SPINORS AND ESSENTIAL DIMENSION

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ABSTRACT. We prove that spin groups act generically freely on various spinor modules, in the sense of group schemes and in a way that does not depend on the characteristic of the base field. As a consequence, we extend the surprising calculation of the essential dimension of spin groups and half-spin groups in characteristic zero by Brosnan–Reichstein–Vistoli (Annals of Math., 2010) and Chernousov–Merkurjev (Algebra & Number Theory, 2014) to fields of characteristic different from 2.

1. INTRODUCTION

The essential dimension of an algebraic group G is, roughly speaking, the number of parameters needed to specify a G-torsor. Since the notion was introduced in [BR97] and [RY00], there have been many papers calculating the essential dimension of various groups, such as [KM03], [CS06], [Flo08], [KM08], [GR09], [Mer10], [BM12], [LMMR13], etc. (See [Mer15a], [Mer13], or [Rei10] for a survey of the current state of the art.) For connected groups, the essential dimension of G tends to be less than the dimension of G as a variety; for semisimple adjoint groups this is well known[1]. Therefore, the discovery by Brosnan–Reichstein–Vistoli in [BRV10] that the essential dimension of the spinor group Spin_n grows exponentially as a function of n (whereas $\dim \mathrm{Spin}_n$ is quadratic in n), was startling. Their results, together with refinements for n divisible by 4 in [Mer09] and [CM14], determined the essential dimension of Spin_n for $n > 14$ if $\mathrm{char}\, k = 0$. One goal of the present paper is to extend this result to all characteristics except 2.

Generically free actions. The source of the characteristic zero hypothesis in [BRV10] is that the upper bound relies on a fact about the action of spin groups on spinors that is only available in the literature in case the field k has characteristic zero. Recall that a group G acting on a vector space V is said to act *generically freely* if there is a dense open subset U of V such that, for every $K \supseteq k$ and every $u \in U(K)$, the stabilizer in G of u is the trivial group scheme. We prove:

Theorem 1.1. *Suppose $n > 14$. Then Spin_n acts generically freely on the spin representation if $n \equiv 1, 3 \bmod 4$; a half-spin representation if $n \equiv 2 \bmod 4$; or a direct sum of the vector representation and a half-spin representation if $n \equiv 0 \bmod 4$. Furthermore, if $n \equiv 0 \bmod 4$ and $n \geq 20$, then HSpin_n acts generically freely on a half-spin representation.*

Throughout, we write Spin_n for the split spinor group, which is the simply connected cover (in the sense of linear algebraic groups) of the split group SO_n. To

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[1]See [GG15a] for a proof that works regardless of the characteristic of the field.

be precise, the *vector representation* is the map $\mathrm{Spin}_n \to \mathrm{SO}_n$, which is uniquely defined up to equivalence unless $n = 8$. For n not divisible by 4, the kernel μ_2 of this representation is the unique central μ_2 subgroup of Spin_n.

For n divisible by 4, the natural action of Spin_n on the spinors is a direct sum of two inequivalent representations, call them V_1 and V_2, each of which is called a *half-spin* representation. The center of Spin_n in this case contains two additional copies of μ_2, namely the kernels of the half-spin representations $\mathrm{Spin}_n \to \mathrm{GL}(V_i)$, and we write HSpin_n for the image of Spin_n (which does not depend on i). For $n \geq 12$, HSpin_n is not isomorphic to SO_n.

Theorem 1.1 is known under the additional hypothesis that $\mathrm{char}\, k = 0$, see [AP71, Th. 1] for $n \geq 29$ and [Pop88] for $n \geq 15$. The proof below is independent of the characteristic zero results, and so gives an alternative proof.

We note that Guerreiro proved that the generic stabilizer in the Lie algebra $\mathfrak{spin}_n$, acting on a (half) spin representation, is central for $n = 22$ and $n \geq 24$, see Tables 6 and 9 of [Gue97]. At the level of group schemes, this gives the weaker result that the generic stabilizer is finite étale. Regardless, we recover these cases quickly, see §3; the longest part of our proof concerns the cases $n = 18$ and 20.

Although Theorem 1.1 is stated and proved for split groups, it quickly implies analogous results for non-split forms of these groups, see [Löt13, §4] for details.

Generic stabilizer in Spin_n for small n. For completeness, we list the stabilizer in Spin_n of a generic vector for $6 \leq n \leq 14$ in Table 1. The entries for $n \leq 12$ and $\mathrm{char}\, k \neq 2$ are from [Igu70]; see sections 7–9 below for the remaining cases. The case $n = 14$ is particularly important due to its relationship with the structure of 14-dimensional quadratic forms with trivial discriminant and Clifford invariant (see [Ros99a], [Ros99b], [Gar09], and [Mer15b]), so we calculate the stabilizer in detail in that case.

n	$\mathrm{char}\, k \neq 2$	$\mathrm{char}\, k = 2$	n	$\mathrm{char}\, k \neq 2$	$\mathrm{char}\, k = 2$
6	$(\mathrm{SL}_3) \cdot (\mathbb{G}_\mathrm{a})^3$	same	11	SL_5	$\mathrm{SL}_5 \rtimes \mathbb{Z}/2$
7	G_2	same	12	SL_6	$\mathrm{SL}_6 \rtimes \mathbb{Z}/2$
8	Spin_7	same	13	$\mathrm{SL}_3 \times \mathrm{SL}_3$	$(\mathrm{SL}_3 \times \mathrm{SL}_3) \rtimes \mathbb{Z}/2$
9	Spin_7	same	14	$G_2 \times G_2$	$(G_2 \times G_2) \rtimes \mathbb{Z}/2$
10	$(\mathrm{Spin}_7) \cdot (\mathbb{G}_\mathrm{a})^8$	same			

TABLE 1. Stabilizer sub-group-scheme in Spin_n of a generic vector in an irreducible (half) spin representation for small n.

Essential dimension. We recall the definition of essential dimension. For an extension K of a field k and an element a in the Galois cohomology set $H^1(K, G)$, we define $\mathrm{ed}(x)$ to be the minimum of the transcendence degree of K_0/k for $k \subseteq K_0 \subseteq K$ such that x is in the image of $H^1(K_0, G) \to H^1(K, G)$. The *essential dimension* of G, denoted $\mathrm{ed}(G)$, is defined to be $\max \mathrm{ed}(x)$ as x varies over all extensions K/k and all $x \in H^1(K, G)$. There is also a notion of *essential p-dimension* for a prime p. The essential p-dimension $\mathrm{ed}_p(x)$ is the minimum of $\mathrm{ed}(\mathrm{res}_{K'/K}\, x)$ as K' varies over finite extensions of K such that p does not divide $[K' : K]$, where $\mathrm{res}_{K'/K} : H^1(K, G) \to H^1(K', G)$ is the natural map. The essential p-dimension of G, $\mathrm{ed}_p(G)$, is defined to be the minium of $\mathrm{ed}_p(x)$ as K and x vary; trivially,

$\mathrm{ed}_p(G) \le \mathrm{ed}(G)$ for all p and G, and $\mathrm{ed}_p(G) = 0$ if for every K every element of $H^1(K,G)$ is killed by some finite extension of K of degree not divisible by p.

Our Theorem 1.1 gives upper bounds on the essential dimension of Spin_n and HSpin_n regardless of the characteristic of k. Combining these with the results of [BRV10], [Mer09], [CM14], and [Löt13] quickly gives the following, see §6 for details.

Corollary 1.2. *For $n > 14$ and $\operatorname{char} k \neq 2$,*

$$\mathrm{ed}_2(\mathrm{Spin}_n) = \mathrm{ed}(\mathrm{Spin}_n) = \begin{cases} 2^{(n-1)/2} - \frac{n(n-1)}{2} & \text{if } n \equiv 1,3 \bmod 4; \\ 2^{(n-2)/2} - \frac{n(n-1)}{2} & \text{if } n \equiv 2 \bmod 4; \text{ and} \\ 2^{(n-2)/2} - \frac{n(n-1)}{2} + 2^m & \text{if } n \equiv 0 \bmod 4 \end{cases}$$

where 2^m is the largest power of 2 dividing n in the final case. For $n \ge 20$ and divisible by 4,

$$\mathrm{ed}_2(\mathrm{HSpin}_n) = \mathrm{ed}(\mathrm{HSpin}_n) = 2^{(n-2)/2} - \frac{n(n-1)}{2}.$$

Combining this with the calculation of $\mathrm{ed}(\mathrm{Spin}_n)$ for $n \le 14$ by Markus Rost in [Ros99a] and [Ros99b] (see also [Gar09]), we find for $\operatorname{char} k \neq 2$:

n	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\mathrm{ed}(\mathrm{Spin}_n)$	0	0	4	5	5	4	5	6	6	7	23	24	120	103	341	326

Notation. Let G be an affine group scheme of finite type over a field k, which we assume is algebraically closed and of characteristic different from 2. (If G is additionally smooth, then we say that G is an *algebraic group*.) If G acts on a variety X, the stabilizer G_x of an element $x \in X(k)$ is a sub-group-scheme of G with R-points

$$G_x(R) = \{g \in G(R) \mid gx = x\}$$

for every k-algebra R.

If $\mathrm{Lie}(G) = 0$ then G is finite and étale. If additionally $G(k) = 1$, then G is the trivial group scheme $\operatorname{Spec} k$.

For a representation $\rho\colon G \to \mathrm{GL}(V)$ and elements $g \in G(k)$ and $x \in \mathrm{Lie}(G)$, we denote the fixed spaces by $V^g := \ker(\rho(g) - 1)$ and $V^x := \ker(\mathrm{d}\rho(x))$.

We use fraktur letters such as $\mathfrak{g}$, $\mathfrak{spin}_n$, etc., for the Lie algebras $\mathrm{Lie}(G)$, $\mathrm{Lie}(\mathrm{Spin}_n)$, etc.

2. Fixed spaces of elements

Fix some $n \ge 6$. Let V be a (half) spin representation for Spin_n, of dimension $2^{\lfloor (n-1)/2 \rfloor}$.

Proposition 2.1. *For $n \ge 6$:*

1. *For all noncentral $x \in \mathfrak{spin}_n$, $\dim V^x \le \frac{3}{4} \dim V$.*
2. *For all noncentral $g \in \mathrm{Spin}_n$, $\dim V^g \le \frac{3}{4} \dim V$.*

If $n > 8$, $\operatorname{char} k \neq 2$, and $g \in \mathrm{Spin}_n$ is noncentral semisimple, then $\dim V^g \le \frac{5}{8} \dim V$.

In the proof, in case $\operatorname{char} k \neq 2$, we view SO_n as the group of matrices

$$\mathrm{SO}_n(k) = \{A \in \mathrm{SL}_n(k) \mid SA^\top S = A^{-1}\},$$

where S is the matrix 1's on the "second diagonal", i.e., $S_{i,n+1-i} = 1$ and the other entries of S are zero. The intersection of the diagonal matrices with SO_n are a maximal torus. For n even, one finds elements of the form $(t_1, t_2, \ldots, t_{n/2}, t_{n/2}^{-1}, \ldots, t_1^{-1})$, and we abbreviate these as $(t_1, t_2, \ldots, t_{n/2}, \ldots)$. Explicit formulas for a triality automorphism σ of Spin_8 of order 3 are given in [Gar98, §1], and for $g = (t_1, t_2, t_3, t_4, \ldots) \in \mathrm{Spin}_8$ the elements $\sigma(g)$ and $\sigma^2(g)$ have images in SO_8

$$\varepsilon\left(\sqrt{\tfrac{t_1t_2t_3}{t_4}}, \sqrt{\tfrac{t_1t_4t_2}{t_3}}, \sqrt{\tfrac{t_1t_3t_4}{t_2}}, \sqrt{\tfrac{t_1}{t_2t_3t_4}}, \ldots\right) \quad \text{and} \quad \varepsilon\left(\sqrt{t_1t_2t_3t_4}, \sqrt{\tfrac{t_1t_2}{t_3t_4}}, \sqrt{\tfrac{t_1t_3}{t_2t_4}}, \sqrt{\tfrac{t_2t_3}{t_1t_4}}, \ldots\right), \tag{2.2}$$

where $\varepsilon = \pm 1$ is the only impecision in the expression.

Proof. For (1), in the Jordan decomposition $x = s + n$ where s is semisimple, n is nilpotent, and $[s, n] = 0$, we have $V^x \subseteq V^s \cap V^n$, so it suffices to prove (1) for x nilpotent or semisimple.

Suppose first that x is a root element. If $n = 6$, then $\mathfrak{spin}_n \cong \mathfrak{sl}_4$ and V is the natural representation of $\mathfrak{sl}_4$, so we have the desired equality. For $n > 6$, the module restricted to so_{n-1} is either irreducible or the direct sum of two half spins and so the result follows.

If x is nonzero nilpotent, then we may replace x by a root element in the closure of $(\mathrm{Ad}\, G)x$. If x is noncentral semisimple, choose a root subgroup U_α of SO_n belonging to a Borel subgroup B such that x lies in $\mathrm{Lie}(B)$ and does not commute with U_α. Then for all $y \in \mathrm{Lie}(U_\alpha)$ and all scalars λ, $x + \lambda y$ is in the same $\mathrm{Ad}(\mathrm{SO}_n)$-orbit as x and y is in the closure of the set of such elements; replace x with y. If x is nonzero nilpotent, then root elements are in the closure of $\mathrm{Ad}(G)x$.

(In case n is divisible by 4, the natural map $\mathfrak{spin}_n \to \mathfrak{hspin}_n$ is an isomorphism on root elements. It follows that for noncentral $x \in \mathfrak{hspin}_n$, $\dim V^x \leq \frac{3}{4} \dim V$ by the same argument.)

For (2), we may assume that g is unipotent or semi simple. If g is unipotent, then by taking closures, we may pass to root elements and argue as for x in the Lie algebra.

If g is semi simple, we actually prove a slightly stronger result: *all* eigenspaces have dimension at most $\frac{3}{4} \dim V$.

Suppose now that n is even. The image of g in SO_n can be viewed as an element of $\mathrm{SO}_{n-2} \times \mathrm{SO}_2$, where it has eigenvalues (a, a^{-1}) in SO_2. Replacing if necessary g with a multiple by an element of the center of Spin_n, we may assume that g is in the image of $\mathrm{Spin}_{n-2} \times \mathrm{Spin}_2$. Then $V = V_1 \oplus V_2$ where the V_i are distinct half spin modules for Spin_{n-2} and the Spin_2 acts on each (since they are distinct and Spin_2 commutes with Spin_{n-2}). By induction every eigenspace of g has dim at most $\frac{3}{4} \dim V_i$ and the Spin_2 component of g acts as a scalar, so this is preserved.

If n is odd, then the image of g in SO_n has eigenvalue 1 on the natural module, so is contained in a SO_{n-1} subgroup. Replacing if necessary g with gz for some z in the center of G, we may assume that g is in the image of Spin_{n-1} and the claim follows by induction.

For the final claim, view g as an element in the image of $(g_1, g_2) \in \mathrm{Spin}_8 \times \mathrm{Spin}_{n-8}$. The result is clear unless in some 8-dimensional image of Spin_8, g_1 has diagonal image $(a, 1, 1, 1, \ldots) \in \mathrm{SO}_8$. We may assume that the image of g has prime order in $\mathrm{GL}(V)$.

If g_1 has odd order or order 4, then as in (2.2), on the other two 8 dimensional representations of Spin_8 it has no fixed space and each eigenspace of dimension at most 4. So on the sum of any two of these representations the largest eigenspace is at most 10-dimensional (out of 16), and the claim follows.

If g_1 has order 2, then it maps to an element of order 4 in the other two 8-dimensional representations via (2.2) and again the same argument applies. □

The proposition will feed into the following elementary lemma, which resembles [AP71, Lemma 4] and [Gue97, §3.3].

Lemma 2.3. *Let V be a representation of a semisimple algebraic group G over an algebraically closed field k.*

(1) *If for every unipotent $g \in G$ and every noncentral semisimple $g \in G$ of prime order we have*

$$\dim V^g + \dim g^G < \dim V, \tag{2.4}$$

then for generic $v \in V$, $G_v(k)$ is central in $G(k)$.

(2) *Suppose $\operatorname{char} k = p > 0$ and let $\mathfrak{h}$ be a G-invariant subspace of $\mathfrak{g}$. If for every nonzero $x \in \mathfrak{g} \setminus \mathfrak{h}$ such that $x^{[p]} \in \{0, x\}$ we have*

$$\dim V^x + \dim(\mathrm{Ad}(G)x) < \dim V, \tag{2.5}$$

then for generic $v \in V$, $\mathrm{Lie}(G_v) \subseteq \mathfrak{h}$.

We will apply this to conclude that G_v is the trivial group scheme for generic v, so the hypothesis on $\operatorname{char} k$ in (2) is harmless. When $\operatorname{char} k = 0$, the conclusion of (1) suffices.

Proof. For (1), see [GG15b, §10] or adjust slightly the following proof of (2). For $x \in \mathfrak{g}$, define

$$V(x) := \{v \in V \mid \text{there is } g \in G(k) \text{ s.t. } xgv = 0\} = \bigcup_{g \in G(k)} gV^x.$$

Define $\alpha : G \times V^x \to V$ by $\alpha(g, w) = gw$, so the image of α is precisely $V(x)$. The fiber over gw contains (gc^{-1}, cw) for $\mathrm{Ad}(c)$ fixing x, and so $\dim V(x) \le \dim(\mathrm{Ad}(G)x) + \dim V^x$.

Let $X \subset \mathfrak{g}$ be the set of nonzero $x \in \mathfrak{g} \setminus \mathfrak{h}$ such that $x^{[p]} \in \{0, x\}$; it is a union of finitely many G-orbits. (Every toral element — i.e., x with $x^{[p]} = x$ — belongs to $\mathrm{Lie}(T)$ for a maximal torus T in G by [BS66], and it is obvious that there are only finitely many conjugacy classes of toral elements in $\mathrm{Lie}(T)$.) Now $V(x)$ depends only on the G-orbit of X (because $V^{\mathrm{Ad}(g)x} = gV^x$), so the union $\cup_{x \in X} V(x)$ is a finite union. As $\dim V(x) < \dim V$ by the previous paragraph, the union $\cup V(x)$ is contained in a proper closed subvariety Z of V, and for every v in the (nonempty, open) complement of Z, $\mathfrak{g}_v$ does not meet X.

For each $v \in (V \setminus Z)(k)$ and each $y \in \mathfrak{g}_v$, we can write y as

$$y = y_n + \sum_{i=1}^{r} \alpha_i y_i, \quad [y_n, y_i] = [y_i, y_j] = 0 \text{ for all } i, j$$

such that $y_1, \ldots, y_r \in \mathfrak{g}_v$ are toral, and $y_n \in \mathfrak{g}_v$ satisfies $y_n^{[p]} = 0$, see [SF88, Th. 3.6(2)]. Thus y_n and the $y_1, \ldots, y_r$ are in $\mathfrak{h}$ by the previous paragraph. □

Note that, in proving Theorem 1.1, we may assume that k is algebraically closed (and so this hypothesis in Lemma 2.3 is harmless). Indeed, suppose G is an algebraic group acting on a vector space V over a field k. Fix a basis $v_1, \ldots, v_n$ of V and consider the element $\eta := \sum t_i v_i \in V \otimes k(t_1, \ldots, t_n) = V \otimes k(V)$ for indeterminates $t_1, \ldots, t_n$; it is a sort of generic point of V. Certainly, G acts generically freely on V over k if and only if the stabilizer $(G \times k(V))_v$ is the trivial group scheme, and this statement is unchanged by replacing k with an algebraic closure. That is, G acts generically freely on V over k if and only if $G \times K$ acts generically freely on $V \otimes K$ for K an algebraic closure of k.

3. Proof of Theorem 1.1 for $n > 20$

Suppose $n > 2$, and put V for a (half) spin representation of Spin_n. Recall that

$$\dim \mathrm{Spin}_n = r(2r-1) \quad \text{and} \quad \dim V = 2^{r-1} \quad \text{if } n = 2r$$

whereas

$$\dim \mathrm{Spin}_n = 2r^2 + r \quad \text{and} \quad \dim V = 2^r \quad \text{if } n = 2r+1$$

and in both cases $\operatorname{rank} \mathrm{Spin}_n = r$. Proposition 2.1 gives an upper bound on $\dim V^g$ for noncentral g, and certainly the conjugacy class of g has dimension at most $\dim \mathrm{Spin}_n - r$. If we assume $n \geq 21$ and apply these, we obtain (2.4) and consequently the stabilizer S of a generic $v \in V$ has $S(k)$ central in $\mathrm{Spin}_n(k)$. Repeating this with the Lie algebra $\mathfrak{spin}_n$ (and $\mathfrak{h}$ the center of $\mathfrak{spin}_n$) we find that $\mathrm{Lie}(S)$ is central in $\mathfrak{spin}_n$. For n not divisible by 4, the representation $\mathrm{Spin}_n \to \mathrm{GL}(V)$ restricts to a closed embedding on the center of Spin_n, so S is the trivial group scheme as claimed in Theorem 1.1.

For n divisible by 4, we conclude that HSpin_n acts generically freely on V (using that Proposition 2.11 holds also for $\mathfrak{hspin}_n$). As the kernel μ_2 of $\mathrm{Spin}_n \to \mathrm{HSpin}_n$ acts faithfully on the vector representation W, it follows that Spin_n acts generically freely on $V \oplus W$, completing the proof of Theorem 1.1 for $n > 20$.

4. Proof of Theorem 1.1 for $n \leq 20$ and characteristic $\neq 2$

In this section we assume that $\operatorname{char} k \neq 2$, and in particular the Lie algebra $\mathfrak{spin}_n$ (and $\mathfrak{hspin}_n$ in case n is divisible by 4) is naturally identified with so_n.

Case $n = 18$ or 20. Take V to be a half-spin representation of $G = \mathrm{Spin}_n$ (if $n = 18$) or $G = \mathrm{HSpin}_n$ (if $n = 20$). To prove Theorem 1.1 for these n, it suffices to prove that G acts generically freely on V, which we do by verifying the inequalities (2.4) and (2.5).

Nilpotents and unipotents. Let $x \in \mathfrak{g}$ with $x^{[p]} = 0$. The argument for unipotent elements of G is essentially identical (as we assume $\operatorname{char} k \neq 2$) and we omit it.

If, for a particular x, we find that the centralizer of x has dimension > 89 (if $n = 18$) or > 62 (if $n = 20$), then $\dim(\mathrm{Ad}(G)x) < \frac{1}{4} \dim V$ and we are done by Proposition 2.1.

For x nilpotent, the most interesting case is where x is has partition $(2^{2t}, 1^{n-2t})$ for some t. If $n = 20$, then such a class has centralizer of dimension at least 100, and we are done. If $n = 18$, we may assume by similar reasoning that $t = 3$ or 4. The centralizer of x has dimension ≥ 81, so $\dim(\mathrm{Ad}(G)x) \leq 72$. We claim that $\dim V^x \leq 140$; it suffices to prove this for an element with $t = 3$, as the element with $t = 4$ specializes to it. View it as an element in the image of $\mathrm{so}_9 \times \mathrm{so}_9 \to \mathrm{so}_{18}$

where the first factor has partition $(2^4, 1)$ and the second has partition $(2^2, 1^5)$. Now, triality on so_8 sends elements with partition 2^4 to elements with partition 2^4 and $(3, 1^5)$ — see for example [CM93, p. 97] — consequently the $(2^4, 1)$ in so_9 acts on the spin representation of so_9 as a $(3, 2^4, 1^5)$. Similarly, the $(2^2, 1^5)$ acts on the spin representation of so_9 as $(2^4, 1^8)$. The action of x on the half-spin representation of so_{18} is the tensor product of these, and we find that $\dim V^x \le 140$ as claimed.

Suppose x is nilpotent and has a Jordan block of size at least 5. An element with partition $(5, 1)$ in so_6 is a regular nilpotent in $\mathfrak{sl}_4$ with 1-dimensional kernel. Using the tensor product decomposition as in the proof of Proposition 2.1, we deduce that an element $y \in \mathrm{so}_n$ with partition $(5, 1^{n-5})$ has $\dim V^y \le \frac{1}{4} \dim V$, and consequently by specialization $\dim V^x \le \frac{1}{4} \dim V$. As $\dim(\mathrm{Ad}(G)x) \le \dim G - \operatorname{rank} G < \frac{3}{4} \dim V$, the inequality is verified for this x.

Now suppose x is nilpotent and all Jordan blocks have size at most 4, so it is a specialization of $(4^4, 1^2)$ if $n = 18$ or 4^5 if $n = 20$. These classes have centralizers of dimension 41 and 50 respectively, hence $\dim(\mathrm{Ad}(G)x) < \frac{1}{2} \dim V$. If x has at least two Jordan blocks of size at least 3, then x specializes to $(3^2, 1^{n-6})$; as triality sends elements with partition $(3^2, 1^2)$ to elements with the same partition, we find $\dim V^x \le \frac{1}{2} \dim V$. We are left with the case where x has partition $(3, 2^{2t}, 1^{n-2t-3})$ for some t. If $t = 0$, then the centralizer of x has dimension 121 or 154 and we are done. If $t > 0$, then x specializes to y with partition $(3, 2^2, 1^{n-7})$. As triality on so_8 leaves the partition $(3, 2^2, 1)$ unchanged, we find $\dim V^x \le \dim V^y \le \frac{1}{2} \dim V$, as desired, completing the verification of (2.5) for x nilpotent.

Semisimple elements in $\mathrm{Lie}(G)$. For $x \in \mathrm{so}_n$ semisimple, the most interesting case is when x is diagonal with entries $(a^t, (-a)^t, 0^{n-2t})$ where exponents denote multiplicity and $a \in k^\times$. The centralizer of x is $\mathrm{GL}_t \times \mathrm{SO}_{n-2t}$, so $\dim(\mathrm{Ad}(\mathrm{SO}_n)x) = \binom{n}{2} - t^2 - \binom{n-2t}{2}$. This is less than $\frac{1}{4} \dim V$ for $n = 20$, settling that case. For $n = 18$, if $t = 1$ or 2, x is in the image of an element $(a, -a, 0, 0)$ or $(a/2, a/2, -a/2, -a/2)$ in $\mathfrak{sl}_4 \cong \mathrm{so}_6$, and the tensor product decomposition gives that $\dim V^x \le \frac{1}{2} \dim V$ and again we are done. If $t > 2$, we consider a nilpotent $y = \left(\begin{smallmatrix} 0 & Y \\ 0 & 0 \end{smallmatrix}\right)$ not commuting with x where Y is 9-by-9 and y specializes to a nilpotent y' with partition $(2^4, 1^8)$. Such a y' acts on V as 16 copies of $(3, 2^4, 1^5)$, hence $\dim V^{y'} = 160$. By specializing x to y as in the proof of Proposition 2.1, we find $\dim V^x \le 160$ and again we are done.

Semisimple elements in G. Let $g \in G(k)$ be semisimple, non-central, and of prime order. If $n = 20$, then $\dim g^G \le 180 < \frac{3}{8} \dim V$ and we are done by Proposition 2.1. So assume $n = 18$. If we find that the centralizer of g has dimension > 57, then $\dim g^G < \frac{3}{8} \dim V$ and we are done by Proposition 2.1.

If g has order 2, then it maps to an element of order 2 in SO_{18} whose centralizer is no smaller than $\mathrm{SO}_8 \times \mathrm{SO}_{10}$ of dimension 73, and we are done. So assume g has odd prime order. We divide into cases depending on the image $\overline{g} \in \mathrm{SO}_{18}$ of g.

If $\overline{g}$ has at least 5 distinct eigenvalues, then either it has at least 6 distinct eigenvalues $a, a^{-1}, b, b^{-1}, c, c^{-1}$, or it has 4 distinct eigenvalues that are not equal to 1, and the remaining eigenvalue is 1. In the latter case set $c = 1$. View g as the image of $(g_1, g_2) \in \mathrm{Spin}_6 \times \mathrm{Spin}_{12}$ where g_1 maps to a diagonal $(a, b, c, c^{-1}, b^{-1}, a^{-1})$ in SO_6, a regular semisimple element. Therefore, the eigenspaces of the image of g_1

under the isomorphism $\mathrm{Spin}_6 \cong \mathrm{SL}_4$ are all 1-dimensional and the tensor decomposition argument shows that $\dim V^g \leq \frac{1}{4}\dim V$. As $\dim g^G \leq 144 < \frac{3}{4}\dim V$, we are done in this case.

If $\overline{g}$ has exactly 4 eigenvalues, then the centralizer of $\overline{g}$ is at least as big as $\mathrm{GL}_4 \times \mathrm{GL}_5$ of dimension 41, so $\dim g^G \leq 112 < \frac{1}{2}\dim V$. Viewing g as the image of $(g_1, g_2) \in \mathrm{Spin}_8 \times \mathrm{Spin}_{10}$ such that the image $\overline{g}_1$ of g_1 in SO_8 exhibits all 4 eigenvalues, then $\overline{g}_1$ has eigenspaces all of dimension 2 or of dimensions 3, 3, 1, 1. Considering the possible images of $\overline{g}_1$ as in (2.2), each eigenspaces in each of the 8-dimensional representations is at most 4, so $\dim V^g \leq \frac{1}{2}\dim V$ and this case is settled.

In the remaining case, $\overline{g}$ has exactly 2 nontrivial (i.e., not 1) eigenvalues a, a^{-1}. If 1 is not an eigenvalue of $\overline{g}$, then the centralizer of $\overline{g}$ is GL_9 of dimension 81, and we are done. If the eigenspaces for the nontrivial eigenvalues are at least 4-dimensional, then we can take g to be the image of $(g_1, g_2) \in \mathrm{Spin}_{10} \times \mathrm{Spin}_8$ where g_1 maps to $(a, a, a, a, 1, \ldots) \in \mathrm{SO}_{10}$. The images of $(a, a, a, a, \ldots) \in \mathrm{SO}_8$ as in (2.2) are $(a, a, a, a^{-1}, \ldots)$ and $(a^2, 1, 1, 1, \ldots)$, so the largest eigenspace of g_1 on a half-spin representation is 6, so $\dim V^g \leq \frac{3}{8}\dim V$. As the conjugacy class of a regular element has dimension $144 < \frac{5}{8}\dim V$, this case is complete. Finally, if $\overline{g}$ has eigenspaces of dimension at most 2 for a, a^{-1}, then $\dim g^G \leq 58 < \frac{3}{8}\dim V$ and the $n = 18$ case is complete.

Case $n = 17$ or 19. For $n = 17$ or 19, the spin representation of Spin_n can be viewed as the restriction of a half-spin represenation of the overgroup HSpin_{n+1}. We have already proved that this representation of HSpin_{n+1} is generically free.

Case $n = 15$ or 16. We use the general fact:

Lemma 4.1. *Let G be a quasi-simple algebraic group and H a proper closed subgroup of G and X finite. Then for generic $g \in G$, $H \cap gXg^{-1} = H \cap X \cap Z(G)$.*

Proof. For each $x \in X \setminus Z(G)$, note that $W(x) := \{g \in G \mid x^g \in H\}$ is a proper closed sub variety of G and, since X is finite, $\cup W(x)$ is also proper closed. Thus for an open subset of g in G, $g(X \setminus Z(G))g^{-1}$ does not meet H. □

Lemma 4.2. *Let $G = \mathrm{HSpin}_{16}$ and V a half-spin representation over a field k of characteristic $\neq 2$. The generic stabilizer in the group is finite (of order 2^8) and the generic stabilizer in the Lie algebra is 0.*

See for example [Vin76] or [Lev09] for generalities on representations such as those appearing in the preceding lemma.

Proof of Lemma 4.2. Consider $\mathrm{Lie}(E_8) = \mathrm{Lie}(G) \oplus V$ where these are the eigenspaces of an involution in E_8. That involution inverts a maximal torus T of E_8 and so there is maximal Cartan subalgebra $\mathfrak{t} = \mathrm{Lie}(T)$ on which the involution acts as -1. As the natural map $G \times \mathfrak{t} \to V$ is dominant [Lev09, Lemma 0.2], a generic element $\tau \in \mathfrak{t}$ is a generic element of V. Since $\mathfrak{t}$ misses $\mathrm{Lie}(G)$, the annihilator of τ in $\mathrm{Lie}(G)$ is 0 as claimed.

The stabilizer in E_8 of a generic point $\tau \in \mathfrak{t}$ is the maximal torus T and the annihilator of τ in $\mathrm{Lie}(E_8)$ is $\mathfrak{t}$. Therefore, G_v is contained in the normalizer of T, which is an extension of the Weyl group of E_8 by the elements of T that commute with the involution (i.e., the 2-torsion in T); it follows that $G_v(k) \cong \mu_2(k)^8$. □

The proof used that $\operatorname{char} k \neq 2$. For $\operatorname{char} k = 2$ the generic stabilizer in Spin_{16} has been determined by Eric Rains [Rai].

Corollary 4.3. *If* $\operatorname{char} k \neq 2$*, then* Spin_{15} *acts generically freely on* V*.*

Proof. Of course the Lie algebra does because this is true for $\mathrm{Lie}(\mathrm{Spin}_{16})$.

For the group, a generic stabilizer is $\mathrm{Spin}_{15} \cap X$ where X is a generic stabilizer in Spin_{16}. Now X is finite and meets the center of Spin_{16} in the kernel of $\mathrm{Spin}_{16} \to \mathrm{HSpin}_{16}$, whereas Spin_{15} injects in to HSpin_{16}. Therefore, by Lemma 4.1 a generic conjugate of X intersect Spin_{15} is trivial. □

Corollary 4.4. *If* $\operatorname{char} k \neq 2$*, then* Spin_{16} *acts generically freely on* $V \oplus W$*, where* V *is a half-spin and* W *is the natural (*16*-dimensional) module.*

Proof. Now the generic stabilizer is already 0 for the Lie algebra on V whence on $V \oplus W$.

In the group Spin_{16}, a generic stabilizer is conjugate to $X^g \cap \mathrm{Spin}_{15}$ where X is the finite stabilizer on V and as in the proof of the previous corollary, this is generically trivial. □

5. Proof of Theorem 1.1 for $n \leq 20$ and characteristic 2

To complete the proof of Theorem 1.1, it remains to prove, in case $\operatorname{char} k = 2$, that the following representations $G \to \mathrm{GL}(V)$ are generically free:

(1) $G = \mathrm{Spin}_{15}$, Spin_{17}, Spin_{19} and V is a spin representation.
(2) $G = \mathrm{Spin}_{18}$ and V is a half-spin representation.
(3) $G = \mathrm{Spin}_{16}$ or Spin_{20} and V is a direct sum of the vector representation and a half-spin representation.
(4) $G = \mathrm{HSpin}_{20}$ and V is a half-spin representation.

By applying the same techniques as in the previous section or by referring to [GLL16], we see that the group of k-points $G_v(k)$ of the stabilizer of a generic $v \in V$ is the trivial group. It remains to check that $\mathrm{Lie}(G_v) = 0$, which can be checked computationally as follows. Pick any field F of characteristic 2 and any $w \in V$ and compute the stabilizer $\mathfrak{g}_w$. (This can be done easily in various modern computer algebra systems.) In each case, one can find a w such that $\mathfrak{g}_w = 0$, completing the proof of Theorem 1.1. □

6. Proof of Corollary 1.2

For n not divisible by 4, the (half) spin representation Spin_n is generically free by Theorem 1.1, so by, e.g., [Mer13, Th. 3.13] we have:

$$\mathrm{ed}(\mathrm{Spin}_n) \leq \dim V - \dim \mathrm{Spin}_n .$$

This gives the upper bound on $\mathrm{ed}(\mathrm{Spin}_n)$ for n not divisible by 4. For $n = 16$, we use the same calculation with V the direct sum of the vector representation of Spin_{16} and a half-spin representation. For $n \geq 20$ and divisible by 4, Theorem 1.1 gives that $\mathrm{ed}(\mathrm{HSpin}_n)$ is at most the value claimed; the argument in [CM14, Th. 2.2] (referring now to [Löt13] instead of [BRV10] for the stacky essential dimension inequality) establishes the upper bound on $\mathrm{ed}(\mathrm{Spin}_n)$ for $n \geq 20$ and divisible by 4.

It is trivially true that $\mathrm{ed}_2(\mathrm{Spin}_n) \le \mathrm{ed}(\mathrm{Spin}_n)$. Finally, that $\mathrm{ed}_2(\mathrm{Spin}_n)$ is at least the expression on the right side of the display was proved in [BRV10, Th. 3-3(a)] for n not divisible by 4 and in [Mer09, Th. 4.9] for n divisible by 4; the lower bound on $\mathrm{ed}_2(\mathrm{HSpin}_n)$ is from [BRV10, Remark 3-10]. □

7. Spin_n for $6 \le n \le 12$ and characteristic 2

Suppose now that $6 \le n \le 12$ and $\mathrm{char}\, k = 2$. Let us now calculate the stabilizer in Spin_n of a generic vector v in a (half) spin representation, which will justify those entries in Table 1. For $n = 6$, the $\mathrm{Spin}_6 \cong \mathrm{SL}_4$ and the representation is the natural representation. For $n = 8$, the half-spin representation is indistinguishable from the vector representation $\mathrm{Spin}_8 \to \mathrm{SO}_8$ and again the claim is clear.

For the remaining n, we verify that the k-points $(\mathrm{Spin}_n)_v(k)$ of the generic stabilizer are as claimed, i.e., that the claimed group scheme is the reduced sub-group-scheme of $(\mathrm{Spin}_n)_v$. The cases $n = 9, 11, 12$ are treated in [GLMS97, Lemma 2.11] and the case $n = 10$ is [Lie87, p. 496].

For $n = 7$, view Spin_7 as the stabilizer of an anisotropic vector in the vector representation of Spin_8; it contains a copy of G_2. As a G_2-module, the half-spin representation of Spin_8 is self-dual and has composition factors of dimensions 1, 6, 1, so G_2 fixes a vector in V. As G_2 is a maximal closed connected subgroup of Spin_7, it is the identity component of the reduced subgroup of $(\mathrm{Spin}_7)_v$.

We have verified that the reduced sub-group-scheme of $(\mathrm{Spin}_n)_v$ agrees with the corresponding entry, call it S, in Table 1. We now proceed as in §5 and find a w such that $\dim(\mathfrak{spin}_n)_w = \dim S$, which shows that $(\mathrm{Spin}_n)_v$ is smooth, completing the proof of Table 1 for $n \le 12$.

8. Spin_{13} and Spin_{14} and characteristic $\ne 2$

In this section, we determine the stabilizer in Spin_{14} and Spin_{13} of a generic vector in the (half) spin representation V of dimension 64. We assume that $\mathrm{char}\, k \ne 2$ and k is algebraically closed.

Let C_0 denote the trace zero subspace of an octonion algebra with quadratic norm N. We may view the natural representation of SO_{14} as a sum $C_0 \oplus C_0$ endowed with the quadratic form $N \oplus -N$. This gives an inclusion $G_2 \times G_2 \subset \mathrm{SO}_{14}$ that lifts to an inclusion $G_2 \times G_2 \subset \mathrm{Spin}_{14}$. There is an element of order 4 in SO_{14} such that conjugation by it interchanges the two copies of G_2 — the element of order 2 in the orthogonal group with this property has determinant -1 — so the normalizer of $G_2 \times G_2$ in $\mathrm{SO}_{14}(k)$ is isomorphic to $((G_2 \times G_2) \rtimes \mu_4)(k)$ and in Spin_{14} it is $((G_2 \times G_2) \rtimes \mu_8)(k)$.

Viewing V as an internal Chevalley module for Spin_{14} (arising from the embedding of Spin_{14} in E_8), it follows that Spin_{14} has an open orbit in $\mathbb{P}(V)$. Moreover, the unique $(G_2 \times G_2)$-fixed line kv in V belongs to this open orbit, see [Pop80, p. 225, Prop. 11], [Ros99a], or [Gar09, §21]. That is, for H the reduced sub-group-scheme of $(\mathrm{Spin}_{14})_v$, $H^\circ \supseteq G_2 \times G_2$. By dimension count this is an equality. A computation analogous to the one in the preceding paragraph shows that the idealizer of $\mathrm{Lie}(G_2 \times G_2)$ in so_{14} is $\mathrm{Lie}(G_2 \times G_2)$ itself, hence $\mathrm{Lie}((\mathrm{Spin}_{14})_v) = \mathrm{Lie}(H^\circ)$, i.e., $(\mathrm{Spin}_{14})_v$ is smooth. It follows from the construction above that the stabilizer of kv in Spin_{14} is all of $(G_2 \times G_2) \rtimes \mu_8$ (as a group scheme). The element of order 2 in μ_8 is in the center of Spin_{14} and acts as -1 on V, so the stabilizer of v is $G_2 \times G_2$ as claimed in Table 1.

Now fix a vector $(c, c') \in C_0 \oplus C_0$ so that $N(c)$, $N(c')$ and $N(c) - N(c')$ are all nonzero. The stabilizer of (c, c') in Spin_{14} is a copy of Spin_{13}, and the stabilizer of v in Spin_{13} is its intersection with $G_2 \times G_2$, i.e., the product $(G_2)_c \times (G_2)_{c'}$. Each term in the product is a copy of SL_3 (see for example [KMRT98, p. 507, Exercise 6]), as claimed in Table 1. (On the level of Lie algebras and under the additional hypothesis that $\mathrm{char}\, k = 0$, this was shown by Kac and Vinberg in [GV78, §3.2].)

9. Spin_{13} AND Spin_{14} AND CHARACTERISTIC 2

We will calculate the stabilizer in Spin_n of a generic vector in an irreducible (half-)spin representation for $n = 13, 14$ over a field k of characteristic 2.

Proposition 9.1. *The stabilizer in* Spin_{14} *(over a field k of characteristic 2) of a generic vector in a half-spin representation is the group scheme* $(G_2 \times G_2) \rtimes \mathbb{Z}/2$.

We use the following construction. Let $X \supset R$, V_1, V_2 be vector spaces endowed with quadratic forms q_X, $q_R := q_X|_R$, q_1, q_1 such that q_R is totally singular; q_X, q_1, and q_2 are nonsingular; R is a maximal totally singular subspace of X; and there exist isometric embeddings $f_i : (X, q_X) \hookrightarrow (V_i, q_i)$. For example, one could take V_1 and V_2 to be copies of an octonion algebra C, R to be the span of the identity element 1_C, and X to be a quadratic étale subalgebra of C. There is a natural quadratic form on the pushout $(V_1 \oplus V_2)/(f_1 - f_2)(X)$; if we write $V_i \cong V_i' \perp f_i(X)$, then the quadratic space is isomorphic to $V_1' \perp V_2' \perp X$. We can perform a similar construction where the role of V_i is played by the codimension-1 subspace $f_i(R)^\perp$ and the pushout is $(f_1(R)^\perp \oplus f_2(R)^\perp)/(f_1 - f_2)(R)$, giving a homomorphism of algebraic groups $B_{\ell_1} \times B_{\ell_2} \to B_{\ell_1+\ell_2}$ where $2\ell_i + 2 = \dim V_i$.

Proof of Proposition 9.1. The 7-dimensional Weyl module of the split G_2 gives an embedding $G_2 \hookrightarrow \mathrm{SO}_7$. Combining this with the construction in the previous paragraph gives maps

$$G_2 \times G_2 \to \mathrm{SO}_7 \times \mathrm{SO}_7 \to \mathrm{SO}_{13} \to \mathrm{SO}_{14}$$

which lift to maps where every SO is replaced by Spin.

Put V for a half-spin representation of Spin_{14}. It restricts to the spin representation of Spin_{13}. Calculating the restriction of the weights of V to $\mathrm{Spin}_7 \times \mathrm{Spin}_7$ using the explicit description of the embedding, we see that V is the tensor product of the 8-dimensional spin representations of Spin_7. By triality, the restriction of one of the spin representations to G_2 is the action of G_2 on the octonions C, which is a uniserial module with 1-dimensional socle S (spanned by the identity element in C) and 7-dimensional radical, the Weyl module of trace zero octonions. The restriction of $V = C \otimes C$ to the first copy of G_2 is eight copies of C, so has an 8-dimensional fixed space $S \otimes C$. As $(S \otimes C)^{1 \times G_2} = S \otimes S$, we find that $S \otimes S$ is the unique line in V stabilized by $G_2 \times G_2$.

We now argue that the Spin_{14}-orbit of $S \otimes S$ is open in $\mathbb{P}(V)$. To see this, by [Röh93], it suffices to verify that $G_2 \times G_2$ is not contained in the Levi subgroup of a parabolic subgroup of Spin_{14}. This is easily verified; the most interesting case is where the Levi has type A_6, and $G_2 \times G_2$ cannot be contained in such because the restriction of V to A_6 has composition factors of dimension 1, 7, 21, and 35. We conclude that every nonzero $v \in S \otimes S$ is a generic vector in V and $(\mathrm{Spin}_{14})_v$ has dimension 28.

If one constructs on a computer the representation V of the Lie algebra $\mathfrak{spin}_{14}$ over a finite field F of characteristic 2, then it is a matter of linear algebra to calculate the dimension of the stabilizer $(\mathfrak{spin}_{14})_x$ of a random vector $x \in V$. One finds for some x that the stabilizer has dimension 28, which is the minimum possible, so by semicontinuity of dimension $\dim((\mathfrak{spin}_{14})_v) = 28 = \dim(G_2 \times G_2)$. That is, $(\mathrm{Spin}_{14})_v$ is smooth with identity component $G_2 \times G_2$. Consequently we may compute $(\mathrm{Spin}_{14})_v$ by determining its K-points for K an algebraic closure of k. The map $\mathrm{Spin}_{14}(K) \to \mathrm{SO}_{14}(K)$ is an isomorphism of concrete groups. The normalizer of $(G_2 \times G_2)(K)$ in the latter group is $(G_2 \times G_2)(K) \rtimes \mathbb{Z}/2$, where the nonidentity element $\tau \in \mathbb{Z}/2$ interchanges the two copies of $\mathrm{SO}_7(K)$, hence of $G_2(K)$. As τ normalizes $(G_2 \times G_2)(K)$, it leaves the fixed subspace $S \otimes S \otimes K = Kv$ invariant, and we find a homomorphism $\chi \colon \mathbb{Z}/2 \to \mathbb{G}_\mathrm{m}$ given by $\tau v = \chi(\tau)v$ which must be trivial because $\mathrm{char}\, K = 2$. □

The above proof, which is somewhat longer than some alternatives, was chosen because of the details it provides on the embedding of $G_2 \times G_2$ in Spin_{14}.

Proposition 9.2. *The stabilizer in* Spin_{13} *(over a field of characteristic 2) of a generic vector in the spin representation is the group scheme* $(\mathrm{SL}_2 \times \mathrm{SL}_2) \rtimes \mathbb{Z}/2$.

Proof. We imitate the argument used in §8. View Spin_{13} as $(\mathrm{Spin}_{14})_y$ for an anisotropic y in the 14-dimensional vector representation of Spin_{14}. That representation, as a representation of Spin_{13}, has socle ky and radical $y^\perp$. Let v be a generic element of the spin representation V of Spin_{13}. Our task is to determine the group

$$(\mathrm{Spin}_{13})_v = (\mathrm{Spin}_{14})_y \cap (\mathrm{Spin}_{14})_v. \tag{9.3}$$

The stabilizer $(\mathrm{Spin}_{14})_v$ described above is contained in a copy $(\mathrm{Spin}_{14})_e$ of Spin_{13} where ke is the radical of the 13-dimensional quadratic form given by the pushout construction. As v is generic, y and e are in general position, so tracing through the pushout construction we see that the intersection (9.3) contains the product of 2 copies of the stabilizer in G_2 of a generic octonion z. The quadratic étale subalgebra of C generated by z has normalizer $\mathrm{SL}_3 \rtimes \mathbb{Z}/2$ in G_2, hence the stabilizer of z is SL_3. We conclude that, for K an algebraic closure of k, the group of K-points of $(\mathrm{Spin}_{13})_v$ equals that of the claimed group, hence the stabilizer has dimension 16. Calculating with a computer as in the proof for Spin_{14}, we find that $\dim(\mathfrak{spin}_{13})_v \leq 16$, and therefore the stabilizer of v is smooth as claimed. □

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