. *Suppose that H is a quadratic polynomial map in dimension n over a field K of any characteristic whatsoever, such that $\operatorname{rk} \mathcal{J}H \le 2$ and $\mathcal{J}H$ is nilpotent. Then $\mathcal{J}H$ is conjugation similar to a triangular matrix.*

Proof. Let $M = \mathcal{J}H$. Suppose first that $\operatorname{rk} M = 1$. From theorem 2.1, it follows that there exists $S, T \in \mathrm{GL}_n(K)$, such that $\tilde{M} := SMT$ satisfies one of the following:

- *$\tilde{M}$ is as in (1) of theorem 2.1.*
  Then $\tilde{M}$, $T^{-1}MT$ and $\mathcal{J}\big(T^{-1}H(Tx)\big) = T^{-1}M|_{x=Tx}T$ are lower triangular, because only their first columns are nonzero. So M is conjugation similar to a triangular matrix.

- *$\tilde{M}$ is as in (2) of theorem 2.1.*
  Then $\tilde{M}$, SMS^{-1} and $\mathcal{J}\big(SH(S^{-1}x)\big) = SM|_{x=S^{-1}x}S^{-1}$ are upper triangular, because only their first columns are nonzero. So M is conjugation similar to a triangular matrix.

Suppose next that $\operatorname{rk} M = 2$. From theorem 2.3, it follows that there exists $S, T \in \mathrm{GL}_n(K)$, such that $\tilde{M} := SMT$ satisfies one of the following:

- *$\tilde{M}$ is as in (1) of theorem 2.3.*
  Then only the first two columns of $\tilde{M}$, $T^{-1}MT$ and $\mathcal{J}\big(T^{-1}H(Tx)\big) = T^{-1}M|_{x=Tx}T$ are nonzero. From (i) and (ii) of lemma 3.1, it follows that the leading 2×2 principal minor matrix of $T^{-1}MT$ and $T^{-1}M|_{x=Tx}T$ are conjugation similar to a triangular matrix. Hence M is conjugation similar to a triangular matrix as well.

- *Only the first three rows of $\tilde{M}$ may be nonzero.*
  Then only the first three rows of $\tilde{M}$, SMS^{-1} and $\mathcal{J}\big(SH(S^{-1}x)\big) = SM|_{x=S^{-1}x}S^{-1}$ are nonzero. Hence the leading principal 3×3 minor matrix N of SMS^{-1} is nilpotent.

  In order to show that M is conjugation similar to a triangular matrix, it suffices to show that SMS^{-1} is conjugation similar to a triangular matrix. From corollary 1.4, we deduce that it suffices to show that N is conjugation similar to a triangular matrix. For that purpose, we distinguish three cases.

  rk $\boldsymbol{N(0) = 0}$. Then we can replace M by the result of substituting $x_i = x_i+1$ in M for some i, to obtain $\operatorname{rk} N(0) \neq 0$, because of the following. M becomes $\mathcal{J}(H|_{x_i=x_i+1})$, which is a Jacobian matrix as well, and the linear part of N is not affected. So N and hence M as well is conjugation similar to a triangular matrix before replacing M if N is conjugation similar to a triangular matrix after replacing M.

  rk $\boldsymbol{N(0) = 1}$. Then the Jordan normal form of $N(0)$ is equal to that of $N(0)$ in (iii) of lemma 3.1. So we may assume that $N(0)$ is as in (iii) of lemma 3.1. It follows from (iii) of lemma 3.1 that N is conjugation similar to a triangular matrix.

  rk $\boldsymbol{N(0) \geq 2}$. Then the Jordan normal form of $N(0)$ is equal to (that of) $N(0)$ in (iv) of lemma 3.1. So we may assume that $N(0)$ is as in (iv) of lemma 3.1. Assume that N is not upper triangular. Then it follows from (iv) of lemma 3.1 that N is similar to an antisymmetric matrix $\tilde{N}$ with only zeroes on the diagonal, such that the constant part of $\tilde{N}$ is of the form of the right hand side of (2.1) and the last row of $\tilde{N}$ is $(0\ b\ 0)$ for some nonzero linear form b.

  We may assume that $\tilde{N}$ is the leading principal 3×3 minor matrix of $\tilde{M}$. Since the leading principal 2×2 minor of $\tilde{N}(0)$ has full rank, we can adapt T such that the first and the second row of $\tilde{M}(0)$ will be equal to $(0\ -1\ 0\ 0\ \cdots\ 0)$ and $(1\ 0\ 0\ 0\ \cdots\ 0)$ respectively.

  Looking at the constant parts of the 3×3 minor determinants, we see that the third and subsequent entries of the third row of $\tilde{M}(0)$ are zero. Looking at the linear parts of the 3×3 minor determinants, we see that the second entry of the third row of $\tilde{M}$ is the only nonzero entry in that row. The same holds for the Jacobian matrix of $\tilde{H} := SH(Tx)$, because $\mathcal{J}\tilde{H} = \tilde{M}_{x=Tx}$ on account of (1.2).

  It follows that the second entry of the third row of $\mathcal{J}\tilde{H}$ is of the form λx_2, where $\lambda \neq 0$ because $\lambda x_2 = b|_{x=Tx}$. So $\frac{1}{2}\lambda x_2^2$ appears as a term in $\tilde{H}_3$. In particular, $\frac{1}{2} \in K$. Since $\mathcal{J}_{x_1,x_2,x_3}(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3) = \tilde{N}|_{x=Tx}$ is antisymmetric, it follows from lemma 2.2 that $\deg_{x_2} \tilde{H}_3 \le \deg_{x_1,x_2,x_3}(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3) \le 1$. Contradiction. □

In the proof of [PC, Lem. 4], it is shown that $\mathcal{J}H^2 = 0$ implies $(\mathcal{J}H)(x) \cdot (\mathcal{J}H)(y) = 0$ if H is quadratic homogeneous and $\frac{1}{2} \in K$, where $y = (y_1, y_2, \ldots, y_n)$ is another n-tuple of indeterminates. The maps

$$H = \big(0, x_1, x_1^2, x_1x_2 - \tfrac{1}{2}x_3\big)$$

and

$$H = \big(0, 0, 0, x_2x_3, x_3x_1, x_1x_2, x_1x_4 + x_2x_5 + x_3x_6\big)$$

show that the conditions that H is (quadratic) homogeneous and $\frac{1}{2} \in K$ are necessary respectively.

Theorem 3.3. *Suppose that H is a quadratic polynomial map in dimension n over a field K of any characteristic whatsoever, such that $\mathcal{J}H^2 = 0$. Then the following holds.*

(i) *$\mathcal{J}H$ is conjugation similar to a triangular matrix.*

(ii) *If $\frac{1}{2} \in K$ and H is homogeneous, then*

$$(\mathcal{J}H)(x) \cdot (\mathcal{J}H)(y) = 0$$

where $y = (y_1, y_2, \ldots, y_n)$ is another n-tuple of indeterminates.

(iii) *If $\frac{1}{2} \in K$ and H is not (necessarily) homogeneous, then*

$$(\mathcal{J}H)(x) \cdot (\mathcal{J}H)(y) \cdot (\mathcal{J}H)(z) = 0$$

where $z = (z_1, z_2, \ldots, z_n)$ is yet another n-tuple of indeterminates.

Proof. Let $y = (y_1, y_2, \ldots, y_n)$ be an n-tuple of indeterminates. From $\deg \mathcal{J}H \le 1$, it follows that

$$(\mathcal{J}H)\big(tx + (1-t)y\big) = t(\mathcal{J}H)(x) + (1-t)(\mathcal{J}H)(y)$$

Taking squares on both sides, we deduce that

$$0 = t(1-t)\big((\mathcal{J}H)(x) \cdot (\mathcal{J}H)(y) + (\mathcal{J}H)(y) \cdot (\mathcal{J}H)(x)\big)$$

Consequently,

$$(\mathcal{J}H)(y) \cdot (\mathcal{J}H)(x) = -(\mathcal{J}H)(x) \cdot (\mathcal{J}H)(y) \tag{3.1}$$

(i) We assume that H is homogeneous of degree 2, because due to proposition 1.3, a reduction to this case will be similar to the proof of (iii) below. If $n = 1$, then $\mathcal{J}H$ is triangular, so assume that $n \geq 2$. Let

$$Z = \begin{pmatrix} Z_{11} & Z_{12} & \cdots & Z_{1n} \\ Z_{21} & Z_{22} & \cdots & Z_{2n} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ Z_{n1} & Z_{n2} & \cdots & Z_{nn} \end{pmatrix}$$

From (ii) of [PC, Prop. 3] and (3.1), it follows that

$$(\mathcal{J}H)(Ze_1) \cdot (\mathcal{J}H)(Ze_2) \cdot \cdots \cdot (\mathcal{J}H)(Ze_n) \cdot \big(Ze_i\big)$$
$$= (\mathcal{J}H)(Ze_i) \cdot (\mathcal{J}H)(Ze_i) \cdot \big(\vdots\big) = 0$$

Consequently

$$(\mathcal{J}H)(Ze_1) \cdot (\mathcal{J}H)(Ze_2) \cdot \cdots \cdot (\mathcal{J}H)(Ze_n) \cdot Z = 0$$

As $\operatorname{rk} Z = n$, $(\mathcal{J}H)(Ze_1) \cdot (\mathcal{J}H)(Ze_2) \cdot \cdots \cdot (\mathcal{J}H)(Ze_n) = 0$. Since we can substitute elements of any field $L \supseteq K$ in the indeterminates of Z, it follows from proposition 1.3 that $\mathcal{J}H$ is conjugation similar to a triangular matrix.

(ii) This is shown in the proof of [PC, Lem. 4].

(iii) Let $\bar{H}$ be the quadratic part of H. Notice that

$$\prod_{i=1}^{3} (\mathcal{J}H)(Ze_i) = \prod_{i=1}^{3} \big((\mathcal{J}\bar{H})(Ze_i) + (\mathcal{J}H)(0)\big) \tag{3.2}$$

and that every term of the expansion of the right hand side of (3.2) either has two factors $(\mathcal{J}H)(0)$, or two distinct factors $(\mathcal{J}\bar{H})(Ze_i)$ and $(\mathcal{J}\bar{H})(Ze_j)$, where $1 \leq i \leq j \leq 3$.

From (3.1), it follows that

$$\begin{aligned}\big((\mathcal{J}\bar{H})(x)\big)^2 &= \big((\mathcal{J}H)(x) - (\mathcal{J}H)(0)\big)^2 \\ &= \big((\mathcal{J}H)(x)\big)^2 + \big((\mathcal{J}H)(0)\big)^2 = 0\end{aligned}$$

From (ii), we subsequently deduce that

$$(\mathcal{J}\bar{H})(x) \cdot (\mathcal{J}\bar{H})(y) = 0 \tag{3.3}$$

Furthermore, it follows from (3.1) that

$$\begin{aligned}(\mathcal{J}\bar{H})(x) \cdot (\mathcal{J}H)(0) &= \big((\mathcal{J}H)(x) - (\mathcal{J}H)(0)\big) \cdot (\mathcal{J}H)(0) \\ &= -(\mathcal{J}H)(0) \cdot \big((\mathcal{J}H)(x) - (\mathcal{J}H)(0)\big) \\ &= -(\mathcal{J}H)(0) \cdot (\mathcal{J}\bar{H})(x)\end{aligned}$$

Consequently, the factors of the terms of the expansion of the right hand side of (3.2) anticommute. From (3.3) and $\big((\mathcal{J}H)(0)\big)^2 = 0$, we deduce that every term of the expansion of the right hand side of (3.2) equals zero. So $(\mathcal{J}H)(Ze_1) \cdot (\mathcal{J}H)(Ze_2) \cdot (\mathcal{J}H)(Ze_3) = 0$. □

The conclusions of (ii) and (iii) of theorem 3.3 can be reformulated as properties of a triangular matrix to which JH is conjugation similar, see [dB1, Th. 2.1] and [dB1, Cor. 2.2]. Using this reformulation more generally, one can deduce the following.

Proposition 3.4. *Let H be a polynomial map, such that $\mathcal{J}H$ is conjugation similar to a triangular matrix. If $\mathcal{J}H$ is nilpotent and $r = \operatorname{rk} \mathcal{J}H$, then*

$$(\mathcal{J}H)(Ze_1) \cdot (\mathcal{J}H)(Ze_2) \cdot \cdots \cdot (\mathcal{J}H)(Ze_r) \cdot (\mathcal{J}H)(Ze_{r+1}) = 0$$

where Z is as in the proof of (i) of theorem 3.3.

It follows that the conclusions of (ii) and (iii) of theorem 3.3 can be added to theorem 2.1 and theorem 2.3 respectively as well.

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