

# TRIPLE CRYSTAL ACTION IN FOCK SPACES

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ABSTRACT. We make explicit a triple crystal structure on higher level Fock spaces, by investigating at the combinatorial level the actions of two affine quantum groups and of a Heisenberg algebra. To this end, we first determine a new indexation of the basis elements that makes the two quantum group crystals commute. Then, we define a so-called Heisenberg crystal, commuting with the other two. This gives new information about the representation theory of cyclotomic rational Cherednik algebras, relying on some recent results of Shan and Vasserot and of Losev. In particular, we give an explicit labelling of their finite-dimensional simple modules.

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## 1. INTRODUCTION

Since Ariki's proof [1] of the LLT conjecture [16], it is understood that higher level Fock spaces representations of  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$  have, via categorification, a very important impact on our understanding of some structures related to complex reflection groups. More precisely, if  $\mathcal{F}_{\mathbf{s}_l, e}$  is the level  $l$  Fock space representation of  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$  with multicharge  $\mathbf{s}_l$  and  $V(\mathbf{s}_l)$  the irreducible highest weight submodule of  $\mathcal{F}_{\mathbf{s}_l, e}$  of weight  $\Lambda_{\mathbf{s}_l}$  (determined by  $\mathbf{s}_l$ ), then one can compute the decomposition numbers for the corresponding Ariki-Koike algebra by specialising at  $q = 1$  Kashiwara's canonical basis of  $V(\mathbf{s}_l)$ .

The Fock space itself is no longer irreducible, but one can however define a canonical basis for it, which turns out to give, at  $q = 1$ , the decomposition numbers of a corresponding  $q$ -Schur algebra, as was proved by Varagnolo and Vasserot [27], hence generalising Ariki's result.

The introduction of quiver Hecke algebras by Rouquier [21] and by Khovanov and Lauda [15] has shed some new light about the role of the parameter  $q$ . In fact, quiver Hecke algebras are graded, and graded versions of these results (which do not require to specialise  $q$  at 1) hold for these structures, see [2].

Moreover, Ariki's categorification theorem also permits to interpret the Kashiwara crystal of  $V(\mathbf{s}_l)$  as the branching rule for the associated Ariki-Koike algebra. Shan has proved in [23] that the crystal of the whole Fock space is also categorified by a branching rule, but for another structure, namely a corresponding cyclotomic rational Cherednik algebra.

Very recently, Dudas, Varagnolo and Vasserot [3] have proved a similar result suggested by Gerber, Hiss and Jacon [7] in the context of finite unitary groups. In this case, there is a notion of parabolic (or Harish-Chandra) induction for unipotent representations which, provided one works with the appropriate Levi subgroups, defines a branching graph which also categorifies the crystal of  $\mathcal{F}_{\mathbf{s}_l, e}$ .

Therefore, the study of  $\mathcal{F}_{\mathbf{s}_l, e}$ , and in particular of its crystal structure (which yields the theory of canonical bases in Kashiwara's theory [14]) is crucial for approaching fundamental problems in the representation theory of many classical algebraic structures.

In Uglov's paper [26], canonical bases of higher level Fock spaces have been thoroughly studied, generalising Leclerc and Thibon's results in level one in [17] and [18]. In his work, the Fock space is identified to a subspace of the space of semi-infinite  $q$ -wedges  $\Lambda^s$ . This space  $\Lambda^s$  is essentially the direct sum of all Fock spaces  $\mathcal{F}_{\mathbf{s}_l, e}$  over all  $l$ -charges  $\mathbf{s}_l$  whose components sum up to  $s$ . Three algebras act on  $\Lambda^s$ : two quantum groups, namely  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$  and  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ , where  $p = -1/q$ , and a Heisenberg algebra  $\mathcal{H}$ . A fundamental result is that these three actions are pairwise commutative [26, Proposition 4.6], and that the Fock space can be decomposed in a very simple way: it suffices to act on the empty multipartition by the three algebras for some restricted values of the multicharge to reach any vector [26, Theorem 4.8].

This emphasizes the relevance of considering  $\Lambda^s$  not only as a  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -module, but also as a  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -module and as a  $\mathcal{H}$ -module. This triple module structure is well-defined because one has a three natural ways to index the elements of  $\Lambda^s$ : either by partitions, by  $l$ -partitions, or by  $e$ -partitions. The action of  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$  is understood provided the correspondence between  $l$ -partitions and  $e$ -partitions is known, which is the case (it is explicit and based on taking  $e$ -quotients and "modified"  $l$ -quotients). This double quantum group structure is referred to as "level-rank" duality, and has been investigated in particular in the works of Rouquier, Shan, Varagnolo and Vasserot [22] and Webster [28].

The action of  $\mathcal{H}$  has been less studied, but has recently been proved to have important applications when it comes to the representation theory of rational Cherednik algebras. More precisely, Shan and Vasserot [24] have categorified the Heisenberg action on the Fock space, and used it to characterise finite-dimensional simple modules for the corresponding cyclotomic Cherednik algebra. In a recent preprint [20], Losev has given a combinatorial interpretation of this categorical action, but without using Uglov's approach to the Fock space. Finally, in the context of [7] and [3], some combinatorial observations suggest that  $\mathcal{H}$  might play a role in the study of unipotent representations of finite unitary groups, by relating the notion of weak cuspidality to the classical one.

However, Uglov's work says very little about crystals, and it is not clear how level-rank duality nor the action of  $\mathcal{H}$  is expressed at the crystal level.

The aim of this paper two-fold. First, complete Uglov's study of higher level Fock spaces at the crystal level. This is achieved by explicitly determining a triple crystal structure which yields a nice decomposition of the whole crystal in the spirit of [26, Proposition 4.6] and [26, Theorem 4.8]. This requires to make explicit the  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -crystal structure commuting with the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal, and to define an appropriate notion of *Heisenberg crystal* which shall commute with both affine quantum group crystals. Second, place it into the context of the representation theory of Cherednik

algebras to deduce new results using explicit combinatorics. In order to do this, we must in particular prove a compatibility with a recent result of Losev [20]. Throughout the paper, we will give a significant amount of examples to illustrate the various notions and procedures that we introduce.

This article contains the following main points. In Section 2, we recall the important combinatorial notions used in all the paper, in particular Uglov’s algorithms which permit to juggle the different indexations of the semi-infinite  $q$ -wedges. This section does not contain any new material, however we take some time to introduce all notions carefully.

In Section 3, we recall the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -module structure on higher level Fock space which was explicated in [11]. Note that this requires an order on  $i$ -boxes of multipartitions which gives the so-called “Uglov” realisation of the Fock space, which is not consensual in the literature (cf Remark 4.9). We also introduce the conjugation procedure, which is capital, and the correspondence (3.7). Essentially, it enables to put a new  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ - and  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -structure on the Fock space, which is the appropriate one for our purpose of studying crystals.

Section 4 gives a crystal version of level-rank duality. The  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -crystal graph rule commuting with the classic  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal is explicated (Theorem 4.8). The crystal operators of  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$  therefore give  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal isomorphisms in level  $l$  Fock spaces, adding to the list in [6]. It is explicit and easy to describe on  $l$ -partitions. The use of Correspondence (3.7) is indispensable.

In Section 5, we study “doubly highest weight vertices”, that is to say, vertices that are simultaneously highest weight vertices in the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal and in the  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -crystal. We use a result of Jacon and Lecouvey [9, Theorem 5.9] to characterise these multipartitions, and then give some essential properties. The proofs there are quite technical and require a careful analysis of the correspondence (3.7).

Section 6 is devoted to defining the Heisenberg crystal. We start by introducing maps  $\tilde{b}_{-\kappa}$  and  $\tilde{b}_{\sigma}$  which shift periods to the left and to the right respectively in the abacus representation of a multipartition. These are simultaneously  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ - and  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -crystal isomorphisms (Theorems 6.9 and 6.14). The Heisenberg crystal is then defined as a graph where the arrows are given by the action of “Heisenberg operators” which are refined versions of  $\tilde{b}_{-\kappa}$  and  $\tilde{b}_{\sigma}$ . We end with a decomposition theorem (Theorem 6.21) which is an analogue of [26, Theorem 4.8].

Finally, Section 7 relates the various results of the previous sections to the representation theory of rational Cherednik algebras. We first give a general interpretation of the crystal level-rank duality. Then, we show that Losev’s independent results on the crystal version of the Heisenberg action [20] are compatible with those of Section 6, this is Theorem 7.6. This enables us to give an explicit characterisation of the finite-dimensional simple modules for the cyclotomic rational Cherednik algebras using the notion of FLOTW multipartitions (Theorem 7.7).

2. GENERAL COMBINATORICS

2.1. Charged multipartitions.

Let  $l$  be a positive integer. An  $l$ -charge (or simply multicharge) is an  $l$ -tuple  $\mathbf{s}_l = (s_1, \dots, s_l)$  of integers. An  $l$ -partition (or simply multipartition) is an  $l$ -tuple  $\boldsymbol{\lambda}_l = (\lambda^1, \dots, \lambda^l)$  of partitions. One considers that a partition has an infinite number of size zero parts. The set of partition will be denoted by  $\Pi$  and the set of  $l$ -partitions by  $\Pi_l$ . The rank of  $\boldsymbol{\lambda}_l$  is the sum of the ranks of the partitions  $\lambda^j$ ,  $j = 1, \dots, l$ . A charged  $l$ -partition is the data of an  $l$ -charge  $\mathbf{s}_l$  and an  $l$ -partition  $\boldsymbol{\lambda}_l$ , denoted  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ . It can be represented by an  $l$ -tuple of Young diagrams (corresponding to  $\boldsymbol{\lambda}_l$ ), whose boxes  $(a, b, j)$  (where  $a$  is the row of the box,  $b$  is its column and  $j$  its component) are filled with the integers  $b - a + s_j$ . This integer is called the *content* of the box  $(a, b, j)$ .

*Example 2.1.* Take  $l = 2$ ,  $\mathbf{s}_l = (-1, 2)$  and  $\boldsymbol{\lambda}_l = (2.1, 1^2)$ . Then

$$|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle = \left( \begin{array}{|c|c|} \hline -1 & 0 \\ \hline -2 & \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} \right)$$

2.2. Abacci.

Equivalently, one can use  $\mathbb{Z}$ -graded abacci to represent a charged multipartition, following [10]. Given a charged  $l$ -partition  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ , we can compute, for each  $j = 1 \dots, l$ , the numbers  $\beta_k^j = \lambda_k^j + s_j - k + 1$  for  $k \geq 0$  where the  $\lambda_k^j$  ( $k \geq 1$ ) are the parts of  $\lambda^j$ . For each  $j$ , this give a set of  $\beta$ -numbers for  $\lambda_j$  in the sense of [10], which is infinite by the convention that  $\lambda^j$  has an infinite number of size zero parts. Note that these  $\beta$ -numbers are precisely the “virtual” contents that appear just to the right of the border of  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  in its Young diagram representation.

Formally, we define the abacus  $\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$  to be the subset of  $\{1, \dots, l\} \times \mathbb{Z}$  defined by

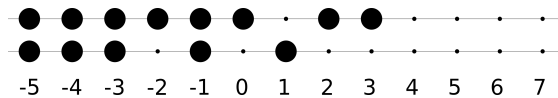
$$\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l) = \{(j, \beta_k^j), j \in \{1, \dots, l\}, k \geq 0\}.$$

We can depict  $\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$  by drawing  $l$  horizontal  $\mathbb{Z}$ -graded rows, numbered from bottom to top, and by drawing a bead on row  $j$  at position  $\beta_k^j$  for all  $j = 1, \dots, l$  and for all  $k \geq 0$ .

*Example 2.2.* Take the same values as Example 2.1. Then the  $\beta$ -numbers are given by

$$\begin{aligned} j = 2 : & \quad (\dots, -5, -4, -3, -2, -1, 0, 2, 3) \\ j = 1 : & \quad (\dots, -5, -4, -3, -1, 1) \end{aligned}$$

and we get the following abacus



From this abacus, one recovers the  $l$ -charge by shifting all beads to the left and by looking at the position of the rightmost bead on each row; and the partition  $\lambda^j$  (for

all  $j = 1 \dots, l$ ) by counting the number of empty spots to the left of each bead on the  $j$ -th row.

### 2.3. Uglov's algorithms.

In this section, we want  $l \geq 2$  and we fix another integer  $e \geq 2$ . Following Uglov [26], we explain a way to associate to a charged  $l$ -partition a charged 1-partition, as well as a charged  $e$ -partition.

Consider the  $l$ -abacus representing a charged  $l$ -partition  $|\lambda_l, \mathbf{s}_l\rangle$ . Divide it into rectangles  $R_k$ , with  $k \in \mathbb{Z}$ , of size  $e \times l$  such that each rectangle contains the positions  $(j, (k-1)e+1), (j, (k-1)e+2), \dots, (j, ke)$  for some  $k \in \mathbb{Z}$  and for all  $j \in \{1, \dots, l\}$ . Then for  $(j, c) \in R_k$ , set  $\tau_l^{-1}(j, c) = (1, c - e(j-1) + elk)$ . Then one can show (see [26]) that  $\tau_l^{-1}$  is a bijection between  $\{1, \dots, l\} \times \mathbb{Z}$  and  $1 \times \mathbb{Z}$ , and we denote  $\tau_l$  its inverse (whose formula can also be explicated). In fact,  $\tau_l^{-1}(\mathcal{A}(\lambda_l, \mathbf{s}_l))$  is a 1-abacus representing a charged partition, which we denote  $|\lambda, s\rangle$ . It is easy to see that  $s = \sum_{j=1}^l s_j$ .

Starting from  $|\lambda, s\rangle$ , we can define a variant of  $\tau_l$  which uses  $e$ . For  $(1, c) \in \mathcal{A}(\lambda, s)$ , set  $\tau_e(1, c) = ((-c) \bmod e + 1, (c - c \bmod e)/e + 1)$ . Then  $\tau_e$  is also bijection between  $1 \times \mathbb{Z}$  and  $\{1, \dots, e\} \times \mathbb{Z}$ . In fact,  $\tau_e(\mathcal{A}(\lambda, s))$  is an  $e$ -abacus representing a charged  $e$ -partition, which we denote  $|\lambda_e, \mathbf{s}_e\rangle$ , and we also have  $s = \sum_{i=1}^e s_i$  if  $\mathbf{s}_e = (s_1, \dots, s_e)$ .

*Example 2.3.* We illustrate these procedures on Example 2.1. Take for instance  $e = 3$ . We see in Figure 1 that  $\tau_l$  “stacks horizontally” the elements of  $\mathcal{A}(\lambda, s)$  into rectangles of height  $l$  and width  $e$ , and that  $\tau_e$  “stacks vertically” the elements of  $\mathcal{A}(\lambda, s)$  into rectangles of height  $e$  and width  $l$ . Notice also that the composition  $\tau_e \circ \tau_l^{-1}$  consists in flipping each rectangle through the diagonal joining the top left corner to the bottom right corner, and then “regluing” the rectangles to get an abacus.

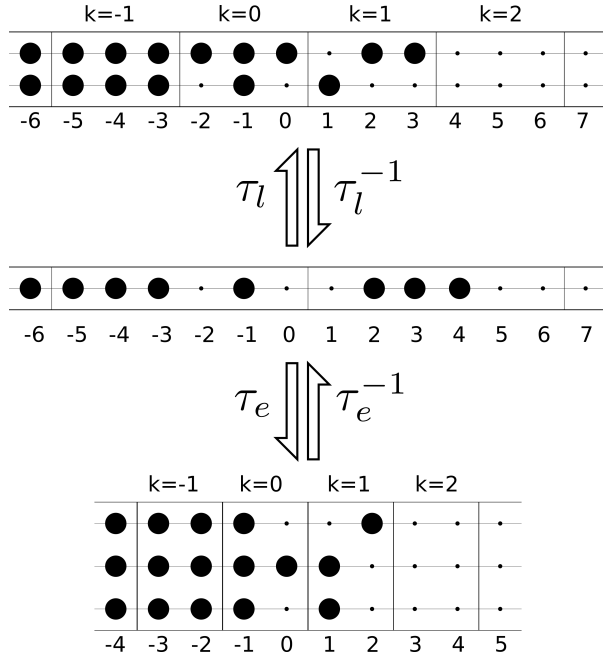


FIGURE 1. The bijections  $\tau_l$  and  $\tau_e$

*Remark 2.4.* Notice that shifting a bead one step to the left in  $|\lambda_e, \mathbf{s}_e\rangle$  amounts to removing an  $e$ -ribbon in  $|\lambda, s\rangle$ . In fact, the bijection  $\tau_e$  gives the  $e$ -quotient (in the sense of [10]) of the partition  $|\lambda, s\rangle$ . The  $e$ -core of  $\lambda$  is obtained after shifting all beads of  $|\lambda_e, \mathbf{s}_e\rangle$  to the left and computing the associated partition using  $\tau_e^{-1}$ . Note finally that this does not hold for  $\tau_l$  (the definitions of  $\tau_l$  and  $\tau_e$  are different).

**2.4. Addable/removable boxes, residues, and order on boxes.**

We keep the notations of the previous section. Recall that the content of a box  $\gamma = (a, b, j)$  of a multipartition  $|\lambda_l, \mathbf{s}_l\rangle$  is the integer  $\text{cont}(\gamma) = b - a + s_j$ . The *residue* of  $\gamma$  is the integer

$$\text{res}(\gamma) = \text{cont}(\gamma) \pmod{e}.$$

For  $i \in \{0, \dots, e - 1\}$ ,  $\gamma$  is called an  $i$ -box if  $\text{res}(\gamma) = i$ .

A box  $\gamma$  is called *removable* for  $\lambda_l$  if  $\gamma$  is a box of  $\lambda_l$  and if  $\lambda_l \setminus \{\gamma\}$  is still a multipartition. Similarly, it is called *addable* if  $\lambda_l \cup \{\gamma\}$  is still a multipartition. In the abacus, this corresponds to a bead which can be shifted one step to the left (respectively to the right). As seen in Remark 2.4, removing (respectively adding) a box in  $|\lambda_e, \mathbf{s}_e\rangle$  corresponds to removing (respectively adding) an  $e$ -ribbon in  $|\lambda, s\rangle = \tau_e^{-1}(|\lambda_e, \mathbf{s}_e\rangle)$ .

For a charged  $l$ -partition  $|\lambda_l, \mathbf{s}_l\rangle$  and  $i \in \{0, \dots, e - 1\}$ , there is a total order on the set of its removable and addable  $i$ -boxes defined by

$$(2.1) \quad \gamma < \gamma' \Leftrightarrow \begin{cases} \text{cont}(\gamma) < \text{cont}(\gamma') & \text{or} \\ \text{cont}(\gamma) = \text{cont}(\gamma') & \text{and } j > j' \end{cases}$$

where  $\gamma = (a, b, j)$  and  $\gamma' = (a', b', j')$ .

For charged  $l$ -partitions  $|\lambda_l, \mathbf{s}_l\rangle$  and  $|\mu_l, \mathbf{s}_l\rangle$  such that  $\mu_l = \lambda_l \cup \{\gamma\}$  where  $\gamma$  is an addable  $i$ -box of  $|\lambda_l, \mathbf{s}_l\rangle$ , we define the quantities

(2.2)

$$N_i(|\lambda_l, \mathbf{s}_l\rangle) = \#\{\text{addable } i\text{-boxes of } |\lambda_l, \mathbf{s}_l\rangle\} \\ - \#\{\text{removable } i\text{-boxes of } |\lambda_l, \mathbf{s}_l\rangle\}$$

$$N_i^{<}(|\lambda_l, \mathbf{s}_l\rangle, |\mu_l, \mathbf{s}_l\rangle) = \#\{\text{addable } i\text{-boxes } \gamma' \text{ of } |\lambda_l, \mathbf{s}_l\rangle \text{ such that } \gamma' < \gamma\} \\ - \#\{\text{removable } i\text{-boxes } \gamma' \text{ of } |\mu_l, \mathbf{s}_l\rangle \text{ such that } \gamma' < \gamma\}$$

$$N_i^{>}(|\lambda_l, \mathbf{s}_l\rangle, |\mu_l, \mathbf{s}_l\rangle) = \#\{\text{addable } i\text{-boxes } \gamma' \text{ of } |\lambda_l, \mathbf{s}_l\rangle \text{ such that } \gamma' > \gamma\} \\ - \#\{\text{removable } i\text{-boxes } \gamma' \text{ of } |\mu_l, \mathbf{s}_l\rangle \text{ such that } \gamma' > \gamma\}.$$

### 3. MODULE STRUCTURES ON THE FOCK SPACE

#### 3.1. $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -action on the level $l$ Fock space.

**Notation 3.1.** For  $s \in \mathbb{Z}$  and  $N \in \mathbb{Z}_{>0}$ , we write

$$\mathbb{Z}^N(s) = \left\{ (x_1, \dots, x_N) \in \mathbb{Z}^N \mid \sum_{k=1}^N x_k = s \right\}.$$

In all what follows, we fix  $s \in \mathbb{Z}$ ,  $l, e \in \mathbb{Z}_{\geq 2}$  and  $q$  be an indeterminate. Set also  $p = -q^{-1}$ . For each  $l$ -charge  $\mathbf{s}_l = (s_1^l, \dots, s_l^l) \in \mathbb{Z}^l(s)$ , consider the level  $l$  Fock space

$$\mathcal{F}_{\mathbf{s}_l, e} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \bigoplus_{\lambda_l \vdash_l n} \mathbb{C}(q) |\lambda_l, \mathbf{s}_l\rangle.$$

**Theorem 3.2** ([11]). *The space  $\mathcal{F}_{\mathbf{s}_l, e}$  is an integrable  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -module with respect to the following action:*

$$(3.3) \quad \begin{aligned} t_i |\lambda_l, \mathbf{s}_l\rangle &= q^{N_i(|\lambda_l, \mathbf{s}_l\rangle)} |\lambda_l, \mathbf{s}_l\rangle \\ e_i |\lambda_l, \mathbf{s}_l\rangle &= \sum_{\text{res}(\lambda_l \setminus \mu_l) = i} q^{-N_i^{<}(|\lambda_l, \mathbf{s}_l\rangle, |\mu_l, \mathbf{s}_l\rangle)} |\mu_l, \mathbf{s}_l\rangle \\ f_i |\lambda_l, \mathbf{s}_l\rangle &= \sum_{\text{res}(\mu_l \setminus \lambda_l) = i} q^{N_i^{>}(|\mu_l, \mathbf{s}_l\rangle, |\lambda_l, \mathbf{s}_l\rangle)} |\mu_l, \mathbf{s}_l\rangle \end{aligned}$$

*Remark 3.3.* These formulas arise from the choice of a coproduct  $\Delta$  on  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ . Namely, we have here chosen  $\Delta(t_i) = t_i \otimes t_i$ ,  $\Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i$  and  $\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i$ . In [11], the choice of  $\Delta$  is different. We refer to [13, Section 1.4] for the different possible conventions.

In the sequel, we will also use level  $e$  Fock spaces  $\mathcal{F}_{\mathbf{s}_e, l}$ , where  $\mathbf{s}_e = (s_1^e, \dots, s_e^e) \in \mathbb{Z}^e(s)$  is an  $e$ -charge. They are endowed with the structure of an integrable  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -module via the formulas 3.3 replacing  $q$  by  $p$  and exchanging  $e$  and  $l$ . For clarity, we might further want to use the notation  $\dot{t}_j, \dot{e}_j, \dot{f}_j$  for the generators of  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ .



### 3.2. Uglov's decomposition of Fock spaces.

In Uglov [26], Fock spaces are realised as submodules of  $\Lambda^s$ , the space of semi-infinite  $q$ -wedges. There are several ways to index the basis elements of  $\Lambda^s$ , namely:

- by charged partitions  $|\lambda, s\rangle$ , by identifying the semi-infinite  $q$ -wedge  $u_{k_1} \wedge u_{k_2} \wedge \dots$  with the partition  $\lambda = (\lambda_1, \lambda_2, \dots)$  where  $\lambda_i = k_i - (s + 1 - i)$  for all  $i$ ,
- by charged  $l$ -partitions  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ , via the bijection  $\tau_l$  described in 2.3,
- by charged  $e$ -partitions  $|\boldsymbol{\lambda}_e, \mathbf{s}_e\rangle$ , via the bijection  $\tau_e$  described in 2.3.

According to [26, Section 4.2], there is an action of  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ , of  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ , and of a Heisenberg algebra  $\mathcal{H}$  on  $\Lambda^s$ . Moreover, the action of  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$  (respectively  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ ) on  $\Lambda^s$  induces, via the indexation by  $l$ -partitions (respectively  $e$ -partitions), an action on  $\mathcal{F}_{\mathbf{s}_l, e}$  (respectively  $\mathcal{F}_{\mathbf{s}_e, l}$ ). These actions are precisely the one of Section 3.1. In fact, we have

$$(3.4) \quad \Lambda^s = \bigoplus_{\mathbf{s}_l \in \mathbb{Z}^l(s)} \mathcal{F}_{\mathbf{s}_l, e} \quad \text{and} \quad \Lambda^s = \bigoplus_{\mathbf{s}_e \in \mathbb{Z}^e(s)} \mathcal{F}_{\mathbf{s}_e, l}.$$

#### Notation 3.4.

We denote  $A_{l, e}(s) = \{(s_1^l, \dots, s_l^l) \in \mathbb{Z}^l(s) \mid s_1^l \leq \dots \leq s_l^l < s_1^l + e\}$ .

**Theorem 3.5** ([26, Proposition 4.6 and Theorem 4.8]).

- (1) The actions of  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ ,  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$  and  $\mathcal{H}$  on  $\Lambda^s$  pairwise commute.
- (2) We have the decomposition

$$(3.5) \quad \Lambda^s = \bigoplus_{\mathbf{s}_l \in A_{l, e}(s)} \mathcal{U}'_q(\widehat{\mathfrak{sl}}_e) \otimes \mathcal{H} \otimes \mathcal{U}'_p(\widehat{\mathfrak{sl}}_l) |\emptyset_l, \mathbf{s}_l\rangle.$$

### 3.3. Conjugating multipartitions.

In this section, we modify the indexation of  $q$ -wedges by charged  $e$ -partitions. For a partition  $\lambda$ , denote  $\lambda'$  its conjugate. Using the indexation of semi-infinite  $q$ -wedges by charged partitions, define an anti-linear isomorphism as follows

$$(3.6) \quad \begin{array}{ccc} \Lambda^s & \longrightarrow & \Lambda^{-s} \\ |\lambda, s\rangle & \longmapsto & |\lambda', -s\rangle \\ q & \longmapsto & q^{-1} \end{array}$$

This is an involution of  $\bigoplus_{s \in \mathbb{Z}} \Lambda^s$ . We write  $u'$  for the image of  $u \in \Lambda^s$ .

*Remark 3.6.* Since  $p = -q^{-1}$ , we also have that  $p' = p^{-1}$ .

The new indexation is given by the following procedure:

$$(3.7) \quad \begin{array}{ccccccc} \bigoplus_{\mathbf{s}_l \in \mathbb{Z}^l(s)} \mathcal{F}_{\mathbf{s}_l, e} & \xrightleftharpoons[\tau_l]{\tau_l^{-1}} & \Lambda^s & \xrightleftharpoons[\prime]{\prime} & \Lambda^{-s} & \xrightleftharpoons[\tau_e^{-1}]{\tau_e} & \bigoplus_{\mathbf{s}_e \in \mathbb{Z}^e(s)} \mathcal{F}_{\mathbf{s}_e, l} \\ |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle & \longleftrightarrow & |\lambda, s\rangle & \longleftrightarrow & |\lambda', -s\rangle & \longleftrightarrow & |\boldsymbol{\lambda}'_e, \mathbf{s}'_e\rangle \end{array}$$

where  $\tau_l$  and  $\tau_e$  are the isomorphisms induced by the bijection of Section 2.3. Here, we have decided to use the notation  $|\lambda'_e, \mathbf{s}'_e\rangle$  for the charged  $e$ -partition indexing the semi-infinite  $q$ -wedge, so that it is compatible with that of Section 2.3. Namely if  $|\lambda_e, \mathbf{s}_e\rangle = \tau_e(|\lambda, \mathbf{s}\rangle)$ , then  $\lambda'_e$  is the “conjugate” of  $\lambda_e$  (that is to say, conjugate each component of  $\lambda_e$  and reverse it), and  $\mathbf{s}'_e = (-s_e^e, \dots, -s_2^e, -s_1^e)$  where  $(s_1^e, \dots, s_e^e) = \mathbf{s}_e$ . In other words, the conjugation commutes with  $\tau_e$ . It also commutes with  $\tau_l$ . We set

$$T_{l,e} = \tau_e \circ (\cdot)' \circ \tau_l^{-1} \quad \text{and} \quad T_{e,l} = T_{l,e}^{-1} = \tau_l \circ (\cdot)' \circ \tau_e^{-1}.$$

*Example 3.7.* Take  $l = 3$ ,  $e = 4$ ,  $\lambda_l = (5.1, 3.1, 1)$  and  $\mathbf{s}_l = (0, -1, 1)$ .

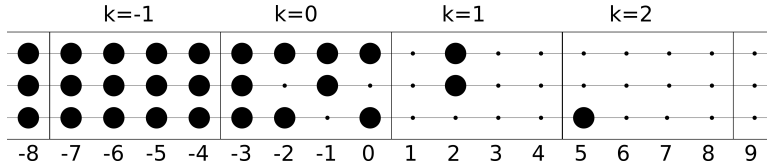


FIGURE 2. The abacus  $\mathcal{A}(\lambda_l, \mathbf{s}_l)$

Applying the procedure described above, we get the following abacus.

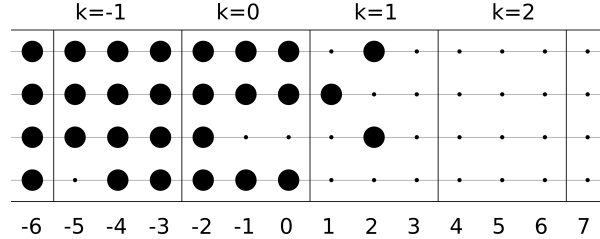


FIGURE 3. The abacus  $\mathcal{A}(\lambda'_e, \mathbf{s}'_e)$

Hence we have  $T_{l,e} = |\lambda'_e, \mathbf{s}'_e\rangle = |(1^5, 3, \emptyset, 1), (-1, -1, 1, 1)\rangle$ .

**Proposition 3.8.** *The action of the Chevalley operators on conjugate charged  $l$ -partitions is given by the following rule.*

$$(3.8) \quad \begin{aligned} e_{-i} |\lambda'_l, \mathbf{s}'_l\rangle &= q^{-N_i(|\lambda_l, \mathbf{s}_l\rangle)+1} (e_i |\lambda_l, \mathbf{s}_l\rangle)' \\ f_{-i} |\lambda'_l, \mathbf{s}'_l\rangle &= q^{N_i(|\lambda_l, \mathbf{s}_l\rangle)+1} (f_i |\lambda_l, \mathbf{s}_l\rangle)', \end{aligned}$$

for all  $i = 0 \dots, e - 1$  and where indices are understood to be modulo  $e$ .

Using Theorem 3.2, one recovers an explicit formula. As usual, we have a similar result for the Chevalley operators  $\hat{e}_j$  and  $\hat{f}_j$  of  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ .

*Proof.* One way to see it is to use the explicit action of  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$  in terms of removable and addable  $i$ -boxes of Theorem 3.2. Now, it is clear that the conjugation isomorphism (3.6) maps a removable (respectively addable)  $i$ -box of  $|\lambda_l, \mathbf{s}_l\rangle$  to a removable (respectively addable)  $(-i)$ -box of  $|\lambda'_l, \mathbf{s}'_l\rangle$ , and that it reverses the way these boxes are ordered. The result follows.  $\square$

Note that the power of  $q$  appearing can be interpreted as the action of the element  $t_i \in \mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$  according to Theorem 3.2. This way, we recover the claim of [26, Proposition 5.10].

**Theorem 3.9.** *The claim of Theorem 3.5 is also valid when the action of  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$  is computed with respect to the indexation (3.7).*

*Proof.* Theorem 3.5 says that the Chevalley operators of  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$  and  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$  commute on  $\Lambda^s$ . when computed on the basis elements via the correspondence  $|\lambda_l, \mathbf{s}_l\rangle \leftrightarrow |\lambda_e, \mathbf{s}_e\rangle$ . Because of the formulas of Proposition 3.8, they still commute when we use the correspondence  $|\lambda_l, \mathbf{s}_l\rangle \leftrightarrow |\lambda'_e, \mathbf{s}'_e\rangle$ .  $\square$

#### 4. TWO COMMUTING CRYSTALS

##### 4.1. Lower and upper crystal operators.

According to the works of Kashiwara, there are two kinds of crystal operators: *upper* and *lower*, giving rise to upper and lower crystal bases (at  $q = 0$ ). In the original paper [12], upper crystal operators are introduced. In [13], the distinction is made between lower and upper operators, and a relation between upper and lower crystal bases is given. Essentially, each definition is related to the choice of a coproduct on the quantum group. Maybe the most comprehensive exposition is to be found in [14], where Kashiwara furthermore defines lower/upper crystal bases at  $q = 0$  and at  $q = \infty$ . In the literature, the convention of Remark 3.3, and therefore the use of lower crystal operators, is usually preferred.

The original definition of upper crystal operators given by Kashiwara [13, Section 2.4] is

$$(4.9) \quad \begin{aligned} \tilde{e}_i^{up} &= (q_i t_i \Delta_i)^{-1/2} e_i \\ \tilde{f}_i^{up} &= (q_i t_i^{-1} \Delta_i)^{-1/2} f_i \end{aligned} \quad \text{where } \Delta_i = q_i^{-1} t_i + q_i t_i + (q_i - q_i^{-1})^2 e_i f_i - 2.$$

There is a similar definition of the lower crystal operators  $\tilde{e}_i^{low}$  and  $\tilde{f}_i^{low}$ .

Note that there is an alternative definition of the crystal operators, introduced see [13, Section 2.2], which is often used in the literature. The use of either of the versions is conventional, and, importantly, everything coincides at  $q = 0$  (respectively  $q = \infty$ ). In this paper, we prefer to use the definitions (4.9), because in this realisation, the crystal operators are directly defined in terms of the generators  $t_i, e_i, f_i$ . This is not the case for the alternative realisation: the definition then depends on the representation and is less explicit. Since the aim of this section is to make explicit two crystal structures that commute, the original definition is more suitable because we already know that the Chevalley operators of the two considered quantum groups commute by Theorems 3.5 and 3.9. Notice also that our preferred realisation is also the one which was used by Jimbo, Misra, Miwa and Okado [11] to make explicit the crystal graph of higher level Fock spaces in terms of multipartitions.

*Remark 4.1.* At  $q = 0$ , upper and lower crystal operators coincide, and we use the notation  $\tilde{e}_i, \tilde{f}_i$ . This is relevant when it comes to crystal bases.

#### 4.2. Crystal graph of the Fock space.

We do not recall here the definition of lower and upper crystal bases  $(L, B)$  at  $q = 0$  (respectively at  $q = \infty$ ) for integrable  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -modules, and refer to Kashiwara [14]. Here,  $L$  is the so-called crystal lattice,  $B$  is a basis of  $L/qL$  (respectively  $L/q^{-1}L$ ) and the lower and upper crystal operators induce endomorphisms of  $L/qL$  (respectively  $L/q^{-1}L$ ). By Remark 4.1, we denote them simply by  $\tilde{e}_i, \tilde{f}_i$ .

**Theorem 4.2** ([11, Theorem 3.7]). *Let  $A_q \subset \mathbb{Q}(q)$  be the ring of rational functions in  $q$  without pole at 0. The crystal basis of  $\mathcal{F}_{\mathbf{s}_l, e}$  is the pair  $(L_e, B_e)$  where*

$$\begin{aligned} L_e &= \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \bigoplus_{\boldsymbol{\lambda}_l \vdash_l n} A_q |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle \\ B_e &= \{ |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle \pmod{qL_e} ; \boldsymbol{\lambda}_l \vdash_l n \text{ for some } n \in \mathbb{Z}_{\geq 0} \} \end{aligned}$$

For simplicity, we will call  $B_e$  the *crystal* of  $\mathcal{F}_{\mathbf{s}_l, e}$ . Thanks to this theorem, we can identify the set of charged  $l$ -partitions (which is the standard basis of  $\mathcal{F}_{\mathbf{s}_l, e}$ ) with the crystal of  $\mathcal{F}_{\mathbf{s}_l, e}$ . We will do this in the rest of the paper.

The crystal operators induce a graph structure on  $B_e$  by drawing an arrow  $\boldsymbol{\lambda}_l \xrightarrow{i} \boldsymbol{\mu}_l$  whenever  $|\boldsymbol{\mu}_l, \mathbf{s}_l\rangle = \tilde{f}_i |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ , called the *crystal graph* of  $\mathcal{F}_{\mathbf{s}_l, e}$ . Moreover, for a charged  $l$ -partition  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle \in B_e$ , we denote  $B_e(|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle)$  the connected component of  $B_e$  containing  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ .

In order to describe the crystal graph of  $\mathcal{F}_{\mathbf{s}_l, e}$  combinatorially, we need to introduce the notion of *good* boxes for  $l$ -partitions.

Fix  $i \in \{0, \dots, e-1\}$ , and let  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  be a charged  $l$ -partition. Recall that we have defined in Section 2.4 a total order  $<$  on the set of removable and addable  $i$ -boxes of  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ . List the addable and removable  $i$ -boxes of  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  in increasing order with respect to  $<$ , and encode each addable  $i$ -box by a sign  $+$  and each removable  $i$ -box by a sign  $-$ . This yields a word in the letters  $+$  and  $-$ , denoted  $w_i(|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle)$  (or simply  $w_i$ ) and called the  *$i$ -word* of  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ . Now, delete recursively the subwords of the form  $(-+)$  in  $w_i$ , in order to obtain a word of the form  $(+)^{\alpha}(-)^{\beta}$ , denoted  $\widehat{w}_i(|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle)$  (or simply  $\widehat{w}_i$ ) and called the *reduced  $i$ -word* of  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ .

**Definition 4.3.** *The good addable (respectively removable)  $i$ -box of  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  is the box corresponding to the leftmost sign  $-$  (respectively the rightmost sign  $+$ ) in  $\widehat{w}_i$ .*

**Theorem 4.4** ([11, Theorem 3.8]). *We have  $|\boldsymbol{\mu}_l, \mathbf{s}_l\rangle = \tilde{f}_i |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  if and only if  $\boldsymbol{\mu}_l$  is obtained from  $\boldsymbol{\lambda}_l$  by adding its good addable  $i$ -box (if it exists).*

*Example 4.5.* We look again at Example 3.7. The reduced  $i$ -words for  $i = 0, 1, 2, 3$  are

$$\begin{aligned}\widehat{w}_0 &= ++- \\ \widehat{w}_1 &= +- \\ \widehat{w}_2 &= ++ \\ \widehat{w}_3 &= +- \end{aligned}$$

The action of the crystal operators is then depicted in the following abacus:

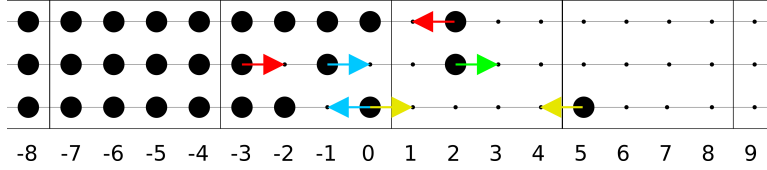


FIGURE 4. The action of  $\tilde{e}_0$  and  $\tilde{f}_0$  (yellow),  $\tilde{e}_1$  and  $\tilde{f}_1$  (red),  $\tilde{f}_2$  (green),  $\tilde{e}_3$  and  $\tilde{f}_3$  (blue)

By level-rank duality, one can switch the roles of  $e$  and  $l$  to describe the crystal graph of the representation  $\mathcal{F}_{s_e, l}$  of  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ . The crystal operators appearing are denoted by  $\tilde{e}_j$  and  $\tilde{f}_j$ .

*Remark 4.6.* Because of the combinatorial definition of  $\tau_e$ , one sees that the action of a crystal operator of  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$  on the corresponding partition is to remove/add a “good”  $l$ -ribbon. The definition of  $\tau_l$  being different, this does not hold for  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ .

### 4.3. Commutation of the crystal operators.

In order to obtain two commuting Kashiwara crystals, we need to work with crystal bases at  $q = 0$ . The crystal graph of  $\mathcal{F}_{s_e, l}$  being defined at  $p = 0$  (i.e. at  $q = \infty$ ), we use the conjugation isomorphism defined in (3.6), which maps  $p$  to  $p^{-1}$ , and use the crystal graph rule of Theorem 4.4 on conjugate  $e$ -partitions in  $\mathcal{F}_{s'_e, l}$ .

**Lemma 4.7.** *The conjugate of the lower crystal basis of  $\mathcal{F}_{s_e, l}$  at  $p = 0$  is the upper crystal basis of  $\mathcal{F}_{s'_e, l}$  at  $q = 0$ .*

*Proof.* By Theorem 4.2, we know the crystal basis of  $\mathcal{F}_{s_e, l}$  at  $p = 0$ . Applying the conjugation isomorphism to  $(L_l, B_l)$ , we obtain a pair  $(L'_l, B'_l)$  where

$$\begin{aligned} L'_l &= \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \bigoplus_{\lambda'_e \vdash_e n} A_{p^{-1}} |\lambda'_e, s'_e\rangle \\ B'_l &= \{ |\lambda'_e, s'_e\rangle \pmod{p^{-1}L'_l}; \lambda'_e \vdash_e n \text{ for some } n \in \mathbb{Z}_{\geq 0} \}, \end{aligned}$$

which is exactly the crystal basis of  $\mathcal{F}_{s'_e, l}$  at  $p = \infty$ . Moreover, we have  $A_{p^{-1}} = A_q$  and  $L'_l/p^{-1}L'_l = L'_l/qL'_l$ . Hence,  $(L'_l, B'_l)$  is the crystal basis of  $\mathcal{F}_{s'_e, l}$  at  $q = 0$ .

More precisely, it sends the lower crystal basis to the upper crystal basis because by definition, one passes from one to the other by reversing the tensor product rule, see [14]. This amounts to reversing the order on  $j$ -boxes and exchanging  $j$  and  $-j$  in the

action of  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ , which is exactly what the conjugation does in light of Proposition 3.8  $\square$

**Theorem 4.8.** *For all  $i \in \{0, e - 1\}$  and all  $j \in \{0, l - 1\}$ , the crystal operators  $\tilde{e}_i, \tilde{f}_i$  and  $\tilde{e}_j, \tilde{f}_j$  commute when computed with respect to the indexation (3.7).*

As a consequence, the procedure (3.7) combined with the crystal graph rule for level  $e$  Fock spaces gives rise to a  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal isomorphism in the sense of [6].

*Proof.* By Theorem 3.9, the Chevalley operators of  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$  and  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$  commute on  $\Lambda^s$  when we use the correspondence  $|\lambda_l, \mathbf{s}_l\rangle \leftrightarrow |\lambda'_e, \mathbf{s}'_e\rangle$ .

Now, by Lemma 4.7, the upper crystal basis  $(L'_l, B'_l)$  of  $\mathcal{F}_{\mathbf{s}'_e, l}$  at  $q = 0$  is known, and coincides with the lower crystal basis  $(L_e, B_e)$  of  $\mathcal{F}_{\mathbf{s}_l, e}$  by identifying  $|\lambda_l, \mathbf{s}_l\rangle$  and  $|\lambda'_e, \mathbf{s}'_e\rangle$ . We can then look at the action of the crystal operators  $\tilde{e}_i, \tilde{f}_i$  and  $\tilde{e}_j, \tilde{f}_j$  on this basis.

Using their definition in terms of the Chevalley generators (4.9), which commute for this indexation, we are ensured that the  $\tilde{e}_i, \tilde{f}_i$  commute with the  $\tilde{e}_j, \tilde{f}_j$  provided we use the correspondence  $|\lambda_l, \mathbf{s}_l\rangle \leftrightarrow |\lambda'_e, \mathbf{s}'_e\rangle$ .  $\square$

*Remark 4.9.* It should be possible to state this theorem by sticking with Uglov's indexation  $|\lambda_l, \mathbf{s}_l\rangle \leftrightarrow |\lambda_e, \mathbf{s}_e\rangle$  and introducing some alternative crystal operators, whose combinatorial rule would be described in a similar way as that of Theorem 6.24 but using a "reverse" order on  $i$ -boxes. This approach is sketched in [2, Remark 3.17], but with another realisation of the Fock space. More precisely, they use the so-called *Kleshchev* realisation of the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -module  $\mathcal{F}_{\mathbf{s}_l, e}$ , in opposition to the *Uglov* realisation used in this paper. They differ by the order on  $i$ -boxes (cf Section 2.4) used to define the action of  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$  (the Kleshchev order does not require the notion of residue, and can be seen as an asymptotic order of the Uglov order, see [5]). As a consequence, the Kleshchev level  $l$  Fock space becomes a tensor product of level 1 Fock spaces, which is not our case.

*Example 4.10.* We take the same values as in Example 3.7. The action of the different crystal operators of  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$  on  $|\lambda'_e, \mathbf{s}'_e\rangle$  is depicted on its abacus as follows:

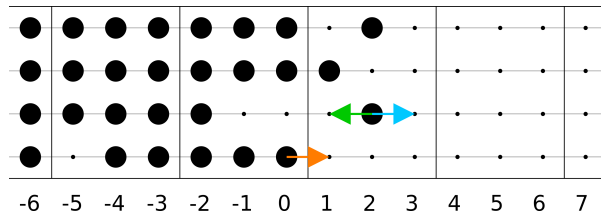


FIGURE 5. The action of  $\tilde{f}_0$  (orange),  $\tilde{e}_1$  (green) and  $\tilde{f}_2$  (blue) on  $\mathcal{A}(\lambda'_e, \mathbf{s}'_e)$

On the  $l$ -abacus representing  $|\lambda_l, \mathbf{s}_l\rangle$ , this gives the following picture:

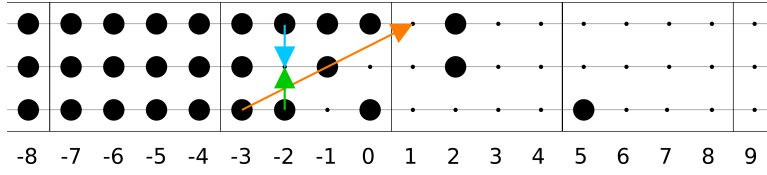


FIGURE 6. The action of  $\tilde{f}_0$  (orange),  $\tilde{e}_1$  (green) and  $\tilde{f}_2$  (blue) on  $\mathcal{A}(\lambda_l, \mathbf{s}_l)$

Take for instance  $|\mu_l, \mathbf{r}_l\rangle = \tilde{f}_0|\lambda_l, \mathbf{s}_l\rangle$ .

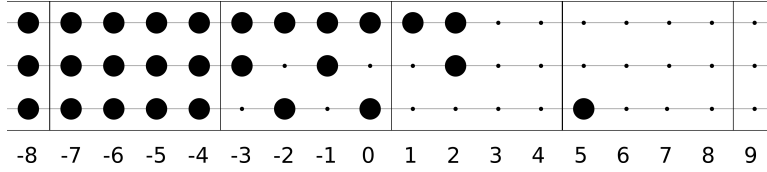


FIGURE 7. The abacus representing the  $l$ -partition  $|\mu_l, \mathbf{r}_l\rangle = \tilde{f}_0|\lambda_l, \mathbf{s}_l\rangle$

Then the reduced  $i$ -words ( $i = 0, 1, 2, 3$ ) for  $|\mu_l, \mathbf{r}_l\rangle$  are

$$(4.10) \quad \begin{aligned} \widehat{w}_0 &= ++- \\ \widehat{w}_1 &= +- \\ \widehat{w}_2 &= ++ \\ \widehat{w}_3 &= +- \end{aligned}$$

These are exactly the  $i$ -words for  $|\lambda_l, \mathbf{s}_l\rangle$ , see Example 4.5.

*Remark 4.11.* Consider the action of the dual crystal operators  $\tilde{e}_j$ . On the  $l$ -abacus, it either shifts a bead one step down (if  $j \neq 0$ ), or  $l$  steps down and  $e$  steps left (if  $j = 0$ ). So these are particular cases of *elementary operations* in the sense of [7, Section 7.3] if  $l = 2$ . This observation actually inspired this part of this paper in the first place: elementary operations are  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal isomorphisms, and they resemble dual crystal operators, so this suggested that there should be a version of the dual crystal operators that commute with the crystal operators  $\tilde{e}_i$  and  $\tilde{f}_i$ . This is indeed made explicit in Theorem 4.8.

There is another already known combinatorial procedure resembling the action of  $\tilde{e}_j$ , namely Tingley's tightening procedure on abacci [25, Definition 3.8]. This is however fundamentally different, in that tightening an abacus consists of systematically applying the procedure described above for all beads, regardless of the order on  $i$ -boxes in the corresponding multipartition.

## 5. DOUBLY HIGHEST WEIGHT VERTICES

### 5.1. A combinatorial characterisation of the $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -highest weight vertices.

According to [9], the highest weight vertices for the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal structure are precisely the charged  $l$ -partitions whose abacus is totally periodic. Respectively, the

same holds for the  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -crystal structure and  $e$ -partitions. Let us recall the notion of totally periodic multipartition, cf [9, Definition 2.2].

Consider the abacus  $\mathcal{A}$  representing a charged multipartition  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ . The first  $e$ -period in  $\mathcal{A}$  is, if it exists, the sequence

$$P = ((j_1, \beta_1), \dots, (j_e, \beta_e))$$

of  $e$  beads in  $\mathcal{A}$  such that

- $\beta_1$  is the greatest  $\beta$ -number appearing in  $\mathcal{A}$ ,
- $\beta_i = \beta_{i-1} - 1$  for all  $i = 2, \dots, e$ ,
- $j_i \leq j_{i-1}$  for all  $i = 2, \dots, e$ ,
- for all  $i = 1, \dots, e$ , there does not exist  $(j_0, \beta_i) \in \mathcal{A}$  such that  $j_0 \leq k_i$ .

The first period of  $\mathcal{A} \setminus P$ , if it exists, is called the second period of  $\mathcal{A}$ . We define similarly the  $k$ -th period of  $\mathcal{A}$  by induction.

The abacus  $\mathcal{A}$  is said to be *totally  $e$ -periodic* if it has infinitely many  $e$ -periods. In this case, there exists a non-negative integer  $N$  such that the abacus obtained from  $\mathcal{A}$  by removing its first  $N$  periods corresponds to the empty multipartition. We call an  $e$ -period  $P$  *trivial* if

$$(j, \beta) \in P \quad \Rightarrow \quad (j, \beta - c) \in \mathcal{A} \text{ for all } c \in \mathbb{Z}_{>0}.$$

In other words, a period is trivial if it encodes only size zero parts.

The key property of periods is the following:

**Proposition 5.1.** *An  $e$ -period does not contribute in the computation of the reduced  $i$ -words (for all  $i = 0, \dots, e - 1$ ).*

*Proof.* Let  $P = ((j_1, \beta_1), \dots, (j_e, \beta_e))$  be the first  $e$ -period in  $\mathcal{A} = \mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$ . Fix a residue  $i \in \{0, \dots, e - 1\}$  and look at the  $i$ -word (respectively reduced  $i$ -word)  $w_i$  (respectively  $\widehat{w}_i$ ) for  $\mathcal{A}$  on the one hand, and the  $i$ -word (respectively reduced  $i$ -word)  $v_i$  (respectively  $\widehat{v}_i$ ) for  $\mathcal{A} \setminus P$  on the other hand. Let us show that  $\widehat{w}_i = \widehat{v}_i$ . Suppose first that  $i = \beta_1 \pmod{e}$ . Then  $(j_1, \beta_1)$  corresponds to a rightmost sign  $+$  in  $w_i$ . Now, either  $(j_e, \beta_e - 1) \in \mathcal{A}$ , in which case  $(j_e, \beta_e - 1) \in \mathcal{A} \setminus P$ ,  $(j_e, \beta_e - 1)$  corresponds to a rightmost sign  $+$  in  $v_i$ ; or  $(j_e, \beta_e - 1) \notin \mathcal{A}$ , in which case  $(j_e, \beta_e)$  corresponds to a sequence  $(-+)$  in  $w_i$ , which simplifies in  $\widehat{w}_i$ . So in both cases,  $\widehat{w}_i = \widehat{v}_i$ .

Suppose now that  $i \neq \beta_1 \pmod{e}$ . Then there is an element  $(j_{k_0}, \beta_{k_0}) \in P$  such that  $i = \beta_{k_0} \pmod{e}$ . If  $j_{k_0+1} = j_{k_0}$ , this means that  $(j_{k_0}, \beta_{k_0})$  and  $(j_{k_0+1}, \beta_{k_0+1})$  do not contribute in  $w_i$ , and therefore neither in  $\widehat{w}_i$ . In the case where  $j_{k_0+1} < j_{k_0}$ , then either  $(j_{k_0}, \beta_{k_0} - 1) \in \mathcal{A}$ , in which case there is a rightmost  $+$  in  $w_i$ , corresponding to  $(j_{k_0+1}, \beta_{k_0+1})$ , which also exists in  $v_i$ , corresponding to  $(j_{k_0}, \beta_{k_0} - 1) \in P$ ; or  $(j_{k_0}, \beta_{k_0} - 1) \notin \mathcal{A}$ , in which case there is a sequence  $(-+)$  in  $P$ , corresponding to the beads  $(j_{k_0}, \beta_{k_0})$  and  $(j_{k_0+1}, \beta_{k_0} - 1)$ , which simplifies in  $\widehat{w}_i$ . Again, in both cases,  $\widehat{w}_i = \widehat{v}_i$ .  $\square$

**Theorem 5.2** ([9, Theorem 5.9]). *The charged  $l$ -partition  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  is a highest weight vertex in the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal if and only if  $\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$  is totally  $e$ -periodic.*



This result also holds by switching  $e$  and  $l$  and replacing  $q$  by  $p$ .

**Corollary 5.3.** *The charged partition  $|\lambda, s\rangle$  is a highest weight vertex in both the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal and the  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -crystal if and only if:*

- (1)  $\mathcal{A}(\lambda_l, s_l)$  is totally  $e$ -periodic, and
- (2)  $\mathcal{A}(\lambda'_e, s'_e)$  is totally  $l$ -periodic.

Such an element is called a doubly highest weight vertex.

*Proof.* This is a direct consequence of Theorem 5.2 together with Theorem 4.8.  $\square$

*Example 5.4.* Take  $l = 3$ ,  $e = 2$ ,  $\lambda_l = (3, 3.1, 1)$  and  $s_l = (-1, 0, 0)$ . Then we have  $\lambda'_e = (2^2.1^3, 2.1^3)$  and  $s'_e = (0, 1)$ .

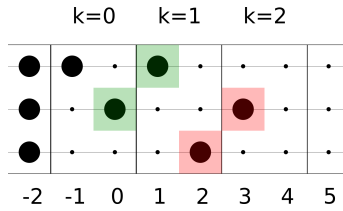


FIGURE 8. The abacus  $\mathcal{A}(\lambda_l, s_l)$

One sees that  $\mathcal{A}(\lambda_l, s_l)$  has two  $e$ -periods, and is totally  $e$ -periodic. The first period corresponds to parts of size 3 (colored in red) and the second to parts of size 1 (colored in green).

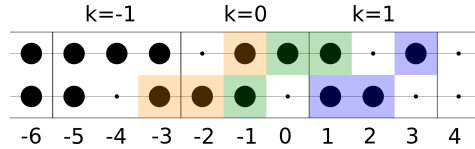


FIGURE 9. The abacus  $\mathcal{A}(\lambda'_e, s'_e)$

Similarly,  $\mathcal{A}(\lambda'_e, s'_e)$  has three  $l$ -periods, and is totally  $l$ -periodic. The first period corresponds to parts of size 2 (blue) and the next two to parts of size 1 (green, orange). The other periods are trivial.

Therefore, the associated charged partition  $|\lambda, s\rangle = |(10.8.4.2), -1\rangle$  is a doubly highest weight vertex.

**5.2. Properties of doubly highest weight vertices.**

We now list some properties of such charged partitions. In what follows, we let  $|\lambda, s\rangle$  be a doubly highest weight vertex.

**Lemma 5.5.**

- (1) *In  $\mathcal{A}(\lambda_l, s_l)$  (respectively  $\mathcal{A}(\lambda'_e, s'_e)$ ), all beads of a given period correspond to parts of the same size. We then denote  $S_k(l)$  (respectively  $S_k(e)$ ) the part size corresponding to the  $k$ -th period in  $\mathcal{A}(\lambda_l, s_l)$  (respectively  $\mathcal{A}(\lambda'_e, s'_e)$ ).*

- (2) Let  $N(l)$  (respectively  $N(e)$ ) be the number of non-trivial periods in  $\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$  (respectively  $\mathcal{A}(\boldsymbol{\lambda}'_e, \mathbf{s}'_e)$ ). We have  $\#\{S_k(l); 1 \leq k \leq N(l)\} = \#\{S_k(e); 1 \leq k \leq N(e)\}$ .

*Proof.* Consider the first  $e$ -period  $P$  of  $\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$ . There are only empty spots to the right of  $P$  as represented in Figure 10.

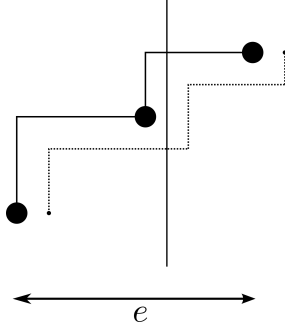


FIGURE 10. The first period of  $\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$ .

Using the correspondence (3.7), we get the abacus  $\mathcal{A}(\boldsymbol{\lambda}'_e, \mathbf{s}'_e)$  in which the empty spots neighbouring  $P$  in  $\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$  give rise to some beads that have to form an  $l$ -period, call it  $P'$ , since  $\mathcal{A}(\boldsymbol{\lambda}'_e, \mathbf{s}'_e)$  is totally  $l$ -periodic. Therefore, the pattern of Figure 11 appears in  $\mathcal{A}(\boldsymbol{\lambda}'_e, \mathbf{s}'_e)$ . Moreover,  $\mathcal{A}(\boldsymbol{\lambda}'_e, \mathbf{s}'_e)$  is full to the left of  $P'$  (i.e. there are no more empty spots, only beads). In other words,  $P'$  is the first trivial  $l$ -period of  $\mathcal{A}(\boldsymbol{\lambda}'_e, \mathbf{s}'_e)$ .

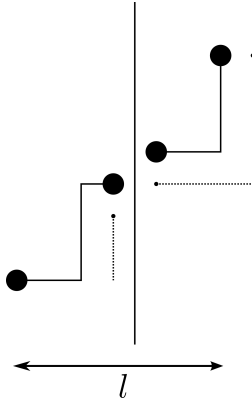


FIGURE 11. The pattern in  $\mathcal{A}(\boldsymbol{\lambda}'_e, \mathbf{s}'_e)$  that arises from the first period of  $\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$ : this is its first trivial period.

Now, assume that the beads of  $P$  do not all encode parts of the same size. That implies that there exists two different positive integers  $M$  and  $N$  (without loss of generality,  $M > N$ ) such that  $M$  is the part encoded by, say, the beads of the form  $(j, \beta) \in P$  for some row index  $j$ , and  $N$  is the part encoded by the beads of the

form  $(j', \beta) \in P$  for some row index  $j'$ . Choose such pair  $(j, j')$  such that the  $|j - j'|$  is minimal. Write  $\delta = M - N > 0$ , so that there are  $\delta$  more empty spots in row  $j$  than in row  $j'$ . Therefore, there are  $\delta$  beads in  $\mathcal{A}(\boldsymbol{\lambda}'_e, \mathbf{s}'_e)$  which do not belong to an  $l$ -period. So  $\mathcal{A}(\boldsymbol{\lambda}'_e, \mathbf{s}'_e)$  is not totally  $l$ -periodic, which is a contradiction. By induction on the number of non-trivial  $e$ -periods in  $\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$ , this is the case for all  $e$ -period.

Obviously, the same holds for the  $l$ -periods in  $\mathcal{A}(\boldsymbol{\lambda}'_e, \mathbf{s}'_e)$  by symmetry, and we have proved Point (1).

Let  $P_k$  be the  $k$ -th  $e$ -period in  $\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$ , for some  $1 \leq k \leq N(l)$ . Then by (1), it corresponds to a part  $S_k(l)$  of  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ . Now, by definition of  $T_{l,e}$  (Section 3.6), it corresponds to  $S_k(l)$   $l$ -periods in  $\mathcal{A}(\boldsymbol{\lambda}'_e, \mathbf{s}'_e)$ , denoted  $P'_{k_1}, \dots, P'_{k_{S_k(l)}}$ , encoding the same part, that is to say

$$S_{k_1}(e) = S_{k_1}(e) = \dots = S_{k_{S_k(l)}}(e).$$

Therefore, each such  $P_k$  encodes a part  $S_k(l)$  which corresponds to a unique part  $S_{k_1}(e)$  in the  $e$ -abacus. This proves point (2).  $\square$

**Corollary 5.6.**

- (1) *The multiplicity of each part in  $\boldsymbol{\lambda}_l$  (respectively  $\boldsymbol{\lambda}'_e$ ) is divisible by  $e$  (respectively  $l$ ).*
- (2) *The rank of  $\lambda$  is divisible by  $el$ .*

*Proof.*

- (1) This is straightforward from Lemma 5.5 (1), since each part of  $\boldsymbol{\lambda}_l$  (respectively  $\boldsymbol{\lambda}'_e$ ) is read off the abacus by looking at each bead of the non-trivial periods.
- (2) By Point (1) above, the multiplicity of each part in  $\boldsymbol{\lambda}'_e$  is divisible by  $l$ . Using Remark 2.4 and the definition of the correspondence 3.7, we can see  $\lambda'$  as the partition with  $e$ -quotient  $\boldsymbol{\lambda}'_e$  and  $e$ -core determined by  $\mathbf{s}'_e$ . Therefore the rank of  $\lambda'$  is divisible by  $el$ , hence so is that of  $\lambda$ .  $\square$

Recall the definition of the domain  $A_{l,e}(s)$  given in Notation 3.4.

**Proposition 5.7.** *We have*

$$\mathbf{s}_l \in A_{l,e}(s) \quad \text{and} \quad \mathbf{s}'_e \in A_{e,l}(s).$$

*Proof.* Recall that the multicharge is read from the abacus by shifting all beads to the left and looking at the index of the rightmost bead in each row of the resulting abacus. By Lemma 5.5, all periods in a doubly highest weight vertex correspond to the same part. Therefore, a doubly highest weight charged  $l$ -partition  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  is obtained from  $|\emptyset_l, \mathbf{s}_l\rangle$  by shifting whole  $e$ -periods to the right. This is a key fact and will be used in what comes next. Therefore, it suffices to prove the claim for vertices of the form  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ . In this case, let us observe the corresponding charged

$e$ -partition  $|\lambda'_e, s'_e\rangle$  defined via Formula (3.7). By definition of  $T_{l,e}$ , if  $\mathbf{s}_l \notin A_{l,e}(s)$ , then:

- either there exists an index  $j$  such that  $s_j > s_{j+1}$ , in which case the difference  $\delta = s_j - s_{j+1}$  creates  $\delta$  empty spots in the  $l$ -abacus which are beads in the  $e$ -abacus but do not form an  $l$ -period,
- or there exists two indices  $j$  and  $j'$  such that  $j' < j$  and  $s_{j'} < s_j - e$ , in which case the difference  $s_j - s_{j'}$  also gives beads in the corresponding  $e$ -abacus which do not form an  $l$ -period.

In both cases,  $\mathcal{A}(\lambda'_e, s'_e)$  is not totally  $l$ -periodic, which is a contradiction, so the claim is proved.  $\square$

**5.3. Shifting periods in abacci.** We now consider the crucial procedure of shifting periods one step to the left. Let  $P$  be an  $e$ -period in  $\mathcal{A}(\lambda_l, \mathbf{s}_l)$  which is shiftable one step to the left. By Lemma 5.5 above, it is equivalent to say that there exists  $(j, \beta) \in P$  such that  $(j, \beta - 1) \notin \mathcal{A}(\lambda_l, \mathbf{s}_l)$  (i.e. there is one empty spot left adjacent to some bead in  $P$ ). Then by the same observation as in the proof of Lemma 5.5, depicted in Figures 10 and 11, there is a corresponding  $l$ -period  $P'$  in  $\mathcal{A}(\lambda'_e, s'_e)$  which is shiftable one step to the left. Define then

$$(5.11) \quad \begin{aligned} \varphi_{l,P} : \mathcal{A}(\lambda_l, \mathbf{s}_l) &\longrightarrow \{1, \dots, l\} \times \mathbb{Z} \\ (j, c) &\longmapsto \begin{cases} (j, c - 1) & \text{if } (j, c) \in P \\ (j, c) & \text{otherwise.} \end{cases} \end{aligned}$$

The image of  $\mathcal{A}(\lambda_l, \mathbf{s}_l)$  under  $\varphi_{l,P}$  is the  $l$ -abacus obtained from  $\mathcal{A}(\lambda_l, \mathbf{s}_l)$  by shifting  $P$  one step to the left. We define  $\varphi_{e,P'}$  similarly, that is to say, the map shifting  $P'$  one step to the left in the  $e$ -abacus  $\mathcal{A}(\lambda'_e, s'_e)$ .

**Lemma 5.8.** *We have*

$$\varphi_{l,P} = T_{e,l} \circ \varphi_{e,P'} \circ T_{l,e}.$$

Note that the use of  $T_{l,e}$  and its inverse only means that we look at the action of  $\varphi_{l,P}$  on the  $e$ -abacus using Indexation 3.7. In fact, this lemma claims that shifting an  $l$ -period of  $\mathcal{A}(\lambda'_e, s'_e)$  one step to the left amounts to shifting an  $e$ -period of  $\mathcal{A}(\lambda_l, \mathbf{s}_l)$  one step to the left. Of course, the same holds by switching  $\mathcal{A}(\lambda'_e, s'_e)$  and  $\mathcal{A}(\lambda_l, \mathbf{s}_l)$  and  $e$  and  $l$ . In particular, this procedure is always well defined when the considered period is the last non-trivial period. This will be used in Section 6.1.

*Proof.* Assume, without loss of generality, that  $l \leq e$ . We have already explained how an  $l$ -period  $P'$  corresponds to a given  $e$ -period  $P$ . Now, write

$$P = \{(j_k, \beta_k); k \in \{1, \dots, e\}\} \subseteq \mathcal{A}(\lambda_l, \mathbf{s}_l).$$

By definition of  $\varphi_{l,P}$  (5.11),  $\varphi_{l,P}$  only affects  $P$ , namely  $\varphi_{l,P}(P)$  is an  $e$ -period  $\widehat{P}$  defined by

$$\widehat{P} = \{(j_k, \beta_k - 1); k \in \{1, \dots, e\}\}.$$

If  $k$  is such that  $j_k = j_{k+1}$  (which is a case that necessarily happens if  $e > l$ ), then  $\beta_{k+1} = \beta_k - 1$ . Moreover,

$$\begin{aligned} \{(j_k, \beta_k), (j_k, \beta_{k+1})\} &\xrightarrow{\varphi_{l,P}} \{(j_k, \beta_k - 1), (j_k, \beta_{k+1} - 1)\} \\ &= \{(j_k, \beta_{k+1}), (j_k, \beta_k - 2)\}, \end{aligned}$$

so  $\varphi_{l,P}$  fixes  $(j_k, \beta_{k+1})$ . Therefore,  $\varphi_{l,P}$  fixes all elements of  $P$  that are on the same row but one, so it moves exactly  $e$  beads, and in fact these are the beads of  $P'$  and they are moved in  $\mathcal{A}(\lambda'_e, s'_e)$  one step to the left. This is precisely the action of  $\varphi_{e,P'}$  on  $\mathcal{A}(\lambda'_e, s'_e)$  by definition.  $\square$

*Example 5.9.* To illustrate the phenomenon explained in the proof of Lemma 5.8, look at Example 5.4. We get the following picture.

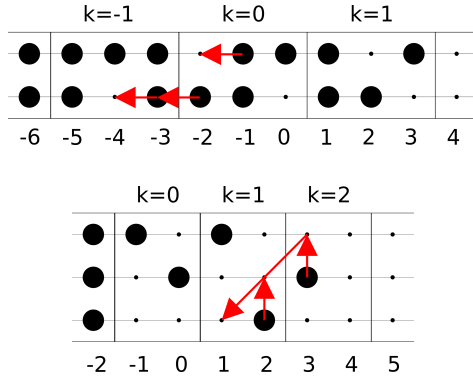


FIGURE 12. Shifting the last non-trivial period of  $\mathcal{A}(\lambda'_e, s'_e)$  one step to the left, and its representation on  $\mathcal{A}(\lambda_l, s_l)$

We see that on  $\mathcal{A}(\lambda_l, s_l)$ , the action depicted with the red arrows actually corresponds to shifting a period of  $\mathcal{A}(\lambda_l, s_l)$  (the first one) one step to the left.

#### 5.4. The partition $\kappa$ .

Denote  $\mathcal{S}_l = \{S_k(l); 1 \leq k \leq N(l)\}$  and  $\mathcal{S}_e = \{S_k(e); 1 \leq k \leq N(e)\}$ . The elements of  $\mathcal{S}_l$  (respectively  $\mathcal{S}_e$ ) are the different non-zero size parts of  $\lambda_l$  (respectively  $\lambda'_e$ ), see Lemma 5.5 (2). Note that we have

$$S_k(l) < S_{k-1}(l) \quad \text{for all } k \in \{2, \dots, N(l)\}.$$

Similarly,

$$S_k(e) < S_{k-1}(e) \quad \text{for all } k \in \{2, \dots, N(e)\}.$$

For  $S_k(l)$  in  $\mathcal{S}_l$  (respectively  $S_k(e)$  in  $\mathcal{S}_e$ ), denote  $M_l(S_k(l))$  (respectively  $M_e(S_k(e))$ ) the multiplicity of the non-zero part  $S_k(l)$  (respectively  $S_k(e)$ ) in the  $l$ -partition  $\lambda_l$  (respectively the  $e$ -partition  $\lambda'_e$ ). By Corollary 5.6,  $M_l(S_k(l))$  (respectively  $M_e(S_k(e))$ ) is divisible by  $e$  (respectively  $l$ ). Let  $m_l(S_k(l))$  (respectively  $m_e(S_k(e))$ ) be the integer  $M_l(S_k(l))/e$  (respectively  $M_e(S_k(e))/l$ ).

Set

$$(5.12) \quad \begin{aligned} \kappa_l &= (S_1(l)^{m_l(S_1(l))}, S_2(l)^{m_l(S_2(l))}, \dots, S_{N(l)}(l)^{m_l(S_{N(l)}(l))}) \\ \kappa_e &= (S_1(e)^{m_e(S_1(e))}, S_2(e)^{m_e(S_2(e))}, \dots, S_{N(e)}(e)^{m_e(S_{N(e)}(e))}) \end{aligned}$$

*Remark 5.10.* Equivalently (and maybe more simply),  $\kappa_l$  can be defined as the ordered multiset  $\{S_k(l); 1 \leq k \leq N(l)\}$  (and similarly for  $\kappa_e$ ).

**Proposition 5.11.** *The sequences  $\kappa_l$  and  $\kappa_e$  are partitions, and  $\kappa_e = \kappa'_l$ .*

*Proof.* Because  $S_k(l) < S_{k-1}(l)$  for all  $k \in \{2, \dots, N(l)\}$  and  $S_k(e) < S_{k-1}(e)$  for all  $k \in \{2, \dots, N(e)\}$  as already observed,  $\kappa_l$  and  $\kappa_e$  are partitions. Further, each period  $P$  in the  $l$ -abacus corresponds to a period  $P'$  in the  $e$ -abacus, and therefore the partition  $\kappa_e$  can be read off the partition  $\kappa_l$ . In fact, by definition of the correspondence  $T_{l,e}$  (3.7), it is obtained by conjugating the original partition  $\kappa_l$ .  $\square$

*Example 5.12.* Take the charged multipartition in Example 5.4. We have  $\mathcal{S}_l = \{3, 1\}$ , with  $M_l(3) = 2$  and  $M_l(1) = 2$ . Similarly, we have  $\mathcal{S}_e = \{2, 1\}$  with  $M_e(2) = 3$  and  $M_e(1) = 6$ . We get  $\kappa_l = (3, 1) = \square\square$  and  $\kappa_e = (2, 1^2) = \square\square$ . Note that using the multiset definition of  $\kappa_l$  and  $\kappa_e$  (Remark 5.10), we have directly  $\kappa_l = \mathcal{S}_l = \{3, 1\}$  and  $\kappa_e = \mathcal{S}_e = \{2, 1, 1\}$ .

We decide to rename  $\kappa = \kappa_l$ .

*Remark 5.13.* Note that  $\kappa$  depends on  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ . In fact, it induces two maps

$$\begin{aligned} \kappa : \quad B_e &\longrightarrow \Pi \\ |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle &\longmapsto \kappa \end{aligned}$$

and

$$\begin{aligned} \kappa : \quad B'_l &\longrightarrow \Pi \\ |\boldsymbol{\lambda}'_e, \mathbf{s}'_e\rangle &\longmapsto \kappa'. \end{aligned}$$

In the rest, we want to use the notation  $\kappa(|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle)$ , or simply  $\kappa(\boldsymbol{\lambda}_l)$  (or  $\kappa(\lambda)$ ). Importantly, note that the map  $\kappa$  is surjective: starting from a partition  $\sigma$ , it is easy to construct a doubly highest weight  $l$ -partition (respectively  $e$ -partition)  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  (respectively  $|\boldsymbol{\lambda}'_e, \mathbf{s}'_e\rangle$ ) such that  $\kappa(\lambda) = \sigma$ , so  $\kappa$  is surjective.

Moreover, if we restrict  $\kappa$  to the set of doubly highest weight vertices, it is clearly injective since two doubly highest weight  $l$ -partitions with different  $\kappa$  are different. So  $\kappa$  restricted to the set of doubly highest weight vertices is a bijection.

We end this section on a refinement of Corollary 5.6.

**Corollary 5.14.** *We have*

- (1)  $|\boldsymbol{\lambda}_l| = e|\kappa|$  and  $|\boldsymbol{\lambda}'_e| = l|\kappa|$
- (2)  $|\lambda| = e|\kappa|$

*Proof.*

- (1) The partition  $\kappa$  encodes the position of the non-trivial  $e$ -periods in  $\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$ . Each  $e$ -period consists of  $e$  beads, so that  $|\boldsymbol{\lambda}_l| = e|\kappa|$ . Similarly,  $|\boldsymbol{\lambda}'_e| = l|\kappa|$ .

- (2) As in the proof of Corollary 5.6, we use Remark 2.4, which ensures that  $|\lambda| = e|\boldsymbol{\lambda}'_e| = el|\kappa|$  by (1).

□

## 6. THE HEISENBERG CRYSTAL

The aim of this section is to obtain a “crystal version” of Theorem 3.5. More precisely, we want to construct *crystal Heisenberg operators*, such that

- (1) they induce maps between  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ - and  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -crystals which commute with the two kinds of crystal operators. Such maps are called *double crystal isomorphisms*.
- (2) every charged  $l$ -partition can be obtained from the empty partition charged by  $\mathbf{s}_l \in A_{l,e}(s)$  for some  $s \in \mathbb{Z}$  by applying some sequence of Kashiwara crystal operators of  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$  and  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$  and of Heisenberg crystal operators.

**Notation:** If  $\phi : B_e \rightarrow B_e$  is a map between crystals, then we will also write  $\phi$  for the map from  $B'_l$  to  $B'_l$  as well as for the map going from the set of charged partitions to itself induced by the correspondence (3.7).

### 6.1. The maps $\tilde{b}_{-\kappa}$ .

**Definition 6.1.** Let  $|\lambda, s\rangle$  be a charged partition which is a doubly highest weight vertex. We identify  $|\lambda, s\rangle$  with the charged  $l$ -partition  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  and the charged  $e$ -partition  $|\boldsymbol{\lambda}'_e, \mathbf{s}'_e\rangle$  using (3.7). Define

$$\tilde{b}_{-1}|\lambda, s\rangle = |\mu, s\rangle \quad \text{and} \quad \tilde{b}'_{-1}|\lambda, s\rangle = |\nu, s\rangle$$

where  $|\mu, s\rangle$  (respectively  $|\nu, s\rangle$ ) is identified with  $|\boldsymbol{\mu}_l, \mathbf{s}_l\rangle$  (respectively  $|\boldsymbol{\nu}'_e, \mathbf{s}'_e\rangle$ ) using (3.7) where

- $\mathcal{A}(\boldsymbol{\nu}'_e, \mathbf{s}'_e)$  is obtained from  $\mathcal{A}(\boldsymbol{\lambda}'_e, \mathbf{s}'_e)$  by shifting its last non-trivial period one step to the left.
- $\mathcal{A}(\boldsymbol{\mu}_l, \mathbf{s}_l)$  is obtained from  $\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$  by shifting its last non-trivial period one step to the left.

*Remark 6.2.* Remember that we had defined a map  $\varphi_{l,P}$  in (5.11) which shifts the period  $P$  one step to the left in the  $l$ -abacus. Therefore, identifying abacci and charged multipartitions,  $\tilde{b}_{-1} = \varphi_{l,P}$  where  $P$  is the last non-trivial  $e$ -period of  $\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$  (which we have noticed earlier is well-defined). Similarly,  $\tilde{b}'_{-1} = \varphi_{e,Q}$  where  $Q$  is the last non-trivial  $l$ -period of  $\mathcal{A}(\boldsymbol{\lambda}'_e, \mathbf{s}'_e)$ .

Note that by Remark 2.4 together with Lemma 5.8, both  $\tilde{b}_{-1}$  and  $\tilde{b}'_{-1}$  act on  $|\lambda, s\rangle$  by removing  $l$   $e$ -ribbons.

*Example 6.3.* In Example 5.9, we have depicted the action of  $\tilde{b}_{-1}$  on both the  $e$ -abacus and the  $l$ -abacus. In terms of multipartitions, we have

$$\begin{aligned} \tilde{b}_{-1}|(3, 3.1, 1), (-1, 0, 0)\rangle &= |(3, 3, \emptyset), (-1, 0, 0)\rangle, \text{ i.e.} \\ \tilde{b}_{-1}|(10.8.4.2), -1\rangle &= |(10.8), -1\rangle \end{aligned}$$

and

$$\begin{aligned}\tilde{b}'_{-1}|(2^2.1^3, 2.1^3), (0, 1)\rangle &= |(2^2.1, 2.1^2), (0, 1)\rangle, \text{ i.e.} \\ \tilde{b}'_{-1}|(10.8.4.2), -1\rangle &= |(6.6.4.2), -1\rangle\end{aligned}$$

Recall that we have denoted  $B_e$  the crystal graph of the Fock space  $\mathcal{F}_{s_l, e}$  and  $B'_l$  the crystal graph of the Fock space  $\mathcal{F}_{s'_e, l}$  in Section 4.3. We have two induced maps between crystals which we denote the same way:

$$(6.13) \quad \begin{array}{ccc} \tilde{b}_{-1} : B_e & \longrightarrow & B_e \\ |\nu_l, \mathbf{s}_l\rangle & \longmapsto & \tilde{b}_{-1}|\nu_l, \mathbf{s}_l\rangle \end{array} \quad \begin{array}{ccc} \tilde{b}'_{-1} : B'_l & \longrightarrow & B'_l \\ |\nu'_e, \mathbf{s}'_e\rangle & \longmapsto & \tilde{b}'_{-1}|\nu'_e, \mathbf{s}'_e\rangle, \end{array}$$

where  $\tilde{b}_{-1}(|\nu_l, \mathbf{s}_l\rangle)$  is computed as follows:

- (1) Find the highest weight vertex in the connected component of  $B_e$  containing  $|\nu_l, \mathbf{s}_l\rangle$  (either recursively via a sequence of Kashiwara crystal operators, or more explicitly via the algorithm exposed in [6, Remark 6.4]).
- (2) Use the correspondence (3.7) to get an element of  $B'_l$ , and find the highest weight vertex in the connected component of  $B'_l$  of this  $e$ -partition. This is a doubly highest weight vertex in view of Theorem 4.8.
- (3) Apply  $\tilde{b}_{-1}$  using again the correspondence (3.7).
- (4) Do the reverse operations of Points (2) and (1). The resulting  $l$ -partition is  $\tilde{b}_{-1}|\nu_l, \mathbf{s}_l\rangle$ .

The map  $\tilde{b}'_{-1} : B'_l \longrightarrow B'_l$  is defined similarly, switching indexations using (3.7) and replacing  $\tilde{b}_{-1}$  by  $\tilde{b}'_{-1}$  in the above procedure.

**Theorem 6.4.** *The maps  $\tilde{b}_{-1}$  and  $\tilde{b}'_{-1}$  are double crystal isomorphisms.*

*Proof.* By the ad hoc construction above, it suffices to check that this holds for the doubly highest vertices.

By Lemma 5.8, each of these two maps act on a doubly highest weight vertex by shifting an  $e$ -period one step to the left in the  $l$ -abacus and by shifting an  $l$ -period one step to the left in the  $e$ -abacus. Now, by Proposition 5.1, the reduced  $i$ -word of  $|\lambda_l, \mathbf{s}_l\rangle$  is the same as the reduced  $i$ -word of  $\tilde{b}_{-1}|\lambda_l, \mathbf{s}_l\rangle$  for all  $i = 0 \dots, e - 1$ ; and the reduced  $j$ -word of  $|\lambda'_e, \mathbf{s}'_e\rangle$  is the same as the reduced  $j$ -word of  $\tilde{b}'_{-1}|\lambda'_e, \mathbf{s}'_e\rangle$  for all  $j = 0 \dots, l - 1$ . Hence,  $\tilde{b}_{-1}$  is a double crystal isomorphism. The same holds for  $\tilde{b}'_{-1}$ .  $\square$

By extension, we define  $\tilde{b}_{-z} : B_e \longrightarrow B_e$  by saying that, for a doubly highest weight vertex  $|\lambda_l, \mathbf{s}_l\rangle \in B_e$ ,  $\tilde{b}_{-z}|\lambda_l, \mathbf{s}_l\rangle$  is obtained from  $|\lambda_l, \mathbf{s}_l\rangle$  by shifting its last non-trivial  $e$ -period  $z$  steps to the left, if possible. We extend this to  $B_e$  in the same way as  $\tilde{b}_{-1}$  (Formula (6.13) above). We define similarly  $\tilde{b}'_{-z} : B'_l \longrightarrow B'_l$ .

*Remark 6.5.* One sees that  $\tilde{b}_{-z}$  is the  $z$ -fold composition of  $\tilde{b}_{-1}$  if and only if  $S_{N(l)}(l) = z$  (see Lemma 5.5, i.e. if and only if the last non-trivial  $e$ -period of  $\lambda_l$  corresponds to a part  $z$ ). We have the similar property for  $\tilde{b}'_{-z}$ .



**Definition 6.6.** For a partition  $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_t)$ , we define  $\tilde{b}_{-\sigma} : B_e \longrightarrow B_e$  and  $\tilde{b}'_{-\sigma} : B'_l \longrightarrow B'_l$  through the formulas

$$(6.14) \quad \begin{aligned} \tilde{b}_{-\sigma} &= \tilde{b}_{-\sigma_1} \circ \tilde{b}_{-\sigma_2} \circ \dots \circ \tilde{b}_{-\sigma_t} \quad \text{and} \\ \tilde{b}'_{-\sigma} &= \tilde{b}'_{-\sigma_1} \circ \tilde{b}'_{-\sigma_2} \circ \dots \circ \tilde{b}'_{-\sigma_t}. \end{aligned}$$

When one of the  $\tilde{b}_{-\sigma_k}$  is not well defined on  $(\tilde{b}_{-\sigma_{k+1}} \circ \dots \circ \tilde{b}_{-\sigma_t})|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ , we set  $\tilde{b}_{-\sigma}|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle = 0$  (and similarly for  $\tilde{b}'_{-\sigma}$ ).

Remember that the partition  $\kappa$  (respectively  $\kappa'$ ) associated to  $|\lambda, s\rangle$  (see 5.12) is written  $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_t)$  where each  $\kappa_i$  is a certain  $S_k(l)$  (respectively  $\kappa' = (\kappa'_1, \kappa'_2, \dots, \kappa'_u)$  where each  $\kappa'_i$  is a certain  $S_k(e)$ ). So because of Remark 6.5,  $\tilde{b}_{-\kappa}$  and  $\tilde{b}'_{-\kappa'}$  are well-defined on  $|\lambda, s\rangle$ .

**Proposition 6.7.**

- (1) If  $|\lambda, s\rangle$  is a doubly highest weight partition, then  $\tilde{b}_{-\kappa}|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle = |\boldsymbol{\emptyset}_l, \mathbf{s}_l\rangle$  and  $\tilde{b}'_{-\kappa'}|\boldsymbol{\lambda}'_e, \mathbf{s}'_e\rangle = |\boldsymbol{\emptyset}_e, \mathbf{s}'_e\rangle$ .
- (2) The following diagram commutes

$$\begin{array}{ccc} B_e & \xrightarrow{T_{l,e}} & B'_l \\ \tilde{b}_{-\kappa} \downarrow & & \downarrow \tilde{b}'_{-\kappa'} \\ B_e & \xrightarrow{T_{l,e}} & B'_l \end{array}$$

where here, the partition  $\kappa$  depends on the chosen multipartition (see Remark 5.13). Therefore, we write  $\tilde{b}_{-\kappa} = \tilde{b}'_{-\kappa'}$ .

*Proof.*

- (1) By definition of  $\tilde{b}'_{-\sigma}$  on  $B'_l$  and by construction of  $\kappa$ , it is straightforward the  $e$ -partition  $\tilde{b}'_{-\kappa'}|\boldsymbol{\lambda}'_e, \mathbf{s}'_e\rangle$  is empty (and  $\tilde{b}_{-\kappa}$  does not modify the  $e$ -charge). Similarly, we have  $\tilde{b}_{-\kappa}|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle = |\boldsymbol{\emptyset}_l, \mathbf{s}_l\rangle$ .
- (2) Because of Proposition 5.7,  $T_{l,e}$  maps  $|\boldsymbol{\emptyset}_l, \mathbf{s}_l\rangle$  to  $|\boldsymbol{\emptyset}_e, \mathbf{s}_e\rangle$ . In other words, the  $e$ -partition associated to the  $|\boldsymbol{\emptyset}_l, \mathbf{s}_l\rangle$  such that  $\mathbf{s}_l \in A_{l,e}(s)$  via the correspondence (3.7) is also empty. Together with Point (1), we get the commutativity of the diagram.

□

**Notation 6.8.** If  $|\lambda, s\rangle$  is a doubly highest weight vertex, we will use the notation  $|\bar{\lambda}, s\rangle$  for the charged partition  $\tilde{b}_{-\kappa(\lambda)}|\lambda, s\rangle$ .

**Theorem 6.9.** The map  $\tilde{b}_{-\kappa}$  is a double crystal isomorphism.

*Proof.* This is a direct consequence of the definition of  $\tilde{b}_{-\kappa}$  together with Theorem 6.4 and Remark 6.5. □

## 6.2. The inverse maps.

Recall that we have introduced the notion of trivial period in Section 5.1.

**Definition 6.10.** Let  $|\lambda, s\rangle$  be a charged partition which is a doubly highest weight vertex. We identify  $|\lambda, s\rangle$  with the charged  $l$ -partition  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  and the charged  $e$ -partition  $|\boldsymbol{\lambda}'_e, \mathbf{s}'_e\rangle$  using (3.7). Define

$$\tilde{b}_1|\lambda, s\rangle = |\mu, s\rangle \quad \text{and} \quad \tilde{b}'_1|\lambda, s\rangle = |\nu, s\rangle$$

where  $|\mu, s\rangle$  (respectively  $|\nu, s\rangle$ ) is identified with  $|\boldsymbol{\mu}_l, \mathbf{s}_l\rangle$  (respectively  $|\boldsymbol{\nu}'_e, \mathbf{s}'_e\rangle$ ) using (3.7) where

- $\mathcal{A}(\boldsymbol{\nu}'_e, \mathbf{s}'_e)$  is obtained from  $\mathcal{A}(\boldsymbol{\lambda}'_e, \mathbf{s}'_e)$  by shifting its first trivial period one step to the right.
- $\mathcal{A}(\boldsymbol{\mu}_l, \mathbf{s}_l)$  is obtained from  $\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$  by shifting its first trivial period one step to the right.

We extend this definition and write, for a positive integer  $z$ ,  $\tilde{b}_z|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  to be the  $l$ -partition obtained by shifting the first trivial period of  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$   $z$  steps to the right. Similarly,  $\tilde{b}'_z|\boldsymbol{\lambda}'_e, \mathbf{s}'_e\rangle$  to be the  $e$ -partition obtained by shifting the first trivial period of  $|\boldsymbol{\lambda}'_e, \mathbf{s}'_e\rangle$   $z$  steps to the right. Finally, for a partition  $\sigma = (\sigma_1, \dots, \sigma_t)$ , we define  $\tilde{b}_\sigma = \tilde{b}_{\sigma_t} \circ \dots \circ \tilde{b}_{\sigma_1}$  and  $\tilde{b}'_\sigma = \tilde{b}'_{\sigma_t} \circ \dots \circ \tilde{b}'_{\sigma_1}$ . When this is not well-defined, we set again  $\tilde{b}_\sigma|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle = 0$  (respectively  $\tilde{b}'_\sigma|\boldsymbol{\lambda}'_e, \mathbf{s}'_e\rangle = 0$ ). All of these maps induce maps between crystals  $B_e \rightarrow B_e$  or  $B'_l \rightarrow B'_l$  by the procedure explained in (6.13).

*Remark 6.11.* By definition of  $\kappa$  in Section 5.12 and Proposition 6.7, it is clear that for all charged partition  $|\lambda, s\rangle$  which is a doubly highest weight vertex,

$$|\lambda, s\rangle = \tilde{b}_\kappa|\bar{\lambda}, s\rangle.$$

So, it is enough to understand the connected components in  $B_e$  and  $B'_l$  containing  $|\boldsymbol{\emptyset}_l, \mathbf{s}_l\rangle$  and  $|\boldsymbol{\emptyset}_e, \mathbf{s}'_e\rangle$  respectively. This is the case we consider in the following proposition.

*Remark 6.12.* The maps  $\tilde{b}_\sigma$  and  $\tilde{b}_{-\sigma}$  are defined so that they are inverse to each other, that is

$$\tilde{b}_\sigma \circ \tilde{b}_{-\sigma} = \text{Id}_{B_e} = \tilde{b}_{-\sigma} \circ \tilde{b}_\sigma$$

and

$$\tilde{b}'_\sigma \circ \tilde{b}'_{-\sigma} = \text{Id}_{B'_l} = \tilde{b}'_{-\sigma} \circ \tilde{b}'_\sigma,$$

whenever the first identities make sense.

As a consequence of Remarks 6.11 and 6.12, we see that  $\tilde{b}_\sigma|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle \neq 0$  if and only if the first part of  $\sigma$  is not greater than the last part of  $\kappa = \kappa(|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle)$ . In this case, we have

$$(6.15) \quad \begin{aligned} \tilde{b}_\sigma|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle &= (\tilde{b}_\eta \circ \tilde{b}_{-\kappa})|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle \\ &= \tilde{b}_\eta|\boldsymbol{\emptyset}_l, \mathbf{s}_l\rangle, \end{aligned}$$

where  $\eta$  is the partition obtained by adding the parts of  $\sigma$  to  $\kappa$ .

**Proposition 6.13.** *For all partition  $\sigma$ , we have  $\tilde{b}_\sigma|\bar{\lambda}, s\rangle = \tilde{b}'_{\sigma'}|\bar{\lambda}, s\rangle$ .*

*Proof.* First of all,  $|\bar{\lambda}, s\rangle$  is a doubly highest weight vertex for all  $s \in \mathbb{Z}$ , which ensures that  $\tilde{b}_\sigma|\bar{\lambda}, s\rangle$  and  $\tilde{b}'_{\sigma'}|\bar{\lambda}, s\rangle$  are well-defined. In fact, the  $l$ -partition and the  $e$ -partition corresponding are empty by Proposition 6.7, and the  $l$ -charge (respectively  $e$ -charge) is an element of  $A_{l,e}(s)$  (respectively  $A_{e,l}(s)$ ) by Proposition 5.7.

By Lemma 5.8, the action of  $\tilde{b}_1$  on the  $l$ -abacus (shifting its first trivial  $e$ -period one step to the right) corresponds to shifting an  $l$ -period one step to the right in the  $e$ -abacus. Since this  $e$ -abacus corresponds to the empty  $e$ -partition, it has only trivial  $l$ -periods, and one can only shift its first trivial  $l$ -period to the right. This forces  $\tilde{b}_1$  to be the same as  $\tilde{b}'_1$ . Hence, the result holds for  $\sigma = (1)$ . In fact, one can look directly at the action of  $\tilde{b}_z$  ( $= \tilde{b}_\sigma$  with  $\sigma = (z)$ ) on the empty  $l$ -partition. Using the combinatorial definition of the correspondence (3.7), one sees that moving the first trivial  $e$ -period in the empty  $l$ -abacus  $z$  steps to the right creates  $z$   $l$ -periods in the  $e$ -abacus which are obtained from the empty  $e$ -abacus by recursively shifting its first period one step to the right. In other terms, it corresponds to applying  $\tilde{b}'_1 \circ \dots \circ \tilde{b}'_1$  (with  $z$  factors) to the empty  $e$ -abacus. That is to say,  $\tilde{b}_{(z)} = \tilde{b}'_{(1^z)}$ . Therefore, the result holds for  $\sigma = (z)$ . Similarly, it holds for  $\sigma = (1^z)$ . Using the same observation, we deduce that for an arbitrary  $\sigma$ , the map  $\tilde{b}_\sigma$  acts on the empty  $l$ -abacus exactly like  $\tilde{b}'_{\sigma'}$  acts on the empty  $e$ -abacus, with the identification (3.7).  $\square$

**Theorem 6.14.** *The map  $\tilde{b}_\sigma$  is a double crystal isomorphism.*

*Proof.* It is similar to that of Theorem 6.9. What  $\tilde{b}_\sigma$  does is shift periods (to the right) in the  $l$ -abacus and  $e$ -abacus, so by Proposition 5.1 the reduced words for either quantum group structure are unchanged.  $\square$

*Remark 6.15.* In the level 1 case, Leclerc and Thibon [18] have made explicit the action of some elements  $S_\sigma \in \mathcal{H}$ , defined from the basis of Schur functions in the space of symmetric functions, on the canonical basis of the Fock space, see [18, Theorem 6.9]. This induces an action of  $\mathcal{H}$  at the combinatorial level, i.e. on the crystal on partitions: the operator  $S_\sigma$  acts on a partition  $\lambda$  by adding  $e$  times each part of  $\sigma$  in  $\lambda$ . For instance, if  $e = 3$ ,

$$S_{\begin{array}{c} \square \\ \square \end{array}} \left( \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \square \end{array} \right) = \begin{array}{cccccccc} \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square \end{array} .$$

In the 1-abacus representing  $\lambda$ , this amounts to shifting recursively the first trivial  $e$ -period  $\sigma_k$  steps to the right, where  $\sigma = (\sigma_1, \sigma_2, \dots)$ . So this is exactly the same procedure as our map  $\tilde{b}_\sigma$ . Hence, these maps  $\tilde{b}_\sigma$  can be interpreted as generalisations

of the operators  $S_\sigma$  coming from the action of  $\mathcal{H}$  in the level 1 case. However, throughout this paper,  $l = 1$  is not allowed. In fact, in the level 1 case, the structure of the Fock space is somewhat different since there is only one quantum group and the Heisenberg algebra  $\mathcal{H}$  acting.

### 6.3. Definition of the Heisenberg crystal.

We can now define an oriented colored graph structure on the set of charged partitions, by setting  $|\lambda, s\rangle \xrightarrow{c} |\mu, s\rangle$  if  $\kappa(\mu)$  is obtained from  $\kappa(\lambda)$  by adding a box  $(a, b)$  such that  $b - a = c$ . As usual, we define it on doubly highest weight vertices and we extend it as in (6.13). We call it the *Heisenberg crystal*, or simply the  $\mathcal{H}$ -crystal, of  $\Lambda^s$ .

*Remark 6.16.* The rule for drawing an arrow in the Heisenberg crystal is in fact the  $\mathcal{U}_q(\mathfrak{sl}_\infty)$ -crystal graph rule on  $\{|\kappa(\lambda), 0\rangle; \lambda \in \Pi\}$ , which is equal to  $\{|\sigma, 0\rangle; \sigma \in \Pi\}$  by the surjectivity of  $\kappa$  explained in Remark 5.13. Hence, one can see the Heisenberg crystal as the preimage under the map  $\kappa$  of the  $\mathcal{U}_q(\mathfrak{sl}_\infty)$ -crystal on the set of partitions, which justifies the terminology “crystal”.

Now, observe that the procedure  $|\lambda, s\rangle \xrightarrow{c} |\mu, s\rangle$  is in fact a composition of maps  $\tilde{b}_{\pm\sigma}$ , namely

$$|\lambda, s\rangle \xrightarrow{\tilde{b}_{-\kappa(\lambda)}} |\bar{\lambda}, s\rangle \xrightarrow{\tilde{b}_{\kappa(\mu)}} |\mu, s\rangle.$$

This is a generalisation of Formula (6.15).

Therefore, we call the map

$$(6.16) \quad \tilde{b}_{1,c} = \tilde{b}_{\kappa(\mu)} \circ \tilde{b}_{-\kappa(\lambda)}$$

*Heisenberg crystal operator*, and there is an arrow  $|\lambda, s\rangle \xrightarrow{c} |\mu, s\rangle$  in the Heisenberg crystal if and only if  $|\mu, s\rangle = \tilde{b}_{1,c}|\lambda, s\rangle$ . This is an analogous result to Theorem 6.24, in the sense that the Heisenberg crystal graph is explicitly described in combinatorial terms (via an explicit formula of the Heisenberg crystal operator). In fact,  $\tilde{b}_{1,c}$  is an analogue for  $\mathcal{H}$  of the Kashiwara crystal operator  $\tilde{f}_i$  for  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ .

*Remark 6.17.* The map  $\tilde{b}_{1,c}$  can be seen as a “weighted” version of the map  $\tilde{b}_1$  (Definition 6.10), in the sense that it shifts an  $e$ -period one step to the right in the  $l$ -abacus, which is determined by  $c$  (and is not necessarily the first trivial one).

*Remark 6.18.* By Remark 2.4, a Heisenberg crystal operator acts on a charged partition by adding an  $e$ -ribbon.

Each  $l$ -charge  $\mathfrak{s}_l \in A_{l,e}(s)$  determines a connected component of the  $\mathcal{H}$ -crystal. A source vertex in the  $\mathcal{H}$ -crystal is called a *highest weight vertex* (by analogy with the quantum group case): it is a charged partition  $|\lambda, s\rangle$  such that  $\tilde{b}_{-\sigma}|\lambda, s\rangle = 0$  for all  $\sigma \in \Pi$ . In other terms, the highest weight vertices in the  $\mathcal{H}$ -crystal are the elements of the form  $|\bar{\lambda}, s\rangle$  for some partition  $\lambda$ . The number of arrows necessary to go from  $|\bar{\lambda}, s\rangle$  to  $|\lambda, s\rangle$  in the  $\mathcal{H}$ -crystal is called the *depth* of  $|\lambda, s\rangle$  and is equal to  $|\kappa(\lambda)|$ .

By definition, a map  $\tilde{b}_\sigma$  (with  $\sigma$  a partition) is a composition of maps of the form  $\tilde{b}_z$  with  $z$  positive integer. We can now give an alternative description of  $\tilde{b}_\sigma$  using composition of Heisenberg crystal operators. Let  $\{\gamma_k; k = 1, \dots, |\sigma|\}$  be the set of boxes of  $\sigma$ , ordered from bottom to top, and from right to left. If  $\gamma_k = (a_k, b_k)$  (row and column indices), then write  $c_k = b_k - a_k$ . In particular, one always has  $c_{|\sigma|} = 0$ . We have

$$(6.17) \quad \tilde{b}_\sigma = \tilde{b}_{1,c_1} \circ \tilde{b}_{1,c_2} \circ \dots \circ \tilde{b}_{1,c_{|\sigma|}}.$$

**Theorem 6.19.** *The Heisenberg crystal operators simultaneously commute with the  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -crystal operators and with the  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -crystal operators when computed with respect to Indexation (3.7).*

*Proof.* By definition, the Heisenberg crystal operators are a composition of a map  $\tilde{b}_{-ka}$  and a map  $\tilde{b}_\sigma$ . Both these maps are double crystal-isomorphisms by Theorems 6.9 and 6.14. This proves the claim.  $\square$

To sum up, we have constructed a new crystal structure, so that we have in total three crystal structures on the space  $\Lambda^s$ :

- a  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal,
- a  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -crystal,
- an  $\mathcal{H}$ -crystal,

which are all explicit and pairwise commute provided one uses the correspondence (3.7) to switch between the different indexations.

#### 6.4. The decomposition theorem.

**Notation 6.20.** *Let  $r$  (respectively  $t$ ) be a non-negative integer and, let  $i_1, \dots, i_r$  (respectively  $j_1, \dots, j_t$ ) be elements of  $\{0, \dots, e-1\}$  (respectively  $\{0, \dots, l-1\}$ ). We denote*

$$\tilde{F}_{(i_1 \dots i_r)} = \tilde{f}_{i_1} \dots \tilde{f}_{i_r} \quad \text{and} \quad \tilde{F}_{(j_1 \dots j_t)} = \tilde{f}_{j_1} \dots \tilde{f}_{j_t}.$$

The following theorem says that every charged  $l$ -partition is obtained from the empty  $l$ -partition charged by an element of  $A_{l,e}(s)$  by applying some crystal operators of  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ , of  $\mathcal{H}$ , and of  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ . So this is an analogue of Theorem 3.5 at the crystal level.

**Theorem 6.21.** *For all charged  $l$ -partition  $|\lambda_l, \mathbf{s}_l\rangle$ , there exist  $r, t \in \mathbb{Z}_{\geq 0}$ ,  $i_1, \dots, i_r \in \{0, \dots, e-1\}$ ,  $j_1, \dots, j_t \in \{0, \dots, l-1\}$ , and a partition  $\sigma$  such that*

$$|\lambda_l, \mathbf{s}_l\rangle = (\tilde{F}_{(j_1 \dots j_t)} \circ \tilde{b}_\sigma \circ \tilde{F}_{(i_1 \dots i_r)}) |\emptyset_l, \tilde{\mathbf{s}}_l\rangle,$$

for some  $\tilde{\mathbf{s}}_l \in A_{l,e}(s)$ .

Here, we have implicitly used the correspondence (3.7) to switch between the indexations by  $l$ -partitions, partitions, and  $e$ -partitions. Note that by Identity (6.17), the map  $\tilde{b}_\sigma$  in the middle is indeed a composition of Heisenberg crystal operators.

*Proof.* We identify as usual the  $l$ -,  $e$ -, and 1-partitions using (3.7). Starting from  $|\lambda, s\rangle$ , one first goes back in the  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -crystal to the highest weight vertex, say  $|\nu, s\rangle$ . One then computes  $\tilde{b}_{-\kappa(\nu)}|\nu, s\rangle = |\bar{\nu}, s\rangle$ . Finally, one can go back in the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal to the highest weight vertex. By Theorem 6.19, the order of these operations does not matter, and by Proposition 6.7, the resulting  $l$ -partition is empty, and charged by an element of  $A_{l,e}(s)$  according to Proposition 5.7.  $\square$

### 6.5. An application using FLOTW multipartitions.

A consequence of Theorem 6.21 is the existence of a labelling of each charged  $l$ -partition by a triple consisting of a particular  $l$ -partition, a partition and a particular  $e$ -partition. More precisely, let us introduce the convenient class of *FLOTW* multipartitions.

**Definition 6.22.** *Let  $\lambda_l = (\lambda^1, \dots, \lambda^l)$  be an  $l$ -partition and  $\mathbf{s}_l = (s_1, \dots, s_l)$  be an  $l$ -charge in  $A_{l,e}(s)$  (cf Notation 3.4). For  $j \in \{1, \dots, l\}$ , write  $\lambda^j = (\lambda_1^j, \lambda_2^j, \dots)$ . We call  $|\lambda_l, \mathbf{s}_l\rangle$  FLOTW if the two following conditions are satisfied.*

- (1)  $\bullet$   $\lambda_k^j \geq \lambda_{k+s_{j+1}-s_j}^{j+1} \quad \forall j \in \{1, \dots, l-1\}$  and  $\forall k \geq 1$ , and
  - $\bullet$   $\lambda_k^l \geq \lambda_{k+e+s_1-s_l}^1, \forall k \geq 1$ .
- (2) For all  $\alpha > 0$ , the residues of the rightmost boxes of the parts of size  $\alpha$  do not cover  $\{0, \dots, e-1\}$ .

Denote by  $\Psi_{\mathbf{s}_l}$  the set of FLOTW  $l$ -partitions with  $\mathbf{s}_l$ , and by  $\Psi_l$  the set of all FLOTW  $l$ -partitions (i.e. without specifying the  $l$ -charge).

*Remark 6.23.* Throughout this paper, we have assumed that  $l > 1$ . This definition is however still valid when  $l = 1$ . In this case,  $l$ -partitions are simply partitions, and the FLOTW partitions are precisely the  $e$ -regular partitions (and in this case, the charge is insignificant).

**Theorem 6.24** ([4, Theorem 2.10]). *The vertices of the connected component of the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal graph of  $\mathcal{F}_{\mathbf{s}_l, e}$  containing  $|\emptyset_l, \mathbf{s}_l\rangle$  are the FLOTW  $l$ -partitions.*

The relevance of this theorem is that a priori, the vertices in the crystal graph of  $\mathcal{F}_{\mathbf{s}_l, e}$  are computable, but only have a recursive definition: one starts with the highest weight vertex and recursively applies some crystal operators of the form  $\tilde{f}_i$ ; whereas the FLOTW  $l$ -partitions have a more explicit (in particular non-recursive) combinatorial definition.

*Example 6.25.* Take  $e = 4$ ,  $l = 2$  and  $\mathbf{s}_l = (0, 1)$ . Then the elements of  $\Psi_{\mathbf{s}_l}$  of rank 4 are

$$\begin{array}{cccc} (\emptyset, \boxed{123}) & (\boxed{0}, \boxed{1}) & (\boxed{0}, \boxed{12}) & (\boxed{0}, \boxed{1}) \\ (\boxed{0}, \emptyset) & (\boxed{01}, \boxed{1}) & (\boxed{01}, \emptyset) & (\boxed{012}, \emptyset). \end{array}$$

**Theorem 6.26.** *There is a one-to-one correspondence*

$$B_e \xleftarrow{1:1} \Psi_l \times \Pi \times \Psi_e.$$

*Proof.* Theorem 6.21 says that one can always decompose any element of  $B_e$  as follows

$$|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle = (\tilde{F}_{(j_1 \dots j_t)} \circ \tilde{b}_\sigma \circ \tilde{F}_{(i_1 \dots i_r)}) |\emptyset_l, \tilde{\mathbf{s}}_l\rangle.$$

Now, by Theorem 6.24, the  $l$ -partition  $\tilde{F}_{(i_1 \dots i_r)} |\emptyset_l, \tilde{\mathbf{s}}_l\rangle$  is FLOTW, i.e. an element of  $\Psi_l$ . Denote it  $|\boldsymbol{\mu}_l, \mathbf{r}_l\rangle$ . Similarly  $\tilde{F}_{(j_1 \dots j_t)} |\emptyset_e, \tilde{\mathbf{s}}'_e\rangle \in \Psi_e$ , and we denote it  $|\boldsymbol{\nu}_e, \mathbf{t}_e\rangle$ . Therefore, we get a bijection

$$\begin{array}{ccc} B_e & \xleftarrow{1:1} & \Psi_l \times \Pi \times \Psi_e \\ |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle & \longleftarrow & (|\boldsymbol{\mu}_l, \mathbf{r}_l\rangle, \sigma, |\boldsymbol{\nu}_e, \mathbf{t}_e\rangle). \end{array}$$

□

*Remark 6.27.* In [6, Remark 6.5], we have constructed an analogue of the Robinson-Schensted correspondence, which maps bijectively an element  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle \in B_e$  to a pair consisting of an FLOTW  $l$ -partition  $|\boldsymbol{\mu}_l, \mathbf{r}_l\rangle$  and a combinatorial “recording data”  $(\underline{\mathcal{Q}}, \underline{\alpha})$ . The FLOTW  $l$ -partition is precisely the one appearing in the above theorem. Hence,  $\Pi \times \Psi_e$  can be identified to the set of objects  $(\underline{\mathcal{Q}}, \underline{\alpha})$ . It would be interesting to make this relation explicit.

## 7. APPLICATION TO THE REPRESENTATION THEORY OF CYCLOTOMIC RATIONAL CHEREDNIK ALGEBRAS

There is a close connection between Uglov’s combinatorial level  $l$  Fock space  $\mathcal{F}_{\mathbf{s}_l, e}$  and cyclotomic rational Cherednik algebras. For a fixed non-negative integer  $n$ , one associates the Cherednik algebra  $H_{\mathbf{c}, n}$  with parameter  $\mathbf{c} = (-\frac{1}{e}, \mathbf{s}_l)$  arising from the complex reflection group  $G(l, 1, n) = (\mathbb{Z}/l\mathbb{Z})^n \rtimes \mathfrak{S}_n$  (this is the so-called cyclotomic case). The parameter  $\mathbf{c}$  is sometimes expressed differently in the literature. For some background, one can refer to e.g. [23].

There is a corresponding category  $\mathcal{O}$ , see [8] for its definition, denoted  $\mathcal{O}_{\mathbf{c}, n}$ , and one can consider, for  $n$  varying, all categories  $\mathcal{O}_{\mathbf{c}, n}$  together. Denote it  $\mathcal{O}_{\mathbf{c}}$ . The simple objects in  $\mathcal{O}_{\mathbf{c}}$  are parametrised by the elements of  $\text{Irr}(G(l, 1, n))$  for  $n$  varying, i.e. by  $l$ -partitions.

It is known that the Fock space plays an important role in the representation theory of  $H_{\mathbf{c}, n}$ , with  $n$  varying, via categorification phenomena. In particular, the crystal of  $\mathcal{F}_{\mathbf{s}_l, e}$  reflects the branching rule on  $H_{\mathbf{c}, n}$  with  $n$  varying, where the Kashiwara operators  $\tilde{e}_i$  (respectively  $\tilde{f}_i$ ) reflects the parabolic restriction (respectively induction) in  $\mathcal{O}_{\mathbf{c}}$ , see Shan [23, Theorem 6.3] and Losev [19, Theorem 5.1].

Moreover, the action of the Heisenberg algebra (cf Section 3.2) has also been categorified by Shan and Vasserot [24], and some of the associated combinatorics has been recently studied by Losev [20].

### 7.1. Interpretation of the crystal level-rank duality.

We have seen in Section 4 how Uglov’s level-rank duality is expressed at the crystal level, via the double crystal structure arising from the two quantum group actions. More precisely, the two crystals commute provided one works with the indexation (3.7) (Theorem 4.8). At the categorical level, it is known that level-rank duality

reflects the Koszul duality between the corresponding categories  $\mathcal{O}$ , denoted  $\mathcal{O}_e$  for the category  $\mathcal{O}$  associated to  $\mathcal{F}_{s_l, e}$  and  $\mathcal{O}_l$  for that associated to  $\mathcal{F}_{s_e, l}$ . The first results in this direction are due to Rouquier, Shan, Varagnolo and Vasserot [22] and Webster [28].

More precisely, the Koszul duality sends a simple object of  $\mathcal{O}_e$  to a tilting object in  $\mathcal{O}_l$ . The categorical crystal on the simples modules in  $\mathcal{O}_e$  arises from the abelian action of  $\widehat{\mathfrak{sl}}_e$ . Similarly, one constructs a categorical crystal on the simple modules in  $\mathcal{O}_l$  by considering the action of  $\widehat{\mathfrak{sl}}_l$ , and the Koszul duality yields a categorical crystal on the tilting modules in  $\mathcal{O}_e$ .

At the combinatorial level, the simple modules in  $\mathcal{O}_e$  correspond to the dual canonical basis of  $\mathcal{F}_{s_l, e}$ , whose associated crystal requires the specialisation at  $q^{-1} = 0$ ; and the tilting modules correspond to the canonical basis, whose associated crystal requires the specialisation at  $q = 0$ . In order for the two crystals to commute, they have to be specialisations at the same value of  $q$  (e.g.  $q = 0$ ), so one must use the bijection between the labels of the simples and tilting modules in  $\mathcal{O}_e$ .

Theorem 4.8 suggests that this bijection is given by the conjugation of  $l$ -partitions. Note that this is coherent with [26, Proposition 5.14], which states that the dual canonical basis is related to the (lower) canonical basis of  $\mathcal{F}_{s_l, e}$  through conjugation of  $l$ -partitions.

## 7.2. Propagation in the $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -crystal and compatibility with the results of Losev.

In [20], Losev has introduced a combinatorial recipe to compute a so-called  $\mathfrak{sl}_\infty$ -crystal<sup>1</sup> on the set of charged  $l$ -partitions which reflects, at a combinatorial level, an abstract crystal structure on the set of classes of simple objects in the category  $\mathcal{O}_e$ , arising from the action of the Heisenberg algebra at a categorical level (whose existence goes back to Shan and Vasserot [24]).

This recipe consists of two ingredients:

- An explicit description of some operators  $\tilde{a}_\sigma$  (parametrised by partitions  $\sigma$ , first introduced in [24]) on charged  $l$ -partitions, in the case where the  $l$ -charge is *asymptotic*.
- A formula for *wall-crossing bijections*, that permits to pass from the asymptotic case to the general case.

Notice that the formula for these wall-crossing bijections is unfortunately not very explicit. Moreover, these ingredients are introduced for highest weight vertices in the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal; however, the commutation of this  $\mathfrak{sl}_\infty$ -crystal with the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal ensures that one can extend it to the whole set of partitions (see [20, Remark 5.4]). Finally, Losev does not use the combinatorial level-rank duality at all, so there is no triple crystal structure involved in this recipe. In this section, we will show that Losev's  $\mathfrak{sl}_\infty$ -crystal coincides with the Heisenberg crystal introduced in Section 6.2 above.

<sup>1</sup>This terminology is justified by the same kind of arguments as that of Remark 6.16.



Let  $j_0 \in \{1, \dots, l\}$ . An  $l$ -charge  $\mathbf{s}_l$  is called  $j_0$ -asymptotic if there exists a positive integer  $N$  such that  $s_{j_0} > s_j + N$  for all  $j \in \{1, \dots, l-1\}, j \neq j_0$ . Actually, in what follows, we will consider the maximal such  $N$  for simplicity. In this case, we will also call a charged an element  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  *asymptotic (charged) multipartition*.

**Lemma 7.1.** *Let  $\mathbf{s}_l$  be an  $j_0$ -asymptotic  $l$ -charge. If  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  is a highest weight vertex in the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal such that  $|\boldsymbol{\lambda}_l| \leq N$ , then there exists a partition  $\theta = (\theta_1, \theta_2, \dots)$  such that  $\lambda^{j_0} = (\theta_1^e, \theta_2^e, \dots)$ .*

*Proof.* Because of Theorem 5.2, we know that  $\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$  is totally  $e$ -periodic. In view of the condition on the rank of  $\boldsymbol{\lambda}_l$ , the first periods of  $\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$  consist only of elements of the form  $(j, \beta)$  with  $j = j_0$  (in other terms, the first periods are entirely included in the  $j_0$ -th row of the abacus). Hence, the partition  $\lambda^{j_0}$  is of the form  $(\theta_1^e, \theta_2^e, \dots)$  for some non-negative integers  $\theta_i$ .  $\square$

Notice that this partition  $\theta$  is constructed in a similar way as the partition  $\kappa$  for doubly highest weight vertices (except that for  $\theta$ , one focuses exclusively on the  $j_0$ -th component of  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ ). We will show in Theorem 7.6 that  $\theta$  is in fact the partition  $\kappa$  associated to the corresponding doubly highest weight vertex.

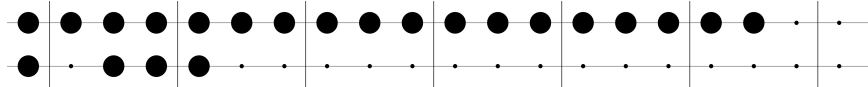
Let us now recal the result of Losev that is relevant in our context. Let  $\mathbf{s}_l$  be an  $j_0$ -asymptotic  $l$ -charge, and  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  be a highest weight vertex in the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal such that  $|\boldsymbol{\lambda}_l| + e|\theta| \leq N$  (cf Lemma 7.1 above). The following is [20, Section 5.1.2 and Proposition 5.3].

**Theorem 7.2.**

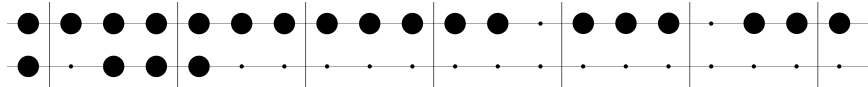
- (1) *The depth of  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  in the  $\mathfrak{sl}_\infty$ -crystal is equal to the rank of  $\theta$ .*
- (2) *If  $\theta = \emptyset$  and  $\sigma = (\sigma_1, \sigma_2, \dots)$  is a partition such that  $|\boldsymbol{\lambda}_l| + e|\sigma| \leq N$ , then  $\tilde{a}_\sigma(|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle) = |\boldsymbol{\mu}_l, \mathbf{s}_l\rangle$ , where  $\mu^j = \lambda^j$  for all  $j \neq j_0$ , and  $\mu^{j_0} = (\sigma_1^e, \sigma_2^e, \dots)$ .*
- (3) *The  $\mathfrak{sl}_\infty$ -crystal commutes with the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal.*

Here, we have slightly “rephrased” the original result of Losev. In particular, the notion of asymptoticity must be reversed in order to be compatible with the language of Fock space, as well as the convention on “multiplicating/dividing” of partitions by  $e$  (one recovers Losev’s convention by conjugating, see also [20, Section 5.5]).

*Example 7.3.* Take  $l = 2$ ,  $\boldsymbol{\lambda}_l = (1^3, \emptyset)$ ,  $\mathbf{s}_l = (0, 14)$ ,  $e = 3$  and  $\sigma = (2, 1)$ . So we have  $j_0 = 2$  and  $N = s_2 - s_1 - 1 = 13$ . Since  $|\boldsymbol{\lambda}_l| + e|\sigma| = 3 + 3 \cdot 3 = 12$ , we are in the conditions of the theorem. The abacus of  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  is



Applying  $\tilde{a}_\sigma$ , we get the following abacus



So we see that  $\tilde{a}_\sigma$  acts by shifting periods in  $\lambda^{j_0}$  to the right according to  $\sigma$ : the first period is shifted two steps and the second one step.

One first thing to notice is that Losev's formula for  $\tilde{a}_\sigma$  is similar to the formula of the operators  $\tilde{b}_\sigma$  of Section 6 (shifting  $e$ -periods to the right). However, one sees that the property of being asymptotic is somehow antagonistic to the property of being a doubly highest weight vertex. More precisely, a doubly highest weight vertex can never be asymptotic; and conversely, an asymptotic multipartition can never be a doubly highest weight vertex (except for the trivial cases). This is clear for instance looking at Proposition 5.7. Still, we have explained how to extend the definition of the new operators to the whole set of partitions, in (6.13). In this section, we will show that the operator  $\tilde{a}_\sigma$  actually coincides with  $\tilde{b}_\sigma$  for all partition  $\sigma$ . Moreover, we show that the partition  $\theta$  arising in the asymptotic case is in fact equal to the partition  $\kappa$  arising in the doubly highest weight vertex case. Note that the maps  $\tilde{b}_{-1}$  and  $\tilde{b}_{-\kappa}$  implicitly then corresponds to taking the inverse maps to  $\tilde{a}_{(1)}$  and  $\tilde{a}_\kappa$ .

For every asymptotic charged  $l$ -partition  $|\lambda_l, \mathbf{s}_l\rangle$  which is a highest weight vertex in the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal, one can consider the corresponding doubly highest weight vertex. One can apply to it an operator  $\tilde{b}_{-\kappa}$ , and go back in the  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -crystal to get the corresponding  $l$ -partition  $|\mu_l, \mathbf{r}_l\rangle = \tilde{b}_{-\kappa}|\lambda_l, \mathbf{s}_l\rangle$  (cf Procedure (6.13)). In fact, the propagation in the  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -crystal turns out to have a nice description: acting by  $\tilde{b}_{-\kappa}$  and by  $\tilde{b}_\sigma$  on  $|\lambda_l, \mathbf{s}_l\rangle$  is combinatorially “the same” as acting on doubly highest weight vertices (i.e. shifting  $e$ -periods to the left), as is stated in the next proposition.

**Proposition 7.4.** *Let  $|\lambda_l, \mathbf{s}_l\rangle$  be a highest weight vertex in the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal. Write  $|\tilde{\lambda}_l, \tilde{\mathbf{s}}_l\rangle$  for the corresponding doubly highest weight vertex, and set  $\kappa = \kappa(|\tilde{\lambda}_l, \tilde{\mathbf{s}}_l\rangle) = (\kappa_1, \kappa_2, \dots, \kappa_{N(l)})$ . Then  $|\lambda_l, \mathbf{s}_l\rangle$  has at least  $N(l)$  non-trivial periods, and*

- (1)  $\tilde{b}_{-\kappa}$  acts on  $|\lambda_l, \mathbf{s}_l\rangle$  by shifting the  $k$ -th period of  $\mathcal{A}(\lambda_l, \mathbf{s}_l)$  by  $\kappa_k$  steps to the left, for all  $k = 1, \dots, N(l)$ , starting from  $k = N(l), \dots$  and finishing by  $k = 1$ ,
- (2) if  $\kappa = \emptyset$ , then for all partition  $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_r)$ ,  $\tilde{b}_\sigma$  acts on  $|\lambda_l, \mathbf{s}_l\rangle$  by shifting the  $k$ -th period of  $\mathcal{A}(\lambda_l, \mathbf{s}_l)$  by  $\sigma_k$  steps to the right, for all  $k = 1, \dots, r$ , starting from  $k = r, \dots$  and finishing by  $k = 1$ .

*Remark 7.5.* Note that in the case  $|\lambda_l, \mathbf{s}_l\rangle = |\tilde{\lambda}_l, \tilde{\mathbf{s}}_l\rangle$ , this procedure coincides with the procedure for  $\tilde{b}_{-\kappa}$  described in Definition 6.6. The subtlety here is that we might have more than  $N(l)$  non-trivial periods in  $|\lambda_l, \mathbf{s}_l\rangle$ , so we have to modify the statement. This being said, the action remains very explicit.

*Proof.* We have already explained in Section 4.3 how the crystal operators  $\tilde{f}_j$  of  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$  act on the  $l$ -abacus, see Theorem 4.8 and Example 4.10. They are  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal isomorphisms, and in fact they transform an  $e$ -period  $P = ((j_k, \beta_k))_{k=1, \dots, e}$  of  $\mathcal{A}(\tilde{\lambda}_l, \tilde{\mathbf{s}}_l)$  into another  $e$ -period  $P' = ((j'_k, \beta'_k))_{k=1, \dots, e}$ , where either

- $(j'_k, \beta'_k) = (j_k, \beta_k)$  for all  $k$  but one, denoted  $k_0$ , for which  $(j'_{k_0}, \beta'_{k_0}) = (j_{k_0} - 1, \beta_{k_0})$ , or
- $(j'_k, \beta'_k) = (j_k, \beta_k)$  for all  $k > 1$  and  $(j'_1, \beta'_1) = (l, \beta_e + e)$  (in which case  $j_e = 1$ ).

This is true because  $\mathcal{A}(\tilde{\lambda}_l, \tilde{s}_l)$  is totally  $e$ -period (and so are all elements in the  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -crystal). We see directly that such a procedure can only preserve the number of periods, or make it increase, proving the opening statement.

Let us prove Point (1). We first need to show that it is possible to apply the shifting procedure in the proposition, that is to say, that each element of the  $N(l)$ -th period has at least  $\kappa_{N(l)}$  empty spots to its left, and so on (formally,  $(j, \beta) \in P_{N(l)} \Rightarrow (j, \beta - a) \notin \mathcal{A}(\lambda_l, \mathbf{s}_l) \forall a = 1, \dots, \kappa_{N(l)}$ , and so on, if  $P_k$  denotes the  $k$ -th period in  $\mathcal{A}(\lambda_l, \mathbf{s}_l)$ ). By contradiction, suppose that applying the operator  $\tilde{f}_j$  to a highest weight vertex in the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal moves a bead of a period  $P_k$  in the abacus to a spot  $(j, \beta)$  ( $j \in \{1, \dots, l\}$  and meaning  $\tilde{f}_0$  if  $j = l$ ) such that a period  $P_{k'}$ , with  $k > k'$ , contains an element of the form  $(j, \beta')$  and has exactly  $\kappa_{k'}$  empty spots to its left. In this case, the element  $(j, \beta')$  creates a  $-$  in the  $j$ -word  $\dot{w}_j(|\lambda'_e, \mathbf{s}'_e\rangle)$ , which directly simplifies with the  $+$  created by the bead that is moved by  $\tilde{f}_j$ , which is a contradiction.

In fact, this procedure indeed gives the crystal action of the Heisenberg algebra for the highest weight vertices in the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal. It suffices to notice that shifting the considered  $e$ -periods preserves the reduced  $j$ -words  $\dot{w}_j$ , because this amounts to potentially make subwords of the form  $(-+)$  collapse. In addition, one observes that  $\tilde{f}_j$  acts on the modified  $l$ -abacus by moving the bead corresponding to the bead of the original abacus  $\mathcal{A}(|\lambda_l, \mathbf{s}_l\rangle)$  which is moved by  $\tilde{f}_j$ . This is the same as applying the procedure to  $\tilde{f}_j|\lambda_l, \mathbf{s}_l\rangle$ . Because  $\tilde{b}_{-\kappa}$  is a  $\mathcal{U}'_p(\widehat{\mathfrak{sl}}_l)$ -crystal isomorphism (Theorem 6.9), this procedure is indeed the action of  $\tilde{b}_{-\kappa}$  on  $l$ -abacci.

Using the exact same arguments and looking at the reverse procedure, Point (2) is also proved.  $\square$

**Theorem 7.6.** *The Heisenberg crystal coincides with Losev's  $\mathfrak{sl}_\infty$ -crystal.*

*Proof.* It suffices to show that  $\tilde{b}_\sigma = \tilde{a}_\sigma$  for all  $\sigma \in \Pi$ , and that  $\theta = \kappa$ . In fact, we first show that the  $\tilde{b}_\sigma$  and  $\tilde{a}_\sigma$  coincide on highest weight vertices in the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal which are  $j_0$ -asymptotic for some  $j_0 \in \{1, \dots, l\}$ . This is enough because we know that in both cases, the maps commute with the crystal operators of  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$  (Theorem 6.19 for  $\tilde{b}_\sigma$  and Theorem 7.2 for  $\tilde{a}_\sigma$ ). Every such charged  $l$ -partition  $|\lambda_l, \mathbf{s}_l\rangle$  is obtained from a doubly highest weight vertex by applying a sequence of Kashiwara crystal operators  $\tilde{f}_{j_1} \tilde{f}_{j_2} \dots \tilde{f}_{j_r}$ . By Proposition 7.4, we know how these operators act. In the asymptotic case, if  $|\lambda_l| + e|\sigma| \leq N$  (where the  $N$  comes from the asymptotic property), applying  $\tilde{b}_\sigma$  only affects the  $j_0$ -th row of  $\mathcal{A}(\lambda_l, \mathbf{s}_l)$ . Moreover, the shifting procedure on abacci described in Proposition 7.4

is exactly Losev's formula for  $\tilde{a}_\sigma$  on charged  $l$ -partitions, see Theorem 7.2. So we have  $\tilde{b}_\sigma = \tilde{a}_\sigma$  for all partition  $\sigma$ .

Similarly, the action of  $\tilde{b}_{-\kappa}$  is entirely described on the  $j_0$ -th row of  $\mathcal{A}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$ , and the procedure of Proposition 7.4 on abacci is in this case precisely the reverse procedure of Losev's formula for  $\tilde{a}_\theta$  on  $l$ -partitions, with  $\theta = \kappa$ . Therefore,  $\kappa = \theta$ . In particular, the depth of  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  in the Heisenberg crystal is by definition  $|\kappa| = |\theta|$ .  $\square$

Therefore, we can now use the results of Losev [20] and Shan and Vasserot [24] on the Heisenberg crystal.

### 7.3. A combinatorial characterisation of finite-dimensional simple modules.

One of the important results of Shan and Vasserot in [24] is Proposition 5.18, which gives a characterisation of the finite-dimensional simple modules for cyclotomic rational Cherednik algebras. They show that this property is equivalent to being "primitive". Combinatorially, this amounts to saying that the  $l$ -partition labelling this module is simultaneously a highest weight vertex in the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal and in the Heisenberg crystal, see e.g. [20, Section 5.1.1].

Using the results of Section 6.4, we can give an explicit combinatorial description of these  $l$ -partitions. For this, recall that we have introduced the notion of FLOTW  $e$ -partitions in Definition 6.22, and that we can use the correspondence (3.7) between  $l$ -partitions charged by  $\mathbf{s}_l$  and  $e$ -partitions charged by  $\mathbf{s}'_e$ .

**Theorem 7.7.** *A simple  $H_{\mathbf{c},n}$ -module is finite-dimensional if and only if it is labelled by an  $l$ -partition  $\boldsymbol{\lambda}_l$  of rank  $n$  such that  $|\boldsymbol{\lambda}'_e, \mathbf{s}'_e\rangle$  is an FLOTW  $e$ -partition.*

*Proof.* As already explained, a simple  $H_{\mathbf{c},n}$ -module is finite-dimensional if and only if it is labelled by an  $l$ -partition  $\boldsymbol{\lambda}_l$  of rank  $n$  such that  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  is a highest weight vertex in the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal and  $|\lambda, \mathbf{s}\rangle$  is a highest weight vertex in the Heisenberg crystal, i.e.  $\lambda = \bar{\lambda}$ .

Assume first that  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  is a highest weight vertex in the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal and  $|\boldsymbol{\lambda}, \mathbf{s}\rangle$  is a highest weight vertex in the Heisenberg crystal. Then with the notation of Theorem 6.21, we have  $(i_1, \dots, i_r) = \emptyset$  and  $\sigma = \emptyset$ , thus

$$|\boldsymbol{\lambda}'_e, \mathbf{s}'_e\rangle = \tilde{F}_{(j_1, \dots, j_k)} |\emptyset_e, \tilde{\mathbf{s}}'_e\rangle$$

where  $\tilde{\mathbf{s}}'_e \in A_{e,l}(s)$ . By Theorem 6.24,  $|\boldsymbol{\lambda}'_e, \mathbf{s}'_e\rangle \in \Psi_e$ , i.e. is FLOTW.

Conversely, if  $|\boldsymbol{\lambda}'_e, \mathbf{s}'_e\rangle$  is an FLOTW  $e$ -partition, then there exist some  $(j_1, \dots, j_k) \in \{0, \dots, l-1\}^k$  such that  $|\boldsymbol{\lambda}'_e, \mathbf{s}'_e\rangle = \tilde{F}_{(j_1, \dots, j_k)} |\emptyset_e, \tilde{\mathbf{s}}'_e\rangle$ , for some  $\tilde{\mathbf{s}}'_e \in A_{e,l}(s)$ . Then for all  $\sigma \in \Pi$ , we have

$$\begin{aligned} \tilde{b}_{-\sigma} |\boldsymbol{\lambda}'_e, \mathbf{s}'_e\rangle &= (\tilde{b}_{-\sigma} \circ \tilde{F}_{(j_1, \dots, j_k)}) |\emptyset_e, \tilde{\mathbf{s}}'_e\rangle \\ &= (\tilde{F}_{(j_1, \dots, j_k)} \circ \tilde{b}_{-\sigma}) |\emptyset_e, \tilde{\mathbf{s}}'_e\rangle \quad \text{by Theorem 6.19} \\ &= 0. \end{aligned}$$

and for all  $i \in \{0, \dots, e-1\}$ ,

$$\begin{aligned} \tilde{e}_i |\boldsymbol{\lambda}'_e, \mathbf{s}'_e\rangle &= (\tilde{e}_i \circ \tilde{F}_{(j_1, \dots, j_k)}) |\emptyset_e, \tilde{\mathbf{s}}'_e\rangle \\ &= (\tilde{F}_{(j_1, \dots, j_k)} \circ \tilde{e}_i) |\emptyset_e, \tilde{\mathbf{s}}'_e\rangle \text{ by Theorem 4.8} \\ &= 0. \end{aligned}$$

So  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$  is a highest weight vertex in the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -crystal and  $|\lambda, s\rangle$  is a highest weight vertex in the Heisenberg crystal.  $\square$

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