ABSENCE OF CARTAN SUBALGEBRAS FOR HECKE VON NEUMANN ALGEBRAS

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ABSTRACT. For a right-angled Coxeter system (W, S) and $q > 0$, let $\mathcal{M}_q$ be the associated Hecke von Neumann algebra, which is generated by self-adjoint operators $T_s, s \in S$ satisfying the Hecke relation $(\sqrt{q}\, T_s - q)(\sqrt{q}\, T_s + 1) = 0$ as well as suitable commutation relations. Under the assumption that (W, S) is reduced and $|S| \geq 3$ it was proved by Garncarek [Gar15] that $\mathcal{M}_q$ is a factor (of type II_1) for a range $q \in [\rho^{-1}, \rho]$ and otherwise $\mathcal{M}_q$ is the direct sum of a II_1-factor and $\mathbb{C}$.

In this paper we prove (under the same natural conditions as Garncarek) that $\mathcal{M}_q$ is non-injective, has the weak-$*$ completely bounded approximation property and is a strongly solid algebra. Consequently $\mathcal{M}_q$ cannot have a Cartan subalgebra.

1. Introduction

Hecke algebras are one-parameter deformations of group algebras of a Coxeter group. They were the fundament for the theory of quantum groups [Jim86], [Kas95] and have remarkable applications in the theory of knot invariants [Jon85] as was shown by V. Jones. A wide range of applications of Coxeter groups and their Hecke deformations can be found in [Dav08]. In [Dym06] (see also [Dav08, Section 19]) Dymara introduced the von Neumann algebras generated by Hecke algebras. Many important results were then obtained (see also [DDJB07]) for these Hecke von Neumann algebras, including their dimension theory, cohomology and L^2-Betti-numbers. In this paper we investigate the approximation properties of Hecke von Neumann algebras as well as their Cartan subalgebras (here we mean the notion of a Cartan subalgebra in the von Neumann algebraic sense which we recall in Section 6 and not the Lie algebraic notion).

Let us recall the following definition. Let $q > 0$ and let W be a right-angled Coxeter group with generating set S (see Section 2). The associated Hecke algebra is a $*$-algebra generated by $T_s, s \in S$ which satisfies the relation:

$$(\sqrt{q}\, T_s - q)(\sqrt{q}\, T_s + 1) = 0, \qquad T_s^* = T_s \qquad \text{and} \qquad T_s T_t = T_t T_s,$$

for $s, t \in S$ with $st = ts$. Hecke algebras carry a canonical faithful tracial vector state (the vacuum state) and therefore generate a von Neumann algebra $\mathcal{M}_q$ under its GNS construction. It was recently proved by Garncarek [Gar15] that if (W, S) is reduced (see Section 2) and $|S| \geq 3$, the von Neumann algebra $\mathcal{M}_q$ is a factor in case $q \in [-\rho, \rho]$ where ρ is the radius of convergence of the fundamental power series (2.2). If $q \notin [-\rho, \rho]$ then $\mathcal{M}_q$ is the direct sum of a II_1 factor and $\mathbb{C}$.

The first aim of this paper is to determine approximation properties of $\mathcal{M}_q$ (assuming the same natural conditions as Garncarek). We first show that $\mathcal{M}_q$ is a

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non-injective von Neumann algebra and therefore falls outside Connes' classification of hyperfinite factors [Con76]. Secondly we show that $\mathcal{M}_q$ has the weak-$*$ completely bounded approximation property (wk-$*$ CBAP). This means that there exists a net of finite rank maps on $\mathcal{M}_q$ that is uniformly bounded and converges to the identity in the point σ-weak topology. In case $q = 1$ the algebra $\mathcal{M}_q$ is the group von Neumann algebra of a right-angled Coxeter group. In this case the result was known and can be proved using techniques that now became standard and origin from [Haa78]. We refer to [BoSp94], [BrOz08], [Oza08] for these and related results. In this context we also mention the parallel results for q-Gaussian algebras with $-1 < q < 1$: factoriality by Ricard [Ric05], non-injectivity by Nou [Nou04] and the completely contractive approximation property by Avsec [Avs11]. The latter paper also obtains strong solidity, see below. Some of these results were preceded by the same result for a smaller range of the parameter q by others; see references in these papers. Another important achievement concerning the approximation properties of operator algebras was obtained by Houdayer and Ricard [HoRi11] who settled the approximation properties of free Araki-Woods factors (including the non-almost periodic case). For our Hecke von Neumann algebra $\mathcal{M}_q$ we summarize:

Theorem A. Let $q > 0$.

(1) Let (W, S) be a reduced right-angled Coxeter system with $|S| \geq 3$. Then $\mathcal{M}_q$ is non-injective.
(2) For a general right-angled Coxeter system (W, S) the associated Hecke von Neumann algebra $\mathcal{M}_q$ has the wk-$*$ CBAP.

Obviously non-injectivity and the wk-$*$ CBAP of Theorem A have different proofs. However the two proofs each borrow some ideas from [RiXu06] where Ricard and Xu proved that weak amenability with constant 1 is preserved by taking free products of discrete groups. In order to prove non-injectivity we first obtain a Khintchine inequality for Hecke algebras. We show that this Khintchine inequality leads to a contradiction in case $\mathcal{M}_q$ were to be injective. For the wk-$*$ CBAP we first obtain cb-estimates for radial multipliers and then use estimates of word length projections (see Proposition 5.17) going back to Haagerup [Haa78].

Our second aim is the study of Cartan subalgebras of the Hecke von Neumann algebra $\mathcal{M}_q$. Recall that a Cartan subalgebra of a II_1-factor is by definition a maximal abelian subalgebra whose normalizer generates the II_1-factor itself. Cartan subalgebras arise typically in crossed products of free ergodic probability measure preserving actions of discrete groups on a probability measure space. In fact Cartan subalgebras always come from some orbit equivalence class in the following sense: for a separable II_1 factor $\mathcal{M}$ any Cartan subalgebra $\mathcal{A} \subseteq \mathcal{M}$ gives rise to a standard probability measure space X and an orbit equivalence class $\mathcal{R}$ with cocycle σ such that $(\mathcal{A} \subseteq \mathcal{M}) \simeq (L^\infty(X) \subseteq \mathcal{R}(X, \sigma))$. We refer to [FeMo77] for details. Cartan subalgebras can be used to obtain further fascinating rigidity results, see [PoVa14], [IPV13] for two very prominent illustrations of this: the first showing that (suitable) actions of free groups on probability spaces remember the number of generators of the group; the second showing that certain group von Neumann algebras completely remember the group (W*-superrigidity). These results typically rely on the *uniqueness* of a Cartan subalgebra. Other applications can be found in prime factorization theorems [PoOz04], [HoIs15] in which the *absence* of Cartan algebras plays a crucial role.

In [OzPo10] Ozawa and Popa were able to find the first classes of von Neumann algebras that do not have a Cartan subalgebra, namely the free group von Neumann algebras. These results generlize to a larger class of groups, see for instance [PoVa14]. Another result concerning the absence of Cartan subalgebras was obtained by Isono [Iso15] in which he proves that free orthogonal/unitary quantum groups do not have a Cartan subalgebra. In order to do so Isono first put earlier results from [Oza04] and [PoVa14] into a general von Neumann framework. In particular he proposed condition $(\mathrm{AO})^+$ – generalizing Ozawa's condition (AO) [Oza04] by assuming the existence of a certain ucp lift. Then [Iso15] proves that condition $(\mathrm{AO})^+$ together with the wk-$*$ CBAP implies strong solidity of a von Neumann algebra. The notion of strong solidity is recalled in Definition 6.1. Using Isono's result we are able to show the following.

Theorem B. Let $q > 0$. Let (W, S) be a reduced right-angled Coxeter system with $|S| \geq 3$. The associated Hecke von Neumann algebra $\mathcal{M}_q$ is strongly solid.

In turn as $\mathcal{M}_q$ is non-injective by Theorem A we are able to derive the result announced in the title of this paper.

Corollary C. Let $q > 0$. For a reduced Coxeter system (W, S) with $|S| \geq 3$ the associated Hecke von Neumann algebra $\mathcal{M}_q$ does not have a Cartan subalgebra.

Structure. In Section 2 we introduce Hecke von Neumann algebras and some basic algebraic properties. Lemma 2.7 is absolutely crucial as each of the results in this paper rely in their own way on this decomposition lemma. In Section 3 we obtain universal properties of Hecke von Neumann algebras. In Section 4 we prove that $\mathcal{M}_q$ is non-injective. In Section 5 we find approximation properties of $\mathcal{M}_q$ and conclude Theorem A. Finally Section 6 proves the strong solidity result of Theorem B from which Corollary C shall easily follow.

Convention. Let X be a set and let $A, B \subseteq X$. We will briefly write $A \backslash B$ for $A \backslash (A \cap B)$.

2. Notation and preliminaries

Standard result on operator spaces can be found in [EfRu00], [Pis02]. Standard references for von Neumann algebras are [StZs75] and [Tak79]. Recall that *ucp* stands for unital completely positive.

2.1. **Coxeter groups.** A *Coxeter group* W is a group that is generated by a finite set S and which satisfies the relation

$$(st)^{m(s,t)} = 1,$$

for some constant $m(s,t) \in \{1, 2, \ldots, \infty\}$ with $m(s,t) = m(t,s) \geq 2, s \neq t$ and $m(s,s) = 1$. The constant $m(s,t) = \infty$ means that no relation is imposed, so that s, t are free variables. The Coxeter group W is called right-angled if either $m(s,t) = 2$ or $m(s,t) = \infty$ for all $s, t \in S, s \neq t$ and this is the only case we need in this paper. Therefore we assume from now on that W is a right-angled Coxeter group with generating set S. The pair (W, S) is also called a Coxeter system.

Let $\mathbf{w} \in W$ and suppose that $\mathbf{w} = w_1 \ldots w_n$ with $w_i \in S$. The representing expression $w_1 \ldots w_n$ is called reduced if whenever also $\mathbf{w} = w'_1 \ldots w'_m$ with $w'_i \in S$

then $n \leq m$, i.e. the expression is of minimal length. In that case we will write $|\mathbf{w}| = n$. Reduced expressions are not necessarily unique (only if $m(s,t) = \infty$ whenever $s \neq t$), but for each $\mathbf{w} \in W$ we may pick a reduced expression which we shall call minimal.

Convention: For $\mathbf{w} \in W$ we shall write w_i for the minimal representative $\mathbf{w} = w_1 \ldots w_n$.

To the pair (W,S) we associate a graph Γ with vertex set $V\Gamma = S$ and edge set $E\Gamma = \{(s,t) \mid m(s,t) = 2\}$. A subgraph Γ_0 of Γ is called *full* if the following property holds: $\forall s,t \in V\Gamma_0$ with $(s,t) \in E\Gamma$ we have $(s,t) \in E\Gamma_0$. A clique in Γ is a full subgraph in which every two vertices share an edge. We let $\mathsf{Cliq}(\Gamma)$ denote the set of cliques in Γ. To keep the notation consistent with the literature the empty graph is in $\mathsf{Cliq}(\Gamma)$ by convention (in this paper we shall often exclude the empty graph from $\mathsf{Cliq}(\Gamma)$ explicitly or treat it as a special case to keep some of the arguments more transparent). We let $\mathsf{Cliq}(\Gamma, l)$ be the set of cliques with l vertices.

Definition 2.1. A Coxeter system (W,S) is called *reduced* if the complement of Γ is connected.

2.2. **Hecke von Neumann algebras.** Let (W,S) be a right-angled Coxeter system. Let $q > 0$. By [Dav08, Proposition 19.1.1] there exists a unique unital $*$-algebra $\mathbb{C}_q(\Gamma)$ generated by a basis $\{\widetilde{T}_{\mathbf{w}} \mid \mathbf{w} \in W\}$ satisfying the following relations. For every $s \in S$ and $\mathbf{w} \in W$ we have:

$$\widetilde{T}_s \widetilde{T}_{\mathbf{w}} = \begin{cases} \widetilde{T}_{s\mathbf{w}} & \text{if } |s\mathbf{w}| > |\mathbf{w}|, \\ q\widetilde{T}_{s\mathbf{w}} + (1-q)\widetilde{T}_{\mathbf{w}} & \text{otherwise,} \end{cases}$$

$$\widetilde{T}_{\mathbf{w}}^* = \widetilde{T}_{\mathbf{w}^{-1}}.$$

We define normalized elements $T_{\mathbf{w}} = q^{-|\mathbf{w}|/2} \widetilde{T}_{\mathbf{w}}$. Then for $\mathbf{w} \in W$ and $s \in S$,

$$T_s T_{\mathbf{w}} = \begin{cases} T_{s\mathbf{w}} & \text{if } |s\mathbf{w}| > |\mathbf{w}|, \\ T_{s\mathbf{w}} + pT_{\mathbf{w}} & \text{otherwise,} \end{cases}, \tag{2.1}$$

where

$$p = \frac{q-1}{\sqrt{q}}.$$

There is a natural positive linear tracial map τ on $\mathbb{C}_q(\Gamma)$ satisfying $\tau(T_{\mathbf{w}}) = 0, \mathbf{w} \neq 1$ and $\tau(1) = 1$. Let $L^2(\mathcal{M}_q)$ be the Hilbert space given by the closure of $\mathbb{C}_q(\Gamma)$ with respect to $\langle x, y \rangle = \tau(y^* x)$ and let $\mathcal{M}_q$ be the von Neumann algebra generated by $\mathbb{C}_q(\Gamma)$ acting on $L^2(\mathcal{M}_q)$. τ extends to a state on $\mathcal{M}_q$ and $L^2(\mathcal{M}_q)$ is its GNS space with cyclic vector $\Omega := T_e$. $\mathcal{M}_q$ is called the *Hecke von Neumann algebra* at parameter q associated to the Coxeter system (W,S).

Theorem 2.2 (see [Gar15])**.** *Let (W,S) be a reduced Coxeter system and suppose that $|S| \geq 3$. Let ρ be the radius of convergence of the fundamental power series:*

$$\sum_{k=0}^{\infty} |\{\mathbf{w} \in W \mid |\mathbf{w}| = k\}| z^k. \tag{2.2}$$

For every $q \in [-\rho^{-1}, \rho]$ the von Neumann algebra $\mathcal{M}_q$ is a factor. For $q > 0$ not in $[-\rho^{-1}, \rho]$ the von Neumann algebra $\mathcal{M}_q$ is the direct sum of a factor and $\mathbb{C}$.

As $\mathcal{M}_q$ posesses a normal faithful tracial state the factors appearing in Theorem 2.2 are of type II_1.

For the analysis of $\mathcal{M}_q$ we shall in fact need $\mathcal{M}_1$ which is the group von Neumann algebra of the Coxeter group W. It can be represented on $L^2(\mathcal{M}_q)$. Indeed, let $T^{(1)}_{\mathbf{w}}$ denote the generators of $\mathcal{M}_1$ as in (2.1) and let $T_{\mathbf{w}}$ be the generators of $\mathcal{M}_q$. Set the unitary map,

$$U : L^2(\mathcal{M}_1) \to L^2(\mathcal{M}_q) : T^{(1)}_{\mathbf{w}}\Omega \to T_{\mathbf{w}}\Omega.$$

In this paper we shall always assume that $\mathcal{M}_1$ is represented on $L^2(\mathcal{M}_q)$ by the identification $\mathcal{M}_1 \to \mathcal{B}(L^2(\mathcal{M}_q)) : x \mapsto UxU^*$. Note that this way

$$T^{(1)}_{\mathbf{v}}(T_{\mathbf{w}}\Omega) = T_{\mathbf{vw}}\Omega. \tag{2.3}$$

For $\mathbf{w} \in W$ we shall write $P_{\mathbf{w}}$ for the projection of $L^2(\mathcal{M}_q)$ onto the closure of the space spanned linearly by $\{T_{\mathbf{v}}\Omega \mid |\mathbf{w}^{-1}\mathbf{v}| = |\mathbf{v}| - |\mathbf{w}|\}$. For $\Gamma_0 \in \mathsf{Cliq}(\Gamma)$ we shall write $P_{V\Gamma_0}$ for $P_{\mathbf{w}}$ where $\mathbf{w} \in W$ is the product of all vertex elements of Γ_0 and $|V\Gamma_0|$ for the number of elements in $V\Gamma_0$. Similarly we shall write $P_{\mathbf{v}V\Gamma_0}$ for $P_{\mathbf{w}}$ where $\mathbf{w} \in W$ is the product of $\mathbf{v}$ with all vertex elements of Γ_0.

Remark 2.3 (Creation and annihilation arguments)**.** Note that for $\mathbf{w}, \mathbf{v} \in W$ saying that $|\mathbf{w}^{-1}\mathbf{v}| = |\mathbf{v}| - |\mathbf{w}|$ just means that the start of $\mathbf{v}$ contains the word $\mathbf{w}$. Throughout the paper we say that $s \in W$ acts by means of a creation operator on $\mathbf{v} \in W$ if $|s\mathbf{v}| = |\mathbf{v}| + 1$. It acts as an annihilation operator if $|s\mathbf{v}| = |\mathbf{v}| - 1$. For $\mathbf{v}, \mathbf{w} \in W$ we may always decompose $\mathbf{w} = \mathbf{w}'\mathbf{w}''$ such that $|\mathbf{w}| = |\mathbf{w}'| + |\mathbf{w}''|, |\mathbf{w}''\mathbf{v}| = |\mathbf{v}| - |\mathbf{w}''|$ and $|\mathbf{wv}| = |\mathbf{v}| - |\mathbf{w}''| + |\mathbf{w}'|$. That is $\mathbf{w}$ first acts as by means of annihilations of the letters of $\mathbf{w}''$ and then $\mathbf{w}'$ acts as a creation operator on $\mathbf{w}''\mathbf{v}$. We will use such arguments without further reference.

The following Lemma 2.7 together with Lemma 2.4 say that $T_{\mathbf{w}}$ decomposes in terms of a sum of operators that first act by annihilation (this is $T^{(1)}_{\mathbf{u}''}$) then a diagonal action (this is the projection $P_{\mathbf{u}V\Gamma_0}$) and finally by creation (this is $T^{(1)}_{\mathbf{u}'}$). This decomposition is crucial for *each* of our main results.

Definition 2.4. Let $\mathbf{w} \in W$. Let $A_{\mathbf{w}}$ be the set of triples $(\mathbf{w}', \Gamma_0, \mathbf{w}'')$ with $\mathbf{w}', \mathbf{w}'' \in W$ and $\Gamma_0 \in \mathsf{Cliq}(\Gamma)$ such that: (1) $\mathbf{w} = \mathbf{w}'V\Gamma_0\mathbf{w}''$, (2) $|\mathbf{w}| = |\mathbf{w}'| + |V\Gamma_0| + |\mathbf{w}''|$, (3) Γ_0 is not the empty graph, (4) if $s \in S$ commutes with $V\Gamma_0$ then $|\mathbf{w}'s| > |\mathbf{w}'|$.

Lemma 2.5. *For $(\mathbf{w}', \Gamma_0, \mathbf{w}'') \in A_{\mathbf{w}}$ there exists $\mathbf{u}, \mathbf{u}', \mathbf{u}'' \in W$ such that*

$$T^{(1)}_{\mathbf{w}'} P_{V\Gamma_0} T^{(1)}_{\mathbf{w}''} = T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0} T^{(1)}_{\mathbf{u}''}, \tag{2.4}$$

and moreover if $s \in W$ is such that $|\mathbf{u}'s| < |\mathbf{u}'|$ then $|s\mathbf{u}''| > |\mathbf{u}''|$.

Proof. Let $\mathbf{u} \in W$ be the (unique) element of maximal length such that $|\mathbf{w}'\mathbf{u}^{-1}| = |\mathbf{w}'| - |\mathbf{u}|$ and $|\mathbf{uw}''| = |\mathbf{w}''| - |\mathbf{u}|$. Set $\mathbf{u}' = \mathbf{w}'\mathbf{u}^{-1}$ and $\mathbf{u}'' = \mathbf{uw}''$. It then remains to prove (2.4) as the rest of the properties are obvious. For $\mathbf{v} \in W$ such that $|V\Gamma_0\mathbf{w}''\mathbf{v}| = |\mathbf{w}''\mathbf{v}| - |V\Gamma_0|$ we have,

$$\begin{aligned} &T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0} T^{(1)}_{\mathbf{u}''}(T_{\mathbf{v}}\Omega) = T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0}(T_{\mathbf{uw}''\mathbf{v}}\Omega) = T^{(1)}_{\mathbf{u}'}(T_{\mathbf{uw}''\mathbf{v}}\Omega) \\ =&T_{\mathbf{w}'\mathbf{w}''\mathbf{v}}\Omega = T^{(1)}_{\mathbf{w}'} P_{V\Gamma_0}(T_{\mathbf{w}''\mathbf{v}}\Omega) = T^{(1)}_{\mathbf{w}'} P_{V\Gamma_0} T^{(1)}_{\mathbf{w}''}(T_{\mathbf{v}}\Omega). \end{aligned}$$

For $\mathbf{v} \in W$ such that $|V\Gamma_0\mathbf{w}''\mathbf{v}| \neq |\mathbf{w}''\mathbf{v}| - |V\Gamma_0|$ it follows from a similar computation that both sides of (2.4) have $(T_{\mathbf{v}}\Omega)$ in its kernel. $\square$

Remark 2.6. In Lemma 2.5 the property that $|\mathbf{u}'s| < |\mathbf{u}'|$ implies that $|s\mathbf{u}''| > |\mathbf{u}''|$ is equivalent to $|\mathbf{u}'\mathbf{u}''| = |\mathbf{u}'| + |\mathbf{u}''|$. The words $\mathbf{u}'$ and $\mathbf{u}''$ in Lemma 2.5 are not unique: in case $|s\mathbf{u}''| = |\mathbf{u}''| - 1$ and s commutes with $V\Gamma_0$ then we may replace $(\mathbf{u}', \mathbf{u}'')$ by $(\mathbf{u}'s, s\mathbf{u}'')$.

Lemma 2.7. *We have,*

$$T_{\mathbf{w}} = T_{\mathbf{w}}^{(1)} + \sum_{(\mathbf{w}',\Gamma_0,\mathbf{w}'')\in A_{\mathbf{w}}} p^{\#V\Gamma_0} T_{\mathbf{w}'}^{(1)} P_{V\Gamma_0} T_{\mathbf{w}''}^{(1)}, \tag{2.5}$$

where $A_{\mathbf{w}}$ is given in Definition 2.4.

Proof. The proof proceeds by induction on the length of $\mathbf{w}$. If $|\mathbf{w}| = 1$ then $T_{\mathbf{w}} = T_{\mathbf{w}}^{(1)} + pP_{\mathbf{w}}$ by (2.1). Now suppose that (2.5) holds for all $\mathbf{w} \in W$ with $|\mathbf{w}| = n$. Let $\mathbf{v} \in W$ be such that $|\mathbf{v}| = n+1$. Decompose $\mathbf{v} = s\mathbf{w}, |\mathbf{w}| = n, s \in S$. Then,

$$\begin{aligned} T_{\mathbf{v}} =& T_s T_{\mathbf{w}} \\ =& \left(T_s^{(1)} + pP_s\right)\left(T_{\mathbf{w}}^{(1)} + \sum_{(\mathbf{w}',\Gamma_0,\mathbf{w}'')\in A_{\mathbf{w}}} p^{\#V\Gamma_0} T_{\mathbf{w}'}^{(1)} P_{V\Gamma_0} T_{\mathbf{w}''}^{(1)}\right) \\ =& T_{s\mathbf{w}}^{(1)} + pP_s T_{\mathbf{w}}^{(1)} + \sum_{(\mathbf{w}',\Gamma_0,\mathbf{w}'')\in A_{\mathbf{w}}} \left(p^{\#V\Gamma_0} T_{s\mathbf{w}'}^{(1)} P_{V\Gamma_0} T_{\mathbf{w}''}^{(1)} + p^{\#V\Gamma_0+1} P_s T_{\mathbf{w}'}^{(1)} P_{V\Gamma_0} T_{\mathbf{w}''}^{(1)}\right). \end{aligned} \tag{2.6}$$

Now we need to make the following observations.

(1) If $s\mathbf{w}' = \mathbf{w}'s$ then $P_s T_{\mathbf{w}'}^{(1)} = T_{\mathbf{w}'}^{(1)} P_s$. So in that case,

$$P_s T_{\mathbf{w}'}^{(1)} P_{V\Gamma_0} T_{\mathbf{w}''}^{(1)} = T_{\mathbf{w}'}^{(1)} P_s P_{V\Gamma_0} T_{\mathbf{w}''}^{(1)}.$$

Moreover $P_s P_{V\Gamma_0}$ equals $P_{sV\Gamma_0}$ in case s commutes with all elements of $V\Gamma_0$ and 0 otherwise.

(2) In case $s\mathbf{w}' \neq \mathbf{w}'s$ we claim that $P_s T_{\mathbf{w}'}^{(1)} P_{V\Gamma_0} T_{\mathbf{w}''}^{(1)} = 0$. To see this, rewrite $P_s T_{\mathbf{w}'}^{(1)} P_{V\Gamma_0} T_{\mathbf{w}''}^{(1)} = P_s T_{\mathbf{u}'}^{(1)} P_{\mathbf{u}V\Gamma_0} T_{\mathbf{u}''}^{(1)}$ with $\mathbf{u}, \mathbf{u}', \mathbf{u}''$ as in Lemma 2.5. As $s\mathbf{w}' \neq \mathbf{w}'s$ we have $s\mathbf{u}' \neq \mathbf{u}'s$ and/or $s\mathbf{u} \neq \mathbf{u}s$.

(a) Assume $s\mathbf{u}' \neq \mathbf{u}'s$. For $\mathbf{v} \in W$ with $T_{\mathbf{u}''\mathbf{v}}\Omega$ in the range of $P_{\mathbf{u}V\Gamma_0}$,

$$P_s T_{\mathbf{u}'}^{(1)} P_{\mathbf{u}V\Gamma_0} T_{\mathbf{u}''}^{(1)} (T_{\mathbf{v}}\Omega) = P_s T_{\mathbf{u}'\mathbf{u}''\mathbf{v}}\Omega. \tag{2.7}$$

Furthermore, the assertions of Lemma 2.5 imply $|\mathbf{u}'\mathbf{u}V\Gamma_0| = |\mathbf{u}'| + |\mathbf{u}V\Gamma_0|$ and therefore $|\mathbf{u}'\mathbf{u}''\mathbf{v}| = |\mathbf{u}''\mathbf{v}| + |\mathbf{u}'|$ which implies (because $s\mathbf{u}' \neq \mathbf{u}'s$ and $\mathbf{u}'\mathbf{u}''\mathbf{v}$ starts with all letters of $\mathbf{u}'$) that (2.7) is 0. For $\mathbf{v} \in W$ with $T_{\mathbf{u}''\mathbf{v}}\Omega$ not in the range of $P_{\mathbf{u}V\Gamma_0}$ we have $T_{\mathbf{u}'}^{(1)} P_{\mathbf{u}V\Gamma_0} T_{\mathbf{u}''}^{(1)} (T_{\mathbf{v}}\Omega) = 0$. In all we conclude $P_s T_{\mathbf{u}'}^{(1)} P_{\mathbf{u}V\Gamma_0} T_{\mathbf{u}''}^{(1)} = 0$.

(b) Assume $s\mathbf{u}' = \mathbf{u}'s$ but $s\mathbf{u} \neq \mathbf{u}s$. Then $P_s T_{\mathbf{u}'}^{(1)} P_{\mathbf{u}} = T_{\mathbf{u}'}^{(1)} P_s P_{\mathbf{u}} = 0$.

So in all (2.6) gives,

$$\begin{aligned} T_{\mathbf{v}} =& T_{s\mathbf{w}}^{(1)} + pP_s T_{\mathbf{w}}^{(1)} + \sum_{(\mathbf{w}',\Gamma_0,\mathbf{w}'')\in A_{\mathbf{w}}} p^{\#V\Gamma_0} T_{s\mathbf{w}'}^{(1)} P_{V\Gamma_0} T_{\mathbf{w}''}^{(1)} \\ &+ \sum_{(\mathbf{w}',\Gamma_0,\mathbf{w}'')\in A_{\mathbf{w}}, s\mathbf{w}'=\mathbf{w}'s, sV\Gamma_0=V\Gamma_0 s} p^{\#V\Gamma_0+1} T_{\mathbf{w}'}^{(1)} P_{sV\Gamma_0} T_{\mathbf{w}''}^{(1)}, \end{aligned}$$

and in turn an identification of all summands shows that the latter expression equals,

$$T_{s\mathbf{w}}^{(1)} + \sum_{(\mathbf{v}', \Gamma_0, \mathbf{v}'') \in A_{s\mathbf{w}}} p^{\# V\Gamma_0} T_{\mathbf{v}'}^{(1)} P_{V\Gamma_0} T_{\mathbf{v}''}^{(1)}.$$

This concludes the proof. □

2.3. **Group von Neumann algebras.** Let G be a locally compact group with left regular representation $s \mapsto \lambda_s$ and group von Neumann algebra $\mathcal{L}(\mathsf{G}) = \{\lambda_s \mid s \in \mathsf{G}\}''$. We let $A(\mathsf{G})$ be the Fourier algebra consisting of functions $\varphi(s) = \langle \lambda_s \xi, \eta \rangle, \xi, \eta \in L^2(\mathsf{G})$. There is a pairing between $A(\mathsf{G})$ and $\mathcal{L}(\mathsf{G})$ which is given by $\langle \varphi, \lambda(f) \rangle = \int_{\mathsf{G}} f(s)\varphi(s) ds$ which turns $A(\mathsf{G})$ into an operator space that is completely isometrically identified with $\mathcal{L}(\mathsf{G})_*$. We let $M_{\mathcal{CB}}A(\mathsf{G})$ be the space of completely bounded Fourier multipliers of $A(\mathsf{G})$. For $m \in M_{\mathcal{CB}}A(\mathsf{G})$ we let $T_m : \mathcal{L}(\mathsf{G}) \to \mathcal{L}(\mathsf{G})$ be the normal completely bounded map determined by $\lambda(f) \mapsto \lambda(mf)$. The following theorem is due to Bozejko and Fendler [BoFe84] (see also [JNR09, Theorem 4.5]).

Theorem 2.8. *Let $m \in M_{\mathcal{CB}}A(\mathsf{G})$. There exists a unique normal completely bounded map $M_m : \mathcal{B}(L^2(\mathsf{G})) \to \mathcal{B}(L^2(\mathsf{G}))$ that is an $L^\infty(\mathsf{G})$-bimodule homomorphism and such that M_m restricts to $T_m : \lambda(f) \mapsto \lambda(mf)$ on $\mathcal{L}(\mathsf{G})$. Moreover, $\|M_m\|_{\mathcal{CB}} = \|T_m\|_{\mathcal{CB}} = \|m\|_{M_{\mathcal{CB}}A(\mathsf{G})}$.*

3. Universal property and conditional expectations

In this section we establish some standard universal properties for subalgebras of $\mathcal{M}_q$.

Theorem 3.1. *Let $q > 0$ put $p = (q-1)/\sqrt{q}$ and let (W, S) be a right angled Coxeter system with associated Hecke von Neumann algebra $(\mathcal{M}_q, \tau)$. Suppose that $(\mathcal{N}, \tau_{\mathcal{N}})$ is a von Neumann algebra with GNS faithful state τ that is generated by self-adjoint operators $R_s, s \in S$ that satisfy the relations $R_s R_t = R_t R_s$ whenever $m(s,t) = 2$, $R_s^2 = 1 + pR_s, s \in S$ and further $\tau_{\mathcal{N}}(R_{w_1} \dots R_{w_n}) = 0$ for every non-empty reduced word $\mathbf{w} = w_1 \dots w_n \in W$. Then there exists a unique normal $*$-homomorphism $\pi : \mathcal{M}_q \to \mathcal{N}$ such that $\pi(T_s) = R_s$. Moreover $\tau_{\mathcal{N}} \circ \pi = \tau$.*

Proof. The proof is routine, c.f. [CaFi15, Proposition 2.12]. We sketch it here. Let $(L^2(\mathcal{N}), \pi_{\mathcal{N}}, \eta)$ be a GNS construction for $(\mathcal{N}, \tau_{\mathcal{N}})$. As τ is GNS faithful we may assume that $\mathcal{N}$ is represented on $L^2(\mathcal{N})$ via $\pi_{\mathcal{N}}$. We define a linear map $V : L^2(\mathcal{M}_q) \to L^2(\mathcal{N})$ by $V\Omega = \eta$ and

$$V(T_{\mathbf{w}}\Omega) = R_{\mathbf{w}}\eta, \qquad \text{where } \mathbf{w} \in W,$$

and $R_{\mathbf{w}} := R_{w_1} \dots R_{w_n}$. It is easy to check that V is well-defined and isometric. Then $\pi(\,\cdot\,) = V(\,\cdot\,)V^*$ does the job. As $V\Omega = \eta$ we get $\tau_{\mathcal{N}} \circ \pi = \tau$. □

Remark 3.2. Note that the property $T_s^2 = 1 + pT_s, s \in S$ with $p = \frac{q-1}{\sqrt{q}}$ is equivalent to the usual Hecke relation $(\sqrt{q}T_s - q)(\sqrt{q}T_s + 1) = 0$ that appears in the literature.

We shall say that (W', S') is a Coxeter subsystem of (W, S) if $S' \subseteq S$ and $m'(s,t) = m(s,t)$ for all $s, t \in S'$. Here m' is the function on $S' \times S'$ that determines the commutation relations for W', c.f. Section 2.1.

Corollary 3.3. *Let $q > 0$. Let (W', S') be a Coxeter subsystem of a right angled Coxeter system (W, S). Let $\mathcal{M}_q'$ and $\mathcal{M}_q$ be their respective Hecke von Neumann algebras. There exists a trace preserving normal conditional expectation $\mathcal{E} : \mathcal{M}_q \to \mathcal{M}_q'$.*

Proof. Theorem 3.1 implies that $\mathcal{M}_q'$ is a von Neumann subalgebra of $\mathcal{M}_q$ and the canonical trace of $\mathcal{M}_q$ agrees with the one on $\mathcal{M}_q'$. Therefore $\mathcal{M}_q'$ admits a trace preserving normal conditional expectation value, c.f. [Tak03]. □

4. Non-injectivity of $\mathcal{M}_q$

Recall that a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is called injective if there exists a (not necessarily normal) conditional expectation $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{M}$. This means that $\mathcal{E}$ is a completely positive linear map which satisfies $\forall x \in \mathcal{M} : \mathcal{E}(x) = x$.

We prove that $\mathcal{M}_q$ is non-injective. The proof is based on a Khintchine type inequality. For the sake of presentation we shall first prove non-injectivity in the free case, meaning that $m(s,t) = \infty$ whenever $s \neq t$. The proof is conceptually the same as the general case but the notation simplifies quite a lot, making the proof much more accessible.

4.1. **Non-injectivity of $\mathcal{M}_q$: the free case.** In this subsection assume that $m(s,t) = \infty$ for all $s, t \in S, s \neq t$. In particular this means that in Lemma 2.7 every clique appearing in the sum has only 1 vertex. We proceed now as in [RiXu06, Section 2]. Define the following two linear subspaces of $\mathcal{B}(L^2(\mathcal{M}_q))$:

$$L_1 := \operatorname{span}\left\{ P_s T_s^{(1)} P_s^{\perp} \mid s \in S \right\}, \qquad K_1 := \operatorname{span}\left\{ P_s^{\perp} T_s^{(1)} P_s \mid s \in S \right\}. \tag{4.1}$$

Note that as $s^2 = e$ in fact $P_s T_s^{(1)} P_s^{\perp} = T_s^{(1)} P_s^{\perp}$ and $P_s^{\perp} T_s^{(1)} P_s = T_s^{(1)} P_s$. The following Lemma 4.1 is a special case of [RiXu06, Lemma 2.3 and Corollary 2.4].

Lemma 4.1. *We have complete isometric identifications,*

$$\begin{array}{llcll} L_1 & \simeq \left(\mathbb{C}^{\#S}\right)_{\text{column}} & : & T_s^{(1)} P_s^{\perp} & \mapsto e_s, \\ K_1 & \simeq \left(\mathbb{C}^{\#S}\right)_{\text{row}} & : & T_s^{(1)} P_s & \mapsto e_s, \end{array}$$

where the lower scripts indicate the operator space structure of a column and row Hilbert space with orthonormal basis $e_s, s \in S$.

We define $\Sigma_1 = \operatorname{span}\{T_s \mid s \in S\}$ and subsequently:

$$\Sigma_d = \operatorname{span}\{T_{w_1} \otimes \ldots \otimes T_{w_d} \mid \mathbf{w} \in W\},$$

which is contained in the d-fold algebraic tensor copy of Σ_1. There exists a canonical map $\sigma_d : \Sigma_d \to \mathcal{B}(L^2(\mathcal{M}_q)) : T_{w_1} \otimes \ldots \otimes T_{w_d} \mapsto T_{\mathbf{w}}$. For $s \in V\Gamma$ we let $A_s := \mathbb{C}P_s$, i.e. a 1-dimensional operator space. Using $\otimes_h$ for the Haagerup tensor product we set $L_k = (L_1)^{\otimes_h k}, K_k = (K_1)^{\otimes_h k}$ and

$$X_d = \left(\bigoplus_{k=0}^{d} L_k \otimes_h K_{d-k} \right) \bigoplus \left(\bigoplus_{s \in S} \bigoplus_{k=0}^{d-1} L_k \otimes_h A_s \otimes_h K_{d-k-1} \right).$$

Here the sums are understood as ℓ^∞-direct sums of operator spaces. The multiplication maps $L_k \otimes_h K_{d-k} \to \mathcal{B}(L^2(\mathcal{M}_q))$ and $L_k \otimes_h A_s \otimes_h K_{d-k-1} \to \mathcal{B}(L^2(\mathcal{M}_q))$ are completely contractive by the very definition of the Haagerup tensor product. Extending linearly to X_d gives a map $\Pi_d : X_d \to \mathcal{B}(L^2(\mathcal{M}_q))$ with

$$\|\Pi_d\|_{\mathcal{CB}} \leq (d+1) + d\,\#S. \tag{4.2}$$

In fact the tensor amplification

$$(\iota \otimes \Pi_d) : \mathcal{M}_q \otimes_{\min} X_d \to \mathcal{M}_q \otimes_{\min} \mathcal{B}(L^2(\mathcal{M}_q)) \tag{4.3}$$

has complete bound majorized by $(d+1) + d\,\#S$. Define a mapping $j_d : \Sigma_d \to X_d$,

(4.4)

$$\begin{aligned} &T_{w_1} \otimes \ldots \otimes T_{w_d} \mapsto \\ &\quad \left(\bigoplus_{k=0}^{d} T_{w_1} P_{w_1}^{\perp} \otimes \ldots \otimes T_{w_k} P_{w_k}^{\perp} \otimes T_{w_{k+1}} P_{w_{k+1}} \otimes \ldots \otimes T_{w_d} P_{w_d} \right) \\ &\quad \bigoplus \left(\bigoplus_{s \in S} \bigoplus_{k=0}^{d-1} T_{w_1} P_{w_1}^{\perp} \otimes \ldots \otimes T_{w_k} P_{w_k}^{\perp} \otimes P_s \otimes T_{w_{k+2}} P_{w_{k+2}} \otimes \ldots \otimes T_{w_d} P_{w_d} \right), \end{aligned}$$

and extend linearly.

Lemma 4.2. *We have* $\sigma_d = \Pi_d \circ j_d$.

Proof. The lemma follows if we could prove the following equalities, the first one being Lemma 2.7,

(4.5)

$$\begin{aligned} T_{\mathbf{w}} =& (P_{w_1} + P_{w_1}^{\perp}) T_{w_1}^{(1)} (P_{w_1} + P_{w_1}^{\perp}) \ldots (P_{w_d} + P_{w_d}^{\perp}) T_{w_d}^{(1)} (P_{w_d} + P_{w_d}^{\perp}) \\ &+ p \sum_{k=0}^{d-1} (P_{w_1} + P_{w_1}^{\perp}) T_{w_1}^{(1)} (P_{w_1} + P_{w_1}^{\perp}) \ldots (P_{w_k} + P_{w_k}^{\perp}) T_{w_k}^{(1)} (P_{w_k} + P_{w_k}^{\perp}) \times \\ & P_{w_{k+1}} (P_{w_{k+2}} + P_{w_{k+2}}^{\perp}) T_{w_{k+2}}^{(1)} (P_{w_{k+2}} + P_{w_{k+2}}^{\perp}) \ldots (P_{w_d} + P_{w_d}^{\perp}) T_{w_d}^{(1)} (P_{w_d} + P_{w_d}^{\perp}) \\ =& \sum_{k=0}^{d} (T_{w_1} P_{w_1}^{\perp}) \ldots (T_{w_k} P_{w_k}^{\perp}) (T_{w_{k+1}} P_{w_{k+1}}) \ldots (T_{w_d} P_{w_d}) \\ &+ p \sum_{k=0}^{d-1} (T_{w_1} P_{w_1}^{\perp}) \ldots (T_{w_k} P_{w_k}^{\perp}) P_{w_{k+1}} (T_{w_{k+2}} P_{w_{k+2}}) \ldots (T_{w_d} P_{w_d}). \end{aligned}$$

The proof is a creation/annihilation argument as in [RiXu06, Fact 2.6]. First note that from the fact that $w_i^2 = 1$ we obtain that $T_{w_i}^{(1)} P_{w_i} = P_{w_i}^{\perp} T_{w_i}^{(1)} P_{w_i} = P_{w_i}^{\perp} T_{w_i}^{(1)}$ and by taking adjoints $P_{w_i} T_{w_i}^{(1)} = P_{w_i} T_{w_i}^{(1)} P_{w_i}^{\perp} = T_{w_i}^{(1)} P_{w_i}^{\perp}$. Therefore also $P_{w_i}^{\perp} T_{w_i}^{(1)} P_{w_i}^{\perp} = P_{w_i} T_{w_i}^{(1)} P_{w_i} = 0$. Next note that $P_{w_i} P_{w_{i+1}} = 0$. Using these considerations we see that in the left hand side expression of (4.5) all terms are zero except for the ones that remain on the right hand side of (4.5). Indeed consider a product $\prod_{i=1}^{d} T_{\mathbf{w}_i}^{(1)} Q_{\mathbf{w}_i}$ with $Q_{\mathbf{w}_i} = P_{\mathbf{w}_i}$ or $Q_{\mathbf{w}_i} = P_{\mathbf{w}_i}^{\perp}$. If a factor $T_{\mathbf{w}_{i+1}}^{(1)} P_{\mathbf{w}_{i+1}}^{\perp}$ occurs then we must have $Q_{\mathbf{w}_i} = P_{\mathbf{w}_i}^{\perp}$ or this product is zero. This shows that the only non-zero summands in the first term on the left side of (4.5) are the ones appearing in the first summation on the right side. Similarly the second summations on each side of (4.5) may be identified. □

Remark 4.3. The inequality (4.2) can be interpreted as a Khintchine inequality. It is also possible to obtain the reverse Khintchine inequality $\|j_d(x)\| \leq \|\sigma_d(x)\|$ following (almost exactly) the proof of [RiXu06, Theorem 2.5]. The reverse inequality will not be used in this paper.

Lemma 4.4. *Let* A *be a* C^**-algebra and let* $M = (M_{i,j})_{1 \leq i \leq k, 1 \leq j \leq l} \in M_{k,l}(\mathsf{A})$ *be a* $k \times l$*-matrix with entries* $M_{i,j} \in \mathsf{A}$*. Assume* $k > l$*. Suppose that if* $M_{i,j}$ *and* $M_{i,j'}$ *are non-zero then* $j = j'$*. Then,*

$$\|M\|_{M_{k,l}(\mathsf{A})} \leq \sqrt{\| \sum_{i,j} M_{i,j} M_{i,j}^* \|_{\mathsf{A}}}.$$

Proof. The assumption that if $M_{i,j}$ and $M_{i,j'}$ are non-zero then $j = j'$ just means that after possibly permuting the standard basis vectors we may represent the transpose of M by means of the following matrix,

$$\left(\begin{array}{ccccccccc} M_{1,1} & \dots & M_{1,n_1} & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & M_{2,n_1+1} & \dots & M_{2,n_2} & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & M_{k,n_{k-1}+1} & \dots & M_{k,l} \end{array} \right).$$

Then $\|M\|_{M_{k,l}(\mathsf{A})} = \max_i \sqrt{\| \sum_{j=n_{i-1}+1}^{n_i} M_{i,j} M_{i,j}^* \|_{\mathsf{A}}}$, which yields the lemma. □

Theorem 4.5. *Assume that* $|S| \geq 3$ *and* $\forall s,t \in S, m(s,t) = \infty$*. The Hecke von Neumann algebra* $\mathcal{M}_q$ *is not injective.*

Proof. If $\mathcal{M}_q$ were to be injective then we would have for all choices $x_i, y_i \in \mathcal{M}_q, 1 \leq i \leq m$ that

$$\| \sum_{i=1}^m x_i \otimes y_i \| \geq |\tau(\sum_{i=1}^m x_i^* y_i)|, \tag{4.6}$$

c.f. [Was77, Corollary 2]. We show that this contradicts (4.2). Fix $d \in \mathbb{N}$ and find a (finite) sequence $\mathbf{w}^{(i)} \in W$ with $|\mathbf{w}^{(i)}| = 2d$ that satisfies the property that $i = j$ whenever

$$w_1^{(i)} \dots w_d^{(i)} = w_1^{(j)} \dots w_d^{(j)} \text{ or } w_{d+1}^{(i)} \dots w_{2d}^{(i)} = w_{d+1}^{(j)} \dots w_{2d}^{(j)}. \tag{4.7}$$

One can choose such a sequence of length equal to at least 2^{d-1}. (Indeed let $s,t,r \in S$ be three different generators. There are exactly 2^{d-1} reduced words $w_1 \dots w_{d-1} s$ with $w_i \in \{s,t,r\}$. Call these words $\mathcal{A}$. Also there are exactly 2^{d-1} words $t w_{d+2} \dots w_{2d}$ with $w_i \in \{s,t,r\}$. Call these words $\mathcal{B}$. Take some bijection $\varphi : \mathcal{A} \to \mathcal{B}$ and consider the words $\mathbf{w}\varphi(\mathbf{w})$. This results in 2^{d-1} words with property (4.7); in fact the only thing that matters for the proof is that the length of such a sequence is exponential in d). (4.6) then reads

$$\| \sum_{i=1}^{2^{d-1}} T_{\mathbf{w}^{(i)}} \otimes T_{\mathbf{w}^{(i)}} \| \geq 2^{d-1}. \tag{4.8}$$

On the other hand, the Khintchine inequality (4.2) with length $2d$ gives

$$((2d+1) + 2d \,\#S) A_d \geq \| \sum_{i=1}^{2^{d-1}} T_{\mathbf{w}^{(i)}} \otimes T_{\mathbf{w}^{(i)}} \|, \tag{4.9}$$

where A_d is the maximum over the norms of each of the following expressions:

$$\sum_{i=1}^{2^{d-1}} T_{\mathbf{w}^{(i)}} \otimes T_{w_1^{(i)}} P^{\perp}_{w_1^{(i)}} \otimes \dots \otimes T_{w_k^{(i)}} P^{\perp}_{w_k^{(i)}} \otimes T_{w_{k+1}^{(i)}} P_{w_{k+1}^{(i)}} \otimes \dots \otimes T_{w_{2d}^{(i)}} P_{w_{2d}^{(i)}}, \tag{4.10}$$

with $0 \leq k \leq 2d$ and
(4.11)
$$\sum_{i=1}^{2^{d-1}} T_{\mathbf{w}^{(i)}} \otimes T_{w_1^{(i)}} P^{\perp}_{w_1^{(i)}} \otimes \ldots \otimes T_{w_k^{(i)}} P^{\perp}_{w_k^{(i)}} \otimes P_{w_{k+1}^{(i)}} \otimes T_{w_{k+2}^{(i)}} P_{w_{k+2}^{(i)}} \otimes \ldots \otimes T_{w_{2d}^{(i)}} P_{w_{2d}^{(i)}},$$

with $0 \leq k \leq 2d-1, s \in S$. Here these expressions are identified in respectively

$$\mathcal{M}_q \otimes_{\min} L_k \otimes_h K_{2d-k} \simeq \mathcal{M}_q \otimes_{\min} \mathcal{B}(\mathbb{C}^{(\#S)^k}, \mathbb{C}^{(\#S)^{(2d-k)}}), \tag{4.12}$$

with $0 \leq k \leq 2d$ and

$$\mathcal{M}_q \otimes_{\min} A_{w_k} \otimes_{\min} \mathcal{B}(\mathbb{C}^{(\#S)^k}, \mathbb{C}^{(\#S)^{2d-k-1}}),$$

with $0 \leq k \leq 2d-1$. The isomorphism (4.12) is given by

$$\begin{aligned} & x \otimes T^{(1)}_{s_1} P^{\perp}_{s_1} \otimes \ldots \otimes T^{(1)}_{s_k} P^{\perp}_{s_k} \otimes T^{(1)}_{s_{k+1}} P_{s_{k+1}} \otimes \ldots \otimes T^{(1)}_{s_{2d}} P_{s_{2d}} \\ \mapsto & x \otimes \mid e_{s_1} \otimes \ldots e_{s_k} \rangle \langle e_{s_{k+1}} \otimes \ldots \otimes e_{s_{2d}} \mid, \end{aligned}$$

and therefore Condition (4.7) and Lemma 4.4 show that the norm of (4.10) can be upper estimated by:

$$\begin{aligned} & \| \sum_{i=1}^{2^{d-1}} T_{\mathbf{w}^{(i)}} T^*_{\mathbf{w}^{(i)}} \|^{\frac{1}{2}} \leq 2^{(d-1)/2} \sqrt{\max_i \| T_{\mathbf{w}^{(i)}} T^*_{\mathbf{w}^{(i)}} \|} \\ = & 2^{(d-1)/2} \max_i \| T_{\mathbf{w}^{(i)}} \|. \end{aligned}$$

The expression can be upper estimated by $2^{(d-1)/2}\,(1+2dp)$, c.f. Lemma 2.7. A similar argument shows that we may upper estimate the norm of (4.11) with $2^{(d-1)/2}\,(1+2dp)$. Combining this with (4.8) shows that for every d we must have:

$$2^{d-1} \leq 2^{(d-1)/2}\,(1+2dp)((2d+1)+2d\,\#S).$$

As for large d this leads to a contradiction, we conclude that $\mathcal{M}_q$ cannot be injective. □

Corollary 4.6. *Let (W,S) be a right angled Coxeter system with associated Hecke von Neumann algebra $\mathcal{M}_q$. Let (W',S') be a Coxeter subsystem of (W,S) that is free, i.e. $m'(s,t)=\infty$ for all $s,t \in S'$. Then $\mathcal{M}_q$ is non-injective.*

Proof. This follows as the Hecke von Neumann algebra $\mathcal{M}'_q$ of (W',S') is an expected subalgebra of $\mathcal{M}_q$, c.f. Corollary 3.3. If $\mathcal{M}_q$ were to be injective then so would $\mathcal{M}'_q$ which contradicts Theorem 4.5. □

Remark 4.7. Following the argument of Corollary 4.6 we would be able to prove non-injectivity of a general reduced Coxeter system (W,S) with $|S| \geq 3$ if we could prove this for the case that $|S|=3$, say $S=\{r,s,t\}$, and $m(r,s)=\infty, m(s,t)=\infty$ and $m(r,t)=2$. Though that this special case suffices, we derive nevertheless a general Khintchine inequality in the next section as this involves the same modifications.

4.2. **Non-injectivity of $\mathcal{M}_q$: the general case.** We now assume again that (W, S) is a general Coxeter system of a right-angled Coxeter group, i.e. $\forall s, t \in S$ we have that $m(s,t)$ equals either 2 or ∞. Non-injectivity of $\mathcal{M}_q$ follows essentially by the same argument as in the free case. We only need to treat the Khintchine inequality with more care. Therefore we introduce some additional terminology. Firstly, for $s \in S$ we set

$$\mathsf{Link}(s) = \{t \in S \mid m(s,t) = 2\},$$

so these are all vertices in Γ that have distance exactly 1 to s. For a subset $X \subseteq V\Gamma$ we set $\mathsf{Link}(X) = \cap_{s \in X} \mathsf{Link}(s)$. We sometimes regard $\mathsf{Link}(X)$ as a full subgraph of Γ. We let Σ_d be $\mathrm{span}\{T_{w_1} \otimes \ldots \otimes T_{w_d} \mid \mathbf{w} \in W\}$ which is contained in the algebraic tensor product $\Sigma_1^{\otimes d}$ and Σ_1 the linear space spanned by $T_s, s \in S$. Let (W_f, S) be the free Coxeter system which is determined by the same generating set S but with relations $m_f(s,t) = \infty, s \neq t$. Let $\mathcal{M}_q^f$ be the free Hecke von Neumann algebra and $L^2(\mathcal{M}_q^f)$ its GNS space. We define the spaces L_1 and K_1 exactly as in (4.1) but with respect to the Coxeter system (W_f, S). In particular Lemma 4.1 remains valid. It is *not* valid for the system (W, S) which is the reason we need to introduce an extra intertwining argument in this section. Then set $L_k = (L_1)^{\otimes_h k}, K_k = (K_1)^{\otimes_h k}$. Let $\mathsf{Cliq}(\Gamma, l)$ be the set of cliques in Γ with l vertices. For $\Gamma_0 \in \mathsf{Cliq}(\Gamma, l)$ we let $\mathsf{Comm}(\Gamma_0)$ be the set of all pairs $(\Gamma_1, \Gamma_2) \in \mathsf{Cliq}(\mathsf{Link}(\Gamma_0))^2$ such that $V\Gamma_1 \cap V\Gamma_2 = \emptyset$. Let

$$X_d = \bigoplus_{l=0}^{d} \bigoplus_{\Gamma_0 \in \mathrm{Cliq}(\Gamma, l)} \bigoplus_{(\Gamma_1, \Gamma_2) \in \mathsf{Comm}(\Gamma_0)} \bigoplus_{k=0}^{d-l} L_k \otimes_h A_{\Gamma_0} \otimes_h K_{d-k-l}, \tag{4.13}$$

where $A_{\Gamma_0} = \mathbb{C} P^f_{V\Gamma_0} \subseteq \mathcal{B}(L^2(\mathcal{M}_q^f))$; here $P^f_{V\Gamma_0}$ is the projection onto the vectors $T_{\mathbf{v}}\Omega \in L^2_f(\mathcal{M}_q)$ with $\mathbf{v}$ starting with letters $V\Gamma_0$ ordered in minimal order.

Parallel to the free case we shall define a mapping $j_d : \Sigma_d \to X_d$. Let $0 \leq l \leq d, 0 \leq k \leq d - l$ and let $\Gamma_0 \in \mathsf{Cliq}(\Gamma, l), (\Gamma_1, \Gamma_2) \in \mathsf{Comm}(\Gamma_0)$. The image of $T_{w_1} \otimes \ldots \otimes T_{w_d}$ under j_d in the corresponding summand of (4.13) is,

$$\begin{aligned} &T_{w_{\sigma(1)}} (P^f_{w_{\sigma(1)}})^\perp \otimes \ldots \otimes T_{w_{\sigma(k)}} (P^f_{w_{\sigma(k)}})^\perp \otimes P_{V\Gamma_0} \\ &\quad \otimes T_{w_{\sigma(k+l+1)}} (P^f_{w_{\sigma(k+l+1)}}) \otimes \ldots \otimes T_{w_{\sigma(d)}} (P^f_{w_{\sigma(d)}}), \end{aligned} \tag{4.14}$$

provided that there exists a permutation σ of indices such that:

(1) $w_1 \ldots w_d = w_{\sigma(1)} \ldots w_{\sigma(d)}$;
(2) $w_{\sigma(k+1)} \ldots w_{\sigma(k+l)}$ make up all the letters in $V\Gamma_0$;
(3) $|w_{\sigma(1)} \ldots w_{\sigma(k)} s| = k - 1$ whenever $s \in V\Gamma_1$;
(4) $|w_{\sigma(1)} \ldots w_{\sigma(k)} s| = k + 1$ whenever $s \in \mathsf{Link}(\Gamma_0) \backslash V\Gamma_1$;
(5) $|s w_{\sigma(k+l+1)} \ldots w_{\sigma(d)}| = d - k - l - 1$ whenever $s \in V\Gamma_2$;
(6) $|s w_{\sigma(k+l+1)} \ldots w_{\sigma(d)}| = d - k - l + 1$ whenever $s \in \mathsf{Link}(\Gamma_0) \backslash V\Gamma_2$.

In (4.14) we shall assume moreover that $w_{\sigma(1)} \ldots w_{\sigma(k)}$, $w_{\sigma(k+1)} \ldots w_{\sigma(k+l)}$ and $w_{\sigma(k+l+1)} \ldots w_{\sigma(d)}$ are minimal words and if $w_{\sigma(i)} = w_{\sigma(j)}, i < j$ then $\sigma(i) < \sigma(j)$ so that σ is unique. If such a σ does not exist then (4.14) should be read as the zero vector.

Lemma 4.8. *Let $\mathbf{w} \in W$. Let $A_{\mathbf{w}}(k, \Gamma_0)$ be the set of pairs $(\mathbf{w}', \mathbf{w}'')$ with $\mathbf{w} = \mathbf{w}' V\Gamma_0 \mathbf{w}''$, $|\mathbf{w}'| = k$ and $|\mathbf{w}| = |\mathbf{w}'| + |V\Gamma_0| + |\mathbf{w}''|$. For each $\Gamma_1 \in \mathsf{Cliq}(\mathsf{Link}(\Gamma_0))$ there exists at most one $(\mathbf{w}', \mathbf{w}'') \in A_{\mathbf{w}}(k, \Gamma_0)$ with (*) for all $s \in V\Gamma_1, |\mathbf{w}' s| = |\mathbf{w}'| - 1$, for all $s \in \mathsf{Link}(\Gamma_0) \backslash V\Gamma_1, |\mathbf{w}' s| = |\mathbf{w}'| + 1$.*

Proof. Suppose that $(\mathbf{w}_1', \mathbf{w}_1'') \in A_{\mathbf{w}}(k, \Gamma_0)$ and $(\mathbf{w}_2', \mathbf{w}_2'') \in A_{\mathbf{w}}(k, \Gamma_0)$ both satisfy the property $(*)$ of the lemma. Suppose that for some $t \in S$ we have $|\mathbf{w}_1' t| = |\mathbf{w}_1'| - 1$ but $|\mathbf{w}_2' t| = |\mathbf{w}_2'| + 1$. Then we must have $|t\mathbf{w}_1''| = |\mathbf{w}_1'| + 1$ but $|t\mathbf{w}_2''| = |\mathbf{w}_2'| - 1$ and moreover t commutes with $V\Gamma_0$. But property $(*)$ shows that such t that commutes with $V\Gamma_0$ can only either be on the left or on the right, leading to a contradiction. So we must have $\mathbf{w}_1' = \mathbf{w}_2'$ and hence also $\mathbf{w}_1'' = \mathbf{w}_2''$. □

Lemma 4.9. *For every $T_{w_1} \otimes \ldots \otimes T_{w_d} \in \Sigma_d$ we have,*

(4.15)
$$T_{w_1} \ldots T_{w_d} = \sum_{l=0}^{d} \sum_{k=0}^{d-l} \sum_{\Gamma_0 \in \mathit{Cliq}(\Gamma, l)} \sum_{(\Gamma_1, \Gamma_2) \in \mathit{Comm}(\Gamma_0)} p^{\# V\Gamma_0} (T^{(1)}_{w_{\sigma(1)}} P^{\perp}_{w_{\sigma(1)}}) \ldots (T^{(1)}_{w_{\sigma(k)}} P^{\perp}_{w_{\sigma(k)}}) \times P_{V\Gamma_0} (T^{(1)}_{w_{\sigma(k+2)}} P_{w_{\sigma(k+2)}}) \ldots (T^{(1)}_{w_{\sigma(d)}} P_{w_{\sigma(d)}}),$$

where σ (changing over the summation) is as in (1) – (6) *above this lemma. If such σ does not exist then the summand is understood as 0.*

Proof. We first note that we may decompose,

$$T_{w_1} \ldots T_{w_d} = (P_{w_1} + P^{\perp}_{w_1}) T_{w_1} (P_{w_1} + P^{\perp}_{w_1}) \ldots (P_{w_d} + P^{\perp}_{w_d}) T_{w_d} (P_{w_d} + P^{\perp}_{w_d}). \tag{4.16}$$

Therefore consider an expression of the form:

$$Q^{(1)}_{w_1} T_{w_1} Q^{(2)}_{w_1} \ldots Q^{(1)}_{w_d} T_{w_d} Q^{(2)}_{w_d}, \tag{4.17}$$

where $Q^{(j)}_{w_i}$ equals either P_{w_i} or $P^{\perp}_{w_i}$. Throughout the proof we shall assume that (4.17) is non-zero. The following claims show that after possibly interchanging commuting factors in the expression (4.17) we may assume that (4.17) is of a specific form.

Claim 1. The expression (4.17) is after possibly interchanging commuting letters in $w_1 \ldots w_d$ of the form:

$$Q^{(1)}_{w_1} T_{w_1} Q^{(2)}_{w_1} \ldots Q^{(1)}_{w_s} T_{w_s} Q^{(2)}_{w_s} (P^{\perp}_{w_{s+1}} T_{w_{s+1}} P_{w_{s+1}}) \ldots (P^{\perp}_{w_d} T_{w_d} P_{w_d}). \tag{4.18}$$

Moreover, the tail of annihilation operators is maximal in the sense that if for some $i \leq s$ we have $Q^{(2)}_{w_i} = P_{w_i}$ then $Q^{(1)}_{w_i} = P_{w_i}$.

Proof of Claim 1. Suppose that we are given an expression as in (4.18). Suppose that for some $i < s$ we have $Q^{(1)}_{w_i} = P^{\perp}_{w_i}, Q^{(2)}_{w_i} = P_{w_i}$. Then we need to show that w_i commutes with $w_{i+1} \ldots w_s$. To do so we may suppose the index i was chosen maximal. Suppose that w_i and $w_{i+1} \ldots w_s$ do not commute and let w_k be the first letter in $w_{i+1} \ldots w_s$ that does not commute with w_i. Our choice of i yields that $Q^{(1)}_{w_k} = P_{w_k}$ (indeed if $Q^{(1)}_{w_k}$ were to be $P^{\perp}_{w_k}$ then (4.18) is 0 in case $Q^{(2)}_{w_k} = P^{\perp}_{w_k}$ and in case $Q^{(2)}_{w_k} = P_{w_k}$ then i was not maximal). But then (4.18) contains a factor $P_{w_i} P_{w_k} = 0$ which means that (4.18) would be zero which in turn is a contradiction.

Claim 2. The expression (4.17) is after possibly interchanging commuting letters in $w_1 \ldots w_d$ of the form:

$$\begin{aligned} &Q^{(1)}_{w_1} T_{w_1} Q^{(2)}_{w_1} \ldots Q^{(1)}_{w_r} T_{w_r} Q^{(2)}_{w_r} (P_{w_{r+1}} T_{w_{r+1}} P_{w_{r+1}}) \ldots (P_{w_s} T_{w_s} P_{w_s}) \\ &\times \quad (P^{\perp}_{w_{s+1}} T_{w_{s+1}} P_{w_{s+1}}) \ldots (P^{\perp}_{w_d} T_{w_d} P_{w_d}). \end{aligned} \tag{4.19}$$

Moreover, the tail of annihilation and diagonal operators is maximal in the sense that if for some $i \leq r$ we have $Q_{w_i}^{(1)} = P_{w_i}$ then $Q_{w_i}^{(2)} = P_{w_i}^{\perp}$.

Proof of Claim 2. Suppose that we are given a (non-zero) expression as in (4.19). Suppose that for some $i < r$ we have $Q_{w_i}^{(1)} = P_{w_i}, Q_{w_i}^{(2)} = P_{w_i}$. Then we need to show that w_i commutes with $w_{i+1} \ldots w_r$. To do so we may suppose the index $i < r$ was chosen maximal. Suppose that w_i and $w_{i+1} \ldots w_r$ do not commute and let w_k be the first letter in $w_{i+1} \ldots w_r$ that does not commute with w_i. Our choice of i yields that $Q_{w_k}^{(1)} = P_{w_k}$ and $Q_{w_k}^{(2)} = P_{w_k}^{\perp}$. But then (4.19) contains a factor $P_{w_i} P_{w_k} = 0$ which means that (4.19) would be zero. As this is a contradiction the claim follows.

Claim 3. The expression (4.17) is after possibly interchanging commuting letters in $w_1 \ldots w_d$ of the form:

$$
\begin{aligned}
(4.20) \qquad & (P_{w_1} T_{w_1} P_{w_1}^{\perp}) \ldots (P_{w_r} T_{w_r} P_{w_r}^{\perp})(P_{w_{r+1}} T_{w_{r+1}} P_{w_{r+1}}) \ldots (P_{w_s} T_{w_s} P_{w_s}) \\
& \times \quad (P_{w_{s+1}}^{\perp} T_{w_{s+1}} P_{w_{s+1}}) \ldots (P_{w_d}^{\perp} T_{w_d} P_{w_d}).
\end{aligned}
$$

Moreover $w_{r+1} \ldots w_s$ forms a clique.

Proof of Claim 3. This is obvious now from Claim 2 and the fact that $P_{w_i}^{\perp} T_{w_i} P_{w_i}^{\perp} = 0$. As $P_{w_i} P_{w_j}$ is non-zero only if w_i and w_j commute we must have that $w_{r+1} \ldots w_s$ forms a clique.

Remainder of the proof. Note that as $T_w = T_w^{(1)} + p P_w$ we have that (4.17) after possibly interchanging commuting letters in $w_1 \ldots w_d$ equals:

$$
(4.21) \qquad p^{\# V\Gamma_0} (T_{w_1}^{(1)} P_{w_1}^{\perp}) \ldots (T_{w_r}^{(1)} P_{w_r}^{\perp}) P_{V\Gamma_0} (T_{w_{s+1}}^{(1)} P_{w_{s+1}}) \ldots (T_{w_d}^{(1)} P_{w_d}),
$$

where Γ_0 is the clique comprising the letters $w_{s+1} \ldots w_r$ as in Claim 3. Therefore the non-zero terms on the right hand side of (4.16) all occur in the summation (4.15). Note that the 'possible commutations' in each of the claims do not affect the permutation σ in (4.15). That the summands of (4.15) are in 1–1 correspondence with the non-zero terms of (4.16) follows by Lemma 4.8. □

We define

$$
(4.22) \qquad \Pi_d : j_d(\Sigma_d) \to \mathcal{B}(L^2(\mathcal{M}_q)) : j_d(x) \mapsto \sigma_d(x).
$$

As Σ_d is finite dimensional this map is completely bounded and by definition $\sigma_d = \Pi_d \circ j_d$. It remains to obtain control over the complete bound of Π_d in terms of d. This is done by means of the following intertwining lemma.

Lemma 4.10. Π_d *defined in* (4.22) *has complete bound that is majorized by* Cd^2 *for a constant* C *that is independent of* d.

Proof. The proof is an intertwining argument between product maps associated to the general and to the free case. Let us make this precise. Let $L^2(\mathcal{M}_q^f)$ be the GNS space of the Hecke algebra $\mathcal{M}_q^f$ generated by (W_f, S) where again W_f is the 'free' Coxeter group with generating set S and relations $\forall s, t \in S, m_f(s,t) = \infty$. Let $0 \leq l \leq d, 0 \leq k \leq d - l$ and let $\Gamma_0 \in \mathsf{Cliq}(\Gamma, l), (\Gamma_1, \Gamma_2) \in \mathsf{Comm}(\Gamma_0)$. Let,

$$
(4.23) \qquad \Pi_{d,k,l,\Gamma_0,\Gamma_1,\Gamma_2}^{f} : L_k \otimes_h A_{\Gamma_0} \otimes_h K_{d-k-l} \to \mathcal{B}(L^2(\mathcal{M}_q^f))
$$

be the product map. This map is completely bounded as follows from the definition of the Haagerup tensor product.

Definition of the intertwining maps. We define two unitary maps. Note that the second map only differs from the first one at the place we put an exclamation mark.

- We define the intertwining map,

$$\mathcal{Q}_{k,l,\Gamma_0,\Gamma_1} : L^2(\mathcal{M}_q) \to L^2(\mathcal{M}_q^f), \tag{4.24}$$

  by sending a vector $T_{\mathbf{w}}\Omega$ with $|\mathbf{w}| = d$ to $T_{w_{\sigma(1)} \dots w_{\sigma(d)}}\Omega$ where σ is the permutation defined in (1) – (4). Moreover we assume that this σ is chosen such that each of the expressions $\mathbf{w}_{\sigma(k)} \dots \mathbf{w}_{\sigma(1)}$, $\mathbf{w}_{\sigma(k+1)} \dots \mathbf{w}_{\sigma(k+l)}$ and $\mathbf{w}_{\sigma(k+l+1)} \dots \mathbf{w}_{\sigma(d)}$ are minimal (which uniquely determines $\mathcal{Q}_{k,l,\Gamma_0,\Gamma_1}$). If such σ does not exist then $\mathcal{Q}_{k,l,\Gamma_0,\Gamma_1}(T_{\mathbf{w}}\Omega)$ is understood as the zero vector.
- We define the intertwining map,

$$\mathcal{R}_{k,l,\Gamma_0,\Gamma_1} : L^2(\mathcal{M}_q) \to L^2(\mathcal{M}_q^f), \tag{4.25}$$

  by sending a vector $T_{\mathbf{w}}\Omega$ with $|\mathbf{w}| = d$ to $T_{w_{\sigma(1)} \dots w_{\sigma(d)}}$ where σ is the permutation defined in (1) – (4). Moreover we assume that this σ is chosen such that each of the expressions $\mathbf{w}_{\sigma(1)} \dots \mathbf{w}_{\sigma(k)}$ (!), $\mathbf{w}_{\sigma(k+1)} \dots \mathbf{w}_{\sigma(k+l)}$ and $\mathbf{w}_{\sigma(k+l+1)} \dots \mathbf{w}_{\sigma(d)}$ are minimal (which uniquely determines $\mathcal{R}_{k,l,\Gamma_0,\Gamma_1}(T_{\mathbf{w}}\Omega)$). If such σ does not exist then $\mathcal{R}_{k,l,\Gamma_0,\Gamma_1}(T_{\mathbf{w}}\Omega)$ is understood as the zero vector.

Claim. Let $x = T_{w_1} \otimes \dots \otimes T_{w_d} \in \Sigma_d$ and let $x_{d,k,l,\Gamma_0,\Gamma_1,\Gamma_2}$ with $0 \leq l \leq d$, $0 \leq k \leq d-l$, $\Gamma_0 \in \mathsf{Cliq}(\Gamma, l)$ and $(\Gamma_1, \Gamma_2) \in \mathsf{Comm}(\Gamma_0)$ be the corresponding summands of $j_d(x)$ in X_d. We have,

$$\begin{aligned} &\mathcal{R}^*_{k,l,\Gamma_0,\Gamma_1} \Pi^f_{d,k,l,\Gamma_0,\Gamma_1,\Gamma_2}(x_{d,k,l,\Gamma_0,\Gamma_1,\Gamma_2}) \mathcal{Q}_{d-l-k,l,\Gamma_0,\Gamma_2} \\ =&(T^{(1)}_{w_{\sigma(1)}} P^{\perp}_{w_{\sigma(1)}}) \dots (T^{(1)}_{w_{\sigma(k)}} P^{\perp}_{w_{\sigma(k)}}) P_{V\Gamma_0} (T^{(1)}_{w_{\sigma(k+l+1)}} P_{w_{\sigma(k+l+1)}}) \dots (T^{(1)}_{w_{\sigma(d)}} P_{w_{\sigma(d)}}), \end{aligned} \tag{4.26}$$

where σ is defined as in (1) – (6) and the right hand side should be understood as 0 otherwise.

Proof of the Claim. Note that both sides of (4.26) equal 0 if a σ as in the statement of the claim does not exist, c.f. the definition of j_d. So we assume that this is not the case. Take an elementary tensor product $T_{w_1} \otimes \dots \otimes T_{w_d} \in \Sigma_d$. As both sides of (4.26) change in the same way under interchaning w_i and w_{i+1} in case $m(w_i, w_{i+1}) = 2$, we may assume that the tensor product $T_{w_1} \otimes \dots \otimes T_{w_d}$ is ordered in such a way that the permutation σ on the right hand side of (4.26) is trivial. Now take a vector $T_{\mathbf{v}}\Omega$ and set,

$$\mathcal{Q}_{d-l-k,l,\Gamma_0,\Gamma_2}(T_{\mathbf{v}}\Omega) = T^f_{\mathbf{v}'}\Omega,$$

with $v'_i = v_{\sigma(i)}$ and σ as in the definition of $\mathcal{Q}_{d-l-k,l,\Gamma_0,\Gamma_2}$ so that $v'_{d-k-l} \dots v'_1$, $v'_{d-k-l+1} \dots v'_{d-k}$ and $v'_{d-k+1} \dots v'_n$ are minimal. If such σ does not exist then $\mathcal{Q}_{d-l-k,l,\Gamma_0,\Gamma_2}(T_{\mathbf{v}}\Omega) = 0$. We have,

$$\begin{aligned} &\Pi^f_{d,k,l,\Gamma_0,\Gamma_1,\Gamma_2}(x_{d,k,l,\Gamma_0,\Gamma_1,\Gamma_2}) \\ =&(T^f_{w_1} P^{f\,\perp}_{w_1}) \dots (T^f_{w_k} P^{f\,\perp}_{w_k}) P^f_{V\Gamma_0} (T^f_{w_{k+l+1}} P^f_{w_{k+l+1}}) \dots (T^f_{w_d} P^f_{w_d}). \end{aligned}$$

And therefore, if $\mathcal{Q}_{d-l-k,l,\Gamma_0,\Gamma_2}(T_{\mathbf{v}}\Omega)$ is non-zero,

$$\begin{aligned}
&\Pi^f_{d,k,l,\Gamma_0,\Gamma_1,\Gamma_2}(x_{d,k,l,\Gamma_0,\Gamma_1,\Gamma_2})\mathcal{Q}_{d-l-k,l,\Gamma_0,\Gamma_2}(T_{\mathbf{v}}\Omega)\\
=&\langle T^f_{v'_1}\Omega, T^f_{w_d}\Omega\rangle \ldots \langle T^f_{v'_{d-k-l}}\Omega, T^f_{w_{k+l+1}}\Omega\rangle\\
&T^f_{w_1\ldots w_k} T^f_{v'_{d-k-l+1}\ldots v'_{d-k}} T^f_{v'_{d-k+1}\ldots v'_n}\Omega.
\end{aligned} \tag{4.27}$$

On the other hand consider an expression

$$\begin{aligned}
&(T^{(1)}_{w_1}P^{\perp}_{w_1})\ldots(T^{(1)}_{w_k}P^{\perp}_{w_k})P_{V\Gamma_0}\\
&\quad\times (T^{(1)}_{w_{k+l+1}}P_{w_{k+l+1}})\ldots(T^{(1)}_{w_d}P_{w_d})T_{\mathbf{v}}\Omega
\end{aligned} \tag{4.28}$$

Because $w_{k+l+1}\ldots w_d$ starts with $V\Gamma_2$ (since we assumed that (4.26) is non-zero) this expression can only be non-zero if there exists $\mathbf{v}'$ as defined above in which case,

$$\begin{aligned}
(4.28) =&(T^{(1)}_{w_1}P^{\perp}_{w_1})\ldots(T^{(1)}_{w_k}P^{\perp}_{w_k})P_{V\Gamma_0}\\
&\times (T^{(1)}_{w_{k+l+1}}P_{w_{k+l+1}})\ldots(T^{(1)}_{w_d}P_{w_d})T_{\mathbf{v}'}\Omega\\
=&\langle T_{v'_1}\Omega, T_{w_d}\Omega\rangle \ldots \langle T_{v'_{d-k-l}}\Omega, T_{w_{k+l+1}}\Omega\rangle\\
&T_{w_1\ldots w_k}T_{v'_{d-k-l+1}\ldots v'_{d-k}}T_{v'_{d-k+1}\ldots v'_n}\Omega.
\end{aligned} \tag{4.29}$$

Clearly the image of (4.29) under $\mathcal{R}_{k,l,\Gamma_0,\Gamma_1}$ equals (4.27). This concludes the claim.

Remainder of the proof. From Lemma 4.9 we see that Π_d is given by the direct sum of the maps $\mathcal{R}^*_{k,l,\Gamma_0,\Gamma_1}\Pi^f_{d,k,l,\Gamma_0,\Gamma_1,\Gamma_2}(\,\cdot\,)\mathcal{Q}_{d-l-k,l,\Gamma_0,\Gamma_2}$ which are defined on the corresponding summands of X_d. As each of these summands is completely contractive and there are Cd^2 summands for some constant C independent of d, we see that Π_d is completely bounded with complete bound majorized by Cd^2.

□

Theorem 4.11. *Let $q > 0$ and let (W, S) be a reduced Coxeter system with $|S| \geq 3$. The Hecke von Neumann algebra $\mathcal{M}_q$ is not injective.*

Proof. Following Remark 4.7 it suffices to consider the case $|S| = 3$, say $S = \{r, s, t\}$, with commutation relations $m(r, s) = \infty, m(s, t) = \infty, m(r, t) = 2$. The proof of this case is now a mutatis mutandis copy of the proof of Theorem 4.5.

Note that at the beginning of the proof we had to justify that there existed a sequence of words $\mathbf{w}^{(i)}$ of length 2^{d-1} such that (4.7) holds. For the proof it only matters that the maximal possible length of such a sequence is exponential in d. We claim that in the current case there is such a sequence of length at least $2^{\frac{1}{2}(d-1)}$ in case d is odd. Indeed, for d odd we may consider words $a_1sa_3sa_5\ldots a_{d-2}sr$ with $a_i \in \{r, t\}$. Such words form a set $\mathcal{A}$ of size $2^{\frac{1}{2}(d-1)}$. Similarly the words $tsa_1sa_3sa_5\ldots sa_{d-2}$ with $a_i \in \{r, t\}$ form a set $\mathcal{B}$ of size $2^{\frac{1}{2}(d-1)}$. Letting again $\varphi : \mathcal{A} \to \mathcal{B}$ be a bijection and considering the words $\mathbf{w}\varphi(\mathbf{w}), \mathbf{w} \in \mathcal{A}$ shows that there is a sequence satisfying (4.7) of length at least $2^{\frac{1}{2}(d-1)}$.

Next note that the Khintchine inequality applied in (4.9) gets replaced by Lemma 4.10. The rest of the proof of Theorem 4.5 changes mutatis mutandis to the Khintchine decomposition (4.13). □

5. Completely bounded approximation property

We show that for a right angled Coxeter system (W, S) the Hecke von Neumann algebra $\mathcal{M}_q$ has the wk-$*$ CBAP, see Definition 5.18. We first consider radial multipliers and then show that cutting down to radius n has a cb-norm that is at most polynomial in n.

5.1. **Creation/annihilation arguments.** In this section we introduce some auxiliary notation and prove some elementary properties concerning creation/annihilation of letters in words. Let $\mathbf{x}, \mathbf{w} \in W$. We shall write $\mathbf{w} \leq \mathbf{x}$ for saying that $|\mathbf{w}^{-1}\mathbf{x}| = |\mathbf{x}| - |\mathbf{w}|$. Then $\mathbf{w} < \mathbf{x}$ is defined naturally. So $\mathbf{w} \leq \mathbf{x}$ means that $\mathbf{w}$ is obtained from $\mathbf{x}$ by cutting off a tail. An element $\mathbf{v} \in W$ is called a *clique word* in case its letters form a clique. For Λ a clique in W and $\mathbf{v} \in W$ we define $\mathbf{v}(2, \emptyset)$ as the maximal clique Γ_0 such that $|\mathbf{v}V\Gamma_0| = |\mathbf{v}| - |V\Gamma_0|$. Then we set the decomposition $\mathbf{v} = \mathbf{v}(1, \Lambda)\mathbf{v}(2, \Lambda)$ with $|\mathbf{v}| = |\mathbf{v}(1, \Lambda)| + |\mathbf{v}(2, \Lambda)|$ and $\mathbf{v}(2, \Lambda) = \mathbf{v}(2, \emptyset)\backslash\Lambda$ (which uniquely determines $\mathbf{v}(1, \Lambda)$). For $\mathbf{g} \leq \mathbf{x}$ we let $\Lambda_{\mathbf{g},\mathbf{x}}$ be $(\mathbf{x}^{-1}\mathbf{g})(2, \emptyset)$. In other words $\Lambda_{\mathbf{g},\mathbf{x}}$ is the maximal clique that appears at the start of $\mathbf{g}^{-1}\mathbf{x}$. We let $C(\mathbf{g}, \mathbf{x})$ be the collection of $\mathbf{w} \in W$ with $\mathbf{g} \leq \mathbf{w} \leq \mathbf{g}\Lambda_{\mathbf{g},\mathbf{x}}$. Note that $C(\mathbf{g}, \mathbf{x})$ contains at least $\mathbf{g}$ and $\mathbf{g}\Lambda_{\mathbf{g},\mathbf{x}}$ (and the latter elements can be equal). We write $C(\mathbf{g}, +)$ for $\cup_{\mathbf{g} \leq \mathbf{x}} C(\mathbf{g}, \mathbf{x})$.

Example 5.1. Consider the Coxeter system (W, S) with $S = \{r, s, t\}$ in which $m(r, s) = 2$ and $m(r, t) = m(s, t) = \infty$. Consider $\mathbf{v} = trs$. Then $\mathbf{v}(1, \emptyset) = t, \mathbf{v}(2, \emptyset) = rs, \mathbf{v}(1, r) = tr$ and $\mathbf{v}(2, r) = s$. Also $\Lambda_{t,trst} = \{t, tr, ts, trs\}$.

We set, for $\mathbf{g} \leq \mathbf{x}$ and $\Lambda \in \mathsf{Cliq}(\Gamma)$,

$$\alpha_{\mathbf{g},\mathbf{x},\Lambda}(r) = \left(\sum_{\mathbf{v} \in C(\mathbf{g},\mathbf{x})} (-1)^{|\mathbf{g}^{-1}\mathbf{v}|} r^{2|\mathbf{v}(1,\Lambda)| + |\mathbf{v}(2,\Lambda)|} \right)^{\frac{1}{2}}. \tag{5.1}$$

Lemma 5.2. *Let $\mathbf{x}, \mathbf{w} \in W$. Let $\mathbf{w} = \mathbf{w}'\mathbf{w}''$ be the decomposition with $|\mathbf{w}| = |\mathbf{w}'| + |\mathbf{w}''|$ such that $|\mathbf{w}''\mathbf{x}| = |\mathbf{x}| - |\mathbf{w}''|$ and $|\mathbf{w}\mathbf{x}| = |\mathbf{x}| - |\mathbf{w}''| + |\mathbf{w}'|$. Take $(\mathbf{w}'')^{-1} \leq \mathbf{g} \leq \mathbf{x}$. Then, for $\mathbf{v} \in C(\mathbf{g}, \mathbf{x})$,*

$$(\mathbf{w}\mathbf{v})(2, (\mathbf{w}\mathbf{g})(2, \emptyset)\backslash\mathbf{g}(2, \emptyset)) = \mathbf{v}(2, \mathbf{g}(2, \emptyset)\backslash(\mathbf{w}\mathbf{g})(2, \emptyset)) \tag{5.2}$$

and

$$|(\mathbf{w}\mathbf{v})(1, (\mathbf{w}\mathbf{g})(2, \emptyset)\backslash\mathbf{g}(2, \emptyset))| = |\mathbf{v}(1, \mathbf{g}(2, \emptyset)\backslash\mathbf{w}\mathbf{g}(2, \emptyset))| - |\mathbf{w}''| + |\mathbf{w}'|. \tag{5.3}$$

Proof. Let $\mathbf{v} \in C(\mathbf{g}, \mathbf{x})$. The clique $\mathbf{v}(2, \emptyset)$ consists of the clique $\mathbf{g}^{-1}\mathbf{v}$ plus all letters in $\mathbf{g}(2, \emptyset)$ that commute with $\mathbf{g}^{-1}\mathbf{v}$. Therefore $\mathbf{v}(2, \mathbf{g}(2, \emptyset)\backslash(\mathbf{w}\mathbf{g})(2, \emptyset))$ is the clique consisting of $\mathbf{g}^{-1}\mathbf{v}$ plus all letters in $(\mathbf{w}\mathbf{g})(2, \emptyset) \cap \mathbf{g}(2, \emptyset)$ that commute with $\mathbf{g}^{-1}\mathbf{v}$. On the other hand $(\mathbf{w}\mathbf{v})(2, \emptyset)$ consists of the clique $\mathbf{g}^{-1}\mathbf{v}$ together with all letters in $(\mathbf{w}\mathbf{g})(2, \emptyset)$ that commute with $\mathbf{g}^{-1}\mathbf{v}$. Then $(\mathbf{w}\mathbf{v})(2, (\mathbf{w}\mathbf{g})(2, \emptyset)\backslash\mathbf{g}(2, \emptyset))$ equals $\mathbf{g}^{-1}\mathbf{v}$ together with all elements in $(\mathbf{w}\mathbf{g})(2, \emptyset) \cap \mathbf{g}(2, \emptyset)$ that commute with $\mathbf{g}^{-1}\mathbf{v}$. So we conclude (5.2). Therefore,

$$\begin{aligned} &|(\mathbf{w}\mathbf{v})(1, (\mathbf{w}\mathbf{g})(2, \emptyset)\backslash\mathbf{g}(2, \emptyset))| \\ =&|\mathbf{w}\mathbf{v}| - |(\mathbf{w}\mathbf{v})(2, (\mathbf{w}\mathbf{g})(2, \emptyset)\backslash\mathbf{g}(2, \emptyset))| \\ =&|\mathbf{v}| - |\mathbf{w}''| + |\mathbf{w}'| - |\mathbf{v}(2, \mathbf{g}(2, \emptyset)\backslash(\mathbf{w}\mathbf{g})(2, \emptyset))| \\ =&|\mathbf{v}(1, \mathbf{g}(2, \emptyset)\backslash\mathbf{w}\mathbf{g}(2, \emptyset))| - |\mathbf{w}''| + |\mathbf{w}'|. \end{aligned} \tag{5.4}$$

□

Corollary 5.3. *With the same notation as Lemma 5.2 we have:*

$$r^{|\mathbf{w}'|-|\mathbf{w}''|}\alpha_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\backslash(\mathbf{wg})(2,\emptyset)} = \alpha_{\mathbf{wg},\mathbf{wx},\mathbf{wg}(2,\emptyset)\backslash\mathbf{g}(2,\emptyset)}.$$

Proof. This now follows from the definition of $\alpha_{\mathbf{g},\mathbf{x},\Lambda}$ and both (5.2) and (5.3). □

Lemma 5.4. *Let* $\mathbf{x}, \mathbf{w} \in W$ *and decompose* $\mathbf{w} = \mathbf{w}'\mathbf{w}''$ *such that* $|\mathbf{w}| = |\mathbf{w}'| + |\mathbf{w}''|, |\mathbf{w}''\mathbf{x}| = |\mathbf{x}| - |\mathbf{w}''|$ *and* $|\mathbf{wx}| = |\mathbf{x}| - |\mathbf{w}''| + |\mathbf{w}'|$*. Let* $(\mathbf{w}'')^{-1} \leq \mathbf{g} \leq \mathbf{x}$*. Then:*

(1) $\mathbf{g}(2,\emptyset)\backslash(\mathbf{wg})(2,\emptyset) = \mathbf{g}(2,\emptyset)\backslash(\mathbf{w}''\mathbf{g})(2,\emptyset)$*;*

(2) For $\mathbf{v} \in C(\mathbf{g},\mathbf{x})$ *we have*

$$\mathbf{v}(2,\mathbf{v}(2,\emptyset)\backslash(\mathbf{w}''\mathbf{v})(2,\emptyset)) = \mathbf{v}(2,\mathbf{g}(2,\emptyset)\backslash(\mathbf{w}''\mathbf{g})(2,\emptyset)). \tag{5.5}$$

Proof. (1) Because $(\mathbf{w}'')^{-1} \leq \mathbf{g} \leq \mathbf{x}$ we also have $|\mathbf{w}''\mathbf{g}| = |\mathbf{g}| - |\mathbf{w}''|$ and $|\mathbf{wg}| = |\mathbf{g}| - |\mathbf{w}''| + |\mathbf{w}'|$. So $\mathbf{w}'$ creates letters in $\mathbf{w}''\mathbf{g}$ so that $\mathbf{g}(2,\emptyset)\backslash(\mathbf{wg})(2,\emptyset) = \mathbf{g}(2,\emptyset)\backslash(\mathbf{w}''\mathbf{g})(2,\emptyset)$.

(2) Let A be the set of letters in $\mathbf{g}(2,\emptyset)$ that commute with $\mathbf{g}^{-1}\mathbf{v}$. The clique $\mathbf{v}(2,\emptyset)$ consists of $\mathbf{g}^{-1}\mathbf{v} \cup A$. This means that $\mathbf{v}(2,\mathbf{v}(2,\emptyset)\backslash(\mathbf{w}''\mathbf{v})(2,\emptyset))$ consists of $\mathbf{g}^{-1}\mathbf{v} \cup A$ intersected with $(\mathbf{w}''\mathbf{v})(2,\emptyset)$. The intersection of $(\mathbf{w}''\mathbf{v})(2,\emptyset)$ with $\mathbf{g}^{-1}\mathbf{v}$ is $\mathbf{g}^{-1}\mathbf{v}$ so that $\mathbf{v}(2,\mathbf{v}(2,\emptyset)\backslash(\mathbf{w}''\mathbf{v})(2,\emptyset)) = \mathbf{g}^{-1}\mathbf{v} \cup (A \cap (\mathbf{w}''\mathbf{v})(2,\emptyset))$. On the other hand $\mathbf{v}(2,\mathbf{g}(2,\emptyset)\backslash(\mathbf{w}''\mathbf{g})(2,\emptyset))$ equals $\mathbf{g}^{-1}\mathbf{v} \cup (A \cap (\mathbf{w}''\mathbf{g})(2,\emptyset))$ and as $\mathbf{g}(2,\emptyset) \cap (\mathbf{w}''\mathbf{g})(2,\emptyset) = \mathbf{g}(2,\emptyset) \cap (\mathbf{w}''\mathbf{v})(2,\emptyset)$ clearly $(A \cap (\mathbf{w}''\mathbf{v})(2,\emptyset)) = (A \cap (\mathbf{w}''\mathbf{g})(2,\emptyset))$. This proves (5.5).

□

Corollary 5.5. *Let* $\mathbf{x}, \mathbf{w} \in W$ *and decompose* $\mathbf{w} = \mathbf{w}'\mathbf{w}''$ *such that* $|\mathbf{w}| = |\mathbf{w}'| + |\mathbf{w}''|, |\mathbf{w}''\mathbf{x}| = |\mathbf{x}| - |\mathbf{w}''|$ *and* $|\mathbf{wx}| = |\mathbf{x}| - |\mathbf{w}''| + |\mathbf{w}'|$*. Then,*

$$\begin{aligned}&\alpha_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\backslash(\mathbf{wg})(2,\emptyset)}(r)^2 \\ = &\sum_{\mathbf{v}\in C(\mathbf{g},\mathbf{x})} (-1)^{|\mathbf{g}^{-1}\mathbf{v}|} r^{2|\mathbf{v}(1,\mathbf{v}(2,\emptyset)\backslash(\mathbf{wv})(2,\emptyset))|+|\mathbf{v}(2,\mathbf{v}(2,\emptyset)\backslash(\mathbf{wv})(2,\emptyset))|}.\end{aligned}$$

Proof. This directly follows from the definition (5.1) and Lemma 5.4. □

5.2. **Radial multipliers.** In this subsection we construct radial multipliers,

$$\Phi_r : \mathcal{M}_q \to \mathcal{M}_q : T_{\mathbf{w}} \mapsto r^{|\mathbf{w}|}T_{\mathbf{w}}, \qquad 0 < r \leq 1, \tag{5.6}$$

and show that these maps are completely bounded with complete bound uniform in $0 < r \leq 1$. Note that radial multipliers were also considered in for example [Haa10] (and [HoRi11]). The results from [Haa10] typically apply to free products. As our situation is somewhat different we present a self-contained proof of complete boundedness of (5.6) by constructing an explicit Stinespring dilation.

Remark 5.6. Note that we may identify $\ell^2(W)$ with basis $\delta_{\mathbf{x}}, \mathbf{x} \in W$ with $L^2(\mathcal{M}_q)$ with basis $T_{\mathbf{x}}\Omega$. This way $T_{\mathbf{w}}^{(1)}$ acts on $\ell^2(W)$ by means of the left regular representation.

We borrow the following construction from [Oza08]. We let $\mathcal{B}_f(W)$ be the set of finite subsets of W. For $A \in \mathcal{B}_f(W)$ we define $\widetilde{\xi}_A^{\pm}$ to be the vectors in $\ell^2(\mathcal{B}_f(W))$ given by

$$\widetilde{\xi}_A^{+}(\omega) = \begin{cases} 1 & \text{if } \omega \subseteq A, \\ 0 & \text{otherwise,} \end{cases} \qquad \widetilde{\xi}_A^{-}(\omega) = \begin{cases} (-1)^{|\omega|} & \text{if } \omega \subseteq A, \\ 0 & \text{otherwise,} \end{cases}$$

The reader may verify the following lemma using the binomial formula.

Lemma 5.7 (Lemma 4 of [Oza08])**.** *We have* $\|\widetilde{\xi}_A^{\pm}\|^2 = 2^{|A|}$ *and*

$$\langle \widetilde{\xi}_A^{+}, \widetilde{\xi}_B^{-} \rangle = \begin{cases} 0 & A \cap B \neq \emptyset, \\ 1 & \textit{otherwise}. \end{cases}$$

We let $\mathcal{B}$ be the linear span of the elements $P_{\mathbf{w}}, \mathbf{w} \in W$. In particular $Q_{\mathbf{w}} \in \mathcal{B}$, c.f. (5.8) below. We define for $0 < r \leq 1$ and Λ a clique the map,

$$\Phi_r^{\Lambda} : \mathcal{B} \to \mathcal{B}$$

by the prescription $P_e \to P_e$ and $P_{\mathbf{w}} \mapsto r^{2|\mathbf{w}(1,\Lambda)|+|\mathbf{w}(2,\Lambda)|} P_{\mathbf{w}}, |\mathbf{w}| \geq 1$. Define the following Stinespring dilation for $\pm \in \{+, -\}$,

$$V_r^{\pm} : \ell^2(W) \to \ell^2(W) \otimes \ell^2(W) \otimes \ell^2(W) \otimes \ell^2(\mathcal{B}_f(W)),$$

$$\delta_{\mathbf{x}} \mapsto \sum_{\mathbf{g} \leq \mathbf{x}} \sum_{\Lambda \leq \mathbf{g}(2,\emptyset)} \alpha_{\mathbf{g},\mathbf{x},\Lambda}(r) \delta_{\mathbf{g}^{-1}\mathbf{x}} \otimes \delta_{\mathbf{g}} \otimes \delta_{\mathbf{g}(2,\Lambda)} \otimes \widetilde{\xi}_{\Lambda}^{\pm}. \tag{5.7}$$

It may not directly be clear that $V_r^{\pm}$ is bounded but we will soon prove this. We let $Q_{\mathbf{w}}$ be the Dirac delta function at $\mathbf{w} \in W$. We let $P_{\mathbf{w}}$ be the indicator function of the set $\{\mathbf{x} \in W \mid \mathbf{w} \leq \mathbf{x}\}$.

Lemma 5.8. *For* $\mathbf{w} \in W$ *we have,*

$$Q_{\mathbf{w}} = \left(\sum_{\mathbf{v} \in C(\mathbf{w},+)} (-1)^{|\mathbf{w}^{-1}\mathbf{v}|} P_{\mathbf{v}} \right). \tag{5.8}$$

Proof. Let $\mathbf{x} \in W$. In case $\mathbf{w} \not\leq \mathbf{x}$ then,

$$Q_{\mathbf{w}}(\mathbf{x}) = 0 = \left(\sum_{\mathbf{v} \in C(\mathbf{w},+)} (-1)^{|\mathbf{w}^{-1}\mathbf{v}|} P_{\mathbf{v}} \right)(\mathbf{x}).$$

Also clearly,

$$Q_{\mathbf{w}}(\mathbf{w}) = 1 = P_{\mathbf{w}}(\mathbf{w}) = \left(\sum_{\mathbf{v} \in C(\mathbf{w},+)} (-1)^{|\mathbf{w}^{-1}\mathbf{v}|} P_{\mathbf{v}} \right)(\mathbf{x}).$$

In case $\mathbf{w} < \mathbf{x}$ then,

$$\begin{aligned} \left(\sum_{\mathbf{v} \in C(\mathbf{w},+)} (-1)^{|\mathbf{w}^{-1}\mathbf{v}|} P_{\mathbf{v}} \right)(\mathbf{x}) &= \left(\sum_{\mathbf{v} \in C(\mathbf{w},\mathbf{x})} (-1)^{|\mathbf{w}^{-1}\mathbf{v}|} P_{\mathbf{v}} \right)(\mathbf{x}) \\ &= \sum_{\mathbf{v} \in C(\mathbf{w},\mathbf{x})} (-1)^{|\mathbf{w}^{-1}\mathbf{v}|} \end{aligned}$$

and this expression equals 0 by the binomial formula. Indeed, it equals

$$\sum_{l=0}^{|\Lambda_{\mathbf{w},\mathbf{x}}|} \sum_{\mathbf{v} \in C(\mathbf{w},\mathbf{x}), |\mathbf{w}^{-1}\mathbf{v}|=l} (-1)^{|\mathbf{w}^{-1}\mathbf{v}|} = \sum_{l=0}^{|\Lambda_{\mathbf{w},\mathbf{x}}|} \binom{|\Lambda_{\mathbf{w},\mathbf{x}}|}{l} (-1)^{|\mathbf{w}^{-1}\mathbf{v}|} = 0.$$

□

Lemma 5.9. *For* $\mathbf{w} \leq \mathbf{x}$ *in* W *and* Λ *a clique we have,*

$$\Phi_r^{\Lambda}(Q_{\mathbf{w}})(\mathbf{x}) = \alpha_{\mathbf{w},\mathbf{x},\Lambda}(r)^2.$$

Proof. We have,

$$\begin{aligned}
&\Phi_r^\Lambda(Q_{\mathbf{w}})(\mathbf{x})\\
=&\Phi_r^\Lambda\left(\sum_{\mathbf{v}\in C(\mathbf{w},+)}(-1)^{|\mathbf{w}^{-1}\mathbf{v}|}P_{\mathbf{v}}\right)(\mathbf{x})\\
=&\left(\sum_{\mathbf{v}\in C(\mathbf{w},+)}(-1)^{|\mathbf{w}^{-1}\mathbf{v}|}r^{2|\mathbf{w}(1,\Lambda)|+|\mathbf{w}(2,\Lambda)|}P_{\mathbf{v}}\right)(\mathbf{x})\\
=&\sum_{\mathbf{v}\in C(\mathbf{w},\mathbf{x})}(-1)^{|\mathbf{w}^{-1}\mathbf{v}|}r^{2|\mathbf{w}(1,\Lambda)|+|\mathbf{w}(2,\Lambda)|}\\
=&\alpha_{\mathbf{w},\mathbf{x},\Lambda}(r)^2.
\end{aligned}$$

□

Proposition 5.10. *The Stinespring maps $V_r^\pm$ are bounded uniformly in $0 < r \leq 1$.*

Proof. Note that the images of $V^\pm\delta_{\mathbf{x}}, \mathbf{x}\in W$ are orthogonal by the first two tensor legs of (5.7) so that it suffices to show that there exists a constant C such that $\|V^\pm\delta_{\mathbf{x}}\|\leq C$. We have by Lemma 5.7,

$$\begin{aligned}
\|V^\pm\delta_{\mathbf{x}}\|^2 =&\|\sum_{\mathbf{g}\leq\mathbf{x}}\sum_{\Lambda\leq\mathbf{g}(2,\emptyset)}\alpha_{\mathbf{g},\mathbf{x},\Lambda}(r)\delta_{\mathbf{g}^{-1}\mathbf{x}}\otimes\delta_{\mathbf{g}}\otimes\delta_{\mathbf{g}(2,\Lambda)}\otimes\widetilde{\xi}_\Lambda^\pm\|^2\\
=&\sum_{\mathbf{g}\leq\mathbf{x}}\sum_{\Lambda\leq\mathbf{g}(2,\emptyset)}\alpha^2_{\mathbf{g},\mathbf{x},\Lambda}(r)2^{|\Lambda|}\\
\leq&\text{ Const }\times\sum_{\mathbf{g}\leq\mathbf{x}}\sum_{\Lambda\leq\mathbf{g}(2,\emptyset)}\alpha^2_{\mathbf{g},\mathbf{x},\Lambda}(r)\\
\leq&\text{ Const }\times\sum_{\Lambda\in\mathsf{Cliq}(\Gamma)}\sum_{\mathbf{g}\leq\mathbf{x}}\alpha^2_{\mathbf{g},\mathbf{x},\Lambda}(r).
\end{aligned}\tag{5.9}$$

From Lemma 5.9 one sees that

$$\begin{aligned}
\sum_{\mathbf{g}\leq\mathbf{x}}\alpha^2_{\mathbf{g},\mathbf{x},\Lambda}(r)=&\sum_{\mathbf{g}\leq\mathbf{x}}\Phi_r^\Lambda(Q_{\mathbf{g}})(\mathbf{x})\\
=&\Phi_r^\Lambda(1\pm\text{ projections not supported at }\mathbf{x})(\mathbf{x})=1.
\end{aligned}$$

so that (5.9) is bounded with bound uniform in $\mathbf{x}$. □

We are now able to show the existence of suitable radial multipliers as in (5.15).

Lemma 5.11. *Let $\mathbf{x},\mathbf{w}\in W$ and decompose $\mathbf{w}=\mathbf{w}'\mathbf{w}''$ such that $|\mathbf{w}|=|\mathbf{w}'|+|\mathbf{w}''|, |\mathbf{w}''\mathbf{x}|=|\mathbf{x}|-|\mathbf{w}''|$ and $|\mathbf{w}\mathbf{x}|=|\mathbf{x}|-|\mathbf{w}''|+|\mathbf{w}'|$. Then,*

$$\sum_{(\mathbf{w}'')^{-1}\leq\mathbf{g}\leq\mathbf{x}}\alpha^2_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\backslash(\mathbf{w}\mathbf{g})(2,\emptyset)}(r)=r^{2|\mathbf{w}''|}.\tag{5.10}$$

Proof. Define $\Phi_r:\mathcal{B}\to\mathcal{B}$ by

$$P_{\mathbf{v}}\mapsto r^{2|\mathbf{v}(1,\mathbf{v}(2,\emptyset)\backslash(\mathbf{w}\mathbf{v})(2,\emptyset))|+|\mathbf{v}(2,\mathbf{v}(2,\emptyset)\backslash(\mathbf{w}\mathbf{v})(2,\emptyset))|}P_{\mathbf{v}}.$$

As in Lemma 5.9, one checks for the first following equality that for $\mathbf{g} \leq \mathbf{x}$ we have,

$$\begin{aligned}\Phi_r(Q_{\mathbf{g}})(\mathbf{x}) = &\sum_{\mathbf{v}\in C(\mathbf{g},\mathbf{x})} (-1)^{|\mathbf{g}^{-1}\mathbf{v}|} r^{2|\mathbf{v}(1,\mathbf{v}(2,\emptyset)\setminus(\mathbf{wv})(2,\emptyset))|+|\mathbf{v}(2,\mathbf{v}(2,\emptyset)\setminus(\mathbf{wv})(2,\emptyset))|} \\ =&\alpha^2_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\setminus(\mathbf{wg})(2,\emptyset)}(r),\end{aligned} \tag{5.11}$$

where the second equality is Corollary 5.5. Note that writing $\widetilde{\mathbf{w}}$ for $(\mathbf{w}'')^{-1}$ we have $\widetilde{\mathbf{w}}(2, \widetilde{\mathbf{w}}(2,\emptyset)\setminus(\mathbf{w}\widetilde{\mathbf{w}})(2,\emptyset)) = \widetilde{\mathbf{w}}(2,\widetilde{\mathbf{w}}(2,\emptyset))$ which is the empty word. And so $\widetilde{\mathbf{w}}(1, \widetilde{\mathbf{w}}(2,\emptyset)\setminus(\mathbf{w}\widetilde{\mathbf{w}})(2,\emptyset)) = \widetilde{\mathbf{w}}$. Then taking sums and using this in the third equality yields,

$$\begin{aligned}\sum_{(\mathbf{w}'')^{-1}\leq\mathbf{g}\leq\mathbf{x}} \alpha^2_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\setminus(\mathbf{wg})(2,\emptyset)}(r) = &\sum_{(\mathbf{w}'')^{-1}\leq\mathbf{g}\leq\mathbf{x}} \Phi_r(Q_{\mathbf{g}})(\mathbf{x}) \\ =&\Phi_r(P_{(\mathbf{w}'')^{-1}} \pm \text{projections not supported at } \mathbf{x})(\mathbf{x}) \\ =&r^{2|\mathbf{w}''|}P_{(\mathbf{w}'')^{-1}}(\mathbf{x}) \\ =&r^{2|\mathbf{w}''|}.\end{aligned} \tag{5.12}$$

□

Lemma 5.12. *Let $\mathbf{w} \in W$, let $(\mathbf{w}', \Gamma_0, \mathbf{w}'') \in A_{\mathbf{w}}$ and decompose $T^{(1)}_{\mathbf{w}'} P_{V\Gamma_0} T^{(1)}_{\mathbf{w}''} = T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0} T^{(1)}_{\mathbf{u}''}$ as in Lemma 2.5. Then,*

$$\sum_{(\mathbf{u}'')^{-1}\mathbf{u}V\Gamma_0\leq\mathbf{g}\leq\mathbf{x}} \alpha^2_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\setminus(\mathbf{wg})(2,\emptyset)}(r) = r^{2|\mathbf{u}|+2|\mathbf{u}''|+|V\Gamma_0|}. \tag{5.13}$$

Proof. As in Lemma 5.11 define $\Phi_r : \mathcal{B} \to \mathcal{B}$ by

$$P_{\mathbf{v}} \mapsto r^{2|\mathbf{v}(1,\mathbf{v}(2,\emptyset)\setminus(\mathbf{u}'\mathbf{u}''\mathbf{v})(2,\emptyset))|+|\mathbf{v}(2,\mathbf{v}(2,\emptyset)\setminus(\mathbf{u}'\mathbf{u}''\mathbf{v})(2,\emptyset))|} P_{\mathbf{v}}.$$

As in Lemma 5.11 , for $\mathbf{g} \leq \mathbf{x}$ we have,

$$\Phi_r(Q_{\mathbf{g}})(\mathbf{x}) = \alpha^2_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\setminus(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset)}(r). \tag{5.14}$$

From this point set $\widetilde{\mathbf{w}} := (\mathbf{u}'')^{-1}\mathbf{u}V\Gamma_0$. First suppose that $\mathbf{u}$ is the empty word. Then $\widetilde{\mathbf{w}}(2, \widetilde{\mathbf{w}}(2,\emptyset)\setminus(\mathbf{u}'\mathbf{u}''\widetilde{\mathbf{w}})(2,\emptyset)) = V\Gamma_0$ and so $\widetilde{\mathbf{w}}(1, \widetilde{\mathbf{w}}(2,\emptyset)\setminus(\mathbf{u}'\mathbf{u}''\widetilde{\mathbf{w}})(2,\emptyset)) = (\mathbf{u}'')^{-1}$. If $\mathbf{u}$ is not the empty word, then let $s \in W$ be a final letter of $\mathbf{u}$ (i.e. $|\mathbf{u}s| = |\mathbf{u}|-1$). Then s cannot commute with $V\Gamma_0$ as this would violate the equation $T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0} T^{(1)}_{\mathbf{u}''} = T^{(1)}_{\mathbf{w}'} P_{V\Gamma_0} T^{(1)}_{\mathbf{w}''}$. Therefore again $\widetilde{\mathbf{w}}(2, \widetilde{\mathbf{w}}(2,\emptyset)\setminus(\mathbf{u}'\mathbf{u}''\widetilde{\mathbf{w}})(2,\emptyset)) = \widetilde{\mathbf{w}}(2,\emptyset) = V\Gamma_0$ and so $\widetilde{\mathbf{w}}(1, \widetilde{\mathbf{w}}(2,\emptyset)\setminus(\mathbf{u}'\mathbf{u}''\widetilde{\mathbf{w}})(2,\emptyset)) = (\mathbf{u}'')^{-1}$.

Then taking sums and using the previous paragraph for the third equation,

$$\begin{aligned}&\sum_{(\mathbf{u}'')^{-1}\mathbf{u}V\Gamma_0\leq\mathbf{g}\leq\mathbf{x}} \alpha^2_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\setminus(\mathbf{wg})(2,\emptyset)}(r) \\ =&\sum_{(\mathbf{u}'')^{-1}\mathbf{u}V\Gamma_0\leq\mathbf{g}\leq\mathbf{x}} \Phi_r(Q_{\mathbf{g}})(\mathbf{x}) \\ =&\Phi_r(P_{(\mathbf{u}'')^{-1}\mathbf{u}V\Gamma_0} \pm \text{projections not supported at } \mathbf{x})(\mathbf{x}) \\ =&r^{2|\mathbf{u}|+2|\mathbf{u}''|+|V\Gamma_0|}P_{(\mathbf{u}'')^{-1}\mathbf{u}V\Gamma_0}(\mathbf{x}) = r^{2|\mathbf{u}|+2|\mathbf{u}''|+|V\Gamma_0|}.\end{aligned}$$

□

Theorem 5.13. *Let (W,S) be a right angled Coxeter system. Let $0 < r \leq 1$. There exists a normal completely bounded map $\Phi_r : \mathcal{M}_q \to \mathcal{M}_q$ that is determined by the formula:*

$$\Phi_r(T_{\mathbf{w}}) = r^{|\mathbf{w}|} T_{\mathbf{w}}. \tag{5.15}$$

Moreover $\|\Phi_r\|_{\mathcal{CB}} \leq C$ for a constant independent of r.

Proof. We show that Φ_r is given by the Stinespring dilation

$$\begin{aligned} \mathcal{B}(\ell^2(W)) &\to \mathcal{B}(\ell^2(W)), \\ x &\mapsto (V_r^-)^*(1 \otimes x \otimes 1 \otimes 1)V_r^+. \end{aligned}$$

In order to do so we will treat the following cases from which this claim follows using Lemma 2.7.

Claim 1. For $\mathbf{w} \in W$ we have,

$$(V_r^-)^*(1 \otimes T^{(1)}_{\mathbf{w}} \otimes 1 \otimes 1)V_r^+ = r^{|\mathbf{w}|} T^{(1)}_{\mathbf{w}}. \tag{5.16}$$

Proof of the claim. We have,

$$\begin{aligned} &\langle (V_r^-)^*(1 \otimes T^{(1)}_{\mathbf{w}} \otimes 1 \otimes 1)V_r^+ \delta_{\mathbf{x}}, \delta_{\mathbf{y}} \rangle \\ =& \Bigg\langle \sum_{\mathbf{g} \leq \mathbf{x}} \sum_{\Lambda \leq \mathbf{g}(2,\emptyset)} \alpha_{\mathbf{g},\mathbf{x},\Lambda}(r) \delta_{\mathbf{g}^{-1}\mathbf{x}} \otimes T^{(1)}_{\mathbf{w}} \delta_{\mathbf{g}} \otimes \delta_{\mathbf{g}(2,\Lambda)} \otimes \widetilde{\xi}^+_{\Lambda}, \\ & \sum_{\mathbf{h} \leq \mathbf{y}} \sum_{\Lambda' \leq \mathbf{h}(2,\emptyset)} \alpha_{\mathbf{h},\mathbf{y},\Lambda'}(r) \delta_{\mathbf{h}^{-1}\mathbf{y}} \otimes \delta_{\mathbf{h}} \otimes \delta_{\mathbf{h}(2,\Lambda')} \otimes \widetilde{\xi}^-_{\Lambda} \Bigg\rangle. \end{aligned} \tag{5.17}$$

By looking at the first and second tensor leg we see that this expression can only be non-zero if $\mathbf{wg} = \mathbf{h}$ and $\mathbf{g}^{-1}\mathbf{x} = \mathbf{h}^{-1}\mathbf{y}$ so that $\mathbf{wx} = \mathbf{y}$. Then the only summands that can be non-zero are the ones where $\mathbf{g}(2,\Lambda) = \mathbf{h}(2,\Lambda')$ and $\Lambda \cap \Lambda' = \emptyset$. This precisely means that

$$\Lambda = \mathbf{g}(2,\emptyset) \backslash (\mathbf{wg})(2,\emptyset), \qquad \Lambda' = (\mathbf{wg})(2,\emptyset) \backslash \mathbf{g}(2,\emptyset).$$

We decompose $\mathbf{w} = \mathbf{w}'\mathbf{w}''$ with $|\mathbf{w}| = |\mathbf{w}'| + |\mathbf{w}''|$, $|\mathbf{w}''\mathbf{x}| = |\mathbf{x}| - |\mathbf{w}''|$ and $|\mathbf{y}| = |\mathbf{x}| - |\mathbf{w}''| + |\mathbf{w}'|$. So $\mathbf{w}''$ annihilates the first letters of $\mathbf{x}$ and then $\mathbf{w}'$ creates letters at the start of $\mathbf{w}''\mathbf{x}$. We therefore find using the previous remarks for the first equality and then Corollary 5.3 and Lemma 5.11,

$$\begin{aligned} (5.17) =& \sum_{(\mathbf{w}'')^{-1} \leq \mathbf{g} \leq \mathbf{x}} \alpha_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\backslash(\mathbf{wg})(2,\emptyset)}(r) \alpha_{\mathbf{wg},\mathbf{wx},(\mathbf{wg})(2,\emptyset)\backslash\mathbf{g}(2,\emptyset)}(r) \\ =& \sum_{(\mathbf{w}'')^{-1} \leq \mathbf{g} \leq \mathbf{x}} \alpha^2_{\mathbf{g},\mathbf{x},(\mathbf{wg})(2,\emptyset)\backslash\mathbf{g}(2,\emptyset)}(r) r^{|\mathbf{w}'| - |\mathbf{w}''|} \\ =& r^{2|\mathbf{w}''|} r^{|\mathbf{w}'| - |\mathbf{w}''|} = r^{|\mathbf{w}|}. \end{aligned} \tag{5.18}$$

So we conclude that applying the functional $\langle \cdot\, \delta_{\mathbf{x}}, \delta_{\mathbf{y}} \rangle$ to both sides of (5.16) gives the same result.

Claim 2. Let $\mathbf{w} \in W$ and let $(\mathbf{w}', \Gamma_0, \mathbf{w}'') \in A_{\mathbf{w}}$, c.f. Definition 2.4. Then,

$$(V_r^-)^*(1 \otimes T^{(1)}_{\mathbf{w}'} P_{V\Gamma_0} T^{(1)}_{\mathbf{w}''} \otimes 1 \otimes 1)V_r^+ = r^{|\mathbf{w}|} T^{(1)}_{\mathbf{w}'} P_{V\Gamma_0} T^{(1)}_{\mathbf{w}''}. \tag{5.19}$$

Proof of the claim. Let $\mathbf{u}, \mathbf{u}', \mathbf{u}''$ be as in Lemma 4.9 so that $T^{(1)}_{\mathbf{w}'} P_{V\Gamma_0} T^{(1)}_{\mathbf{w}''} = T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0} T^{(1)}_{\mathbf{u}''}$. We have,

$$\begin{aligned}
&\langle (V_r^-)^* (1 \otimes T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0} T^{(1)}_{\mathbf{u}''} \otimes 1 \otimes 1) V_r^+ \delta_{\mathbf{x}}, \delta_{\mathbf{y}} \rangle \\
&= \Bigg\langle \sum_{\mathbf{g} \leq \mathbf{x}} \sum_{\Lambda \leq \mathbf{g}(2,\emptyset)} \alpha_{\mathbf{g},\mathbf{x},\Lambda}(r) \delta_{\mathbf{g}^{-1}\mathbf{x}} \otimes T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0} T^{(1)}_{\mathbf{u}''} \delta_{\mathbf{g}} \otimes \delta_{\mathbf{g}(2,\Lambda)} \otimes \widetilde{\xi}^+_{\Lambda}, \\
&\qquad \sum_{\mathbf{h} \leq \mathbf{y}} \sum_{\Lambda' \leq \mathbf{h}(2,\emptyset)} \alpha_{\mathbf{h},\mathbf{y},\Lambda'}(r) \delta_{\mathbf{h}^{-1}\mathbf{y}} \otimes \delta_{\mathbf{h}} \otimes \delta_{\mathbf{h}(2,\Lambda')} \otimes \widetilde{\xi}^-_{\Lambda} \Bigg\rangle.
\end{aligned} \tag{5.20}$$

By looking at the first and second leg we see that this expression can only be non-zero if $T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0} T^{(1)}_{\mathbf{u}''} \delta_{\mathbf{g}} = \delta_{\mathbf{h}}$ and $\mathbf{g}^{-1}\mathbf{x} = \mathbf{h}^{-1}\mathbf{y}$ so that $T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0} T^{(1)}_{\mathbf{u}''} \delta_{\mathbf{x}} = \delta_{\mathbf{y}}$. In the non-zero case, the choice of $\mathbf{u}'$ and $\mathbf{u}''$ is not necessarily unique, but we may always assume that $|\mathbf{y}| = |\mathbf{x}| - |\mathbf{u}''| + |\mathbf{u}'|$. So $\mathbf{u}''$ annihilates the first letters of $\mathbf{x}$ and then $\mathbf{u}'$ creates letters at the start of $\mathbf{u}''\mathbf{x}$. Then the only summands that can be non-zero are the ones where $\mathbf{g}(2, \Lambda) = \mathbf{h}(2, \Lambda')$ and $\Lambda \cap \Lambda' = \emptyset$. As in (5.18) we therefore find the following. Where we used Lemma 5.11 we use Lemma 5.12 instead. So,

$$\begin{aligned}
(5.20) &= \sum_{(\mathbf{u}'')^{-1}\mathbf{u}V\Gamma_0 \leq \mathbf{g} \leq \mathbf{x}} \alpha_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\setminus(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset)}(r) \alpha_{\mathbf{u}'\mathbf{u}''\mathbf{g},\mathbf{u}'\mathbf{u}''\mathbf{x},(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset)\setminus\mathbf{g}(2,\emptyset)}(r) \\
&= \sum_{(\mathbf{u}'')^{-1}\mathbf{u}V\Gamma_0 \leq \mathbf{g} \leq \mathbf{x}} \alpha^2_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\setminus(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset)}(r) r^{|\mathbf{u}'| - |\mathbf{u}''|} \\
&= r^{2(|\mathbf{u}| + |\mathbf{u}''|) + |\Gamma_0|} r^{|\mathbf{u}'| - |\mathbf{u}''|} = r^{|\mathbf{w}|},
\end{aligned}$$

as $|\mathbf{w}| = |\mathbf{u}'| + |\mathbf{u}''| + 2|\mathbf{u}| + |V\Gamma_0|$. The claim follows again as applying $\langle \cdot \delta_{\mathbf{x}}, \delta_{\mathbf{y}} \rangle$ to both sides of (5.19) yields the same result. □

5.3. **Weak-$*$ completely bounded approximation property.** Let $\mathcal{A}_q$ be the $*$-algebra generated by the operators $T_{\mathbf{w}}, \mathbf{w} \in W$. So $\mathcal{M}_q$ is the σ-weak closure of $\mathcal{A}_q$. We define

$$\Psi_{\leq n} : \mathcal{A}_q \to \mathcal{M}_q : T_{\mathbf{w}} \mapsto \begin{cases} T_{\mathbf{w}} & |\mathbf{w}| \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

We also set $\Psi_n = \Psi_{\leq n} - \Psi_{\leq (n-1)}$. The crucial part we need to prove is that $\Psi_{\leq n}$ is completely bounded with a complete bound that can be upper estimated in n polynomially. In order to do so we first introduce 3 auxiliary maps.

Auxiliary map 1. Recall that $\mathcal{M}_1$ is just the group von Neumann algebra of the right-angled Coxeter group W. For $k \in \mathbb{N}$ define the multiplier,

$$\rho_k(T^{(1)}_{\mathbf{w}}) = \delta_{|\mathbf{w}|,k} T^{(1)}_{\mathbf{w}}.$$

By [Oza08, Theorem 1 (2)] this map is completely bounded and moreover $\|\rho_k\|_{\mathcal{CB}} \leq C(k+1)$ for some constant C independent of k. By the Bozejko-Fendler Theorem 2.8 we may extend ρ_k uniquely to a σ-weakly continuous $L^\infty(W)$-bimodule map $\mathcal{B}(\ell^2(W)) \to \mathcal{B}(\ell^2(W))$ with the same completely bounded norm. Using Lemma 2.7 we see that

$$\Psi_{\leq n} = \sum_{k=0}^{n} \rho_k \circ \Psi_{\leq n}.$$

Auxiliary map 2. Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$. For $z \in \mathbb{T}$ we define,

$$W_z : \ell^2(W) \to \ell^2(W) : \delta_{\mathbf{w}} \mapsto z^{|\mathbf{w}|}\delta_{\mathbf{w}}.$$

We set for $i \in \mathbb{Z}$,

$$\Phi_i : \mathcal{B}(\ell^2(W)) \to \mathcal{B}(\ell^2(W)) : x \mapsto \int_{\mathbb{T}} z^{-i} W_z^* x W_z dz,$$

where the measure is the normalized Lebesgue measure on $\mathbb{T}$. Using Lemma 2.7 we see that

$$\Psi_{\leq n} = \sum_{i=-n}^{n} \Phi_i \circ \Psi_{\leq n}.$$

Auxiliary map 3. For $a \in \mathbb{N}$ we define Stinespring dilations,

$$U_a^{\pm} : \ell^2(W) \to \ell^2(W) \otimes \ell^2(W) \otimes \ell^2(W) \otimes \ell^2(\mathcal{B}_f(W)), \tag{5.21}$$

by mapping $\delta_{\mathbf{x}}$ to (see Section 5.1 for notation),

$$\sum_{\mathbf{g} \leq \mathbf{x}} \sum_{\Lambda \leq \mathbf{g}(2,\emptyset)} \beta^{\pm}_{\mathbf{g},\mathbf{x},\Lambda,a} \delta_{\mathbf{g}^{-1}\mathbf{x}} \otimes \delta_{\mathbf{g}} \otimes \delta_{\mathbf{g}(2,\Lambda)} \otimes \widetilde{\xi}^{\pm}_{\Lambda}.$$

Here

$$\beta^{+}_{\mathbf{g},\mathbf{x},\Lambda,a} = \sum_{\mathbf{v} \in C(\mathbf{g},\mathbf{x})} (-1)^{|\mathbf{g}^{-1}\mathbf{v}|} F_{\Lambda,a}(\mathbf{v}),$$

where $F_{\Lambda,a}(\mathbf{v}) = 1$ if

$$2|\mathbf{v}(1,\Lambda)| + |\mathbf{v}(2,\Lambda)| \leq a,$$

and else $F_{\Lambda,a}(\mathbf{v}) = 0$. We let $\beta^{-}_{\mathbf{g},\mathbf{x},\Lambda,a} = 1$ if $\beta^{+}_{\mathbf{g},\mathbf{x},\Lambda,a} \neq 0$ and $\beta^{-}_{\mathbf{g},\mathbf{x},\Lambda,a} = 0$ otherwise. Then set,

$$\sigma_{a,b}(x) = U_a^{-}(1 \otimes x \otimes 1 \otimes 1)U_b^{+}.$$

The map $U_a^{\pm}$ is bounded as follows from the fact that $\mathbf{g} \mapsto \beta^{+}_{\mathbf{g},\mathbf{x},\Lambda,a}$ finitely supported and in fact this bound is uniform in a.

Lemma 5.14. *Let $\mathbf{x} \in W$. Let $\mathbf{u}', \mathbf{u}'' \in W$ be such that $|\mathbf{u}''\mathbf{x}| = |\mathbf{x}| - |\mathbf{u}''|$, $|\mathbf{u}'\mathbf{u}''\mathbf{x}| = |\mathbf{x}| - |\mathbf{u}''| + |\mathbf{u}'|$ and $(\mathbf{u}'')^{-1} \leq \mathbf{v} \leq \mathbf{x}$. Then,*

$$\begin{aligned} &\sum_{\mathbf{v} \leq \mathbf{g} \leq \mathbf{x}} \beta^{+}_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\backslash(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset),a} \beta^{-}_{\mathbf{u}'\mathbf{u}''\mathbf{g},\mathbf{u}'\mathbf{u}''\mathbf{x},(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset)\backslash\mathbf{g}(2,\emptyset),a-2|\mathbf{u}'|+2|\mathbf{u}''|} \\ &= \begin{cases} 1 & \textit{in case } 2|\mathbf{v}(1,\mathbf{v}(2,\emptyset)\backslash(\mathbf{u}'\mathbf{u}''\mathbf{v})(2,\emptyset))| + |\mathbf{v}(2,\mathbf{v}(2,\emptyset)\backslash(\mathbf{u}'\mathbf{u}''\mathbf{v})(2,\emptyset))| \leq a, \\ 0 & \textit{otherwise.} \end{cases} \end{aligned} \tag{5.22}$$

Proof. By Equations (5.2) and (5.3) for $\mathbf{v} \leq \mathbf{g}$ we get,

$$\begin{aligned} &\beta^{+}_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\backslash(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset),a} \\ =& \sum_{\mathbf{w} \in C(\mathbf{g},\mathbf{x})} F_{\mathbf{g}(2,\emptyset)\backslash(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset),a}(\mathbf{w}) \\ =& \sum_{\mathbf{w} \in C(\mathbf{u}'\mathbf{u}''\mathbf{g},\mathbf{u}'\mathbf{u}''\mathbf{x})} F_{(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset)\backslash\mathbf{g}(2,\emptyset),a-2|\mathbf{u}''|+2|\mathbf{u}'|}(\mathbf{w}) \\ =& \beta^{+}_{\mathbf{u}'\mathbf{u}''\mathbf{g},\mathbf{u}'\mathbf{u}''\mathbf{x},(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset)\backslash\mathbf{g}(2,\emptyset),a-2|\mathbf{u}'|+2|\mathbf{u}''|}. \end{aligned}$$

Therefore also

$$\beta^{-}_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\backslash(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset),a} = \beta^{-}_{\mathbf{u}'\mathbf{u}''\mathbf{g},\mathbf{u}'\mathbf{u}''\mathbf{x},(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset)\backslash\mathbf{g}(2,\emptyset),a-2|\mathbf{u}'|+2|\mathbf{u}''|}.$$

We therefore have that the left hand side of (5.22) equals,

$$\sum_{\mathbf{v}\leq\mathbf{g}\leq\mathbf{x}} \beta^{+}_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\setminus(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset),a}\beta^{-}_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\setminus(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset),a} = \sum_{\mathbf{v}\leq\mathbf{g}\leq\mathbf{x}} \beta^{+}_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\setminus(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset),a}$$

To compute this sum define the mapping $\kappa_a : \mathcal{B} \to \mathcal{B} : P_{\mathbf{v}} \mapsto F_{\mathbf{v}(2,\emptyset)\setminus(\mathbf{u}'\mathbf{u}''\mathbf{v})(2,\emptyset),a}(\mathbf{v})P_{\mathbf{v}}$. As in the proof of Lemma 5.11 one checks that $\kappa_a(Q_{\mathbf{g}})(\mathbf{x}) = \beta^{+}_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\setminus(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset),a}$. And therefore,

$$\begin{aligned}\sum_{\mathbf{v}\leq\mathbf{g}\leq\mathbf{x}} \beta^{+}_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\setminus(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset),a} =& \sum_{\mathbf{v}\leq\mathbf{g}\leq\mathbf{x}} \kappa_a(Q_{\mathbf{g}})(\mathbf{x}) \\ =&\kappa_a(P_{\mathbf{v}} \pm \text{ projections not supported at } \mathbf{x})(\mathbf{x}).\end{aligned}$$

This expression equals 1 if $F_{\mathbf{v}(2,\emptyset)\setminus(\mathbf{u}'\mathbf{u}''\mathbf{v})(2,\emptyset),a}(\mathbf{v}) = 1$ and 0 otherwise which corresponds exactly to the statement of the lemma.

□

Lemma 5.15. *We have for $n \in \mathbb{N}$:*

$$\Psi_{\leq n} = \sum_{i=-n}^{n} \sigma_{n-i,n+i} \circ \Phi_i \circ \Psi_{\leq n}.$$

Proof. Let $T_{\mathbf{w}} \in \mathcal{M}_q$ with $|\mathbf{w}| \leq n$. We need to show that,

$$T_{\mathbf{w}} = \sum_{k=0}^{n}\sum_{i=-n}^{n} \sigma_{n-i,n+i} \circ \Phi_i \circ \rho_k(T_{\mathbf{w}}).$$

We split

$$T_{\mathbf{w}} = T^{(1)}_{\mathbf{w}} + \sum_{(\mathbf{w}',\Gamma_0,\mathbf{w}'')\in A_{\mathbf{w}}} T^{(1)}_{\mathbf{w}'}P_{V\Gamma_0}T^{(1)}_{\mathbf{w}''},$$

and show that $\sum_{k=0}^{n}\sum_{i=-n}^{n} \sigma_{n-i,n+i}\circ\Phi_i\circ\rho_k$ applied to each of these summands acts as the identity. Let us consider the summands $T^{(1)}_{\mathbf{w}'}P_{V\Gamma_0}T^{(1)}_{\mathbf{w}''}$ with $(\mathbf{w}',\Gamma_0,\mathbf{w}'') \in A_{\mathbf{w}}$. Let $\mathbf{u},\mathbf{u}',\mathbf{u}''$ be as in Lemma 2.5 so that $T^{(1)}_{\mathbf{w}'}P_{V\Gamma_0}T^{(1)}_{\mathbf{w}''} = T^{(1)}_{\mathbf{u}'}P_{\mathbf{u}V\Gamma_0}T^{(1)}_{\mathbf{u}''}$. We have

$$\rho_k(T^{(1)}_{\mathbf{u}'}P_{\mathbf{u}V\Gamma_0}T^{(1)}_{\mathbf{u}''}) = \begin{cases} T^{(1)}_{\mathbf{u}'}P_{\mathbf{u}V\Gamma_0}T^{(1)}_{\mathbf{u}''} & \text{if } k = |\mathbf{u}'| + |\mathbf{u}''|, \\ 0 & \text{otherwise.}\end{cases}$$

So assume $k = |\mathbf{u}'| + |\mathbf{u}''|$ so that it remains to show that for $\mathbf{x},\mathbf{y} \in W$,

$$\langle \sum_{i=-n}^{n} \sigma_{n-i,n+i} \circ \Phi_i(T^{(1)}_{\mathbf{u}'}P_{\mathbf{u}V\Gamma_0}T^{(1)}_{\mathbf{u}''})\delta_{\mathbf{x}}, \delta_{\mathbf{y}}\rangle = \langle T^{(1)}_{\mathbf{u}'}P_{\mathbf{u}V\Gamma_0}T^{(1)}_{\mathbf{u}''}\delta_{\mathbf{x}}, \delta_{\mathbf{y}}\rangle. \tag{5.23}$$

As in the proof of Theorem 5.13 both sides are 0 unless $T^{(1)}_{\mathbf{u}'}P_{\mathbf{u}V\Gamma_0}T^{(1)}_{\mathbf{u}''}\delta_{\mathbf{x}} = \delta_{\mathbf{y}}$. In the non-zero case there is a choice for $\mathbf{u}',\mathbf{u}''$ for which $|\mathbf{u}''\mathbf{x}| = |\mathbf{x}| - |\mathbf{u}''|$ and $|\mathbf{u}'\mathbf{u}''\mathbf{x}| = |\mathbf{x}| - |\mathbf{u}''| + |\mathbf{u}'|$. And in this case $i = |\mathbf{u}'| - |\mathbf{u}''|$ is the only non-zero summand. Then we find, as in the proof of Theorem 5.13,

$$
\begin{aligned}
(5.24)\quad & \langle (1\otimes T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0} T^{(1)}_{\mathbf{u}''}\otimes 1 \otimes 1) U^{+}_{n-i}\delta_{\mathbf{x}}, U^{-}_{n+i}\delta_{\mathbf{y}}\rangle \\
=& \langle \sum_{\mathbf{g}\leq \mathbf{x}} \sum_{\Lambda \leq \mathbf{g}(2,\emptyset)} \beta^{+}_{\mathbf{g},\mathbf{x},\Lambda,n-i} \delta_{\mathbf{g}^{-1}\mathbf{x}} \otimes T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0} T^{(1)}_{\mathbf{u}''}\delta_{\mathbf{g}} \otimes \delta_{\mathbf{g}(2,\emptyset)} \otimes \widetilde{\xi}^{+}_{\Lambda}, \\
& \quad \sum_{\mathbf{h}\leq \mathbf{y}} \sum_{\Lambda' \leq \mathbf{h}(2,\emptyset)} \beta^{-}_{\mathbf{h},\mathbf{y},\Lambda',n+i} \delta_{\mathbf{h}^{-1}\mathbf{x}} \otimes \delta_{\mathbf{h}} \otimes \delta_{\mathbf{h}(2,\emptyset)} \otimes \widetilde{\xi}^{-}_{\Lambda}\rangle \\
=& \sum_{(\mathbf{u}'')^{-1}\mathbf{u}V\Gamma_0 \leq \mathbf{g} \leq \mathbf{x}} \beta^{+}_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\backslash(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset),n-i} \beta^{-}_{\mathbf{u}'\mathbf{u}''\mathbf{g},\mathbf{u}'\mathbf{u}''\mathbf{x},(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset)\backslash\mathbf{g}(2,\emptyset),n+i}
\end{aligned}
$$

We claim that this expression is 1 by verifying Lemma 5.14. Indeed set $\mathbf{w} := (\mathbf{u}'')^{-1}\mathbf{u}V\Gamma_0$. First suppose that $\mathbf{u}$ is the empty word. Then $\mathbf{w}(2,\mathbf{w}(2,\emptyset)\backslash(\mathbf{u}'\mathbf{u}''\mathbf{w})(2,\emptyset)) = V\Gamma_0$ and so $\mathbf{w}(1,\mathbf{w}(2,\emptyset)\backslash(\mathbf{u}'\mathbf{u}''\mathbf{w})(2,\emptyset)) = (\mathbf{u}'')^{-1}$. If $\mathbf{u}$ is not the empty word, then let $s\in W$ be a final letter of $\mathbf{u}$ (i.e. $|\mathbf{u}s| = |\mathbf{u}|-1$). Then s cannot commute with $V\Gamma_0$ as this would violate the equation $T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0} T^{(1)}_{\mathbf{u}''} = T^{(1)}_{\mathbf{w}'} P_{V\Gamma_0} T^{(1)}_{\mathbf{w}''}$. Therefore again $\mathbf{w}(2,\mathbf{w}(2,\emptyset)\backslash(\mathbf{u}'\mathbf{u}''\mathbf{w})(2,\emptyset)) = \mathbf{w}(2,\emptyset) = V\Gamma_0$ and so $\mathbf{w}(1,\mathbf{w}(2,\emptyset)\backslash(\mathbf{u}'\mathbf{u}''\mathbf{w})(2,\emptyset)) = (\mathbf{u}'')^{-1}$. Further our constructions give that $|\mathbf{u}''| = \frac{k-i}{2}$ and $2|\mathbf{u}| + |V\Gamma_0| = |\mathbf{w}| - |\mathbf{u}'| - |\mathbf{u}''| = |\mathbf{w}| - k$. So we have,

$$
\begin{aligned}
& 2|\mathbf{w}(1,\mathbf{w}(2,\emptyset)\backslash(\mathbf{u}'\mathbf{u}''\mathbf{w})(2,\emptyset))| + |\mathbf{w}(2,\mathbf{w}(2,\emptyset)\backslash(\mathbf{u}'\mathbf{u}''\mathbf{w})(2,\emptyset))| \\
=& 2|(\mathbf{u}'')^{-1}| + (2|\mathbf{u}| + |V\Gamma_0|) \\
=& 2\frac{k-i}{2} + (|\mathbf{w}| - k) \\
=& |\mathbf{w}| - i \leq n-i,
\end{aligned} \tag{5.25}
$$

so that by Lemma 5.14 we see that (5.24) is 1. So we conclude that (5.23) holds. For the summand $T^{(1)}_{\mathbf{w}}$ instead of $T^{(1)}_{\mathbf{w}'} P_{V\Gamma_0} T^{(1)}_{\mathbf{w}''}$ the proof follows similarly.

□

Lemma 5.16. *We have for $n\in\mathbb{N}, -n\leq i\leq n$:*

$$\sigma_{n-i,n+i}\circ \Phi_i \circ \Psi_{\leq n} = \sigma_{n-i,n+i}\circ\Phi_i.$$

Proof. We need to show that the right hand side applied to $T_{\mathbf{w}}$ with $|\mathbf{w}| > n$ equals 0. Therefore we may look at the summands $T^{(1)}_{\mathbf{w}'} P_{V\Gamma_0} T^{(1)}_{\mathbf{w}''}$ with $(\mathbf{w}',\Gamma_0,\mathbf{w}'')\in A_{\mathbf{w}}$ which can be further decomposed as $T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0} T^{(1)}_{\mathbf{u}''}$ with $\mathbf{u},\mathbf{u}',\mathbf{u}''$ as in Lemma 2.5. The summand $T^{(1)}_{\mathbf{w}}$ can be treated in the same manner. Consider,

$$\langle \sigma_{n-i,n+i}\circ\Phi_i\circ\rho_k(T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0} T^{(1)}_{\mathbf{u}''})\delta_{\mathbf{x}},\delta_{\mathbf{y}}\rangle, \tag{5.26}$$

and this expression is 0 in case $|\mathbf{u}'| + |\mathbf{u}''| \neq k$. If $|\mathbf{u}'| + |\mathbf{u}''| = k$ then

$$(5.26) = \langle \sigma_{n-i,n+i}\circ\Phi_i(T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0} T^{(1)}_{\mathbf{u}''})\delta_{\mathbf{x}},\delta_{\mathbf{y}}\rangle.$$

So (5.26) equals 0 unless $T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0} T^{(1)}_{\mathbf{u}''}\delta_{\mathbf{x}} = \delta_{\mathbf{y}}$. In the latter case there is a choice for $\mathbf{u}',\mathbf{u}''$ for which $|\mathbf{u}''\mathbf{x}| = |\mathbf{x}| - |\mathbf{u}''|$ and $|\mathbf{u}'\mathbf{u}''\mathbf{y}| = |\mathbf{y}| - |\mathbf{u}''| + |\mathbf{u}'|$. In that case

$i = |\mathbf{u}'| - |\mathbf{u}''|$ and as in (5.24),

$$\begin{aligned}(5.27)\\ (5.26) =&\langle (1 \otimes T^{(1)}_{\mathbf{u}'} P_{\mathbf{u}V\Gamma_0} T^{(1)}_{\mathbf{u}''} \otimes 1 \otimes 1) U^{+}_{n-i}\delta_{\mathbf{x}}, U^{-}_{n+i}\delta_{\mathbf{y}}\rangle \\ =& \sum_{(\mathbf{u}'')^{-1}\mathbf{u}V\Gamma_0 \leq \mathbf{g} \leq \mathbf{x}} \beta_{\mathbf{g},\mathbf{x},\mathbf{g}(2,\emptyset)\backslash(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset),n-i}\beta_{\mathbf{u}'\mathbf{u}''\mathbf{g},\mathbf{u}'\mathbf{u}''\mathbf{x},(\mathbf{u}'\mathbf{u}''\mathbf{g})(2,\emptyset)\backslash\mathbf{g}(2,\emptyset),n+i}.\end{aligned}$$

As for $\mathbf{w} := (\mathbf{u}'')^{-1}\mathbf{u}V\Gamma_0$ we have again as in (5.25),

$$2|\mathbf{w}(1,\mathbf{w}(2,\emptyset)\backslash(\mathbf{u}'\mathbf{u}''\mathbf{w})(2,\emptyset))| + |\mathbf{w}(2,\mathbf{w}(2,\emptyset)\backslash(\mathbf{u}'\mathbf{u}''\mathbf{w})(2,\emptyset))| = |\mathbf{w}| - i > n - i,$$

the expression (5.27) is zero by Lemma 5.14. □

Proposition 5.17. *We have $\|\Psi_{\leq n}\|_{\mathcal{CB}} \leq P(n)$ for some polynomial P.*

Proof. By Lemmas 5.15 and 5.16 we have,

$$\begin{aligned}\Psi_{\leq n} &= \sum_{i=-n}^{n} \sigma_{n-i,n+i} \circ \Phi_i \circ \Psi_{\leq n} \\ &= \sum_{i=-n}^{n} \sigma_{n-i,n+i} \circ \Phi_i,\end{aligned}$$

and the right hand side is completely bounded with polynomial bound in n. □

Definition 5.18. A von Neumann algebra $\mathcal{M}$ has the weak-$*$ completely bounded approximation property (wk-$*$ CBAP) if there exists a net of normal finite rank maps $\Phi_i : \mathcal{M} \to \mathcal{M}$ such that $\Phi_i(x) \to x$ in the σ-weak topology and moreover $\sup_i \|\Phi_i\|_{\mathcal{CB}} < \infty$.

Theorem 5.19. *Let (W,S) be a right angled Coxeter system and let $q > 0$. The Hecke von Neumann algebra $\mathcal{M}_q$ has the wk-$*$ CBAP.*

Proof. The proof goes back to Haagerup [Haa78]. Consider the completely bounded map $\Psi_{\leq n} \circ \Phi_r : \mathcal{A}_q \to \mathcal{M}_q$. Clearly as $n \to \infty$ and $r \nearrow 1$ this map converges to the identity in the point σ-weak topology. Furthermore,

$$\|\Psi_{\leq n} \circ \Phi_r\|_{\mathcal{CB}} \leq \|(\Psi_{\leq n} - \mathrm{Id}) \circ \Phi_r\|_{\mathcal{CB}} + \|\Phi_r\|_{\mathcal{CB}} \leq \left(\sum_{i=n+1}^{\infty} r^n \|\Psi_n\|_{\mathcal{CB}} \right) + \|\Phi_r\|_{\mathcal{CB}},$$

which shows using Proposition 5.17 and Theorem 5.13 that we may let $r \nearrow 1$ and then choose $n := n_r$ converging to ∞ such that $\|\Psi_{\leq n_r} \circ \Phi_r\|_{\mathcal{CB}} \leq C$ for some constant.

The map Φ_r is normal by Theorem 5.13. Also $\Psi_{\leq n}$ is normal by a standard argument: indeed using duality and Kaplansky's density theorem one sees that Ψ_n maps $L^1(\mathcal{M}_q) \to L^1(\mathcal{M}_q)$ boundedly. Then taking the dual of this map yields that $\Psi_n : \mathcal{M}_q \to \mathcal{M}_q$ is a normal map. We may extend $\Psi_{\leq n} \circ \Phi_r$ to a normal map $\mathcal{M}_q \to \mathcal{M}_q$. Then using a standard approximation argument yields the result. □

Remark 5.20. Recall that if the finite rank maps in the definition of the wk-$*$ CBAP can be chosen contractive then we call this the weak-$*$ completely contractive approximation property (wk-$*$ CCAP). We do not know if $\mathcal{M}_q$ has the wk-$*$ CCAP (nor we obtain the Haagerup property, see [BrOz08]). If the radial multipliers we constructed are ucp maps then $\mathcal{M}_q$ possesses these properties.

6. Strong solidity

We prove that the II_1 summand of $\mathcal{M}_q$ – see Theorem 2.2 – is a strongly solid von Neumann algebra.

6.1. **Preliminaries on strongly solid algebras.** The *normalizer* of a von Neumann subalgebra $\mathcal{P}$ of $\mathcal{M}$ is defined as $\mathcal{N}_{\mathcal{M}}(\mathcal{P}) = \{u \in \mathcal{U}(\mathcal{M}) \mid u\mathcal{P}u^* = \mathcal{P}\}$. A von Neumann algebra is called *diffuse* if it does not contain minimal projections.

Definition 6.1. A II_1-factor $\mathcal{M}$ is *strongly solid* if for any diffuse injective von Neumann subalgebra $\mathcal{P} \subseteq \mathcal{M}$ the von Neumann algebra generated by the normalizer $\mathcal{N}_{\mathcal{M}}(\mathcal{P})''$ is again injective.

In [OzPo10] Ozawa and Popa proved that free group factors are strongly solid and consequently they could give the first examples of II_1 factors that have no Cartan subalgebras. A general source for strongly solid von Neumann algebras are group von Neumann algebras of groups that have the weak-$*$ completely bounded approximation property and are bi-exact (see [ChSi13], [CSU13], [PoVa14]; we also refer to these sources for the definition of bi-exactness). The following definition and subsequent theorem were then introduced and proved in [Iso15]. For standard forms of von Neumann algebras we refer to [Tak03].

Definition 6.2. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra represented on the standard Hilbert space $\mathcal{H}$ with modular conjugation J. We say that $\mathcal{M}$ satisfies condition $(\mathrm{AO})^+$ if there exists a unital C*-subalgebra $\mathsf{A} \subseteq \mathcal{M}$ that is σ-weakly dense in $\mathcal{M}$ and which satisfies the following two conditions:

1. A is locally reflexive.
2. There exists a ucp map $\theta : \mathsf{A} \otimes_{\min} J\mathsf{A}J \to \mathcal{B}(\mathcal{H})$ such that $\theta(a \otimes b) - ab$ is a compact operator on $\mathcal{H}$.

Theorem 6.3 ([Iso15])**.** *Let $\mathcal{M}$ be a II_1-factor with separable predual. Suppose that $\mathcal{M}$ satisfies condition $(AO)^+$ and has the weak-$*$ completely bounded approximation property. Then $\mathcal{M}$ is strongly solid.*

A maximal abelian von Neumann subalgebra $\mathcal{P} \subseteq \mathcal{M}$ of a II_1 factor $\mathcal{M}$ is called a *Cartan subalgebra* if $\mathcal{N}_{\mathcal{M}}(\mathcal{P})'' = \mathcal{M}$. It is then obvious that if $\mathcal{M}$ is a non-injective strongly solid II_1-factor, then $\mathcal{M}$ cannot contain a Cartan subalgebra. Therefore we will now prove that the Hecke von Neumann algebra $\mathcal{M}_q$ (in the factorial case) satisfies condition $(\mathrm{AO})^+$.

6.2. **Crossed products.** Let A be a C*-algebra that is represented on a Hilbert space $\mathcal{H}$. Let $\alpha : \mathsf{G} \curvearrowright \mathsf{A}$ be a continuous action of a locally compact group G on A. The reduced crossed product $\mathsf{A} \rtimes_r \mathsf{G}$ is the C*-algebra of operators acting on $\mathcal{H} \otimes \ell^2(\mathsf{G})$ generated by operators

$$(6.1)\quad u_g := \sum_{h \in \mathsf{G}} 1 \otimes e_{gh,h}, \quad g \in \mathsf{G}, \qquad \text{and} \qquad \pi(x) := \sum_{h \in \mathsf{G}} h^{-1} \cdot x \otimes e_{h,h}, \quad x \in \mathsf{A}.$$

Here the convergence of the sums should be understood in the strong topology. There is also a universal crossed product $\mathsf{A} \rtimes_u \mathsf{G}$ for which we refer to [BrOz08] (in the case we need it, it turns out to equal the reduced crossed product).

6.3. **Gromov boundary and condition (AO)$^+$.** Let again (W, S) be a Coxeter system. Let Λ be the associated Cayley tree. A geodesic ray starting at a point $\mathbf{w} \in \Lambda$ is a sequence $(\mathbf{w}, \mathbf{w}v_1, \mathbf{w}v_1v_2, \ldots)$ such that $|\mathbf{w}v_1 \ldots v_n| = |\mathbf{w}| + n$. We typically write $\omega = (\omega(0), \omega(1), \ldots)$ for a geodesic ray. Let δW be the Gromov boundary of W which is the collection of all geodesic rays starting at the identitiy of W. δW may be topologized with the smallest topology that contains the open sets $U_{\mathbf{w}} = \{\omega \in \delta W \mid \omega(|\mathbf{w}|) = \mathbf{w}\}$. Then δW is a compact space. We topologize $W \cup \delta W$ with the smallest topology making the functions $P_{\mathbf{w}} \cup \chi_{U_{\mathbf{w}}}$ continuous (here χ is an indicator function). Note that the topology of δW is then inherited from $W \cup \delta W$.

Let $W \curvearrowright W$ be the action by means of left translation. The action extends continuously to $W \cup \delta W$ and then restricts to an action $W \curvearrowright \delta W$. We may pull back this action to obtain $W \curvearrowright C(\delta W)$. As W is a hyperbolic group the action $W \curvearrowright \delta W$ is well-known to be amenable [BrOz08] which implies that $C(\delta W) \rtimes_u W = C(\delta W) \rtimes_r W$ and furthermore this crossed product is a nuclear C*-algebra. Let $f \in C(\delta W)$, let $\widetilde{f}_1, \widetilde{f}_2 \in C(W \cup \delta W)$ be two continuous extensions of f and let f_1 and f_2 be their respective restrictions to W. Then $f_1 - f_2 \in C_0(W)$. That is, the multiplication map $f_1 - f_2$ acting on $\ell^2(W)$ determines a compact operator. So the assignment $f \mapsto f_1$ is a well-defined $*$-homomorphism $C(\delta W) \to \mathcal{B}(\ell^2(W))/\mathcal{K}$ where $\mathcal{K}$ are the compact operators on $\ell^2(W)$. It is easy to check that this map is W-equivariant and thus we obtain a $*$-homomorpism:

$$\pi_1 : C(\delta W) \rtimes_u W \to \mathcal{B}(\ell^2(W))/\mathcal{K}. \tag{6.2}$$

Let again $W \curvearrowright W$ be the action by means of left translation which may be pulled back to obtain an action $W \curvearrowright \ell^\infty(W)$. Let

$$\rho : \ell^\infty(W) \rtimes_r W \to \mathcal{B}(\ell^2(W))$$

be the σ-weakly continuous $*$-isomorphism determined by $\rho : u_{\mathbf{w}} \mapsto T^{(1)}_{\mathbf{w}}$ and $\rho : \pi(x) \mapsto x$ (see [Vae01, Theorem 5.3]). In fact ρ is an injective map (this follows immediately from [Com11, Theorem 2.1] as the operator G in this theorem equals the mutliplicative unitary/struture operator [Tak03, p. 68]). Let $C_\infty(W)$ be the C*-algebra generated by the projections $P_{\mathbf{w}}, \mathbf{w} \in W$. Take $f \in C_\infty(W)$ and let $\widetilde{f}$ be the continuous extension of f to $W \cup \delta W$. The map $f \mapsto \widetilde{f}|_{\delta W}$ determines a $*$-homomorphism $\sigma : C_\infty(W) \to C(\delta W)$ that is W-equivariant. Therefore it extends to the crossed product map

$$\sigma \rtimes_r \mathrm{Id} : C_\infty(W) \rtimes_r W \to C(\delta W) \rtimes_r W.$$

Theorem 6.4. *The von Neumann algebra $\mathcal{M}_q$ satisfies condition (AO)$^+$.*

Proof. We let A_q be the unital C*-subalgebra of $\mathcal{M}_q$ generated by operators $T_{\mathbf{w}}, \mathbf{w} \in W$. It is easy to see that A_q is preserved by the multipliers that we constructed in order to prove that $\mathcal{M}_q$ had the wk-$*$ CBAP, see Section 5 (indeed these were compositions of radial multipliers (5.6) and word length projections). Therefore A_q has the CBAP, hence by the remarks before [HaKr94, Theorem 2.2] it is exact. Therefore A_q is locally reflexive [BrOz08], [Pis02, Chapter 18].

It remains to prove condition (2) of Definition 6.2. By Lemma 2.7 we see that A_q is contained in the C*-subalgebra of $\mathcal{B}(\ell^2(W))$ generated by the operators $P_{\mathbf{w}}, T^{(1)}_{\mathbf{w}}$ with $\mathbf{w} \in W$. So $\rho^{-1}(\mathsf{A}_q)$ is contained in $C_\infty(W) \rtimes_r W$ and therefore we may set

$$\gamma : \mathsf{A}_q \to C(\delta W) \rtimes_r W \qquad \text{as} \qquad \gamma = (\sigma \rtimes_r \mathrm{Id}) \circ \rho^{-1}.$$

The mapping $\pi_2 : J\mathsf{A}_q J \to \mathcal{B}(\ell^2(W))/\mathcal{K} : b \mapsto b$ is a $*$-homomorphism and its image commutes with the image of π_1 of (6.2) (as was argued in [HiGu04, Lemma 6.2.8]). By definition of the maximal tensor product there exists a $*$-homomorphism:

$$(\pi_1 \otimes \pi_2) : (C(\delta W) \rtimes_u W) \otimes_{\max} J\mathsf{A}_q J \to \mathcal{B}(\ell^2(W))/\mathcal{K} : a \otimes JbJ \mapsto \pi_1(a)JbJ.$$

We may now consider the following composition of $*$-homomorphisms:

$$\begin{array}{ccc} \mathsf{A}_q \otimes_{\min} J\mathsf{A}_q J & \xrightarrow{\gamma\otimes\mathrm{id}} & (C(\delta W) \rtimes_r W) \otimes_{\min} J\mathsf{A}_q J \\ & & \downarrow \simeq \\ \mathcal{B}(\ell^2(W))/\mathcal{K} & \xleftarrow[\pi_1\otimes\pi_2]{} & (C(\delta W) \rtimes_u W) \otimes_{\max} J\mathsf{A}_q J. \end{array} \tag{6.3}$$

By construction this map is given by:

$$a \otimes JbJ \mapsto aJbJ + \mathcal{K}, \qquad \text{where} \quad a, b \in \mathsf{A}_q. \tag{6.4}$$

The map π_1 is nuclear because we already observed that $C(\delta W) \rtimes_u W$ is nuclear. Also π_2 is nuclear as it equals $J(\,\cdot\,)J \circ \pi_1 \circ \gamma \circ J(\,\cdot\,)J$. It therefore follows that the mapping $\pi_1 \otimes \pi_2 : (C(\delta W) \rtimes_r W) \otimes_{\min} JA_qJ \to \mathcal{B}(\ell^2(W))/\mathcal{K}$ in diagram (6.3) is nuclear and we may apply the Choi-Effros lifting theorem [ChEf76] in order to obtain a ucp lift $\theta : (C(\delta W) \rtimes_r W) \otimes_{\min} JA_qJ \to \mathcal{B}(\ell^2(W))$. Then $\theta \circ (\gamma \otimes \mathrm{Id})$ together with (6.4) witness the result. □

Corollary 6.5. *Let (W, S) be a reduced Coxeter system with $|S| \geq 3$ and $q > 0$. Then the Hecke von Neumann algebra $\mathcal{M}_q$ has no Cartan subalgebra.*

Proof. This is a consequence of Theorem 6.3 together with Theorems 4.11, 5.19 and 6.4. □

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