IRREDUCIBLE REPRESENTATIONS OF THE CHINESE MONOID

ŁUKASZ KUBAT AND JAN OKNIŃSKI

Abstract. All irreducible representations of the Chinese monoid C_n, of any rank n, over a nondenumerable algebraically closed field K, are constructed. It turns out that they have a remarkably simple form and they can be built inductively from irreducible representations of the monoid C_2. The proof shows also that every such representation is monomial. Since C_n embeds into the algebra $K[C_n]/J(K[C_n])$, where $J(K[C_n])$ denotes the Jacobson radical of the monoid algebra $K[C_n]$, a new representation of C_n as a subdirect product of the images of C_n in the endomorphism algebras of the constructed simple modules follows.

1. Introduction

For a positive integer n the monoid C_n defined by the finite presentation: $C_n = \langle a_1, \dots, a_n \rangle$ with the defining relations

$$a_j a_k a_i = a_k a_j a_i = a_k a_i a_j \qquad \text{for } i \le j \le k \tag{1.1}$$

is referred to as the Chinese monoid of rank n. It is known that each element x of C_n has a unique presentation of the form $x = b_1 b_2 b_3 \cdots b_n$, where

$$\begin{aligned} b_1 &= a_1^{k_{1,1}}, \\ b_2 &= (a_2 a_1)^{k_{2,1}} a_2^{k_{2,2}}, \\ b_3 &= (a_3 a_1)^{k_{3,1}} (a_3 a_2)^{k_{3,2}} a_3^{k_{3,3}}, \\ &\vdots \\ b_n &= (a_n a_1)^{k_{n,1}} (a_n a_2)^{k_{n,2}} \cdots (a_n a_{n-1})^{k_{n,n-1}} a_n^{k_{n,n}}, \end{aligned} \tag{1.2}$$

with all exponents $k_{i,j}$ nonnegative [1]. We call it the canonical form of the element $x \in C_n$. The monoid algebra $K[C_n]$ over a field K, which can be viewed as the unital algebra defined by the algebra presentation determined by the relations (1.1), is called the Chinese algebra of rank n. The Chinese monoid is related to the so called plactic monoid, introduced and studied in [15]. Both constructions are strongly related to Young tableaux, and therefore to several aspects of representation theory and algebraic combinatorics. The latter construction became a classical and powerful tool in representation theory of the full linear group and in the theory of symmetric functions, via the Littlewood-Richardson rule (cf. [8], [13]). It also plays an important role in quantum groups (in the context of crystal bases) and in the area of classical Lie algebras, [6], [14], [16].

The Chinese monoid appeared in the classification of monoids with the growth function coinciding with that of the plactic monoid [7]. One of the motivations for a study of the Chinese monoid is based on an expectation that it might play a similar role as the plactic monoid in several aspects of representation theory, quantum algebras, and in algebraic combinatorics. Combinatorial properties of C_n were studied in detail in [1]. In case $n = 2$, the Chinese and the plactic monoids coincide. The structure of the algebra $K[C_2]$ is described in [3]. In particular, this algebra is prime and semiprimitive, it is not noetherian and it does not satisfy any polynomial identity. For $n = 3$ some information on $K[C_n]$ was obtained in [9]. In particular the Jacobson radical of $K[C_3]$ is nonzero, but it is nilpotent, and the prime spectrum of $K[C_3]$ is pretty well understood. A surprisingly simple form of the minimal prime ideals of the algebra $K[C_n]$, for every n, was established in [10], [4]. Namely, every minimal prime ideal P is of the form $P = \mathrm{Span}_K\{x - y : x, y \in C_n \text{ and } x - y \in P\}$. Hence, in particular $K[C_n]/P \cong K[C_n/\rho_P]$, for the congruence ρ_P on C_n defined by $\rho_P = \{(x, y) \in C_n \times C_n : x - y \in P\}$. We write $P = I_{\rho_P}$ in this case. It was shown that every P is generated by a finite set of elements of the form $x - y$, where x, y are words in the generators $a_1, \dots, a_n$, both of length 2 or both of length 3. Consequently, $K[C_n]/P$ inherits

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the natural $\mathbb{Z}$-gradation and this algebra is again defined by a homogeneous semigroup presentation. In particular, the number of minimal primes P is finite. Moreover, every C_n/ρ_P embeds into the product $B^k \times \mathbb{Z}^l$, for some nonnegative integers k, l, where $\mathbb{Z}$ is the infinite cyclic group and $B = \langle p, q \colon qp = 1\rangle$ is the bicyclic monoid. The latter plays an important role in ring theory and in semigroup theory, [5], [12]. It was also shown that C_n embeds into the product $\prod_P K[C_n]/P$, where P runs over the set of all minimal primes in $K[C_n]$. Hence C_n embeds into some $B^r \times \mathbb{Z}^s$. However, the algebra $K[C_n]$ is not semiprime if $n \geq 3$. Moreover, the description of minimal primes P of $K[C_n]$ allows to prove that every $K[C_n]/P$ is semiprimitive and the Jacobson radical of $K[C_n]$ is nilpotent, and nonzero if $n \geq 3$.

The aim of this paper is to describe all irreducible representations of C_n over a nondenumerable algebraically closed field K. First, one shows that they are infinite dimensional unless the dimension is 1. Then, in our main result, Theorem 3.13, all irreducible representations are constructed. It turns out that they have a remarkably transparent form. In particular, they can be built inductively from irreducible representations of the monoid C_2, that are easy to determine. The proof shows that every such representation is monomial. This is in contrast with the case of representations of the plactic monoid, as recently shown in [2]. Since C_n embeds into $K[C_n]/J(K[C_n])$, where $J(K[C_n])$ denotes the Jacobson radical of the algebra $K[C_n]$ (see [10]), a new representation of C_n as a subdirect product of the images of C_n in the endomorphism algebras of the constructed simple modules also follows.

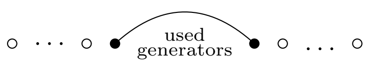
2. Background on minimal prime ideals

Throughout the paper, K will stand for a nondenumerable algebraically closed field, if not stated otherwise. In this section, we recall from [10] the necessary background on minimal prime ideals of the Chinese algebra $K[C_n]$ of rank $n \geq 3$.

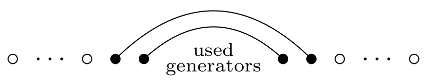
A finite tree D is associated to C_n, whose vertices are diagrams of certain special type. Each diagram d in D determines a congruence $\rho(d)$ on the monoid C_n in such a way that the ideals $I_{\rho(d)}$ corresponding to the leaves d of D are exactly all the minimal prime ideals of $K[C_n]$.

Each diagram d in D is a graph with n vertices, labeled $1, \dots, n$ and corresponding to the generators $a_1, \dots, a_n$ of C_n. For every d in D that is not the root of D there exist $u \leq v$, with $u, v \in \{1, \dots, n\}$, such that the vertices $u, \dots, v$ are marked (colored black) and the corresponding generators $a_u, \dots, a_v$ are called the used generators in d. Some pairs k, l $(k < l)$ of the used generators can be connected with an edge and then we say that such a pair is an arc $\widehat{a_l a_k}$ in d. A given generator can be used in at most one arc. The used (marked) generators not appearing in any arc are called dots. The root of D is the diagram in which none of the generators is used. The first level of the tree D (diagrams connected by an edge of D to the root) consists of $2n-3$ diagrams. There are $n-2$ diagrams with only one of the generators $a_2, \dots, a_{n-1}$ used, and $n-1$ diagrams with exactly two consecutive generators a_s, a_{s+1} used in an arc. Then, if d is in level t of D and it is not a leaf (by a leaf of the tree D we understand a diagram containing an arc of the form $\widehat{a_k a_1}$ or $\widehat{a_n a_k}$, for some k) then it is connected by an edge of D to certain diagrams in level $t+1$ which are obtained from d by adding an arc or adding a dot, according to the following rules:

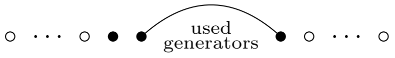
(1) If in the last step of construction of the diagram in level t an arc was added, that is, if the diagram in level t has the form

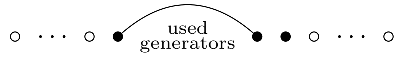


then we can either get, as a diagram in level $t+1$, the diagram

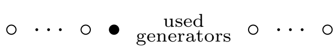


or one of the following two diagrams

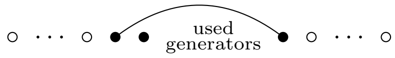


or


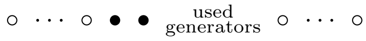
(2) Whereas, if the diagram in level t has the form



then we can either get, as a diagram in level $t+1$, the diagram

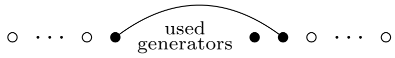


or the following diagram



(3) Similarly, if the diagram in level t has the form

then we can either get, as a diagram in level $t+1$, the diagram

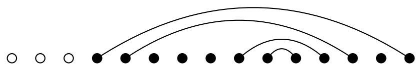


or the following diagram

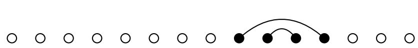
(4) Finally, after a dot in the first level of D only an arc can be added, so after a diagram

the following diagram can only occur

Example 2.1. The diagram in D (for $n = 15$) of the form

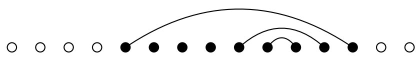


arises, in accordance with the rules mentioned above, in the following steps. First, choose the arc $\widehat{a_{11}a_{10}}$ and then the arc $\widehat{a_{12}a_9}$. This leads to the diagram



Next, choose consecutively three dots a_8, a_7, a_6. This yields the following diagram

Then, choosing the arc $\widehat{a_{13}a_5}$ we get



Finally, choosing the dot a_{14} and then the arc $\widehat{a_{15}a_4}$ leads to the considered diagram (which is a leaf). The full description of D in case $n = 3$ and $n = 4$ is given in Section 4.

We shall also consider several homomorphic images of C_n of type C_n/ρ, where ρ is a congruence on C_n generated by certain pairs of the form (a_ia_j, a_ja_i) and of the form $(a_ia_ja_k, a_{\sigma(i)}a_{\sigma(j)}a_{\sigma(k)})$ for some permutations σ of $\{i, j, k\}$. Then, for the sake of simplicity, the image of a_i in C_n/ρ will also be denoted by a_i. Clearly, in monoids of this type we have a degree function with respect to every generator and we write $\deg_{a_i}(x)$ for the degree of $x \in C_n/\rho$ in a_i. Moreover, by $\deg(x)$ we mean the total degree of x, that is, $\deg(x) = \sum_{i=1}^{n} \deg_{a_i}(x)$.

If $u, v \in \{1, \ldots, n\}$ are such that $u \leq v + 1$ then we define the monoid

$$C_n^{u,v} = \langle a_1, \ldots, a_{u-1}, a_{v+1}, \ldots, a_n \rangle \subseteq C_n,$$

which is the Chinese monoid of rank $u - 1 + n - v$, and its homomorphic image

$$\overline{C}_n^{u,v} = \langle a_1, \ldots, a_{u-1}, a_{v+1}, \ldots, a_n \rangle / \left(\begin{smallmatrix} a_1, \ldots, a_{u-1} \text{ commute} \\ a_{v+1}, \ldots, a_n \text{ commute} \end{smallmatrix} \right),$$

that is, $\overline{C}_n^{u,v} = C_n/\eta$, where η is the congruence on C_n generated by all pairs (a_ia_j, a_ja_i) for $i, j < u$ and all pairs (a_ka_l, a_la_k) for $k, l > v$.

By $\mathbb{Z}$ we mean the (multiplicative) infinite cyclic group, with a generator g.

If d_1 is the diagram with only one used generator a_s, where $1 < s < n$, then we associate to it the homomorphism $\phi_0 \colon C_n \longrightarrow \overline{C}_n^{s,s} \times \mathbb{Z}$ defined by

$$\phi_0(a_i) = \begin{cases} (1, g) & \text{if } i = s, \\ (a_i, 1) & \text{if } i \neq s. \end{cases}$$

The congruence $\mathrm{Ker}(\phi_0)$ on C_n is then generated by the pairs:

$$\begin{aligned} &(a_i a_j, a_j a_i) && \text{for } i, j \le s, \\ &(a_k a_l, a_l a_k) && \text{for } k, l \ge s. \end{aligned}$$

If d_1 is the diagram with only two used generators that form an arc $\widehat{a_{s+1}a_s}$, where $1 \le s < n$, then we associate to it the homomorphism $\psi_0 \colon C_n \longrightarrow \overline{C}_n^{s,s+1} \times B \times \mathbb{Z}$, where $B = \langle p, q : qp = 1 \rangle$ is the bicyclic monoid, defined by

$$\psi_0(a_i) = \begin{cases} (a_i, p, 1) & \text{if } i < s, \\ (1, p, g) & \text{if } i = s, \\ (1, q, 1) & \text{if } i = s+1, \\ (a_i, q, 1) & \text{if } i > s+1. \end{cases}$$

The congruence $\mathrm{Ker}(\psi_0)$ on C_n is then generated by the pairs:

$$\begin{aligned} &(a_i a_j, a_j a_i), && (a_i a_{s+1} a_j, a_j a_{s+1} a_i) && \text{for } i, j \le s, \\ &(a_k a_l, a_l a_k), && (a_k a_s a_l, a_l a_s a_k) && \text{for } k, l > s. \end{aligned}$$

Now, we define

$$\kappa(d_1) \colon C_n \longrightarrow \overline{C}_n^{u,v} \times S_1$$

as $\kappa(d_1) = \phi_0$ (and then $(u, v) = (s, s)$ and $S_1 = \mathbb{Z}$) in case d_1 is the diagram with only one used generator a_s, or $\kappa(d_1) = \psi_0$ (and then $(u, v) = (s, s+1)$ and $S_1 = B \times \mathbb{Z}$) in case d_1 is the diagram with only two used generators that form an arc $\widehat{a_{s+1}a_s}$. Moreover, let $\rho(d_1) = \mathrm{Ker}\,\kappa(d_1)$.

So, the homomorphisms and congruences described above are associated to the $2n-3$ diagrams from the first level of D. The procedure described below allows us to associate (inductively) a homomorphism $\kappa(d) \colon C_n \longrightarrow \overline{C}_n^{u,v} \times (B \times \mathbb{Z})^k \times \mathbb{Z}^l$, where u, v and k, l depend on d, and the congruence $\rho(d) = \mathrm{Ker}\,\kappa(d)$ to every diagram d at the level > 1 of D. However, in contrast to the congruences from the first level, the generators of $\rho(d)$ are much harder to determine, see [4].

Assume that a diagram d_t in level $t \ge 1$ of the tree D has been constructed and it is not a leaf. Assume also that the homomorphism $\kappa(d_t) \colon C_n \longrightarrow \overline{C}_n^{u,v} \times S_t$, where $S_t = (B \times \mathbb{Z})^k \times \mathbb{Z}^l$, together with the congruence $\rho(d_t) = \mathrm{Ker}\,\kappa(d_t)$ have been defined. Here, the indices u, v correspond to the used generators $a_u, \dots, a_v$ in the diagram d_t, whereas the nonnegative integers k, l correspond to the number of arcs and dots, respectively, used in the construction of the diagram d_t. Moreover, let d_{t+1} be a diagram at the level $t+1$ of D that is connected to d_t by an edge in D.

If d_{t+1} is obtained by adding a dot to d_t (then this is either a_{u-1} or a_{v+1}), we have a homomorphism

$$\phi_t \colon \overline{C}_n^{u,v} \times S_t \longrightarrow \overline{C}_n^{u',v'} \times \mathbb{Z} \times S_t$$

given by

$$\phi_t(a_i, x) = \begin{cases} (1, g, x) & \text{if } i = s, \\ (a_i, 1, x) & \text{if } i < u \text{ or } i > v \text{ but } i \ne s, \end{cases}$$

where $s = u-1$ (and then $(u', v') = (u-1, v)$) or $s = v+1$ (and then $(u', v') = (u, v+1)$), depending on which of the two possible dots was added.

If d_{t+1} is obtained by adding an arc to d_t (then this arc is $\widehat{a_{v+1}a_{u-1}}$), we have a homomorphism

$$\psi_t \colon \overline{C}_n^{u,v} \times S_t \longrightarrow \overline{C}_n^{u-1,v+1} \times B \times \mathbb{Z} \times S_t$$

given by

$$\psi_t(a_i, x) = \begin{cases} (a_i, p, 1, x) & \text{if } i < u-1, \\ (1, p, g, x) & \text{if } i = u-1, \\ (1, q, 1, x) & \text{if } i = v+1, \\ (a_i, q, 1, x) & \text{if } i > v+1, \end{cases}$$

and then we put $(u', v') = (u-1, v+1)$. Furthermore, we put $S_{t+1} = \mathbb{Z} \times S_t$ in case d_{t+1} is obtained by adding a dot to d_t, and $S_{t+1} = B \times \mathbb{Z} \times S_t$ in case d_{t+1} is obtained by adding an arc to d_t.

Now, we define

$$\kappa(d_{t+1}) \colon C_n \longrightarrow \overline{C}_n^{u',v'} \times S_{t+1}$$

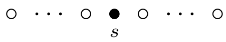
as a composition

$$\kappa(d_{t+1}) = \begin{cases} \phi_t \circ \kappa(d_t) & \text{if } d_{t+1} \text{ is obtained from } d_t \text{ by adding a dot,} \\ \psi_t \circ \kappa(d_t) & \text{if } d_{t+1} \text{ is obtained from } d_t \text{ by adding an arc.} \end{cases}$$

Moreover, let $\rho(d_{t+1}) = \operatorname{Ker} \kappa(d_{t+1})$. Then, of course, $\rho(d_t) \subseteq \rho(d_{t+1})$.

Summarizing, if $d_0, d_1, \ldots, d_m = d$ in a branch in D then $\rho(d_0) \subseteq \rho(d_1) \subseteq \cdots \subseteq \rho(d_m) = \rho(d)$ (by $\rho(d_0)$, for the root d_0 of D, we mean the trivial congruence on C_n). However, if $m > 1$, then as was mentioned at the beginning of this section, generators of the congruence $\rho(d) = \operatorname{Ker} \kappa(d)$ are hard to determine explicitly. Though, if the diagram d is of some special shape (e.g. one of the shapes listed below), then looking at the embedding $C_n/\rho(d) \longrightarrow \overline{C}_n^{u,v} \times (B \times \mathbb{Z})^k \times \mathbb{Z}^l$ (for some u, v and some k, l), induced by the homomorphism $\kappa(d)$, it is quite easy to derive some relations that must hold in $C_n/\rho(d)$ and which will be needed later.

Namely, if d is a diagram of the form

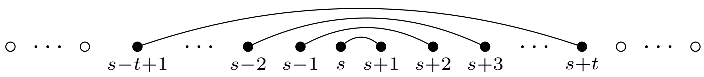


consisting of a single dot a_s, then the following equalities hold in $C_n/\rho(d)$:

$$a_i a_j = a_j a_i \qquad \text{for all } i, j \leq s, \tag{2.1}$$

$$a_k a_l = a_l a_k \qquad \text{for all } k, l \geq s. \tag{2.2}$$

If d is a diagram of the form



consisting of $t > 0$ consecutive arcs $\widehat{a_{s+1}a_s}, \ldots, \widehat{a_{s+t}a_{s-t+1}}$, then the following equalities hold in $C_n/\rho(d)$:

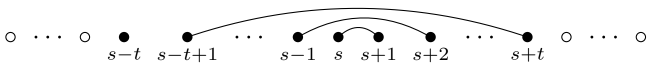
$$a_i a_j = a_j a_i \qquad \text{for all } i, j \leq s, \tag{2.3}$$

$$a_k a_l = a_l a_k \qquad \text{for all } k, l > s, \tag{2.4}$$

$$a_i a_{s+r} a_j = a_j a_{s+r} a_i \qquad \text{for all } i, j \leq s - r + 1, \text{ where } r = 1, \ldots, t, \tag{2.5}$$

$$a_k a_{s-r+1} a_l = a_l a_{s-r+1} a_k \qquad \text{for all } k, l \geq s + r, \text{ where } r = 1, \ldots, t. \tag{2.6}$$

If d is a diagram of the form



consisting of $t > 0$ consecutive arcs $\widehat{a_{s+1}a_s}, \ldots, \widehat{a_{s+t}a_{s-t+1}}$ and a single dot a_{s-t}, then the following equalities hold in $C_n/\rho(d)$:

$$a_i a_j = a_j a_i \qquad \text{for all } i, j \leq s, \tag{2.7}$$

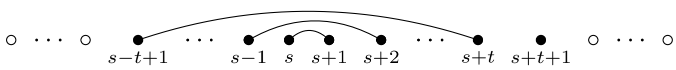
$$a_k a_l = a_l a_k \qquad \text{for all } k, l > s, \tag{2.8}$$

$$a_k a_{s-t} a_l = a_l a_{s-t} a_k \qquad \text{for all } k, l \geq s + t, \tag{2.9}$$

$$a_i a_{s+r} a_j = a_j a_{s+r} a_i \qquad \text{for all } i, j \leq s - r + 1, \text{ where } r = 1, \ldots, t, \tag{2.10}$$

$$a_k a_{s-r+1} a_l = a_l a_{s-r+1} a_k \qquad \text{for all } k, l \geq s + r, \text{ where } r = 1, \ldots, t. \tag{2.11}$$

Similarly, if d is a diagram of the form



consisting of $t > 0$ consecutive arcs $\widehat{a_{s+1}a_s}, \ldots, \widehat{a_{s+t}a_{s-t+1}}$ and a single dot a_{s+t+1}, then equalities dual to (2.7)–(2.11) also can be derived. However, these equalities will not be used explicitly in the paper.

3. Irreducible representations

Our first result shows that infinite dimensional simple $K[C_n]$-modules will be crucial.

Proposition 3.1. *Let $\phi\colon C_n \longrightarrow \operatorname{End}_K(V)$ be an irreducible representation of C_n over a field K.*

1. *If $\phi(a_n a_1) = 0$ then either $\phi(a_n) = 0$ or $\phi(a_1) = 0$.*
2. *If $\dim_K V < \infty$, then $\phi(C_n)$ is commutative, hence $\dim_K V = 1$ if K is algebraically closed.*

Proof. First, we claim that $a_nC_na_1 \subseteq a_na_1C_n$. Let $x \in C_n$. From the canonical form of elements in C_n it follows that $xa_1 = x_j \cdots x_n$ for some $j \in \{1,\dots,n\}$, where $x_j,\dots,x_n \in C_n$ and $x_j \in a_1C_n$ if $j = 1$, and $x_j \in a_ja_1C_n$ if $j > 1$. Since $a_na_ja_1 = a_na_1a_j$, the claim follows.

Suppose that $\phi(a_na_1) = 0$. Then $\phi(a_n)\phi(C_n)\phi(a_1) = \phi(a_n)\phi(a_1)\phi(C_n) = 0$. Since ϕ is irreducible, it follows that either $\phi(a_n) = 0$ or $\phi(a_1) = 0$.

In order to prove the second assertion, consider $z = a_na_1 \in C_n$. If $\phi(z) = 0$ then, by the first part of the proof, ϕ comes from an irreducible representation of C_{n-1}. Hence, the result follows by induction in this case. Otherwise, $\phi(z) \neq 0$ is an invertible element in the simple algebra $R = \mathrm{Span}_K\, \phi(C_n)$, because it is central. Thus, in particular, $\phi(a_n)$ is invertible in R, so the relations (1.1) defining C_n easily imply that $\phi(C_n)$ is commutative and the assertion follows. $\square$

Our next aim is to construct a family of simple left $K[C_n]$-modules in case n is even. Later we shall see that these modules are of special interest, because they are the corner stone of an inductive classification of all simple left modules over the algebra $K[C_n]$.

Proposition 3.2. *Let V be a K-linear space with basis $\{e_{i_1,\dots,i_s} : i_1,\dots,i_s \geq 0\}$ for some $s \geq 1$. Moreover, let $0 \neq \lambda_1,\dots,\lambda_s \in K$ and $n = 2s$. Then the action of $a_1,\dots,a_n \in C_n$ on V defined by*

$$a_je_{i_1,\dots,i_s} = \begin{cases} \lambda_je_{i_1,\dots,i_{j-1},i_j+1,\dots,i_s+1} & \text{if } j \leq s,\\ e_{i_1,\dots,i_{n-j},i_{n-j+1}-1,\dots,i_s-1} & \text{if } j > s \text{ and } i_k > 0 \text{ for all } k > n-j,\\ 0 & \text{if } j > s \text{ and } i_k = 0 \text{ for some } k > n-j \end{cases}$$

makes $V = V(\lambda_1,\dots,\lambda_s)$ a simple left $K[C_n]$-module. Moreover, if $0 \neq \mu_1,\dots,\mu_s \in K$ then we have $V(\lambda_1,\dots,\lambda_s) \cong V(\mu_1,\dots,\mu_s)$ as left $K[C_n]$-modules if and only if $\lambda_i = \mu_i$ for all $i = 1,\dots,s$.

Proof. First, we have to check that the defined action of $a_1,\dots,a_n \in C_n$ on V respects the Chinese relations. So, we have to prove that

$$(a_la_ka_j - a_la_ja_k)V = a_l(a_ka_j - a_ja_k)V = 0$$

and

$$(a_la_ka_j - a_ka_la_j)V = (a_la_k - a_ka_l)a_jV = 0$$

for all $j \leq k \leq l$. Since we have $(a_ka_j - a_ja_k)V = 0$ for all $j,k \leq s$ and $(a_la_k - a_ka_l)V = 0$ for all $k,l > s$, it is enough to show that:

1. $(a_la_ka_j - a_la_ja_k)V = 0$ for all $j \leq s < k \leq l$ such that $j + l \leq n$,
2. $(a_la_ka_j - a_la_ja_k)V = 0$ for all $j \leq s < k \leq l$ such that $j + k \leq n < j + l$,
3. $(a_la_ka_j - a_la_ja_k)V = 0$ for all $j \leq s < k \leq l$ such that $n < j + k$,
4. $(a_la_ka_j - a_ka_la_j)V = 0$ for all $j \leq k \leq s < l$ such that $k + l \leq n$,
5. $(a_la_ka_j - a_ka_la_j)V = 0$ for all $j \leq k \leq s < l$ such that $j + l \leq n < k + l$,
6. $(a_la_ka_j - a_ka_la_j)V = 0$ for all $j \leq k \leq s < l$ such that $j + l > n$.

It is easy to verify that we have, respectively:

(1) If $k < l$ then

$$a_la_ka_je_{i_1,\dots,i_s} = a_la_ja_ke_{i_1,\dots,i_s} = \begin{cases} \lambda_je_{i_1,\dots,i_{j-1},i_j+1,\dots,i_{n-l}+1,i_{n-l+1},\dots,i_{n-k},i_{n-k+1}-1,\dots,i_s-1} \\ \qquad \text{if } i_p > 0 \text{ for all } p > n-k,\\ 0 \qquad \text{otherwise.} \end{cases}$$

Whereas, if $k = l$ then

$$a_la_ka_je_{i_1,\dots,i_s} = a_la_ja_ke_{i_1,\dots,i_s} = \begin{cases} \lambda_je_{i_1,\dots,i_{j-1},i_j+1,\dots,i_{n-k}+1,i_{n-k+1}-1,\dots,i_s-1} \\ \qquad \text{if } i_p > 0 \text{ for all } p > n-k,\\ 0 \qquad \text{otherwise.} \end{cases}$$

(2) If $j + l > n + 1$ then

$$a_la_ka_je_{i_1,\dots,i_s} = a_la_ja_ke_{i_1,\dots,i_s} = \begin{cases} \lambda_je_{i_1,\dots,i_{n-l},i_{n-l+1}-1,\dots,i_{j-1}-1,i_j,\dots,i_{n-k},i_{n-k+1}-1,\dots,i_s-1} \\ \qquad \text{if } i_p > 0 \text{ for all } n-l < p < j,\\ \qquad \text{and } i_q > 0 \text{ for all } q > n-k,\\ 0 \qquad \text{otherwise.} \end{cases}$$

Whereas, if $j+l=n+1$ then

$$a_la_ka_je_{i_1,\dots,i_s}=a_la_ja_ke_{i_1,\dots,i_s}=\begin{cases}\lambda_je_{i_1,\dots,i_{n-k},i_{n-k+1}-1,\dots,i_s-1} & \text{if } i_p>0 \text{ for all } p>n-k,\\ 0 & \text{otherwise.}\end{cases}$$

(3) If $k<l$ and $j+k>n+1$ then

$$a_la_ka_je_{i_1,\dots,i_s}=a_la_ja_ke_{i_1,\dots,i_s}=\begin{cases}\lambda_je_{i_1,\dots,i_{n-l},i_{n-l+1}-1,\dots,i_{n-k}-1,i_{n-k+1}-2,\dots,i_{j-1}-2,i_j-1,\dots,i_s-1} & \\ \qquad \text{if } i_p>0 \text{ for all } n-l<p\le n-k, & \\ \qquad i_q>1 \text{ for all } n-k<q<j, & \\ \qquad \text{and } i_r>0 \text{ for all } r\ge j, & \\ 0 & \text{otherwise.}\end{cases}$$

If $k=l$ and $j+k>n+1$ then

$$a_la_ka_je_{i_1,\dots,i_s}=a_la_ja_ke_{i_1,\dots,i_s}=\begin{cases}\lambda_je_{i_1,\dots,i_{n-k},i_{n-k+1}-2,\dots,i_{j-1}-2,i_j-1,\dots,i_s-1} & \\ \qquad \text{if } i_p>1 \text{ for all } n-k<p<j, & \\ \qquad \text{and } i_q>0 \text{ for all } q\ge j, & \\ 0 & \text{otherwise.}\end{cases}$$

Whereas, if $j+k=n+1$ then

$$a_la_ka_je_{i_1,\dots,i_s}=a_la_ja_ke_{i_1,\dots,i_s}=\begin{cases}\lambda_je_{i_1,\dots,i_{n-l},i_{n-l+1}-1,\dots,i_s-1} & \text{if } i_p>0 \text{ for all } p>n-l,\\ 0 & \text{otherwise.}\end{cases}$$

(4) If $j<k$ then

$$a_la_ka_je_{i_1,\dots,i_s}=a_ka_la_je_{i_1,\dots,i_s}=\lambda_j\lambda_ke_{i_1,\dots,i_{j-1},i_j+1,\dots,i_{k-1}+1,i_k+2,\dots,i_{n-l}+2,i_{n-l+1}+1,\dots,i_s+1}.$$

Whereas, if $j=k$ then

$$a_la_ka_je_{i_1,\dots,i_s}=a_ka_la_je_{i_1,\dots,i_s}=\lambda_j^2e_{i_1,\dots,i_{j-1},i_j+2,\dots,i_{n-l}+2,i_{n-l+1}+1,\dots,i_s+1}.$$

(5) If $k+l>n+1$ then

$$a_la_ka_je_{i_1,\dots,i_s}=a_ka_la_je_{i_1,\dots,i_s}=\lambda_j\lambda_ke_{i_1,\dots,i_{j-1},i_j+1,\dots,i_{n-l}+1,i_{n-l+1},\dots,i_{k-1},i_k+1,\dots,i_s+1}.$$

Whereas, if $k+l=n+1$ then

$$a_la_ka_je_{i_1,\dots,i_s}=a_ka_la_je_{i_1,\dots,i_s}=\lambda_j\lambda_ke_{i_1,\dots,i_{j-1},i_j+1,\dots,i_s+1}.$$

(6) If $j<k$ and $j+l>n+1$ then

$$a_la_ka_je_{i_1,\dots,i_s}=a_ka_la_je_{i_1,\dots,i_s}=\begin{cases}\lambda_j\lambda_ke_{i_1,\dots,i_{n-l},i_{n-l+1}-1,\dots,i_{j-1}-1,i_j,\dots,i_{k-1},i_k+1,\dots,i_s+1} & \\ \qquad \text{if } i_p>0 \text{ for all } n-l<p<j, & \\ 0 & \text{otherwise.}\end{cases}$$

If $j=k$ and $j+l>n+1$ then

$$a_la_ka_je_{i_1,\dots,i_s}=a_ka_la_je_{i_1,\dots,i_s}=\begin{cases}\lambda_j^2e_{i_1,\dots,i_{n-l},i_{n-l+1}-1,\dots,i_{j-1}-1,i_j+1,\dots,i_s+1} & \\ \qquad \text{if } i_p>0 \text{ for all } n-l<p<j, & \\ 0 & \text{otherwise.}\end{cases}$$

Whereas, if $j+l=n+1$ then

$$a_la_ka_je_{i_1,\dots,i_s}=a_ka_la_je_{i_1,\dots,i_s}=\lambda_j\lambda_ke_{i_1,\dots,i_{k-1},i_k+1,\dots,i_s+1}.$$

Now, let us prove that the $K[C_n]$-module V is simple. First, it can be easily verified that for each $j<s$ we have $a_{n-j}a_je_{i_1,\dots,i_s}=\lambda_je_{i_1,\dots,i_{j-1},i_j+1,i_{j+1},\dots,i_s}$ (that is, the action of $a_{n-j}a_j$ on $e_{i_1,\dots,i_s}$ increases the index i_j by one and leaves other indices unchanged) and $a_se_{i_1,\dots,i_s}=\lambda_se_{i_1,\dots,i_{s-1},i_s+1}$ (that is, the action of a_s on $e_{i_1,\dots,i_s}$ increases the index i_s by one and leaves other indices unchanged). Therefore,

$$e_{i_1,\dots,i_s}=(\lambda_1^{-1}a_{n-1}a_1)^{i_1}\cdots(\lambda_{s-1}^{-1}a_{s+1}a_{s-1})^{i_{s-1}}(\lambda_s^{-1}a_s)^{i_s}e_{0,\dots,0}$$

for all $i_1, \dots, i_s \geq 0$. Hence, to prove simplicity of V, it suffices to check that $e_{0,\dots,0} \in K[C_n]v$ for each $0 \neq v \in V$. So, let $0 \neq v = \sum_{i_1,\dots,i_s=0}^{r} \lambda_{i_1,\dots,i_s} e_{i_1,\dots,i_s}$ for some $\lambda_{i_1,\dots,i_s} \in K$ be fixed. Then define

$$\begin{aligned}
m_1 &= \max\{i_1 : \lambda_{i_1,\dots,i_s} \neq 0 \text{ for some } i_2, \dots, i_s\}, \\
m_2 &= \max\{i_2 : \lambda_{m_1,i_2,\dots,i_s} \neq 0 \text{ for some } i_3, \dots, i_s\}, \\
m_3 &= \max\{i_3 : \lambda_{m_1,m_2,i_3,\dots,i_s} \neq 0 \text{ for some } i_4, \dots, i_s\}, \\
&\vdots \\
m_s &= \max\{i_s : \lambda_{m_1,\dots,m_{s-1},i_s} \neq 0\}.
\end{aligned}$$

Because, for each $j < s$, we have

$$a_{n-j+1}a_{j+1}e_{i_1,\dots,i_s} = \begin{cases} \lambda_{j+1}e_{i_1,\dots,i_{j-1},i_j-1,i_{j+1},\dots,i_s} & \text{if } i_j > 0, \\ 0 & \text{if } i_j = 0 \end{cases}$$

(that is, the action of $a_{n-j+1}a_{j+1}$ on $e_{i_1,\dots,i_s}$ decreases the index i_j by one, if possible, and leaves other indices unchanged) and because

$$a_{s+1}e_{i_1,\dots,i_s} = \begin{cases} e_{i_1,\dots,i_{s-1},i_s-1} & \text{if } i_s > 0, \\ 0 & \text{if } i_s = 0 \end{cases}$$

(that is, the action of a_{s+1} on $e_{i_1,\dots,i_s}$ decreases the index i_s by one, if possible, and leaves other indices unchanged), we conclude that

$$e_{0,\dots,0} = (\lambda_{m_1,\dots,m_s}\lambda_2^{m_1} \cdots \lambda_s^{m_{s-1}})^{-1} a_{s+1}^{m_s}(a_{s+2}a_s)^{m_{s-1}} \cdots (a_n a_2)^{m_1} v \in K[C_n]v,$$

as claimed.

Finally, note that isomorphic modules have equal annihilators and $(a_{n-i+1}a_i - \lambda_i)V = 0$ for each $i = 1, \dots, s$. Hence, if also $(a_{n-i+1}a_i - \mu_i)V = 0$ for some $i \in \{1, \dots, s\}$, then $(\lambda_i - \mu_i)V = 0$ and, in consequence, $\lambda_i = \mu_i$. Thus the last part of the proposition also follows. □

It is worth to note that the modules constructed in Proposition 3.2 can be obtained by a successive application of the construction presented in Proposition 3.3, starting with the left $K[C_2]$-module Z with basis $\{e_i : i \geq 0\}$, and with the action of $a_1, a_2 \in C_2$ on Z defined by

$$a_1 e_i = \lambda_1 e_{i+1}, \qquad a_2 e_i = \begin{cases} e_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0 \end{cases}$$

for some $0 \neq \lambda_1 \in K$. (Notice that such a module Z is a straightforward generalization of the classical simple $K[B]$-module, considered for example in [12, p. 195 and Ex. 11.9.].) This is fully explained below.

Proposition 3.3. *Let U be a left $K[C_n^{s,s+1}]$-module with basis $\{e_{i_1,\dots,i_{s-1}} : i_1, \dots, i_{s-1} \geq 0\}$, where $n = 2s$ for some $s \geq 1$. Assume that $(a_i a_j - a_j a_i)U = (a_k a_l - a_l a_k)U = 0$ for all $i, j < s$ and $k, l > s+1$. Assume also that for each $0 \neq u \in U$ and $i_1, \dots, i_{s-1} \geq 0$ there exists $x \in K \cdot C_n^{s,s+1}$ satisfying $xu = e_{i_1,\dots,i_{s-1}}$. Moreover, let V be a K-linear space with basis $\{f_i : i \geq 0\}$. Then, for each $0 \neq \lambda_s \in K$, the action of $a_1, \dots, a_n \in C_n$ on the K-linear space $W = U \otimes_K V$ with basis $\{e_{i_1,\dots,i_s} = e_{i_1,\dots,i_{s-1}} \otimes f_{i_s} : i_1, \dots, i_s \geq 0\}$ defined by*

$$a_j e_{i_1,\dots,i_s} = \begin{cases} a_j e_{i_1,\dots,i_{s-1}} \otimes f_{i_s+1} & \text{if } j < s, \\ \lambda_s e_{i_1,\dots,i_{s-1}} \otimes f_{i_s+1} & \text{if } j = s, \\ e_{i_1,\dots,i_{s-1}} \otimes f_{i_s-1} & \text{if } j = s+1 \text{ and } i_s > 0, \\ a_j e_{i_1,\dots,i_{s-1}} \otimes f_{i_s-1} & \text{if } j > s+1 \text{ and } i_s > 0, \\ 0 & \text{otherwise} \end{cases}$$

makes W a left $K[C_n]$-module such that $(a_i a_j - a_j a_i)W = (a_k a_l - a_l a_k)W = 0$ for all $i, j \leq s$ and $k, l > s$. Moreover, for each $0 \neq w \in W$ and $i_1, \dots, i_s \geq 0$ there exists $x \in K \cdot C_n$ satisfying $xw = e_{i_1,\dots,i_s}$. In particular, W is a simple left $K[C_n]$-module.

Proof. First, it is convenient to define $a_s e_{i_1,\dots,i_{s-1}} = \lambda_s e_{i_1,\dots,i_{s-1}}$ and $a_{s+1} e_{i_1,\dots,i_{s-1}} = e_{i_1,\dots,i_{s-1}}$ for all $i_1, \dots, i_{s-1} \geq 0$. With this notation the action of $a_1, \dots, a_n \in C_n$ on the basis of W can be rewritten as

$$a_j e_{i_1,\dots,i_s} = \begin{cases} a_j e_{i_1,\dots,i_{s-1}} \otimes f_{i_s+1} & \text{if } j \leq s, \\ a_j e_{i_1,\dots,i_{s-1}} \otimes f_{i_s-1} & \text{if } j > s \text{ and } i_s > 0, \\ 0 & \text{if } j > s \text{ and } i_s = 0. \end{cases}$$

It is almost obvious that the defined action respects the Chinese relations not involving a_s and a_{s+1}. Moreover, if $i, j \leq s$ then

$$(a_i a_j - a_j a_i)e_{i_1,\dots,i_s} = (a_i a_j - a_j a_i)e_{i_1,\dots,i_{s-1}} \otimes f_{i_s+2} = 0,$$

hence $(a_i a_j - a_j a_i)W = 0$. Similarly, if $k, l > s$ then

$$(a_k a_l - a_l a_k)e_{i_1,\dots,i_s} = \begin{cases} (a_k a_l - a_l a_k)e_{i_1,\dots,i_{s-1}} \otimes f_{i_s-2} & \text{if } i_s > 1, \\ 0 & \text{if } i_s \leq 1 \end{cases} = 0,$$

hence $(a_k a_l - a_l a_k)W = 0$. Since $(a_{s+1}a_s - \lambda_s)W = 0$, to prove that W is indeed a $K[C_n]$-module it suffices to check that:

(1) $(a_j a_{s+1} a_i - a_{s+1} a_j a_i)W = 0$ for all $i \leq j \leq s$,
(2) $(a_j a_{s+1} a_i - a_j a_i a_{s+1})W = 0$ for all $i \leq s < j$,
(3) $(a_s a_j a_i - a_j a_s a_i)W = 0$ for all $i \leq s < j$,
(4) $(a_j a_i a_s - a_j a_s a_i)W = 0$ for all $s < i \leq j$,
(5) $(a_{s+1} a_{s+1} a_i - a_{s+1} a_i a_{s+1})W = 0$ for all $i \leq s$,
(6) $(a_s a_i a_s - a_i a_s a_s)W = 0$ for all $i > s$.

Let W_0 denote the subspace of W spanned by the set $\{e_{i_1,\dots,i_s} : i_1, \dots, i_{s-1} \geq 0 \text{ and } i_s = 0\}$, and W_+ denote the subspace of W spanned by the set $\{e_{i_1,\dots,i_s} : i_1, \dots, i_{s-1} \geq 0 \text{ and } i_s > 0\}$. Then we have, respectively:

(1) $(a_j a_{s+1} a_i - a_{s+1} a_j a_i)W = (a_j a_{s+1} - a_{s+1} a_j)a_i W = 0$, because $a_i W \subseteq W_+$ for $i \leq s$ and $(a_j a_{s+1} - a_{s+1} a_j)W_+ = 0$ for $j \leq s$.
(2) $(a_j a_{s+1} a_i - a_j a_i a_{s+1})W = a_j(a_{s+1} a_i - a_i a_{s+1})W = 0$, because $(a_{s+1} a_i - a_i a_{s+1})W \subseteq W_0$ for $i \leq s$ and $a_j W_0 = 0$ for $j > s$.
(3) $(a_s a_j a_i - a_j a_s a_i)W = (a_s a_j - a_j a_s)a_i W = 0$, because $a_i W \subseteq W_+$ for $i \leq s$ and $(a_s a_j - a_j a_s)W_+ = 0$ for $j > s$.
(4) $(a_j a_i a_s - a_j a_s a_i)W = a_j(a_i a_s - a_s a_i)W = 0$, because $(a_i a_s - a_s a_i)W \subseteq W_0$ for $i > s$ and $a_j W_0 = 0$ for $j > s$.
(5) $(a_{s+1} a_{s+1} a_i - a_{s+1} a_i a_{s+1})W = a_{s+1}(a_{s+1} a_i - a_i a_{s+1})W = 0$, because $(a_{s+1} a_i - a_i a_{s+1})W \subseteq W_0$ for $i \leq s$, and $a_{s+1} W_0 = 0$.
(6) $(a_s a_i a_s - a_i a_s a_s)W = (a_s a_i - a_i a_s)a_s W = 0$, because $a_s W \subseteq W_+$ and $(a_s a_i - a_i a_s)W_+ = 0$ for $i > s$.

Now, let us fix $0 \neq w \in W$ and $i_1, \dots, i_s \geq 0$. To show that $xw = e_{i_1,\dots,i_s}$ for some $x \in K \cdot C_n$ write $w = \sum_{j_1,\dots,j_s=0}^{r} \lambda_{j_1,\dots,j_s} e_{j_1,\dots,j_s}$, where $\lambda_{j_1,\dots,j_s} \in K$. Define

$$m = \max\{j_s : \lambda_{j_1,\dots,j_s} \neq 0 \text{ for some } j_1, \dots, j_{s-1}\}.$$

Then replacing the vector w by $a_{s+1}^m w$ we may assume that $\lambda_{j_1,\dots,j_s} = 0$ for all $j_1, \dots, j_{s-1} \geq 0$ and $j_s > 0$, that is

$$0 \neq w = \sum_{j_1,\dots,j_{s-1}=0}^{r} \lambda_{j_1,\dots,j_{s-1},0} e_{j_1,\dots,j_{s-1},0} = u \otimes f_0,$$

where $0 \neq u = \sum_{j_1,\dots,j_{s-1}=0}^{r} \lambda_{j_1,\dots,j_{s-1},0} e_{j_1,\dots,j_{s-1}} \in U$. By assumptions on U, there exists $x \in K \cdot C_n^{s,s+1}$ such that $xu = e_{i_1,\dots,i_{s-1}}$. Let $p = \sum_{j>s+1} \deg_{a_j}(x)$ and $q = \sum_{j<s} \deg_{a_j}(x)$. Then

$$\begin{aligned} \lambda_s^{-p-i_s} a_s^{i_s} a_{s+1}^q x a_s^p w &= (\lambda_s^{-1} a_s)^{i_s} a_{s+1}^q x (\lambda_s^{-1} a_s)^p (u \otimes f_0) \\ &= (\lambda_s^{-1} a_s)^{i_s} a_{s+1}^q x (u \otimes f_p) \\ &= (\lambda_s^{-1} a_s)^{i_s} a_{s+1}^q e_{i_1,\dots,i_{s-1},q} \\ &= (\lambda_s^{-1} a_s)^{i_s} e_{i_1,\dots,i_{s-1},0} \\ &= e_{i_1,\dots,i_s}. \end{aligned}$$

Hence the result follows. □

The next result is one of the essential tools used in this section. An easy proof, based on the Density Theorem, can be found in [3].

Proposition 3.4. *Let A be a left primitive algebra over an algebraically closed field F. If $\dim_F A < |F|$ then the algebra A is central (that is, $Z(A) = F$).*

If $w_1, w_2, \dots, w_k \in M$ for a monoid M then we will write $w_1^* w_2^* \cdots w_k^*$ for the set of all elements of the form $w_1^{i_1} w_2^{i_2} \cdots w_k^{i_k} \in M$ with $i_1, i_2, \dots, i_k \geq 0$.

Now, consider a simple left $K[C_n]$-module V with annihilator P. Since P is a prime ideal, it follows that P contains a minimal prime ideal of $K[C_n]$, which is of the form $I_{\rho(d)}$ for some leaf d in D (see Section 2 or [10]). So, it is reasonable to investigate the structure of left primitive ideals of $K[C_n]$ containing ideals coming from diagrams of a particular shape. Our first result in this direction reads as follows.

Proposition 3.5. *Assume that P is a prime ideal of $K[C_n]$ containing the ideal I_ρ, where ρ is the congruence on C_n determined by the diagram*

consisting of $t > 0$ consecutive arcs $\widehat{a_{s+1}a_s}, \dots, \widehat{a_{s+t}a_{s-t+1}}$ (as shown in the picture). Assume additionally that $a_{s+t}a_{s-t+1} \in P$. Then $a_{s+t} \in P$ or $a_{s-t+1} \in P$.

Proof. Let T be the image of C_n in $K[C_n]/P$. Our first aim is to show that $a_{s-t+1}a_{s+t} = 0$ in T. To prove this let us introduce some notation. For any $1 \leq i \leq j \leq n$ let $W_{i,j}$ denote the subset of C_n consisting of all elements of the form $b_i \cdots b_j$ written in the notation of the canonical form (1.2). Moreover, let us adopt the convention that $W_{i,j} = \{1\}$ in case $i > j$. In the following we shall use the same notation for the elements of T.

First, using $a_{s+t}a_{s-t+1} = 0$ in T, note that:

- If $j \leq s-t+1$ then $(a_{s-t+1}a_{s+t})a_j = a_{s+t}a_{s-t+1}a_j$ in C_n. Hence $(a_{s-t+1}a_{s+t})a_j = 0$ in T for all $j \leq s-t+1$.
- If $j < k$ satisfy $j \leq s-t+1$ and $k \leq s+t$ then $(a_{s-t+1}a_{s+t})(a_k a_j) = a_{s-t+1}a_{s+t}a_j a_k = a_{s+t}a_{s-t+1}a_j a_k$ in C_n. Hence $(a_{s-t+1}a_{s+t})(a_k a_j) = 0$ in T for all $j < k$ such that $j \leq s-t+1$ and $k \leq s+t$.

This implies that $a_{s-t+1}a_{s+t}W_{1,s-t+1} = 0$ in T and $a_{s-t+1}a_{s+t}W_{s-t+2,s+t} = a_{s-t+1}a_{s+t}U \cup \{0\}$ in T, where $U = U_1 \cdots U_{2t-1}$ and

$$\begin{aligned}
U_1 &= (a_{s-t+2})^*,\\
U_2 &= (a_{s-t+3}a_{s-t+2})^*(a_{s-t+3})^*,\\
U_3 &= (a_{s-t+4}a_{s-t+2})^*(a_{s-t+4}a_{s-t+3})^*(a_{s-t+4})^*,\\
&\;\;\vdots\\
U_{2t-1} &= (a_{s+t}a_{s-t+2})^*(a_{s+t}a_{s-t+3})^* \cdots (a_{s+t}a_{s+t-1})^*(a_{s+t})^*.
\end{aligned}$$

As a consequence we get

$$a_{s-t+1}a_{s+t}T \subseteq a_{s-t+1}a_{s+t}UW_{s+t+1,n} \cup \{0\}. \tag{3.1}$$

Next, remembering that $a_{s+t}a_{s-t+1} = 0$ in T, we get:

- If $j \geq s+t$ then $a_j(a_{s-t+1}a_{s+t}) = a_j a_{s+t}a_{s-t+1}$ in C_n. Hence $a_j(a_{s-t+1}a_{s+t}) = 0$ in T for all $j \geq s+t$.
- If $j < k$ satisfy $j \leq s-t+1$ and $k \geq s+t$ then $(a_k a_j)(a_{s-t+1}a_{s+t}) = (a_{s-t+1}a_{s+t})(a_k a_j)$ in C_n. Hence also $(a_k a_j)(a_{s-t+1}a_{s+t}) = (a_{s-t+1}a_{s+t})(a_k a_j)$ in T for all $j < k$ such that $j \leq s-t+1$ and $k \geq s+t$.
- If $j < k$ satisfy $j > s-t+1$ and $k \geq s+t$ then $(a_k a_j)(a_{s-t+1}a_{s+t}) = (a_k a_{s-t+1})(a_j a_{s+t}) = (a_j a_{s+t})(a_k a_{s-t+1}) = a_j a_k a_{s+t}a_{s-t+1}$ in C_n. Hence $(a_k a_j)(a_{s-t+1}a_{s+t}) = 0$ in T for all $j < k$ such that $j > s-t+1$ and $k \geq s+t$.

These equalities assure that $W_{s+t+1,n}a_{s-t+1}a_{s+t} \subseteq a_{s-t+1}a_{s+t}T$. Thus, together with (3.1), we obtain

$$a_{s-t+1}a_{s+t}Ta_{s-t+1}a_{s+t} \subseteq a_{s-t+1}a_{s+t}UW_{s+t+1,n}a_{s-t+1}a_{s+t} \subseteq a_{s-t+1}a_{s+t}Ua_{s-t+1}a_{s+t}T. \tag{3.2}$$

Now, choose $1 \neq u \in U$. Then let m be the minimum of those numbers $j \in \{s-t+2, \dots, s+t\}$ such that the generator a_j appears in u. Since $a_{s+t}(a_k a_m) = a_k(a_{s+t}a_m)$ in C_n for all $m < k \leq s+t$, we get $a_{s+t}u \in Ta_{s+t}a_m$. Therefore,

$$a_{s-t+1}a_{s+t}ua_{s-t+1}a_{s+t} \in a_{s-t+1}T(a_{s+t}a_m a_{s-t+1})a_{s+t} = a_{s-t+1}T(a_{s+t}a_{s-t+1}a_m)a_{s+t} = 0. \tag{3.3}$$

Thus (3.3) together with $(a_{s-t+1}a_{s+t})^2 = 0$ in T yield $a_{s-t+1}a_{s+t}Ua_{s-t+1}a_{s+t} = 0$ in T. Hence, as a consequence of (3.2), we get $a_{s-t+1}a_{s+t}Ta_{s-t+1}a_{s+t} = 0$ as well. Since T is a prime semigroup, we conclude that $a_{s-t+1}a_{s+t} = 0$ in T, as desired.

Next, we claim that $a_{s-t+1}Ta_{s+t} = 0$. First, by (2.3) it follows that a_{s-t+1} commutes in T with $a_1, \ldots, a_s$. Hence we get $a_{s-t+1}W_{1,s} = W_{1,s}a_{s-t+1}$ in T. Similarly, by (2.4) it follows that a_{s+t} commutes in T with $a_{s+1}, \ldots, a_n$. Moreover, a_{s+t} commutes in C_n with a_ja_i for all $i < j$ such that $i \le s+t \le j$. Therefore, we get $W_{s+t,n}a_{s+t} = a_{s+t}W_{s+t,n}$ in T, which leads to

$$a_{s-t+1}Ta_{s+t} = W_{1,s}a_{s-t+1}W_{s+1,n}a_{s+t} = W_{1,s}a_{s-t+1}W_{s+1,s+t-1}a_{s+t}W_{s+t,n}.$$

So it is enough to show that $a_{s-t+1}W_{s+1,s+t-1}a_{s+t} = 0$ in T. Further, let us observe that:

- If $j > s$ then $a_ja_{s+t} = a_{s+t}a_j$ in T (by (2.4), because $j > s$ and $s+t > s$),
- If $k > s$ then $(a_ka_s)a_{s+t} = a_{s+t}a_sa_k$ in T (by (2.6) with $r = 1$, because $k > s$ and $s+t > s$),
- If $j < k$ satisfy $s-t < j \le s$ and $j+k > n$ then $(a_ka_j)a_{s+t} = a_{s+t}a_ja_k$ in T (by (2.6) with $r = s-j+1$. Indeed, the assumption $s-t < j \le s$ assures that $1 \le r \le t$. Moreover, $s+r = 2s-j+1 = n-j+1$. Hence, to use (2.6), it only remains to check that $k \ge n-j+1$ and $s+t \ge n-j+1$. Now, the first inequality is a consequence of $j+k > n$, whereas the second one is obtained as follows. Since $s-t < j$, we get $t \ge s-j+1$. Therefore, $s+t \ge 2s-j+1 = n-j+1$. We note that straightforward calculations on indices of this type will be also used in other proofs in this section; however, complete explanations will be skipped.)

These equalities lead to the conclusion that $W_{s+1,s+t-1}a_{s+t} \subseteq Va_{s+t}T$, where $V = V_1 \cdots V_{t-1}$ and

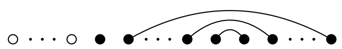
$$\begin{aligned} V_1 &= (a_{s+1}a_1)^*(a_{s+1}a_2)^* \cdots (a_{s+1}a_{s-1})^*, \\ V_2 &= (a_{s+2}a_1)^*(a_{s+2}a_2)^* \cdots (a_{s+2}a_{s-2})^*, \\ V_3 &= (a_{s+3}a_1)^*(a_{s+3}a_2)^* \cdots (a_{s+3}a_{s-3})^*, \\ &\vdots \\ V_{t-1} &= (a_{s+t-1}a_1)^*(a_{s+t-1}a_2)^* \cdots (a_{s+t-1}a_{s-t+1})^*. \end{aligned}$$

Next, we have $a_{s-t+1}(a_ka_j) = a_ja_ka_{s-t+1}$ in T for all $j < k$ such that $j+k \le n+1$ and $s < k \le s+t$. Indeed, here assumptions on j and k can be rewritten as $j \le n-k+1$ and $s-t+1 \le n-k+1$, hence our equality follows by (2.5) with $r = k-s$. Since each element of V is a product of elements a_ka_j with $j < k$ such that $j+k \le n$ and $s < k < s+t$, we conclude that $a_{s-t+1}V \subseteq Ta_{s-t+1}$. Therefore

$$a_{s-t+1}W_{s+1,s+t-1}a_{s+t} \subseteq a_{s-t+1}Va_{s+t}T \subseteq Ta_{s-t+1}a_{s+t}T = 0.$$

This proves the claim. Since T is a prime semigroup, it follows that $a_{s-t+1} = 0$ in T or $a_{s+t} = 0$ in T or, in other words, $a_{s-t+1} \in P$ or $a_{s+t} \in P$. □

Proposition 3.6. *Assume that P is a left primitive ideal of $K[C_n]$ containing the ideal I_ρ, where ρ is the congruence on C_n determined by one of the two diagrams*

 *or* 

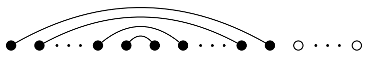
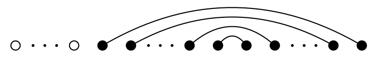
consisting of $t > 0$ consecutive arcs $\widehat{a_{s+1}a_s}, \ldots, \widehat{a_{s+t}a_{s-t+1}}$ and a single dot a_{s-t} or a_{s+t+1} (as shown in the picture). Assume additionally that $a_{s+t}a_{s-t+1} \notin P$. Then $a_{s-t} - \lambda a_{s-t+1} \in P$ or $a_{s+t+1} - \lambda a_{s+t} \in P$ for some $\lambda \in K$, respectively.

Proof. Since the two cases are symmetric, it is enough to consider the former. Let us notice first that the element $a_{s+t}a_{s-t+1}$ is central in $K[C_n]/P$. Indeed, if $i \le s-t+1$, then $a_ia_{s+t}a_{s-t+1} = a_{s-t+1}a_{s+t}a_i = a_{s+t}a_{s-t+1}a_i$ in $K[C_n]/P$ (first equality is a consequence of (2.10) with $r = t$, because $i \le s-t+1$; second equality is valid in C_n). Next, if $s-t+1 < i \le s+t$, then $a_ia_{s+t}a_{s-t+1} = a_{s+t}a_{s-t+1}a_i$ in C_n, hence also in $K[C_n]/P$. Whereas, if $i > s+t$, then $a_ia_{s+t}a_{s-t+1} = a_ia_{s-t+1}a_{s+t} = a_{s+t}a_{s-t+1}a_i$ in $K[C_n]/P$ (first equality holds in C_n; second equality follows from (2.11) with $r = t$, because $i \ge s+t$). Therefore, Proposition 3.4 implies that $a_{s+t}a_{s-t+1} = \mu$ in $K[C_n]/P$ for some $\mu \in K$. Moreover, $\mu \ne 0$ because $a_{s+t}a_{s-t+1} \notin P$. We claim that the element $a_{s+t}a_{s-t}$ is central in $K[C_n]/P$ as well. Indeed, if $j \le s-t$, then $a_ja_{s+t}a_{s-t} = a_{s-t}a_{s+t}a_j = a_{s+t}a_{s-t}a_j$ in $K[C_n]/P$ (first equality holds by (2.10) with $r = t$, because $j \le s-t+1$ and $s-t \le s-t+1$; second equality is valid in C_n). Next, if $s-t < j < s+t$, then $a_ja_{s+t}a_{s-t} = a_{s+t}a_{s-t}a_j$ in C_n, hence also in $K[C_n]/P$. Whereas, if $j \ge s+t$, then $a_ja_{s+t}a_{s-t} = a_{s+t}a_ja_{s-t} = a_{s+t}a_{s-t}a_j$ in $K[C_n]/P$ (first equality holds C_n; second equality is a consequence of (2.9), because $j \ge s+t$). Therefore, by Proposition 3.4, we get $a_{s+t}a_{s-t} = \nu$ in $K[C_n]/P$ for some $\nu \in K$, which leads to

$$\mu a_{s-t} = a_{s+t}a_{s-t+1}a_{s-t} = a_{s+t}a_{s-t}a_{s-t+1} = \nu a_{s-t+1}$$

in $K[C_n]/P$. Hence we get $a_{s-t} - \lambda a_{s-t+1} \in P$, where $\lambda = \mu^{-1}\nu \in K$. □

Proposition 3.7. *Assume that P is a left primitive ideal of $K[C_n]$ containing the ideal I_ρ, where ρ is the congruence on C_n determined by one of the two diagrams*

 *or* 

consisting of $t > 0$ consecutive arcs $\widehat{a_{t+1}a_t}, \ldots, \widehat{a_{2t}a_1}$ or $\widehat{a_{n-t+1}a_{n-t}}, \ldots, \widehat{a_n a_{n-2t+1}}$ (as shown in the picture). Assume additionally that $a_{2t}a_1 \notin P$ or $a_n a_{n-2t+1} \notin P$, respectively. Then $a_{2t+1} - \lambda a_{2t} \in P$ or $a_{n-2t} - \lambda a_{n-2t+1} \in P$ for some $\lambda \in K$, respectively.

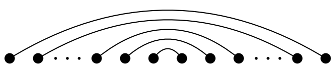
Proof. Since the two cases are symmetric, it suffices to consider the former. First, notice that the element $a_{2t}a_1$ is central in $K[C_n]/P$. Indeed, if $i \leq 2t$, then $a_i a_{2t} a_1 = a_{2t} a_1 a_i$ in C_n, hence also in $K[C_n]/P$. Whereas, if $i > 2t$, then $a_i a_{2t} a_1 = a_i a_1 a_{2t} = a_{2t} a_1 a_i$ (first equality holds in C_n; second equality follows from (2.6) with $r = s = t$, because $i \geq 2t$). Hence, by Proposition 3.4, we get $a_{2t}a_1 = \mu$ in $K[C_n]/P$ for some $\mu \in K$. Moreover, $a_{2t}a_1 \notin P$ implies that $\mu \neq 0$. We claim that the element $a_{2t+1}a_1$ is central in $K[C_n]/P$ as well. Indeed, if $j \leq 2t$, then $a_{2t+1}a_1 a_j = a_j a_{2t+1} a_1$ in C_n, hence also in $K[C_n]/P$. Whereas, if $j > 2t$, then $a_j a_{2t+1} a_1 = a_j a_1 a_{2t+1} = a_{2t+1} a_1 a_j$ in $K[C_n]/P$ (first equality holds in C_n; second equality is a consequence of (2.6) with $r = s = t$, because $j \geq 2t$). Thus, Proposition 3.4 yields $a_{2t+1}a_1 = \nu$ in $K[C_n]/P$ for some $\nu \in K$, and we get

$$\mu a_{2t+1} = a_{2t+1} a_{2t} a_1 = a_{2t+1} a_1 a_{2t} = \nu a_{2t}$$

in $K[C_n]/P$. Hence we get $a_{2t+1} - \lambda a_{2t} \in P$, where $\lambda = \mu^{-1}\nu \in K$. □

Before we proceed to a formulation of the main result of this paper, let us recall some notions and introduce some notation. We say that the element $x \in K[C_n]$ acts regularly on a left $K[C_n]$-module V if $xv \neq 0$ for each $0 \neq v \in V$ (or, in other words, if the annihilator of x in V is equal to zero).

In the following five lemmas we assume that n is even, say $n = 2s$ for some $s \geq 1$. We assume as well that the $K[C_n]$-module V is simple, and its annihilator P contains the ideal I_ρ, where ρ is the congruence on C_n determined by the diagram



consisting of s consecutive arcs $\widehat{a_{s+1}a_s}, \ldots, \widehat{a_n a_1}$ (as shown in the picture). Moreover, we consider the set

$$X = \{a_j a_i : i < s+1 < j \text{ and } i + j > n+1\} \subseteq C_n,$$

and its subsets

$$X_0 = \{x \in X : x \text{ does not act regularly on } V\}, \qquad X_1 = X \setminus X_0.$$

Of course, the sets X_0 and X_1 depend on the module V.

Lemma 3.8. *We have $xy - yx \in P$ for all $x, y \in X$.*

Proof. Let $x = a_j a_i$ and $y = a_l a_k$. If $j = l$ then $xy = yx$ in C_n. Hence, we may assume that $j < l$. If $i \geq k$ then again $xy = yx$ in C_n. So, assume that $i < k$. Summarizing, we are in the situation where $i < k < s+1 < j < l$. Then, by (2.6) with $r = s - i + 1$, we get $a_j a_i a_l = a_l a_i a_j$ in $K[C_n]/P$ (because $l > j > n - i + 1$ so, in particular, $j, l \geq n - i + 1$), which leads to

$$xy = a_j a_i a_l a_k = a_l a_i a_j a_k = a_l a_j a_i a_k = a_l a_k a_j a_i = yx$$

in $K[C_n]/P$. Hence the result follows. □

Lemma 3.9. *Assume that $a_{n-j+1}a_j \notin P$ for each $j = 1, \ldots, s$. If $x \in X_1$ then $x - \mu \in P$ for some $0 \neq \mu \in K$.*

Proof. First, notice that each element $a_{n-j+1}a_j$ for $j = 1, \ldots, s$ is central in $K[C_n]/P$. Indeed, if $i \leq j$, then $a_i a_{n-j+1} a_j = a_j a_{n-j+1} a_i = a_{n-j+1} a_j a_i$ in $K[C_n]/P$ (first equality is a consequence of (2.5) with $r = s - j + 1$, because $i \leq j$; second equality is valid in C_n). Next, if $j < i \leq n - j + 1$, then a_i commutes with $a_{n-j+1}a_j$ in C_n, hence also in $K[C_n]/P$. Whereas, if $i > n - j + 1$, then $a_i a_{n-j+1} a_j = a_i a_j a_{n-j+1} = a_{n-j+1} a_j a_i$ in $K[C_n]/P$ (first equality holds in C_n; second equality follows from (2.6) with $r = s - j + 1$, because $i \geq n - j + 1$). Hence Proposition 3.4 assures that $a_{n-j+1}a_j = \lambda_j$ in $K[C_n]/P$. Since $a_{n-j+1}a_j \notin P$, we get $\lambda_j \neq 0$ for each $j = 1, \ldots, s$. Now, let $x = a_k a_j$. Since we have

$$(a_k a_j)(a_{n-j+1} a_{n-k+1}) = a_{n-j+1} a_k a_j a_{n-k+1} = (a_{n-j+1} a_j)(a_k a_{n-k+1})$$

in C_n and $a_k a_j(a_{n-j+1}a_{n-k+1}a_k a_j - \lambda_j\lambda_{n-k+1})V = 0$, we conclude that x and $y = \nu a_{n-j+1}a_{n-k+1}$, where $\nu = (\lambda_j\lambda_{n-k+1})^{-1} \in K$, are mutual inverses in $K[C_n]/P$.

We claim that each generator a_i, for $i = 1, \ldots, n$, commutes with x or y in $K[C_n]/P$. Indeed, if $i \leq j$ then, by (2.5) with $r = s - j + 1$, we get $a_i a_{n-j+1}a_{n-k+1} = a_{n-k+1}a_{n-j+1}a_i$ in $K[C_n]/P$ (because we have $i \leq j$, and from $j + k > n + 1$ we get also $n - k + 1 \leq j$), so

$$a_i y = \nu a_i a_{n-j+1}a_{n-k+1} = \nu a_{n-k+1}a_{n-j+1}a_i = \nu a_{n-j+1}a_{n-k+1}a_i = y a_i$$

in $K[C_n]/P$. If $j < i \leq n - j + 1$ then from $j + k > n + 1$ we get $n - k + 1 < j < i$, so a_i commutes with $y = a_{n-j+1}a_{n-k+1}$ in C_n, hence in $K[C_n]/P$, too. If $n - j + 1 < i \leq k$ then $j < n - j + 1 < i$ implies that a_i commutes with $x = a_k a_j$ in C_n, hence in $K[C_n]/P$ as well. Whereas, if $i > k$ then, by (2.6) with $r = s - j + 1$, we have $a_i a_j a_k = a_k a_j a_i$ in $K[C_n]/P$ (because from $j + k > n + 1$ we get $i > k \geq n - j + 1$), which yields

$$a_i x = a_i a_k a_j = a_i a_j a_k = a_k a_j a_i = x a_i$$

in $K[C_n]/P$, and the claim follows. In particular, x in central in $K[C_n]/P$. Therefore, Proposition 3.4 guarantees that $x - \mu \in P$ for some $\mu \in K$. Since x acts regularly on V, we must have $\mu \neq 0$. □

Let $x = a_l a_k \in X$ and $y = a_j a_i \in X$. We say that x dominates y (or that y is dominated by x) if $j \leq l$ and $i \leq k$.

Lemma 3.10. *Assume that $a_{n-j+1}a_j \notin P$ for each $j = 1, \ldots, s$. If $y \in X$ is dominated by some $x \in X_1$, then $y \in X_1$.*

Proof. By Lemma 3.9 we know that $x - \mu \in P$ for some $0 \neq \mu \in K$. Moreover, as in Lemma 3.9, we have $a_{n-j+1}a_j = \lambda_j$ in $K[C_n]/P$, where $\lambda_j \neq 0$, for each $j = 1, \ldots, s$. Now, write $x = a_l a_k$ and $y = a_j a_i$. Of course it suffices to consider just two cases. Namely, $(i, j) = (k - 1, l)$ and $(i, j) = (k, l - 1)$. Assume first that $(i, j) = (k - 1, l)$. Then, by (2.6) with $r = s - k + 1$, we have $a_{n-k+2}a_k a_l = a_l a_k a_{n-k+2}$ in $K[C_n]/P$ (because from $k + l > n + 1$ we get $l \geq n - k + 1$ and, of course, $n - k + 2 \geq n - k + 1$). Now, if $v \in V$ and $yv = 0$ then

$$0 = a_{n-k+2}a_k y v = a_{n-k+2}a_k a_l a_{k-1}v = a_l a_k a_{n-k+2}a_{k-1}v = \lambda_{k-1}\mu v,$$

because $a_l a_k = \mu$ and $a_{n-k+2}a_{k-1} = \lambda_{k-1}$ in $K[C_n]/P$. Hence $v = 0$. Finally, let $(i, j) = (k, l - 1)$. If $yv = 0$ for some $v \in V$ then

$$0 = a_l a_{n-l+2} y v = a_l a_{n-l+2}a_{l-1}a_k v = a_l a_{l-1}a_{n-l+2}a_k v = \lambda_{n-l+2}a_l a_k v = \lambda_{n-l+2}\mu v,$$

because $a_{l-1}a_{n-l+2} = \lambda_{n-l+2}$ and $a_l a_k = \mu$ in $K[C_n]/P$. Hence again $v = 0$, and the result follows. □

The statement of the last lemma can be easily visualized if we arrange the elements of X in a triangular matrix, as follows

$$\begin{matrix}
a_n a_2 & a_n a_3 & a_n a_4 & \cdots & a_n a_{s-2} & a_n a_{s-1} & a_n a_s \\
 & a_{n-1}a_3 & a_{n-1}a_4 & \cdots & a_{n-1}a_{s-2} & a_{n-1}a_{s-1} & a_{n-1}a_s \\
 & & a_{n-2}a_4 & \cdots & a_{n-2}a_{s-2} & a_{n-2}a_{s-1} & a_{n-2}a_s \\
 & & & \ddots & \vdots & \vdots & \vdots \\
 & & & & a_{s+4}a_{s-2} & a_{s+4}a_{s-1} & a_{s+4}a_s \\
 & & & & & a_{s+3}a_{s-1} & a_{s+3}a_s \\
 & & & & & & a_{s+2}a_s
\end{matrix} \tag{3.4}$$

Then, for each $x \in X$, elements in X dominated by x constitute a right triangle with x as the vertex of the right angle and with its hypotenuse consisting of elements lying on the diagonal of the above matrix. It is also worth to reformulate Lemma 3.10 in the following way. If $x = a_j a_i \in X_0$ then all elements dominating x also lie in X_0 (these are precisely the elements of the matrix lying inside the rectangle defined by the vertices $a_j a_i$, $a_n a_i$, $a_n a_s$, $a_j a_s$).

Lemma 3.11. *If $x \in X_0$ then for each $v \in V$ there exists $m > 0$ such that $x^m v = 0$.*

Proof. Let $x = a_k a_j$. Since x does not act regularly on V, there exists $0 \neq w \in V$ such that $xw = 0$. Because V is a simple module, we have $V = K[C_n]w$. We claim that for each $z \in C_n$ there exists $l > 0$ such that x commutes with $x^l z$ in $K[C_n]/P$ (in fact, it suffices to take $l = \deg(z)$). Of course this claim implies our lemma. We shall prove the claim by induction on $\deg(z)$. So assume first that $\deg(z) = 1$. Then $z = a_i$ for some $i \in \{1, \ldots, n\}$. If $i < j$ then $xz = a_k a_j a_i = (a_k a_i)a_j$ in C_n. Hence xz commutes with x in C_n, so also in $K[C_n]/P$. Next, if $j \leq i \leq k$ then x commutes with z in C_n, hence also with xz in C_n, and of course in $K[C_n]/P$ as well. Finally, if $i > k$ then, by (2.6) with $r = s - j + 1$, we get

$xz = a_k a_j a_i = (a_i a_j) a_k$ in $K[C_n]/P$ (because $j+k > n+1$ implies $i > k \geq n-j+1$). Thus x commutes with xz in $K[C_n]/P$ as well. Now assume that $\deg(z) > 1$ and write $z = z' a_i$ for some $z' \in C_n$ and some $i \in \{1, \ldots, n\}$. By induction, there exists $l > 0$ such that x commutes with $x^l z'$ in $K[C_n]/P$. Now $x^{l+1} z = x^{l+1} z' a_i = (x^l z')(x a_i)$, so the claim follows, because x commutes in $K[C_n]/P$ with $x^l z'$ and with $x a_i$, hence also with $x^{l+1} z$. □

Lemma 3.12. *Assume that $a_{n-j+1} a_j \notin P$ for each $j = 1, \ldots, s$. If a_{s+1} does not act regularly on V then there exists $0 \neq v \in V$ such that $xv = 0$ for each $x \in X_0$ and $a_j v = 0$ for each $j > s$. In this situation V is spanned as a K-linear space by the set*

$$(a_{n-1}a_1)^*(a_{n-2}a_2)^*(a_{n-3}a_3)^* \cdots (a_{s+3}a_{s-3})^*(a_{s+2}a_{s-2})^*(a_{s+1}a_{s-1})^* a_s^* v.$$

Proof. First, note that for each $j = 1, \ldots, s$ we have $a_{n-j+1} a_j = \lambda_j$ in $K[C_n]/P$ for some $0 \neq \lambda_j \in K$ (this was already proved in Lemma 3.9). Assume for the moment that we already have a vector $0 \neq w \in V$ such that $xw = 0$ for each $x \in X_0$. Then there exists $k > 0$ such that $a_{s+1}^k w = 0 \neq a_{s+1}^{k-1} w$ (the proof of this fact is completely analogous to the proof of Lemma 3.11, so it will be omitted here). Let $v = a_{s+1}^{k-1} w \neq 0$. Since a_{s+1} commutes with each $x \in X$ in C_n, we get $xv = 0$ for each $x \in X_0$. Of course $a_{s+1} v = 0$, and if $j > s+1$ then we have

$$0 = a_j a_s a_{s+1} v = a_j a_{s+1} a_s v = \lambda_s a_j v,$$

which gives $a_j v = 0$, because $\lambda_s \neq 0$. Thus, to finish the proof of the first part of our lemma, it is enough to show that there exists $0 \neq w \in V$ such that $xw = 0$ for each $x \in X_0$. If X_0 is empty then there is nothing to show. Therefore, assume that $X_0 = \{x_1, \ldots, x_d\}$ with $d = |X_0| > 0$. Take $l < d$ and suppose that there exists $0 \neq w_l \in V$ such that $x_1 w_l = \cdots = x_l w_l = 0$. By Lemma 3.11 we know that $x_{l+1}^m w_l = 0 \neq x_{l+1}^{m-1} w_l$ for some $m > 0$. Then define $w_{l+1} = x_{l+1}^{m-1} w_l \neq 0$. Because $x_1, \ldots, x_l$ commute with x_{l+1} in $K[C_n]/P$ (see Lemma 3.8), we get $x_1 w_{l+1} = \cdots = x_l w_{l+1} = x_{l+1} w_{l+1} = 0$. Now, it is clear that after d steps we obtain a nonzero vector $w = w_d \in V$ such that $xw = 0$ for each $x \in X_0$.

Let us proceed to the proof of the last statement of the lemma. So, fix $0 \neq v \in V$ satisfying $xv = 0$ for each $x \in X_0$ and $a_j v = 0$ for each $j > s$. Of course V is spanned as a K-linear space by the set $C_n v$. Hence it suffices to show that for each $x = b_1 \cdots b_n \in C_n$ written in its canonical form (1.2) we have $xv \in K \cdot (a_{n-1}a_1)^* \cdots (a_{s+1}a_{s-1})^* a_s^* v$.

First, by (2.3), we have $a_i a_j = a_j a_i$ in $K[C_n]/P$ for all $i, j \leq s$. Hence, the element $b_1 \cdots b_s$ can be written in $K[C_n]/P$ as an element of the set $a_1^* \cdots a_s^*$. Next, for each $j < s$ we have

$$\lambda_{j+1} a_j = a_{n-j} a_{j+1} a_j = (a_{n-j} a_j) a_{j+1}$$

in $K[C_n]/P$. Thus, we conclude that $a_1^* \cdots a_s^* \subseteq K \cdot (a_{n-1}a_1)^* \cdots (a_{s+1}a_{s-1})^* a_s^*$ in $K[C_n]/P$, which allows us to assume that

$$b_1 \cdots b_s \in (a_{n-1}a_1)^* \cdots (a_{s+1}a_{s-1})^* a_s^*. \tag{3.5}$$

Further, by (2.4), we have $a_k a_l = a_l a_k$ in $K[C_n]/P$ for all $k, l > s$. Therefore, for each $j > s$, the element b_j can be written in $K[C_n]/P$ as an element of the set $(a_j a_1)^* \cdots (a_j a_s)^* a_{s+1}^* \cdots a_j^*$. Because the elements $a_{s+1}, \ldots, a_j$ commute in C_n with each $a_k a_i$, where $i < k$ satisfy $i \leq s$ and $k > j$, we deduce that $a_{s+1}, \ldots, a_j$ commute in $K[C_n]/P$ with all elements $b_{j+1}, \ldots, b_n$. Moreover, $a_{s+1} v = \cdots = a_j v = 0$. These two facts allow us to assume that

$$b_j \in (a_j a_1)^* \cdots (a_j a_s)^* \text{ for each } j > s. \tag{3.6}$$

Next, we claim that for all $i < s < j$ such that $i + j < n$ the equality

$$(\lambda_{i+1} \cdots \lambda_{n-j}) a_j a_i = (a_{n-i} a_i)(a_{n-i-1} a_{i+1})(a_{n-i-2} a_{i+2}) \cdots (a_j a_{n-j}) \tag{3.7}$$

holds in $K[C_n]/P$. We shall prove the claim by induction on $d = n - i - j$. If $d = 1$ then $i + j = n - 1$ and we have

$$(a_{n-i} a_i)(a_j a_{n-j}) = (a_{j+1} a_i)(a_j a_{i+1}) = a_j a_{j+1} a_i a_{i+1} = a_j (a_{j+1} a_{i+1}) a_i = \lambda_{i+1} a_j a_i,$$

because $a_{j+1} a_{i+1} = a_{n-i} a_{i+1} = \lambda_{i+1}$ in $K[C_n]/P$. So assume that $d > 1$, and the claim is true for all $i < s < j$ such that $n - i - j = d$. Our aim is to show that (3.7) holds for all $i < s < j$ such that $n - i - j = d + 1$. Observe that in this case we must have $i + 1 < s$, because otherwise $i \geq s - 1$ and $j \geq s + 1$ give $d + 1 = n - i - j \leq n - (s-1) - (s+1) = 0$, a contradiction. So $i + 1 < s < j$ and $n - (i+1) - j = d$. Therefore, by induction, we get

$$(\lambda_{i+2} \cdots \lambda_{n-j}) a_j a_{i+1} = (a_{n-i-1} a_{i+1}) \cdots (a_j a_{n-j})$$

in $K[C_n]/P$. This equality, together with $a_{n-i}a_{i+1} = \lambda_{i+1}$ in $K[C_n]/P$, lead to

$$\begin{aligned}(\lambda_{i+1}\lambda_{i+2}\cdots\lambda_{n-j})a_ja_i &= (\lambda_{i+2}\cdots\lambda_{n-j})(a_{n-i}a_{i+1})(a_ja_i)\\ &= (\lambda_{i+2}\cdots\lambda_{n-j})a_ja_{n-i}a_{i+1}a_i\\ &= (\lambda_{i+2}\cdots\lambda_{n-j})a_ja_{n-i}a_ia_{j+1}\\ &= (\lambda_{i+2}\cdots\lambda_{n-j})(a_{n-i}a_i)(a_ja_{i+1})\\ &= (a_{n-i}a_i)(a_{n-i-1}a_{i+1})\cdots(a_ja_{n-j})\end{aligned}$$

in $K[C_n]/P$, hence the claim follows. Now, looking at the form (3.6) of b_j and using (3.7) to rewrite the factors a_ja_i with $i < n-j$ (appearing in the form (3.6) of b_j), and also noticing that $a_ja_{n-j+1} = \lambda_{n-j+1}$ in $K[C_n]/P$, we conclude that we may restrict to the situation when

$$b_j \in (a_{n-1}a_1)^*\cdots(a_ja_{n-j})^*\cdot(a_ja_{n-j+2})^*\cdots(a_ja_s)^* \text{ for each } j > s. \tag{3.8}$$

Note that in the form (3.8) of b_j two types of factors appear. First $n-j$ factors are of the form $a_{n-i}a_i$ for $i = 1,\ldots,n-j$, whereas next $s-(n-j+1) = j-s-1$ factors (separated from the first $n-j$ factors by a dot) are of the form a_ja_i, where $i = n-j+2,\ldots,s$. Further, each factor a_ja_i for $n-j+2 \leq i \leq s$, appearing in the form (3.8) of b_j, lies in X and commutes with all factors $a_{n-1}a_1,\ldots,a_{j+1}a_{n-j-1}$ that appear in the form (3.8) of the elements $b_{j+1},\ldots,b_n$. Hence, we can write $b_{s+1}\cdots b_n$ in $K[C_n]/P$ as an element of the set $K\cdot(a_{n-1}a_1)^*\cdots(a_{s+1}a_{s-1})^*\langle X\rangle$, where $\langle X\rangle \subseteq C_n$ denotes the monoid generated by the set $X = X_0\cup X_1$. Since $X_0v = 0$ and $X_1v \subseteq Kv$ (see Lemma 3.9), we get $\langle X\rangle v \subseteq Kv$, which leads to the conclusion that

$$b_{s+1}\cdots b_nv \in K\cdot(a_{n-1}a_1)^*\cdots(a_{s+1}a_{s-1})^*v. \tag{3.9}$$

Finally, (3.5) and (3.9) yield $xv = b_1\cdots b_sb_{s+1}\cdots b_nv \in K\cdot(a_{n-1}a_1)^*\cdots(a_{s+1}a_{s-1})^*a_s^*v$, which ends the proof. □

Now, we are ready to formulate the main result of the paper.

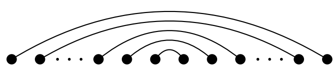
Theorem 3.13. *Let V be a simple left $K[C_n]$-module. Then V is isomorphic to one of the modules constructed in Proposition 3.2 (in this case n must be even) or $xV = 0$, where $x = a_i - \lambda$ for some $i \in \{1,\ldots,n\}$ and $\lambda \in K$, or $x = \lambda a_j - \mu a_{j-1}$ for some $j \in \{2,\ldots,n\}$ and $\lambda,\mu \in K$ not both equal to zero. In the latter case V may be treated as a simple left $K[C_{n-1}]$-module and its structure can be described inductively.*

Proof. Let P denote the annihilator of the module V. Since P is a prime ideal, it follows that P contains a minimal prime ideal of $K[C_n]$, which is of the form $I_{\rho(d)}$ for some leaf $d \in D$ (see Section 2). We also know that the congruence $\rho(d)$ arises as a finite extension $\rho(d_0) \subseteq \rho(d_1) \subseteq \cdots \subseteq \rho(d_m) = \rho(d)$, where each d_j is a diagram in level j of D. In particular, $I_{\rho(d_j)} \subseteq P$ for each $j = 1,\ldots,m$.

First, consider the case in which some diagram d_{j+1} is obtained from d_j by adding a dot. Let us additionally assume that j is minimal with this property. If $j = 0$ then the diagram d_1 consists of a single dot a_i for some $i \in \{2,\ldots,n-1\}$. Hence, by (2.1) and (2.2) with $s = i$, we conclude that a_i is central in $K[C_n]/P$. Therefore, by Proposition 3.4, we have $xV = 0$, where $x = a_i - \lambda$ for some $\lambda \in K$. Whereas, if $j > 0$ then d_j consists of j consecutive arcs $\widehat{a_{s+1}a_s},\ldots,\widehat{a_{s+j}a_{s-j+1}}$ and d_{j+1} arises by adjoining the dot a_{s-j} to d_j or by adjoining the dot a_{s+j+1} to d_j. If $a_{s+j}a_{s-j+1} \in P$ then Proposition 3.5 implies that $xV = 0$, where $x \in \{a_{s+j}, a_{s-j+1}\}$. Whereas, if $a_{s+j}a_{s-j+1} \notin P$ then Proposition 3.6 implies that $xV = 0$, where $x \in \{a_{s-j} - \lambda a_{s-j+1}, a_{s+j+1} - \lambda a_{s+j} : \lambda \in K\}$, and the result also follows.

Now assume that a dot does not appear in the construction of d, but d contains the arc $\widehat{a_ja_1}$, where $j < n$ or the arc $\widehat{a_na_i}$, where $i > 1$. If $a_ja_1 \in P$ or $a_na_i \in P$ then Proposition 3.5 implies that $xV = 0$ for some $x \in \{a_1, a_i, a_j, a_n\}$. Whereas, if $a_ja_1 \notin P$ and $a_na_i \notin P$ then Proposition 3.7 yields $xV = 0$, where $x \in \{a_{j+1} - \lambda a_j, a_{i-1} - \lambda a_i : \lambda \in K\}$, hence the result follows in this situation as well.

Let us observe that if n is odd then one of the cases described above must hold. Therefore, we may assume that $n = 2s$ for some $s \geq 1$. Moreover, it remains to consider the case in which the diagram d



consists of s consecutive arcs $\widehat{a_{s+1}a_s},\ldots,\widehat{a_na_1}$ (as shown in the picture). In this situation we already know (see the proof of Lemma 3.9) that the elements $a_{n-j+1}a_j$ for $j = 1,\ldots,s$ are central in $K[C_n]/P$. Therefore, by Proposition 3.4, we have $a_{n-j+1}a_j = \lambda_j$ in $K[C_n]/P$ for some $\lambda_j \in K$. Moreover, due to Proposition 3.5, we may assume that each $\lambda_j \neq 0$. Further, if a_{s+1} acts regularly on V then the equality $a_{s+1}(a_sa_{s+1} - \lambda_s)V = 0$ implies $(a_sa_{s+1} - \lambda_s)V = 0$. Hence a_s and $\lambda_s^{-1}a_{s+1}$ are mutual inverses in $K[C_n]/P$. Since a_s commutes with $a_1, \ldots, a_s$ in $K[C_n]/P$ (see (2.3)), and a_{s+1} commutes with $a_{s+1}, \ldots, a_n$ in $K[C_n]/P$ (see (2.4)), we conclude that a_s is a central element of $K[C_n]/P$. Thus, again by Proposition 3.4, we conclude that $xV = 0$, where $x = a_s - \lambda$ for some $\lambda \in K$. Therefore, we may assume that a_{s+1} does not act regularly on V. In this situation Lemma 3.12 guarantees that

$$\text{there exists } 0 \neq v \in V \text{ such that } X_0 v = 0 \text{ and } a_j v = 0 \text{ for all } j > s \tag{3.10}$$

(notation introduced before Lemma 3.8 is used here). Moreover, we know that V is spanned as a K-linear space by elements of the set $(a_{n-1}a_1)^* \cdots (a_{s+1}a_{s-1})^* a_s^* v$.

First, assume $X_0 = X$. We claim that in this case elements of the set $(a_{n-1}a_1)^* \cdots (a_{s+1}a_{s-1})^* a_s^* v$ are linearly independent over K. Indeed, suppose on the contrary that

$$\sum_{i_1,\ldots,i_s=0}^{r} \lambda_{i_1,\ldots,i_s} (a_{n-1}a_1)^{i_1} \cdots (a_{s+1}a_{s-1})^{i_{s-1}} a_s^{i_s} v = 0$$

is a nontrivial relation of linear dependence. Then define

$$m_1 = \max\{i_1 : \lambda_{i_1,i_2,\ldots,i_s} \neq 0 \text{ for some } i_2, \ldots, i_s\}.$$

Observe that, by (2.5) with $r = s-1$, we have $a_1 a_{n-1} a_2 = a_2 a_{n-1} a_1$ in $K[C_n]/P$, which yields

$$(a_n a_2)(a_{n-1}a_1) = a_n(a_2 a_{n-1} a_1) = a_n(a_1 a_{n-1} a_2) = (a_n a_1)(a_{n-1} a_2) = \lambda_1 \lambda_2$$

in $K[C_n]/P$. Since $a_n a_2 v = 0$ and because the element $a_n a_2$ commutes in C_n with $a_{n-j} a_j$ for $j > 1$, we get

$$\begin{aligned} 0 &= (a_n a_2)^{m_1} \sum_{i_1,\ldots,i_s=0}^{r} \lambda_{i_1,\ldots,i_s} (a_{n-1}a_1)^{i_1} \cdots (a_{s+1}a_{s-1})^{i_{s-1}} a_s^{i_s} v \\ &= (\lambda_1 \lambda_2)^{m_1} \sum_{i_2,\ldots,i_s=0}^{r} \lambda_{m_1,i_2,\ldots,i_s} (a_{n-2}a_2)^{i_2} \cdots (a_{s+1}a_{s-1})^{i_{s-1}} a_s^{i_s} v. \end{aligned}$$

Assume now that $k < s-1$, the numbers $m_1, \ldots, m_k$ have already been defined, and the equality

$$\sum_{i_{k+1},\ldots,i_s=0}^{r} \lambda_{m_1,\ldots,m_k,i_{k+1},\ldots,i_s} (a_{n-k-1}a_{k+1})^{i_{k+1}} \cdots (a_{s+1}a_{s-1})^{i_{s-1}} a_s^{i_s} v = 0$$

holds with $\lambda_{m_1,\ldots,m_k,i_{k+1},\ldots,i_s} \neq 0$ for some $i_{k+1}, \ldots, i_s$. Put

$$m_{k+1} = \max\{i_{k+1} : \lambda_{m_1,\ldots,m_k,i_{k+1},i_{k+2},\ldots,i_s} \neq 0 \text{ for some } i_{k+2}, \ldots, i_s\}.$$

Then, by (2.5) with $r = s-k-1$, we have $a_{k+2} a_{n-k-1} a_{k+1} = a_{k+1} a_{n-k-1} a_{k+2}$ in $K[C_n]/P$, which yields

$$\begin{aligned} (a_{n-k}a_{k+2})(a_{n-k-1}a_{k+1}) &= a_{n-k}(a_{k+2}a_{n-k-1}a_{k+1}) \\ &= a_{n-k}(a_{k+1}a_{n-k-1}a_{k+2}) \\ &= (a_{n-k}a_{k+1})(a_{n-k-1}a_{k+2}) = \lambda_{k+1}\lambda_{k+2} \end{aligned}$$

in $K[C_n]/P$. Since $a_{n-k}a_{k+2}v = 0$ and because the element $a_{n-k}a_{k+2}$ commutes in C_n with $a_{n-j}a_j$ for $j > k+1$, we get

$$\begin{aligned} 0 &= (a_{n-k}a_{k+2})^{m_{k+1}} \sum_{i_{k+1},\ldots,i_s=0}^{r} \lambda_{m_1,\ldots,m_k,i_{k+1},\ldots,i_s} (a_{n-k-1}a_{k+1})^{i_{k+1}} \cdots (a_{s+1}a_{s-1})^{i_{s-1}} a_s^{i_s} v \\ &= (\lambda_{k+1}\lambda_{k+2})^{m_{k+1}} \sum_{i_{k+2},\ldots,i_s=0}^{r} \lambda_{m_1,\ldots,m_{k+1},i_{k+2},\ldots,i_s} (a_{n-k-2}a_{k+2})^{i_{k+2}} \cdots (a_{s+1}a_{s-1})^{i_{s-1}} a_s^{i_s} v. \end{aligned}$$

Thus, by induction, we conclude that there exist $m_1, \ldots, m_{s-1}$ such that $\lambda_{m_1,\ldots,m_{s-1},i_s} \neq 0$ for some i_s, and

$$\sum_{i_s=0}^{r} \lambda_{m_1,\ldots,m_{s-1},i_s} a_s^{i_s} v = 0.$$

Now, let $m_s = \max\{i_s : \lambda_{m_1,\ldots,m_{s-1},i_s} \neq 0\}$. Since $a_{s+1}a_s = \lambda_s$ in $K[C_n]/P$ and $a_{s+1}v = 0$, we get

$$0 = a_{s+1}^{m_s} \sum_{i_s=0}^{r} \lambda_{m_1,\ldots,m_{s-1},i_s} a_s^{i_s} v = \lambda_s^{m_s} \lambda_{m_1,\ldots,m_s} v,$$

which leads to a false conclusion that $v = 0$. Therefore, the set

$$E = \{e_{i_1,\dots,i_s} = (\lambda_1^{-1} a_{n-1} a_1)^{i_1} \cdots (\lambda_{s-1}^{-1} a_{s+1} a_{s-1})^{i_{s-1}} (\lambda_s^{-1} a_s)^{i_s} v : i_1, \dots, i_s \geq 0\}$$

is a basis of V over K, and one can easily check that the action of $a_1, \dots, a_n \in C_n$ on the basis E agrees with the action of $a_1, \dots, a_n \in C_n$ on the basis of the left $K[C_n]$-module $V(\lambda_1, \dots, \lambda_s)$ defined in Proposition 3.2. Hence we get $V \cong V(\lambda_1, \dots, \lambda_s)$.

Finally, let us consider the last case. Namely, $X_0 \neq X$. This means that some element $a_j a_i \in X$ acts regularly on V (that is, $a_j a_i \in X_1$). Lemma 3.10 assures that we may restrict to the situation in which $i + j = n + 2$ (that is, $a_j a_i$ lies on the diagonal in the matrix notation (3.4) of elements of X). Moreover, we may assume that j is minimal with that property. In this case Lemma 3.10 and the discussion after this lemma imply that all elements $a_l a_k \in X$ with $k > i$ lie in X_0. Because the vector v satisfies $X_0 v = 0$ (see (3.10)), we get, in particular,

$$a_l a_k v = 0 \text{ for all } a_l a_k \in X \text{ with } k > i. \tag{3.11}$$

We also know that V is spanned as a K-linear space by the set

$$(a_{n-1} a_1)^* \cdots (a_j a_{n-j})^* (a_{j-1} a_{n-j+1})^* (a_{j-2} a_{n-j+2})^* \cdots (a_{s+1} a_{s-1})^* a_s^* v.$$

Since $a_j a_i \in X_1$, Lemma 3.9 guarantees that $a_j a_i = \mu_i$ in $K[C_n]/P$ for some $0 \neq \mu_i \in K$. Moreover, we have

$$(a_j a_i)(a_{j-1} a_{i-1}) = a_{j-1} a_j a_i a_{i-1} = a_{j-1} a_j a_{i-1} a_i = (a_{j-1} a_i)(a_j a_{i-1})$$

in C_n. Since $i + j = n + 2$, we get $a_{j-1} a_i = \lambda_i$ and $a_j a_{i-1} = \lambda_{i-1}$ in $K[C_n]/P$. Furthermore, we have $a_j a_i (a_{j-1} a_{i-1} a_j a_i - \lambda_{i-1} \lambda_i) V = 0$. Because $a_j a_i$ acts regularly on V, the last equality yields $(a_{j-1} a_{i-1} a_j a_i - \lambda_{i-1} \lambda_i) V = 0$. Therefore, we conclude that $a_{j-1} a_{n-j+1} = a_{j-1} a_{i-1} = \mu_{i-1}$ in $K[C_n]/P$, where $\mu_{i-1} = \lambda_{i-1} \lambda_i \mu_i^{-1} \neq 0$, hence V is also spanned as a K-linear space by the set

$$(a_{n-1} a_1)^* \cdots (a_j a_{n-j})^* (a_{j-2} a_{n-j+2})^* \cdots (a_{s+1} a_{s-1})^* a_s^* v.$$

We claim that in this case $xV = 0$, where $x = \lambda_i a_j - \mu_i a_{j-1}$. To prove this, fix

$$w = (a_{n-1} a_1)^{i_1} \cdots (a_j a_{n-j})^{i_{n-j}} (a_{j-2} a_{n-j+2})^{i_{n-j+2}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} v \in V,$$

where $i_1, \dots, i_{n-j}, i_{n-j+2}, \dots, i_s \geq 0$. Our aim is to show that $xw = 0$.

Assume first that $j = s + 2$ (and, consequently, $i = s$). Because a_{j-1} and a_j commute in C_n with all elements $a_{n-1} a_1, \dots, a_{s+2} a_{s-2}$, we have

$$a_{j-1} w = (a_{n-1} a_1)^{i_1} \cdots (a_{s+2} a_{s-2})^{i_{s-2}} a_{j-1} a_s^{i_s} v$$

and

$$a_j w = (a_{n-1} a_1)^{i_1} \cdots (a_{s+2} a_{s-2})^{i_{s-2}} a_j a_s^{i_s} v.$$

So it is enough to show that $xw' = 0$, where $w' = a_s^{i_s} v$. But $a_{j-1} v = a_j v = 0$ and $a_{j-1} a_s v = \lambda_i v$, $a_j a_s v = \mu_i v$ imply that

$$a_{j-1} w' = \begin{cases} \lambda_i a_s^{i_s - 1} v & \text{if } i_s > 0, \\ 0 & \text{if } i_s = 0 \end{cases} \qquad \text{and} \qquad a_j w' = \begin{cases} \mu_i a_s^{i_s - 1} v & \text{if } i_s > 0, \\ 0 & \text{if } i_s = 0. \end{cases}$$

Hence the result follows in this case.

Now, let $j > s + 2$. Because a_{j-1} and a_j commute in C_n with all elements $a_{n-1} a_1, \dots, a_j a_{n-j}$, we have

$$a_{j-1} w = (a_{n-1} a_1)^{i_1} \cdots (a_j a_{n-j})^{i_{n-j}} a_{j-1} (a_{j-2} a_{n-j+2})^{i_{n-j+2}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} v$$

and

$$a_j w = (a_{n-1} a_1)^{i_1} \cdots (a_j a_{n-j})^{i_{n-j}} a_j (a_{j-2} a_{n-j+2})^{i_{n-j+2}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} v.$$

So it suffices to check that $xw' = 0$, where $w' = (a_{j-2} a_{n-j+2})^{i_{n-j+2}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} v$. Suppose that $i_{n-j+2} > 0$. Then, remembering that $i + j = n + 2$, we get

$$\begin{aligned} a_{j-1} w' &= a_{j-1} a_{j-2} a_{n-j+2} (a_{j-2} a_{n-j+2})^{i_{n-j+2} - 1} (a_{j-3} a_{n-j+3})^{i_{n-j+3}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} v \\ &= (a_{j-1} a_{n-j+2}) a_{j-2} (a_{j-2} a_{n-j+2})^{i_{n-j+2} - 1} (a_{j-3} a_{n-j+3})^{i_{n-j+3}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} v \\ &= \lambda_i a_{j-2} (a_{j-2} a_{n-j+2})^{i_{n-j+2} - 1} (a_{j-3} a_{n-j+3})^{i_{n-j+3}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} v \end{aligned}$$

and

$$\begin{aligned} a_j w' &= a_j a_{j-2} a_{n-j+2} (a_{j-2} a_{n-j+2})^{i_{n-j+2}-1} (a_{j-3} a_{n-j+3})^{i_{n-j+3}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} v \\ &= (a_j a_{n-j+2}) a_{j-2} (a_{j-2} a_{n-j+2})^{i_{n-j+2}-1} (a_{j-3} a_{n-j+3})^{i_{n-j+3}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} v \\ &= \mu_i a_{j-2} (a_{j-2} a_{n-j+2})^{i_{n-j+2}-1} (a_{j-3} a_{n-j+3})^{i_{n-j+3}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} v, \end{aligned}$$

because $a_{j-1}a_{n-j+2} = a_{j-1}a_i = \lambda_i$ and $a_j a_{n-j+2} = a_j a_i = \mu_i$ in $K[C_n]/P$. Hence $xw' = 0$ in this case. Next, assume that $i_{n-j+2} = 0$. If all $i_{n-j+3} = \cdots = i_{s-1} = 0$ then $w' = a_s^{i_s} v$. Since $j > s+2$, we have $a_{j-1}a_s, a_j a_s \in X$. Thus (3.11) gives $a_{j-1}a_s v = a_j a_s v = 0$, because $s > i$. Moreover, we have $a_{j-1}v = a_j v = 0$. Therefore, $a_{j-1}w' = a_j w' = 0$ and, in consequence, $xw' = 0$. Finally, assume that $i_{n-j+2} = 0$ but $i_k > 0$ for some $k \in \{n-j+3, \ldots, s-1\}$, and choose minimal k with this property. In this situation we have $w' = (a_{n-k}a_k)^{i_k} \cdots (a_{s+1}a_{s-1})^{i_{s-1}} a_s^{i_s} v$. Because $k \geq n-j+3$ then both $j-1$ and j are $\geq n-k$, hence $a_{j-1}a_{n-k}a_k = a_{n-k}(a_{j-1}a_k)$ and $a_j a_{n-k} a_k = a_{n-k}(a_j a_k)$ in C_n. Therefore

$$\begin{aligned} a_{j-1} w' &= a_{j-1} a_{n-k} a_k (a_{n-k} a_k)^{i_k - 1} (a_{n-k-1} a_{k+1})^{i_{k+1}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} v, \\ &= a_{n-k} (a_{j-1} a_k) (a_{n-k} a_k)^{i_k - 1} (a_{n-k-1} a_{k+1})^{i_{k+1}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} v, \\ &= a_{n-k} (a_{n-k} a_k)^{i_k - 1} a_{j-1} a_k (a_{n-k-1} a_{k+1})^{i_{k+1}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} v \end{aligned}$$

and

$$\begin{aligned} a_j w' &= a_j a_{n-k} a_k (a_{n-k} a_k)^{i_k - 1} (a_{n-k-1} a_{k+1})^{i_{k+1}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} v. \\ &= a_{n-k} (a_j a_k) (a_{n-k} a_k)^{i_k - 1} (a_{n-k-1} a_{k+1})^{i_{k+1}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} v. \\ &= a_{n-k} (a_{n-k} a_k)^{i_k - 1} a_j a_k (a_{n-k-1} a_{k+1})^{i_{k+1}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} v. \end{aligned}$$

Further, observe that $k \geq n-j+3$ implies that for each $k < l < s$ we have $j-1 > n-l$, hence $a_{j-1}a_k$ and $a_j a_k$ commute with $a_{n-l}a_l$ in C_n. Since $k < s < j-1$ it is also clear that $a_{j-1}a_k$ and $a_j a_k$ commute in C_n with a_s. Thus

$$\begin{aligned} a_{j-1} w' &= a_{n-k} (a_{n-k} a_k)^{i_k - 1} (a_{n-k-1} a_{k+1})^{i_{k+1}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} a_{j-1} a_k v, \\ a_j w' &= a_{n-k} (a_{n-k} a_k)^{i_k - 1} (a_{n-k-1} a_{k+1})^{i_{k+1}} \cdots (a_{s+1} a_{s-1})^{i_{s-1}} a_s^{i_s} a_j a_k v. \end{aligned}$$

Since $a_{j-1}a_k v = a_j a_k v = 0$ (by (3.11), because $k \geq n-j+3 = i+1$ implies that $a_{j-1}a_k, a_j a_k \in X$ and $k > i$), we get $a_{j-1}w' = a_j w' = 0$ and, in consequence, $xw' = 0$. This finishes the proof. □

Recall that a representation of a monoid M in a K-linear space V is said to be monomial, if V admits a basis E such that for each $w \in M$ and each $e \in E$ there exist $\lambda \in K$ and $f \in E$ such that $we = \lambda f$. As a consequence of Proposition 3.2 and Theorem 3.13 we get the following remarkable result.

Corollary 3.14. *Each irreducible representation of the Chinese monoid C_n is monomial.*

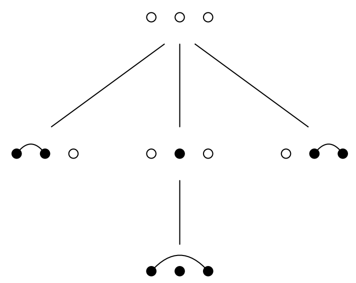
This is in contrast with the results obtained for the, similarly defined, important class of plactic algebras. Namely, in [2] it is shown that the plactic algebra of rank 4 admits irreducible representations which are not monomial. It is also worth to note that all irreducible representations of plactic algebras of rank not exceeding 3 are monomial (see [11]).

4. Illustration of the main theorem for $n \leq 4$

In order to provide more insight into the nature of Theorem 3.13, we interpret it in the case of small values of n. The case $n = 1$ is trivial. Next, it is well known that the Chinese algebra $K[C_2]$ of rank 2 coincides with the plactic algebra of rank 2. Moreover, the irreducible representations of C_2 are easy to describe, as they are induced from irreducible representations of the bicyclic monoid $B \cong C_2/(a_2 a_1 = 1)$. Namely, we have the following result (see [11] for more details).

Remark 4.1. Let V be a simple left $K[C_2]$-module. Then V is 1-dimensional or $V \cong Z$, where Z is the simple left $K[C_2]$-module defined just before Proposition 3.3.

Our next step is to describe all irreducible representations of the monoid C_3. In this case the diagram D has the form



and three leaves of this diagram correspond to the minimal prime ideals of $K[C_3]$:

$$P_1 = (a_2, a_3 \text{ commute, } a_2a_1 \text{ central}),$$
$$P_2 = (a_2 \text{ central}),$$
$$P_3 = (a_1, a_2 \text{ commute, } a_3a_2 \text{ central}).$$

Here, writing for example 'a_2, a_3 commute' in P_1 we mean that P_1 contains the element $a_2a_3 - a_3a_2$. Similarly, writing 'a_2a_1 central' we understand that P_1 contains all elements of the form $a_ia_2a_1 - a_2a_1a_i$ for $i = 1, 2, 3$. The same convention applies to other minimal prime ideals of $K[C_3]$.

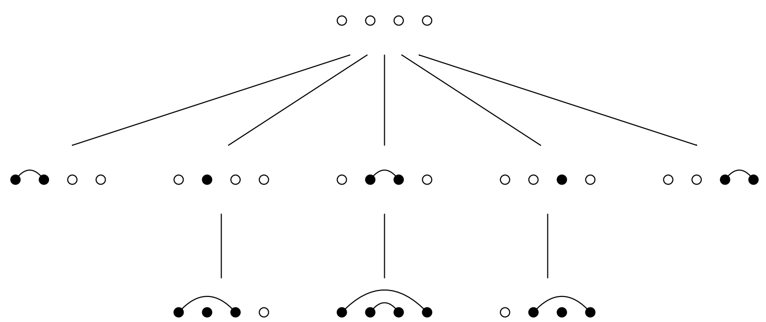
Hence, by Remark 4.1 and the results from Section 3, we get the following classification.

Remark 4.2. Let V be a simple left $K[C_3]$-module. Then V is 1-dimensional or there exists a basis $\{e_i : i \geq 0\}$ of V such that exactly one of the following possibilities holds:

(1) there exist $\lambda, \mu \in K$ such that $\lambda \neq 0$ and $a_1e_i = \lambda e_{i+1}$, $a_2e_i = e_{i-1}$, $a_3e_i = \mu e_{i-1}$ for all $i \geq 0$.
(2) there exist $\lambda, \mu \in K$ such that $\lambda \neq 0$ and $a_1e_i = \lambda e_{i+1}$, $a_2e_i = \mu e_i$, $a_3e_i = e_{i-1}$ for all $i \geq 0$.
(3) there exist $\lambda, \mu \in K$ such that $\mu \neq 0$ and $a_1e_i = \lambda e_{i+1}$, $a_2e_i = \mu e_{i+1}$, $a_3e_i = e_{i-1}$ for all $i \geq 0$.

Note that, to make our statements more compact, we adopted the convention that $e_{-1} = 0$.

Finally, let us describe all irreducible representations of the monoid C_4. In this situation the diagram D has the form



and five leaves of this diagram correspond to the minimal prime ideals of $K[C_4]$:

$$P_1 = (a_2, a_3, a_4 \text{ commute, } a_2a_1, a_3a_1 \text{ central}),$$
$$P_2 = (a_3, a_4 \text{ commute, } a_2, a_3a_1 \text{ central}),$$
$$P_3 = (a_1, a_2 \text{ commute, } a_3, a_4 \text{ commute, } a_3a_2 \text{ central}),$$
$$P_4 = (a_1, a_2 \text{ commute, } a_3, a_4a_2 \text{ central}),$$
$$P_5 = (a_1, a_2, a_3 \text{ commute, } a_4a_2, a_4a_3 \text{ central}).$$

Now, Remark 4.2 together with the results obtained in Section 3 lead to the following classification.

Remark 4.3. Let V be a simple left $K[C_4]$-module. Then V is 1-dimensional or there exists a basis $\{e_{i,j} : i, j \geq 0\}$ of V and $0 \neq \lambda, \mu \in K$ such that

$$a_1e_{i,j} = \lambda e_{i+1,j+1}, \qquad a_2e_{i,j} = \mu e_{i,j+1}, \qquad a_3e_{i,j} = e_{i,j-1}, \qquad a_4e_{i,j} = e_{i-1,j-1}$$

for all $i, j \geq 0$ (with the convention that $e_{i,j} = 0$ if $i = -1$ or $j = -1$), or there exists a basis $\{e_i : i \geq 0\}$ of V such that exactly one of the following possibilities holds:

(1) there exist $\lambda, \mu, \nu \in K$ such that $\lambda \neq 0$ and $a_1e_i = \lambda e_{i+1}$, $a_2e_i = e_{i-1}$, $a_3e_i = \mu e_{i-1}$, $a_4e_i = \nu e_{i-1}$ for all $i \geq 0$.

(2) there exist $\lambda, \mu, \nu \in K$ such that $\lambda \neq 0$ and $a_1 e_i = \lambda e_{i+1}$, $a_2 e_i = \mu e_i$, $a_3 e_i = e_{i-1}$, $a_4 e_i = \nu e_{i-1}$ for all $i \geq 0$.

(3.1) there exist $\lambda, \mu, \nu \in K$ such that $\mu \neq 0$ and $a_1 e_i = \lambda e_{i+1}$, $a_2 e_i = \mu e_{i+1}$, $a_3 e_i = e_{i-1}$, $a_4 e_i = \nu e_{i-1}$ for all $i \geq 0$.

(3.2) there exist $\lambda, \mu, \nu \in K$ such that $\lambda \neq 0$ but $\mu\nu = 0$ and $a_1 e_i = \lambda e_{i+1}$, $a_2 e_i = \mu e_i$, $a_3 e_i = \nu e_i$, $a_4 e_i = e_{i-1}$ for all $i \geq 0$.

(4) there exist $\lambda, \mu, \nu \in K$ such that $\mu \neq 0$ and $a_1 e_i = \lambda e_{i+1}$, $a_2 e_i = \mu e_{i+1}$, $a_3 e_i = \nu e_i$, $a_4 e_i = e_{i-1}$ for all $i \geq 0$.

(5) there exist $\lambda, \mu, \nu \in K$ such that $\nu \neq 0$ and $a_1 e_i = \lambda e_{i+1}$, $a_2 e_i = \mu e_{i+1}$, $a_3 e_i = \nu e_{i+1}$, $a_4 e_i = e_{i-1}$ for all $i \geq 0$.

Note that, as in Remark 4.2, we used the convention that $e_{-1} = 0$. Moreover, it is worth to notice that modules in family (i), for $i = 1, 2, 4, 5$, contain in their annihilators the ideal P_i. Furthermore, modules in both families (3.1) and (3.2) contain P_3 in their annihilators.

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