DERIVATIONS OF A LEAVITT PATH ALGEBRA $W(\ell)$

VIKTOR LOPATKIN*

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Abstract

The aim of this paper is to describe all inner and all outer derivations of Leavitt path algebra via explicit formulas.

Introduction

The Leavitt path algebras were developed by Gene Abrams and Arando Pino [1] and Pere Ara, María A. Moreno and Enrique Pardo [2]. These algebras are an algebraic analog of graph Cuntz–Kreiger C^*-algebra.

In [5] has been proved that the Leavitt path algebra and their generalizations are hereditary algebra. It follows that homology vanish in higher degree. Pere Ara and Guillermo Cortiñas [4] calculated the Hochschild homology of Leavitt path algebras. But they used technique of spectral sequences and from this results does not follow the way of explicit formulas derivations of Leavitt path algebra. In this paper we will describe all derivations via explicit formula. We will give full description of the space of all inner and all outer derivations. The main technique is based on the Gröbner–Shirshov basis[6].

The main technique for the describing all derivations is based on the Gröbner–Shirshov basis and the Composition–Diamond Lemma. If the Gröbner–Shirshov basis for an algebra Λ is known, then the basis $\mathfrak{B}_\Lambda$ for this algebra is also known. It allows us to describe any value of any linear map $f : \Lambda \to \Lambda$ as a decomposition $f(\lambda) = \sum\limits_{x\in\mathfrak{B}_\Lambda} \xi_x(\lambda)x$ via basic elements; surely, in the infinite-dimensional case we have to assume that almost all scalers $\xi_x(\lambda)$ are zero. Since the derivation is a linear map which satisfies the Leibnitz rule, we can use the Gröbner–Shirshov basis for finding exactly the needed conditions for the scalers $\xi_x(\lambda)$. It is our main tool in this paper. The Gröbner–Shirshov basis of Leavitt path algebra $L(\Gamma)$ was found in [3].

The general results of this paper are Theorem 3.1 and Theorem 3.2 which describe all inner and all outer derivations of the Leavitt path algebra $W(\ell)$, respectively.

1 Preliminaries

Here we remind the definition of the Leavitt path algebra and the correspondence terminologies.

Derivations. Let $\mathfrak{A}$ be an arbitrary (non-associative) algebra over the ring R. A *derivation* $\mathscr{D}$ in $\mathfrak{A}$ is a linear map $\mathscr{D} : \mathfrak{A} \to \mathfrak{A}$ satisfying to the Leibnitz rule,

$$\mathscr{D}(xy) = \mathscr{D}(x)y + x\mathscr{D}(y)$$

for any $x, y \in \mathfrak{A}$. It follows that any derivation $\mathscr{D}$ is defined on the generators of algebra $\mathfrak{A}$, moreover, let us assume that the basis $\mathfrak{B}_\mathfrak{A}$ of algebra $\mathfrak{A}$ is known, say $\mathfrak{B}_\mathfrak{A} = \{b_j,\ j \in J\}$, then for any generator x, we can put

$$\mathscr{D}(x) = \sum_{b_i\in\mathfrak{B}_\mathfrak{A}} \xi_{b_i}(x)b_i,$$

where $\xi_{b_i}(x) \in R$ are scalers, and almost all of them are zero.

Now let $\mathfrak{A}$ be an associative algebra and let us fix some element $\lambda \in \mathfrak{A}$, *the inner derivation determined by* λ is a linear map $\mathrm{ad}_\lambda : \mathfrak{A} \to \mathfrak{A}$ which is defined for any generators x as follows

$$\mathrm{ad}_\lambda(x) := \lambda x - x\lambda.$$

It allows to define any inner derivation A for any generator x as follows,

$$A(x) = \sum_{\lambda\in\mathfrak{A}} \zeta_\lambda \mathrm{ad}_\lambda(x),$$

here almost all scalers $\zeta_\lambda \in R$ are zero.

*wickktor@gmail.com

Leavitt path algebra $L(\Gamma)$**.** A directed graph $\Gamma = (V, E, \mathfrak{s}, \mathfrak{r})$ consists of two sets V and E, called vertices and edges respectively, and two maps $\mathfrak{s}, \mathfrak{r} : E \to V$ called *source* and *range* (of edge) respectively. The graph is called *row – finite* if for all vertices $v \in V$, $|\mathfrak{s}^{-1}(v)| < \infty$. A vertex v for which $\mathfrak{s}^{-1}(v)$ is empty is called a *sink*. A *path* $p = e_1 \cdots e_\ell$ in a graph Γ is a sequence of the edges $e_1, \dots, e_\ell \in E$ such that $\mathfrak{r}(e_i) = \mathfrak{s}(e_{i+1})$ for $i = 1, \dots, \ell - 1$. In this case we say that the path p *starts* at the vertex $\mathfrak{s}(e_1)$ and *ends* at the vertex $\mathfrak{r}(e_\ell)$, we put $\mathfrak{s}(p) := \mathfrak{s}(e_1)$ and $\mathfrak{r}(p) := \mathfrak{r}(e_\ell)$. We also set $p_0 := e_1$ and $p_z := e_\ell$, further, we will use the following denoting: we set $p/p_0 := p'$ and $p/p_z = p''$, where the paths p', p'' can be defined as $p_0 p' = p$ and $p'' p_z = p$ respectively. We have also assumed that $p = p_0 \in E$ or $p = p_z \in E$ in this case we set $p/p_0 := \mathfrak{r}(p_0)$ and $p/p_z := \mathfrak{s}(p_z)$ respectively.

Definition 1.1. *Let Γ be a row-finite graph, and let R be an associative ring with unit. The Leavitt path R-algebra $L_R(\Gamma)$ (or shortly $L(\Gamma)$) is the R-algebra presented by the set of generators $\{v, v \in V\}$, $\{e, e^* | e \in E\}$ and the set of relations:*

1) $v_i v_j = \delta_{i,j} v_i$, *for all* $v_i, v_j \in V$;
2) $\mathfrak{s}(e)e = e\mathfrak{r}(e) = e$, $\mathfrak{r}(e)e^* = e^*\mathfrak{s}(e) = e^*$, *for all* $e \in E$;
3) $e^* f = \delta_{e,f}\mathfrak{r}(e)$, *for all* $e, f \in E$;
4) $v = \sum\limits_{\mathfrak{s}(e)=v} ee^*$, *for an arbitrary vertex* $v \in V \setminus \{\text{sinks}\}$.

For an arbitrary vertex $v \in V$ which is not a sink, choose and edge $\widetilde{e} \in E$ such that $\mathfrak{s}(\widetilde{e}) = v$. We will refer to this edge as *special*. In other words, we fix a function $\widetilde{?} : V \setminus \{\text{sinks}\} \to E$ such that $\mathfrak{s}(\widetilde{e}) = v$ for an arbitrary $v \in V \setminus \{\text{sinks}\}$.

In [3] the Gröbner–Shirshov basis has been obtained with respect to the order $<$ on the set of generators $X = V \cup E \cup E^*$. This order is defined as follows: chose an arbitrary well-ordering on the set of vertices V. If e, f are edges and $\mathfrak{s}(e) < \mathfrak{s}(f)$, then $e < f$. It remains to order the edges that have the same source. Let v be a vertex which is not a sink. Let $e_1, \dots, e_\ell$ be all the edges that originate from v. Suppose e_1 is a special edge. We order the edges as follows: $e_1 > e_2 > \dots > e_\ell$. Choose an arbitrary well-ordering on the set E^*. For arbitrary elements $v \in V$, $e \in E^*$, $f^* \in E^*$, we let $v < e < f^*$. Thus the set $X = V \cup E \cup E^*$ is well-ordered.

Theorem 1.1. *[3, Theorem 1] The following elements form a basis of the Leavitt path algebra* $L(\Gamma)$:

(i) *the set of all vertexes* V,
(ii) *the set of all paths* $\mathfrak{P}$,
(iii) *the set* $\mathfrak{P}^* := \{p^* : p \in \mathfrak{P}\}$,
(iv) *a set* $\mathfrak{M}$ *of words of form* wh^*, *where* $w = e_1 \cdots e_n \in \mathfrak{P}$, $h^* = (f_1 \cdots f_m)^* = f_m^* \cdots f_1^* \in \mathfrak{P}^*$, $e_i, f_j \in E$, *are paths that end at the same vertex* $\mathfrak{r}(w) = \mathfrak{r}(h)$, *with the condition that the edges* e_n *and* f_m *are either distinct or equal, but not special.*

Let us describe the Gröbner–Shirshov basis for the Leavitt path algebra $L(\Gamma)$:

$$
\begin{aligned}
& vu = \delta_{v,u} v, \\
& ve = \delta_{v,\mathfrak{s}(e)} e, \quad ev = \delta_{v,\mathfrak{r}(e)} e, \\
& ve^* = \delta_{v,\mathfrak{r}(e)} e^*, \quad e^* v = \delta_{v,\mathfrak{s}(e)} e^*, \\
& ef^* = \delta_{e,f}\mathfrak{r}(e), \quad \sum_{\mathfrak{s}(e)=v} ee^* = v, \\
& ef = \delta_{\mathfrak{r}(e),\mathfrak{s}(f)} ef, \quad e^* f^* = \delta_{\mathfrak{r}(f),\mathfrak{s}(e)} e^* f^*, \quad ef^* = \delta_{\mathfrak{r}(e),\mathfrak{r}(f)} ef^*,
\end{aligned}
$$

here $v, u \in V$ and $e, f \in E$.

Leavitt path algebra $W(\ell)$**.** Let $\ell \geq 1$, and let us consider the graph O_ℓ with ℓ loops (see fig.1). The correspondence Leavitt path algebra $L(O_\ell)$ is denoted by $W(\ell)$.

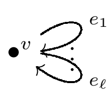

Figure 1: Here the graph O_ℓ is shown. The correspondence Leavitt path algebra is denoted by $W(\ell)$.

Let us denote by Ω the set of all paths of the graph O_ℓ. Then, for the Leavitt path algebra $W(\ell)$, the Theorem 1.1 has the following form

Theorem 1.2. *The following elements form a basis of the Leavitt path algebra $W(\ell)$:*

(i) the vertex $\{v\}$,

(ii) *the set of all paths* Ω,

(iii) *the set* $\{p^* : p \in \Omega\}$,

(iv) *a set* $\mathfrak{M}$ *of words of form* wh^*, *where* $w = e_1 \cdots e_n \in \mathfrak{P}$, $h^* = (f_1 \cdots f_m)^* = f_m^* \cdots f_1^* \in \mathfrak{P}^*$, $e_i, f_j \in E$, *are paths that end at the same vertex* $\mathfrak{r}(w) = \mathfrak{r}(h)$, *with the condition that the if edges* e_n *and* f_m *are either distinct or equal, but not special.*

The Gröbner–Shirshov basis of the Leavitt path algebra $W(\ell)$, *can be described as follows,*

$$vv = v, \tag{1.1}$$

$$ve = ev = e, \qquad e \in E, \tag{1.2}$$

$$ve^* = e^*v = e^*, \qquad e^* \in E^*, \tag{1.3}$$

$$e^*f = \delta_{e,f}v \qquad e, f \in E, \tag{1.4}$$

$$\sum_{i=1}^{\ell} e_i e_i^* = v, \qquad e_1, \ldots, e_\ell \in E. \tag{1.5}$$

Proposition 1.1. *Any derivation* $\mathscr{D}$ *of the Leavitt path algebra* $W(\ell)$ *satisfies the following equations*

$$\begin{aligned}
&\mathscr{D}(v)v + v\mathscr{D}(v) = \mathscr{D}(v), \\
&\mathscr{D}(v)e + v\mathscr{D}(e) = \mathscr{D}(e)v + e\mathscr{D}(v) = \mathscr{D}(e), && e \in E, \\
&\mathscr{D}(v)e^* + v\mathscr{D}(e^*) = \mathscr{D}(e^*)v + e^*\mathscr{D}(v) = \mathscr{D}(e^*), && e^* \in E^*, \\
&\mathscr{D}(e^*)f + e^*\mathscr{D}(f) = \delta_{e,f}\mathscr{D}(v), && e, f \in E, \\
&\sum_{i=1}^{\ell}\Big(\mathscr{D}(e_i)e_i^* + e_i\mathscr{D}(e_i^*)\Big) = \mathscr{D}(v), && e_1, \ldots, e_\ell \in E.
\end{aligned}$$

Proof. It immediately follows from the definition of derivation and the Theorem 1.2. □

Remark 1.1. *Without loss of generality we have assumed that* e_1 *is the special edge.*

2 Derivations of the Leavitt path algebra

Since the basis of the Leavitt path algebra $W(\ell)$ has been described, then any value $\mathscr{D}(x)$ of any linear map $\mathscr{D} : W(\ell) \to W(\ell)$, we can present as

$$\mathscr{D}(x) = \alpha_v(x)v + \sum_{p\in\Omega^*}\Big(\beta_p(x)p + \gamma_p(x)p^*\Big) + \sum_{wh^*\in\mathfrak{M}} \rho_{wh^*}(x)wh^*$$

here $x \in \{v\} \cup E \cup E^*$ and almost all scalers $\alpha, \beta, \gamma, \rho \in R$ are zero.

Proposition 2.1. *Let* $\mathscr{D}$ *be a derivation of Leavitt path algebra* $W(\ell)$, *then* $\mathscr{D}(v) = 0$.

Proof. Using the equation $\mathscr{D}(v)v + v\mathscr{D}(v) = \mathscr{D}(v)$, we get

$$\begin{aligned}
\mathscr{D}(v)v \quad &= \quad \left(\alpha_v(v)v + \sum_{p\in\Omega^*}\Big(\beta_p(v)p + \gamma_p(v)p^*\Big) + \sum_{wh^*\in\mathfrak{M}} \rho_{wh^*}wh^*\right)v = \\
&= \alpha_v(v)v + \sum_{p\in\Omega^*}\Big(\beta_p(v)p + \gamma_p(v)p^*\Big) + \sum_{wh^*\in\mathfrak{M}} \rho_{wh^*}(v)wh^*
\end{aligned}$$

and

$$\begin{aligned}
v\mathscr{D}(v) \quad &= \quad v\left(\alpha_v(v)v + \sum_{p\in\Omega^*}\Big(\beta_p(v)p + \gamma_p(v)p^*\Big) + \sum_{wh^*\in\mathfrak{M}} \rho_{wh^*}(v)wh^*\right) = \\
&= \alpha_v(v)v + \sum_{p\in\Omega^*}\Big(\beta_p(x)p + \gamma_p(x)p^*\Big) + \sum_{wh^*\in\mathfrak{M}} \rho_{wh^*}wh^*
\end{aligned}$$

it follows $\mathscr{D}(v) = 0$, as claimed. □

It follows that we can rewrite all equations in the following form

$$\begin{aligned}
&v\mathscr{D}(e)v = \mathscr{D}(e) \\
&v\mathscr{D}(e^*)v = \mathscr{D}(e^*) \\
&\mathscr{D}(e_i^*)e_j + e_i^*\mathscr{D}(e_j) = 0, \quad 1 \le i, j \le \ell \\
&\sum_{i=1}^{\ell}\Big(\mathscr{D}(e_i)e_i^* + e_i\mathscr{D}(e_i^*)\Big) = 0.
\end{aligned}$$

Theorem 2.1. *Any derivation $\mathscr{D}$ of the Leavitt path algebra $W(\ell)$ can be described as*

$$\mathscr{D}(x) = \begin{cases} 0, \ \text{if } x = v, \\ \alpha_v(x)v + \sum\limits_{p\in\Omega}\Big(\beta_p(x)p + \gamma_p(x)p^*\Big) + \sum\limits_{wh^*\in\mathfrak{M}} \rho_{wh^*}(x)wh^*, \ \text{if } x \in E\cup E^*, \end{cases}$$

where almost all coefficients $\alpha(x), \beta(x), \gamma(x), \rho(x) \in R$ are zero and they satisfy the following equations

$$\gamma_{e_j}(e_i^*) + \beta_{e_i}(e_j) = 0, \tag{2.6}$$

$$\beta_p(e_i^*) + (1-\delta_{1,j})\rho_{pe_je_j^*}(e_i^*) + \beta_{e_ipe_j}(e_i) = 0, \qquad p\in\Omega, \tag{2.7}$$

$$\rho_{pe_j^*}(e_i^*) + \beta_{e_ip}(e_j) = 0, \qquad p\in\Omega,\ p_z\neq e_j, \tag{2.8}$$

$$\alpha_v(e_i^*) + (1-\delta_{1,j})\rho_{e_je_j^*}(e_i^*) + \beta_{e_ie_j}(e_j) = 0, \tag{2.9}$$

$$\gamma_{e_jpe_i}(e_i^*) + \gamma_p(e_i) + (1-\delta_{1,i})\rho_{e_i(pe_i)^*}(e_j) = 0, \qquad p\in\Omega, \tag{2.10}$$

$$\gamma_{e_jp}(e_i^*) + \rho_{e_ip^*}(e_j) = 0, \qquad p\in\Omega,\ p_z\neq e_i, \tag{2.11}$$

$$\alpha_v(e_j) + \gamma_{e_je_i}(e_i^*) + (1-\delta_{1,i})\rho_{e_ie_i^*}(e_j) = 0, \tag{2.12}$$

$$\rho_{w(e_jh)^*}(e_i^*) + \rho_{e_iwh^*}(e_j) = 0, \qquad wh^*\in\mathfrak{M}, \tag{2.13}$$

for any $1\le i,j\le\ell$.

Proof. We have

$$\begin{aligned} v\mathscr{D}(e)v = \mathscr{D}(e) \quad &= \quad v\left(\alpha_v(e)v + \sum_{p\in\Omega^*}\Big(\beta_p(e)p + \gamma_p(e)p^*\Big) + \sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e)wh^*\right)v = \\ &= \alpha_v(e) + \sum_{p\in\Omega^*}\Big(\beta_p(e)p + \gamma_p(e)p^*\Big) + \sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e)wh^*, \end{aligned}$$

further,

$$\begin{aligned} v\mathscr{D}(e^*)v = \mathscr{D}(e^*) \quad &= \quad v\left(\alpha_v(e^*)v + \sum_{p\in\Omega^*}\Big(\beta_p(e^*)p + \gamma_p(e^*)p^*\Big) + \sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e^*)wh^*\right)v = \\ &= \alpha_v(e)v + \sum_{p\in\Omega^*}\Big(\beta_p(e^*)p + \gamma_p(e^*)p^*\Big) + \sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e^*)wh^*. \end{aligned}$$

Let us consider for any $1\le i,j\le\ell$ the equations $\mathscr{D}(e_i^*)e_j + e_i^*\mathscr{D}(e_j) = 0$, we have

$$\begin{aligned} \mathscr{D}(e_i^*)e_j \quad &= \quad \left(\alpha_v(e_i^*)v + \sum_{p\in\Omega^*}\Big(\beta_p(e_i^*)p + \gamma_p(e_i^*)p^*\Big) + \sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e_i^*)wh^*\right)e_j = \\ &= \quad \alpha_v(e_i^*)e_j + \sum_{p\in\Omega^*}\beta_p(e_i^*)pe_j + \sum_{p\in\Omega^*}\gamma_{e_jp}(e_i^*)p^* + \gamma_{e_j}(e_i^*)v + \\ &\quad + \sum_{we_j^*\in\mathfrak{M}}\rho_{we_j^*}(e_i^*)w + \sum_{wh^*\in\mathfrak{M}}\rho_{w(e_jh)^*}(e_i^*)wh^*, \end{aligned}$$

and

$$\begin{aligned} e_i^*\mathscr{D}(e_j) \quad &= \quad e_i^*\left(\alpha_v(e_j)v + \sum_{p\in\Omega^*}\Big(\beta_p(e_j)p + \gamma_p(e_j)p^*\Big) + \sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e_j)wh^*\right) = \\ &= \quad \alpha_v(e_j)e_i^* + \beta_{e_i}(e_j)v + \sum_{p\in\Omega^*}\beta_{e_ip}(e_j)p + \sum_{p\in\Omega^*}\gamma_p(e_j)(pe_i)^* + \\ &\quad + \sum_{e_ih^*\in\mathfrak{M}}\rho_{e_ih^*}(e_j)h^* + \sum_{wh^*\in\mathfrak{M}}\rho_{e_iwh^*}(e_j)wh^*. \end{aligned}$$

Let us add up similar terms,

$$\mathscr{D}(e_i^*)e_j + e_i^*\mathscr{D}(e_j)\Big|_V \quad = \quad \gamma_{e_j}(e_i^*)v + \beta_{e_i}(e_j)v,$$

$$
\begin{aligned}
\mathscr{D}(e_i^*)e_j + e_i^*\mathscr{D}(e_j)\Big|_{\Omega} &= \alpha_v(e_i^*)e_j + \sum_{p\in\Omega}\beta_p(e_i^*)pe_j + \sum_{we_j^*\in\mathfrak{M}}\rho_{we_j^*}(e_i^*)w + \sum_{p\in\Omega}\beta_{e_ip}(e_j)p = \\
&= \sum_{p\in\Omega}\beta_p(e_i^*)pe_j + \sum_{pe_je_j^*\in\mathfrak{M}}\rho_{pe_je_j^*}(e_i^*)pe_j + \sum_{we_j^*\in\mathfrak{M},\, w_z\neq e_j}\rho_{we_j^*}(e_i^*)w + \\
&\quad +\alpha_v(e_i^*)e_j + (1-\delta_{1,j})\rho_{e_je_j^*}(e_i^*)e_j + \beta_{e_ie_j}(e_j)e_j + \\
&\quad +\sum_{p\in\Omega}\beta_{e_ipe_j}(e_j)pe_j + \sum_{p\in\Omega,\, p_z\neq e_j}\beta_{e_ip}(e_j)p = \\
&= \sum_{p\in\Omega}\Big(\beta_p(e_i^*) + (1-\delta_{1,j})\rho_{pe_je_j^*}(e_i^*) + \beta_{e_ipe_j}(e_j)\Big)pe_j + \\
&\quad + \sum_{p\in\Omega,\, p_z\neq e_j}\Big(\rho_{pe_j^*}(e_i^*) + \beta_{e_ip}(e_j)\Big)p + \\
&\quad +\Big(\alpha_v(e_i^*) + (1-\delta_{1,j})\rho_{e_je_j^*}(e_i^*) + \beta_{e_ie_j}(e_j)\Big)e_j,
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{D}(e_i^*)e_j + e_i^*\mathscr{D}(e_j)\Big|_{\Omega^*} &= \alpha_v(e_j)e_i^* + \sum_{p\in\Omega}\gamma_{e_jp}(e_i^*)p^* + \sum_{p\in\Omega}\gamma_p(e_i)(pe_i)^* + \sum_{e_ih^*\in\mathfrak{M}}\rho_{e_ih^*}(e_j)h^* = \\
&= \alpha_v(e_j)e_i^* + \gamma_{e_je_i}(e_i^*)e_i^* + \sum_{p\in\Omega}\gamma_{e_jpe_i}(e_i^*)(pe_i)^* + \sum_{p\in\Omega,\, p_z\neq e_i}\gamma_{e_jp}(e_i^*)p^* + \\
&\quad +\sum_{p\in\Omega}\gamma_p(e_i)(pe_i)^* + (1-\delta_{1,i})\rho_{e_ie_i^*}(e_j)e_i^* + \sum_{e_ie_i^*p^*\in\mathfrak{M}}\rho_{e_i(pe_i)^*}(e_j)(pe_i)^* + \\
&\quad + \sum_{e_ih^*\in\mathfrak{M},\, h_z\neq e_i}\rho_{eh^*}(e_j)h^* = \\
&= \sum_{p\in\Omega}\Big(\gamma_{e_jpe_i}(e_i^*) + \gamma_p(e_i) + (1-\delta_{i,1})\rho_{e_i(pe_i)^*}(e_j)\Big)(pe_i)^* + \\
&\quad + \sum_{p\in\Omega,\, p_z\neq e_i}\Big(\gamma_{e_jp}(e_i^*) + \rho_{e_ip^*}(e_j)\Big)p^* + \\
&\quad +\Big(\alpha_v(e_j) + \gamma_{e_je_i}(e_i^*) + (1-\delta_{1,i})\rho_{e_ie_i^*}(e_j)\Big)e_i^*,
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{D}(e_i^*)e_j + e_i^*\mathscr{D}(e_j)\Big|_{\mathfrak{M}} &= \sum_{wh^*\in\mathfrak{M}}\rho_{w(e_jh)^*}(e_i^*)wh^* + \sum_{wh^*\in\mathfrak{M}}\rho_{e_iwh^*}(e_j)wh^* = \\
&= \sum_{wh^*\in\mathfrak{M}}\Big(\rho_{w(e_jh)^*}(e_i^*) + \rho_{e_iwh^*}(e_j)\Big)wh^*.
\end{aligned}
$$

So, for any $1\leq i,j\leq \ell$, we have the following equations

$$
\begin{aligned}
&\gamma_{e_j}(e_i^*) + \beta_{e_i}(e_j) = 0,\\
&\beta_p(e_i^*) + (1-\delta_{1,j})\rho_{pe_je_j^*}(e_i^*) + \beta_{e_ipe_j}(e_i) = 0, \qquad p\in\Omega,\\
&\rho_{pe_j^*}(e_i^*) + \beta_{e_ip}(e_j) = 0, \qquad p\in\Omega,\ p_z\neq e_j,\\
&\alpha_v(e_i^*) + (1-\delta_{1,j})\rho_{e_je_j^*}(e_i^*) + \beta_{e_ie_j}(e_j) = 0,\\
&\gamma_{e_jpe_i}(e_i^*) + \gamma_p(e_i) + (1-\delta_{1,i})\rho_{e_i(pe_i)^*}(e_j) = 0, \qquad p\in\Omega,\\
&\gamma_{e_jp}(e_i^*) + \rho_{e_ip^*}(e_j) = 0, \qquad p\in\Omega,\ p_z\neq e_i,\\
&\alpha_v(e_j) + \gamma_{e_je_i}(e_i^*) + (1-\delta_{1,i})\rho_{e_ie_i^*}(e_j) = 0,\\
&\rho_{w(e_jh)^*}(e_i^*) + \rho_{e_iwh^*}(e_j) = 0, \qquad wh^*\in\mathfrak{M}.
\end{aligned}
$$

In other hand, let us consider the equation $\sum_{r=1}^{\ell}\Big(\mathscr{D}(e_r)e_r^* + e_r\mathscr{D}(e_r^*)\Big) = 0$. For the special edge e_1, we have

$$
\begin{aligned}
\mathscr{D}(e_1)e_1^* &= \left(\alpha_v(e_1)v + \sum_{p\in\Omega^*}\Big(\beta_p(e_1)p + \gamma_p(e_1)p^*\Big) + \sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e_1)wh^*\right)e_1^* = \\
&= \alpha_v(e_1)e_1^* + \beta_{e_1}(e_1)v - \sum_{k=2}^{\ell}\beta_{e_1}(e_1)e_ke_k^* + \sum_{p\in\Omega}\beta_{pe_1}(e_1)p - \sum_{p\in\Omega}\sum_{k=2}^{\ell}\beta_{pe_1}(e_1)pe_ke_k^* + \\
&\quad + \sum_{p\in\Omega,\, p_z\neq e_1}\beta_p(e_1)pe_1^* + \sum_{p\in\Omega}\gamma_p(e_1)(e_1p)^* + \sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e_1)w(e_1h)^*,
\end{aligned}
$$

and

$$\begin{aligned} e_1\mathscr{D}(e_1^*) &= e_1\left(\alpha_v(e_1^*)v + \sum_{p\in\Omega^*}\Big(\beta_p(e_1^*)p + \gamma_p(e_1^*)p^*\Big) + \sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e_1^*)wh^*\right) = \\ &= \alpha_v(e_1^*)e_1 + \sum_{p\in\Omega}\beta_p(e_1^*)e_1p + \gamma_{e_1}(e_1^*)v - \sum_{k=2}^{\ell}\gamma_{e_1}(e_1^*)e_ke_k^* + \sum_{p\in\Omega}\gamma_{pe_1}(e_1^*)p^* - \\ &\quad -\sum_{p\in\Omega}\sum_{k=2}^{\ell}\gamma_{pe_1}(e_1^*)e_ke_k^*p^* + \sum_{p\in\Omega,\,p_z\neq e_1}\gamma_p(e_1^*)e_1p^* + \sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e_1^*)e_1wh^*, \end{aligned}$$

further, we have

$$\begin{aligned} \mathscr{D}(e_r)e_r^* &= \left(\alpha_v(e_r)v + \sum_{p\in\Omega^*}\Big(\beta_p(e_r)p + \gamma_p(e_r)p^*\Big) + \sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e_r)wh^*\right)e_r^* = \\ &= \alpha_v(e_r)e_r^* + \sum_{p\in\Omega}\beta_p(e_r)pe_r^* + \sum_{p\in\Omega}\gamma_p(e_r)(e_rp)^* + \sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e_r)w(e_rh)^*, \end{aligned}$$

and

$$\begin{aligned} e_r\mathscr{D}(e_r^*) &= e_r\left(\alpha_v(e_r^*)v + \sum_{p\in\Omega}\Big(\beta_p(e_r^*)p + \gamma_p(e_r^*)p^*\Big) + \sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e_r^*)wh^*\right) = \\ &= \alpha_v(e_r^*)e_r + \sum_{p\in\Omega}\beta_p(e_r^*)e_rp + \sum_{p\in\Omega}\gamma_p(e_r^*)e_rp^* + \sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e_r^*)e_rwh^*. \end{aligned}$$

We get

$$\sum_{r=1}^{\ell}\Big(\mathscr{D}(e_r)e_r^* + e_r\mathscr{D}(e_r^*)\Big)\Big|_V = \beta_{e_1}(e_1)v + \gamma_{e_1}(e_1^*)v = 0,$$

it follows $\beta_{e_1}(e_1)v + \gamma_{e_1}(e_1^*) = 0$, it is the equation (2.6). Further,

$$\begin{aligned} \sum_{r=1}^{\ell}\Big(\mathscr{D}(e_r)e_r^* + e_r\mathscr{D}(e_r^*)\Big)\Big|_\Omega &= \sum_{r=1}^{\ell}\alpha_v(e_r^*)e_r + \sum_{p\in\Omega}\beta_{pe_1}(e_1)p + \sum_{r=1}^{\ell}\sum_{p\in\Omega}\beta_p(e_r^*)e_rp = \\ &= \sum_{r=1}^{\ell}\Big(\alpha_v(e_r^*) + \beta_{e_re_1}(e_1)\Big)e_r + \sum_{r=1}^{\ell}\sum_{p\in\Omega}\Big(\beta_{e_rpe_1}(e_1) + \beta_p(e_r^*)\Big)e_rp = \\ &= 0, \end{aligned}$$

it follows

$$\alpha_v(e_r^*) + \beta_{e_re_1}(e_1) = 0, \qquad \beta_{e_rpe_1}(e_1) + \beta_p(e_r^*) = 0,$$

these equations have been already found (see (2.9) and (2.7) respectively). Further,

$$\begin{aligned} \sum_{r=1}^{\ell}\Big(\mathscr{D}(e_r)e_r^* + e_r\mathscr{D}(e_r^*)\Big)\Big|_{\Omega^*} &= \sum_{r=1}^{\ell}\alpha_v(e_r)e_r^* + \sum_{p\in\Omega}\gamma_{pe_1}(e_1^*)p^* + \sum_{r=1}^{\ell}\sum_{p\in\Omega}\gamma_p(e_r)(e_rp)^* = \\ &= \sum_{r=1}^{\ell}\Big(\alpha_v(e_r) + \gamma_{e_re_1}(e_1^*)\Big)e_r^* + \sum_{r=1}^{\ell}\sum_{p\in\Omega}\Big(\gamma_{e_rpe_1}(e_1^*) + \gamma_p(e_r)\Big)(e_rp)^* = \\ &= 0, \end{aligned}$$

then we get

$$\alpha_v(e_r) + \gamma_{e_re_1}(e_1^*)e_r^* = 0, \qquad \gamma_{e_rpe_1}(e_1^*) + \gamma_p(e_r) = 0,$$

these are the equations (2.12) and (2.10) respectively.

$$\begin{aligned} \sum_{r=1}^{\ell}\Big(\mathscr{D}(e_r)e_r^* + e_r\mathscr{D}(e_r^*)\Big)\Big|_{\mathfrak{M}} &= -\sum_{k=2}^{\ell}\beta_{e_1}(e_1)e_ke_k^* - \sum_{p\in\Omega}\sum_{k=2}^{\ell}\beta_{pe_1}(e_1)pe_ke_k^* + \sum_{p\in\Omega,\,p_z\neq e_1}\beta_p(e_1)pe_1^* + \\ &\quad + \sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e_1)w(e_1h)^* - \sum_{k=2}^{\ell}\gamma_{e_1}(e_1^*)e_ke_k^* - \sum_{p\in\Omega}\sum_{k=2}^{\ell}\gamma_{pe_1}(e_1^*)e_ke_k^*p^* + \\ &\quad + \sum_{p\in\Omega,\,p_z\neq e_1}\gamma_p(e_1^*)e_1p^* + \sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e_1^*)e_1wh^* + \sum_{r=2}^{\ell}\sum_{p\in\Omega}\beta_p(e_r)pe_r^* + \\ &\quad + \sum_{r=2}^{\ell}\sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e_r)w(e_rh)^* + \sum_{r=2}^{\ell}\sum_{p\in\Omega}\gamma_p(e_r^*)e_rp^* + \sum_{r=2}^{\ell}\sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e_r^*)e_rwh^*, \end{aligned}$$

Let us add up similar terms,

$$\begin{aligned}
\sum_{r=1}^{\ell}\Big(\mathscr{D}(e_r)e_r^*+e_r\mathscr{D}(e_r^*)\Big)\Big|_{\mathfrak{M}} &= -\sum_{k=2}^{\ell}\Big(\beta_{e_1}(e_1)+\gamma_{e_1}(e_1^*)\Big)e_ke_k^*+\sum_{r=1}^{\ell}\sum_{pe_r^*\in\mathfrak{M}}\beta_p(e_r)pe_r^*+\sum_{r=1}^{\ell}\sum_{e_rp^*\in\mathfrak{M}}\gamma_p(e_r^*)e_rp^*-\\
&\quad-\sum_{k=2}^{\ell}\sum_{p\in\Omega}\beta_{pe_1}(e_1)pe_ke_k^*-\sum_{k=2}^{\ell}\sum_{p\in\Omega}\gamma_{pe_1}(e_1^*)e_k(pe_k)^*+\\
&\quad+\sum_{r=1}^{\ell}\sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e_r)w(e_rh)^*+\sum_{r=1}^{\ell}\sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(e_r^*)e_rwh^*=\\
&= \sum_{k=2}^{\ell}\Big(-\beta_{e_1}(e_1)-\gamma_{e_1}(e_1)+\beta_{e_k}(e_k)+\gamma_{e_k}(e_k^*)\Big)e_ke_k^*+\\
&\quad+\sum_{r=1}^{\ell}\sum_{k=1,k\neq r}^{\ell}\Big(\beta_{e_k}(e_r)+\gamma_{e_r}(e_k^*)\Big)e_ke_r^*+\sum_{r=1}^{\ell}\sum_{pe_r^*\in\mathfrak{M},\,p\notin E}\beta_p(e_r)pe_r^*+\\
&\quad+\sum_{r=1}^{\ell}\sum_{e_rp^*\in\mathfrak{M},\,p\notin E}\gamma_p(e_r^*)e_rp^*-\sum_{k=2}^{\ell}\sum_{p\in\Omega}\beta_{pe_1}(e_1)pe_ke_k^*-\sum_{k=2}^{\ell}\sum_{p\in\Omega}\gamma_{pe_1}(e_1^*)e_k(pe_k)^*+\\
&\quad+\sum_{k=1}^{\ell}\sum_{r=1}^{\ell}\sum_{wh^*\in\mathfrak{M}}\Big(\rho_{e_kwh^*}(e_r)+\rho_{w(e_rh)^*}(e_k^*)\Big)e_kw(e_rh)^*+\\
&\quad+\sum_{k=1}^{\ell}\sum_{r=1}^{\ell}\sum_{e_kh^*\in\mathfrak{M}}\rho_{e_kh^*}(e_r)e_k(e_rh)^*+\sum_{k=1}^{\ell}\sum_{r=1}^{\ell}\sum_{we_k^*\in\mathfrak{M}}\rho_{we_k^*}(e_r^*)e_rwe_k^*=\\
&= 0,
\end{aligned}$$

let us consider a following sum

$$\begin{aligned}
S &= \sum_{r=1}^{\ell}\sum_{pe_r^*\in\mathfrak{M},\,p\notin E}\beta_p(e_r)pe_r^*+\sum_{r=1}^{\ell}\sum_{e_rp^*\in\mathfrak{M},\,p\notin E}\gamma_p(e_r^*)e_rp^*-\\
&\quad-\sum_{k=2}^{\ell}\sum_{p\in\Omega}\beta_{pe_1}(e_1)pe_ke_k^*-\sum_{k=2}^{\ell}\sum_{p\in\Omega}\gamma_{pe_1}(e_1^*)e_k(pe_k)^*+\\
&\quad+\sum_{k=1}^{\ell}\sum_{r=1}^{\ell}\sum_{e_kh^*\in\mathfrak{M}}\rho_{e_kh^*}(e_r)e_k(e_rh)^*+\sum_{k=1}^{\ell}\sum_{r=1}^{\ell}\sum_{we_k^*\in\mathfrak{M}}\rho_{we_k^*}(e_r^*)e_rwe_k^*,
\end{aligned}$$

we have

$$\begin{aligned}
S &= \sum_{r=1}^{\ell}\sum_{pe_r^*\in\mathfrak{M},p\notin E}\beta_p(e_r)pe_r^*-\sum_{k=2}^{\ell}\sum_{p\in\Omega}\beta_{pe_1}(e_1)pe_ke_k^*+\\
&\quad+\sum_{k=1}^{\ell}\sum_{r=1}^{\ell}\sum_{we_k^*\in\mathfrak{M}}\rho_{we_k^*}(e_r^*)e_rwe_k^*+\sum_{r=1}^{\ell}\sum_{e_rp^*\in\mathfrak{M},\,p\notin E}\gamma_p(e_r^*)e_rp^*-\\
&\quad-\sum_{k=2}^{\ell}\sum_{p\in\Omega}\gamma_{pe_1}(e_1^*)e_k(pe_k)^*+\sum_{k=1}^{\ell}\sum_{r=1}^{\ell}\sum_{e_kh^*\in\mathfrak{M}}\rho_{e_kh^*}(e_r)e_k(e_rh)^*=\\
&= \sum_{r=1}^{\ell}\sum_{k=2}^{\ell}\Big(\beta_{e_re_k}(e_k)-\beta_{e_re_1}(e_1)+\rho_{e_ke_k^*}(e_r)\Big)e_re_ke_k^*+\sum_{k=1}^{\ell}\sum_{r=1}^{\ell}\sum_{p\in\Omega,\,p_z\neq e_k}\Big(\beta_{e_rp}(e_k)+\rho_{pe_k^*}(e_r^*)\Big)e_rpe_k^*+\\
&\quad+\sum_{r=1}^{\ell}\sum_{k=2}^{\ell}\sum_{p\in\Omega}\Big(\beta_{e_rpe_k}(e_k)-\beta_{e_rpe_1}(e_1)+\rho_{pe_ke_k^*}(e_r^*)\Big)e_rpe_ke_k^*+\\
&\quad+\sum_{r=1}^{\ell}\sum_{k=2}^{\ell}\Big(\gamma_{e_re_k}(e_k^*)-\gamma_{e_re_1}(e_1^*)+\rho_{e_ke_k^*}(e_r)\Big)e_k(e_re_k)^*+\sum_{r=1}^{\ell}\sum_{k=1}^{\ell}\sum_{p\in\Omega,\,p_z\neq e_k}\Big(\gamma_{e_rp}(e_k^*)+\rho_{e_kp^*}(e_r)\Big)e_k(e_rp)^*+\\
&\quad+\sum_{r=1}^{\ell}\sum_{k=2}^{\ell}\sum_{p\in\Omega}\Big(\gamma_{e_rpe_k}(e_k^*)-\gamma_{e_rp}(e_1^*)+\rho_{e_k(pe_k)^*}(e_r)\Big)e_k(e_rpe_k)^*.
\end{aligned}$$

So, we have

$$\begin{aligned}
\sum_{r=1}^{\ell}\Big(\mathscr{D}(e_r)e_r^*+e_r\mathscr{D}(e_r^*)\Big)\Big|_{\mathfrak{M}} &= \sum_{k=2}^{\ell}\Big(-\beta_{e_1}(e_1)-\gamma_{e_1}(e_1)+\beta_{e_k}(e_k)+\gamma_{e_k}(e_k^*)\Big)e_ke_k^*+\\
&+\sum_{r=1}^{\ell}\sum_{k=1,k\neq r}^{\ell}\Big(\beta_{e_k}(e_r)+\gamma_{e_r}(e_k^*)\Big)e_ke_r^*+\\
&+\sum_{r=1}^{\ell}\sum_{k=2}^{\ell}\Big(\beta_{e_re_k}(e_k)-\beta_{e_re_1}(e_1)+\rho_{e_ke_k^*}(e_r)\Big)e_re_ke_k^*+\\
&+\sum_{k=1}^{\ell}\sum_{r=1}^{\ell}\sum_{p\in\Omega,\,p_z\neq e_k}\Big(\beta_{e_rp}(e_k)+\rho_{pe_k^*}(e_r^*)\Big)e_rpe_k^*+\\
&+\sum_{r=1}^{\ell}\sum_{k=2}^{\ell}\sum_{p\in\Omega}\Big(\beta_{e_rpe_k}(e_k)-\beta_{e_rpe_1}(e_1)+\rho_{pe_ke_k^*}(e_r^*)\Big)e_rpe_ke_k^*+\\
&+\sum_{r=1}^{\ell}\sum_{k=2}^{\ell}\Big(\gamma_{e_re_k}(e_k^*)-\gamma_{e_re_1}(e_1^*)+\rho_{e_ke_k^*}(e_r)\Big)e_k(e_re_k)^*+\\
&+\sum_{r=1}^{\ell}\sum_{k=1}^{\ell}\sum_{p\in\Omega,\,p_z\neq e_k}\Big(\gamma_{e_rp}(e_k^*)+\rho_{e_kp^*}(e_r)\Big)e_k(e_rp)^*+\\
&+\sum_{r=1}^{\ell}\sum_{k=2}^{\ell}\sum_{p\in\Omega}\Big(\gamma_{e_rpe_k}(e_k^*)-\gamma_{e_rp}(e_1^*)+\rho_{e_k(pe_k)^*}(e_r)\Big)e_k(e_rpe_k)^*+\\
&+\sum_{k=1}^{\ell}\sum_{r=1}^{\ell}\sum_{wh^*\in\mathfrak{M}}\Big(\rho_{e_kwh^*}(e_r)+\rho_{w(e_rh)^*}(e_k^*)\Big)e_kw(e_rh)^*=\\
&= 0.
\end{aligned}$$

Let us consider the correspondence equations.

(1) Let us consider the equations

$$-\beta_{e_1}(e_1)-\gamma_{e_1}(e_1)+\beta_{e_k}(e_k)+\gamma_{e_k}(e_k^*)=0,\quad 2\leq k\leq\ell,$$

it is the equations (2.6).

(2) Let us consider the equation

$$\beta_{e_k}(e_r)+\gamma_{e_r}(e_k^*)=0,\qquad 1\leq k\neq r\leq\ell,$$

it is the equation (2.6).

(3) Let us consider the equation

$$\beta_{e_re_k}(e_k)-\beta_{e_re_1}(e_1)-\rho_{e_ke_k^*}(e_r)=0,\qquad 1\leq k\neq r\leq\ell,$$

using (2.9) we get

$$\beta_{e_re_k}(e_k)-\beta_{e_re_1}(e_1)-\rho_{e_ke_k^*}(e_r)=\beta_{e_re_k}(e_k)+\alpha_v(e_r^*)-\rho_{e_ke_k^*}(e_r)=0,$$

it is the equation (2.9).

(4) Let us consider the equation,

$$\beta_{e_rp}(e_k)+\rho_{pe_k^*}(e_r^*)=0,\qquad 1\leq k,r\leq\ell,\ p_z\neq e_k,\ p\in\Omega,$$

it is the equation (2.8).

(5) Let us consider the equation

$$-\beta_{e_rpe_1}(e_1)+\beta_{e_rpe_k}(e_k)+\rho_{pe_ke_k^*}(e_r^*)=0,\qquad 1\leq r\leq\ell,\ 2\leq k\leq\ell,$$

using (2.7) we have $\rho_{pe_ke_k^*}(e_r^*)=-\beta_{e_rpe_k}(e_k)-\beta_p(e_r^*)$, then we get

$$\begin{aligned}
0 &= -\beta_{e_rpe_1}(e_1)+\beta_{e_rpe_k}(e_k)+\rho_{pe_ke_k^*}(e_r^*)=\\
&= -\beta_{e_rpe_1}(e_1)+\beta_{e_rpe_k}(e_k)-\beta_{e_rpe_k}(e_k)-\beta_p(e_r^*)=\\
&= -\beta_{e_rpe_1}(e_1)-\beta_p(e_r^*),
\end{aligned}$$

it is the equation (2.7).

(6) Let us consider the equation

$$\gamma_{e_re_k}(e_k^*)-\gamma_{e_re_1}(e_1^*)+\rho_{e_k(e_re_k)^*}(e_r)=0,\qquad 1\leq r\leq\ell,\ 2\leq k\leq\ell,$$

using (2.12) we get

$$\gamma_{e_r e_k}(e_k^*) - \gamma_{e_r e_1}(e_1^*) + \rho_{e_k e_k^*}(e_r) = \gamma_{e_r e_k}(e_k^*) + \alpha_v(e_r) + \rho_{e_k e_k^*}(e_r) = 0,$$

it is the equation (2.12).

(7) Let us consider the equation,

$$\gamma_{e_r p}(e_k^*) + \rho_{e_k p}(e_r) = 0, \qquad 1 \le r, k \le \ell, \, , p \in \Omega, \, p_z \ne e_z,$$

it is the equation (2.11).

(8) Let us consider the equation

$$-\gamma_{e_r p e_1}(e_1^*) + \gamma_{e_r p e_k}(e_k^*) + \rho_{e_k (p e_k)^*}(e_r) = 0, \qquad 1 \le r \le \ell, \, 2 \le k \le \ell,$$

using (2.10) we get $\rho_{e_k(pe_k)^*}(e_r) = -\gamma_{e_r p e_k}(e_k^*) - \gamma_p(e_k)$ then we have

$$\begin{aligned} 0 &= -\gamma_{e_r p e_1}(e_1^*) + \gamma_{e_r p e_k}(e_k^*) + \rho_{e_k(pe_k)^*}(e_r) = \\ &= -\gamma_{e_r p e_1}(e_1^*) + \gamma_{e_r p e_k}(e_k^*) - \gamma_{e_r p e_k}(e_k^*) - \gamma_p(e_k) = \\ &= -\gamma_{e_r p e_1}(e_1^*) - \gamma_p(e_k), \end{aligned}$$

it is the equation (2.10).

Let us consider the equation

$$\rho_{e_k w h^*}(e_r) + \rho_{w(e_r h)^*}(e_k^*) = 0, \qquad 1 \le k, r \le \ell, \, wh^* \in \mathfrak{M},$$

it is the equation (2.13).

So, we have, for any $1 \le i, j \le \ell$, the following equations,

$$\begin{aligned} &\gamma_{e_j}(e_i^*) + \beta_{e_i}(e_j) = 0, \\ &\beta_p(e_i^*) + (1-\delta_{1,j})\rho_{p e_j e_j^*}(e_i^*) + \beta_{e_i p e_j}(e_i) = 0, && p \in \Omega, \\ &\rho_{p e_j^*}(e_i^*) + \beta_{e_i p}(e_j) = 0, && p \in \Omega, \, p_z \ne e_j, \\ &\alpha_v(e_i^*) + (1-\delta_{1,j})\rho_{e_j e_j^*}(e_i^*) + \beta_{e_i e_j}(e_j) = 0, \\ &\gamma_{e_j p e_i}(e_i^*) + \gamma_p(e_i) + (1-\delta_{1,i})\rho_{e_i(pe_i)^*}(e_j) = 0, && p \in \Omega, \\ &\gamma_{e_j p}(e_i^*) + \rho_{e_i p^*}(e_j) = 0, && p \in \Omega, \, p_z \ne e_i, \\ &\alpha_v(e_j) + \gamma_{e_j e_i}(e_i^*) + (1-\delta_{1,i})\rho_{e_i e_i^*}(e_j) = 0, \\ &\rho_{w(e_j h)^*}(e_i^*) + \rho_{e_i w h^*}(e_j) = 0, && wh^* \in \mathfrak{M}, \end{aligned}$$

as claimed. □

Example 2.1 (Derivations of Laurent polynomial ring)**.** *It is well known that the Leavitt path algebra $W(1)$ is Laurent polynomial ring, i.e.,*

$$L\left(\; \right) \cong R[t, t^{-1}],$$
$$u \longleftrightarrow 1, \quad e \longleftrightarrow t, \quad e^* \longleftrightarrow t^{-1},$$

then the derivations will look like this

$$\mathscr{D}(e) = u \longleftrightarrow \frac{\partial t}{\partial t} = 1, \qquad \mathscr{D}(e^*) = -(ee)^* \longleftrightarrow \frac{\partial t^{-1}}{\partial t} = -(t^2)^{-1},$$
$$\mathscr{D}'(e) = -ee \longleftrightarrow \frac{\partial t}{\partial t^{-1}} = -t^2, \qquad \mathscr{D}'(e^*) = u \longleftrightarrow \frac{\partial t^{-1}}{\partial t^{-1}} = 1,$$

which are classical formulas, indeed, we have

$$\frac{\partial t}{\partial t^{-1}} = \frac{\partial (t^{-1})^{-1}}{\partial t^{-1}} = -(t^{-1})^{-2} = -t^2.$$

Example 2.2. *Let us consider the Leavitt path algebra $W(2)$, and let us consider some derivations, for example we can put*

$$\mathscr{D}(e_1) = \alpha_v(e_1)v, \qquad \mathscr{D}(e_2) = \alpha_v(e_2)v,$$

then we get

$$\mathscr{D}(e_1^*) = -\alpha_v(e_1)(e_1e_1)^* - \alpha_v(e_2)(e_2e_1)^*, \qquad \mathscr{D}(e_2^*) = -\alpha_v(e_1)(e_1e_2)^* - \alpha_v(e_2)(e_2e_2)^*.$$

Let us check the equations

$$\mathscr{D}(e_i^*)e_j + e_i^*\mathscr{D}(e_j) = 0, \qquad 1 \le i, j \le 2,$$
$$\mathscr{D}(e_1)e_1^* + e_1\mathscr{D}(e_1^*) + \mathscr{D}(e_2)e_2^* + e_2\mathscr{D}(e_2^*) = 0,$$

we have

$$
\begin{aligned}
\mathscr{D}(e_1^*)e_1 + e_1^*\mathscr{D}(e_1) &= \Big(-\alpha_v(e_1)(e_1e_1)^* - \alpha_v(e_2)(e_2e_1)^*\Big)e_1 + e_1^*\alpha_v(e_1)v = \\
&= -\alpha_v(e_1)e_1^* + \alpha_v(e_1)e_1^* = 0, \\
\mathscr{D}(e_1^*)e_2 + e_1^*\mathscr{D}(e_2) &= \Big(-\alpha_v(e_1)(e_1e_1)^* - \alpha_v(e_2)(e_2e_1)^*\Big)e_2 + e_1^*\alpha_v(e_2)v = \\
&= -\alpha_v(e_2)e_1^* + \alpha_v(e_2)e_1^* = 0, \\
\mathscr{D}(e_2^*)e_1 + e_2^*\mathscr{D}(e_1) &= \Big(-\alpha_v(e_1)(e_1e_2)^* - \alpha_v(e_2)(e_2e_2)^*\Big)e_1 + e_2^*\alpha_v(e_1)v = \\
&= -\alpha_v(e_1)e_2^* + \alpha_v(e_1)e_2^* = 0, \\
\mathscr{D}(e_2^*)e_2 + e_2^*\mathscr{D}(e_2) &= \Big(-\alpha_v(e_1)(e_1e_2)^* - \alpha_v(e_2)(e_2e_2)^*\Big)e_2 + e_2^*\alpha_v(e_2)v = \\
&= -\alpha_v(e_2)e_2^* + \alpha_v(e_2)e_2^* = 0, \\
\mathscr{D}(e_1)e_1^* + e_1\mathscr{D}(e_1^*) + \mathscr{D}(e_2)e_2^* + e_2\mathscr{D}(e_2^*) &= \alpha_v(e_1)ve_1^* + e_1\Big(-\alpha_v(e_1)(e_1e_1)^* - \alpha_v(e_2)(e_2e_1)^*\Big) + \\
&\quad +\alpha_v(e_2)ve_2^* + e_2\Big(-\alpha_v(e_1)(e_1e_2)^* - \alpha_v(e_2)(e_2e_2)^*\Big) = \\
&= \alpha_v(e_1)e_1^* - \alpha_v(e_1)\big(v - e_2e_2^*\big)e_1^* - \alpha_v(e_2)\big(v - e_2e_2^*\big)e_2^* + \\
&\quad +\alpha_v(e_2)e_2^* - \alpha_v(e_1)e_2(e_1e_2)^* - \alpha_v(e_2)e_2(e_2e_2)^* = \\
&= \alpha_v(e_1)e_1^* - \alpha_v(e_1)e_1^* + \alpha_v(e_1)e_2(e_1e_2)^* - \alpha_v(e_2)e_2^* + \\
&\quad +\alpha_v(e_2)e_2(e_2e_2)^* + \alpha_v(e_2)e_2^* - \alpha_v(e_1)e_2(e_1e_2)^* - \\
&\quad -\alpha_v(e_2)e_2(e_2e_2)^* = 0.
\end{aligned}
$$

3 The inner and the outer derivations of $W(\ell)$

In this section we describe all inner derivations of the Leavitt path algebra $W(\ell)$. We use the standard notations, that is $\mathrm{ad}_x(y) := [x,y] = xy - yx$. From Theorem 1.1 we have for any $v \in V$, $p \in \Omega$ and $wh^* \in \mathfrak{M}$ the following basic elements of $\mathrm{Im}(d_0)$,

$$
\begin{aligned}
\mathrm{ad}_v(-) &= \begin{cases} \mathrm{ad}_v(v) = vv - vv = 0 \\ \mathrm{ad}_v(e) = ve - ev = 0 \\ \mathrm{ad}_v(e^*) = ve^* - e^*v = 0 \end{cases} \\
\mathrm{ad}_p(-) &= \begin{cases} \mathrm{ad}_p(v) = pv - vp = 0 \\ \mathrm{ad}_p(e) = pe - ep \\ \mathrm{ad}_p(e^*) = pe^* - \delta_{p_0,e}(p/p_0) \end{cases} \\
\mathrm{ad}_{p^*}(-) &= \begin{cases} \mathrm{ad}_{p^*}(v) = p^*v - vp^* = 0 \\ \mathrm{ad}_{p^*}(e) = \delta_{p_0,e}(p/p_0)^* - ep^* \\ \mathrm{ad}_{p^*}(e^*) = p^*e^* - e^*p^* \end{cases} \\
\mathrm{ad}_{wh^*}(-) &= \begin{cases} \mathrm{ad}_{wh^*}(v) = wh^*v - vwh^* = 0 \\ \mathrm{ad}_{wh^*}(e) = \delta_{h_0,e}w(h/h_0)^* - (ew)h^* \\ \mathrm{ad}_{wh^*}(e^*) = wh^*e^* - \delta_{w_0,e}(w/w_0)h^*. \end{cases}
\end{aligned}
$$

It follows that any $A \in \mathrm{InnDer}(W(\ell))$, can be presented as follows,

$$
A = \sum_{p\in\Omega}\Big(\nu_p\mathrm{ad}_p + \nu'_p\mathrm{ad}_{p^*}\Big) + \sum_{wh^*\in\mathfrak{M}} \nu''_{wh^*}\mathrm{ad}_{wh^*}.
$$

Theorem 3.1. *Any inner derivation A of the Leavitt path algebra $W(\ell)$ can be described as follows*

$$
A(x) = \begin{cases} 0, \ \textit{if } x = v, \\ \alpha_v(x)v + \sum\limits_{p\in\Omega}\Big(\beta_p(x)p + \gamma_p(x)p^*\Big) + \sum\limits_{wh^*\in\mathfrak{M}} \rho_{wh^*}(x)wh^*, \ \textit{if } x \in E \cup E^*, \end{cases}
$$

where almost all scalers $\beta(x), \gamma(x), \rho(x)$ are zero and they satisfy the following equations,

$$\begin{array}{ll}
\beta_{\underbrace{e_1\cdots e_1}_{t}}(e_1)=\beta_{\underbrace{e_1\cdots e_1}_{t}}(e_1^*)=0, & t\geq 1,\\
\gamma_{\underbrace{e_1\cdots e_1}_{t}}(e_1^*)=\gamma_{\underbrace{e_1\cdots e_1}_{t}}(e_1)=0, & t\geq 1,\\
\gamma_{e_j}(e_i^*)+\beta_{e_i}(e_j)=0, & \\
\beta_p(e_i^*)+(1-\delta_{1,j})\rho_{pe_je_j^*}(e_i^*)+\beta_{e_ipe_j}(e_i)=0, & p\in\Omega,\\
\rho_{pe_j^*}(e_i^*)+\beta_{e_ip}(e_j)=0, & p\in\Omega,\ p_z\neq e_j,\\
\alpha_v(e_i^*)+(1-\delta_{1,j})\rho_{e_je_j^*}(e_i^*)+\beta_{e_ie_j}(e_j)=0, & \\
\gamma_{e_jpe_i}(e_i^*)+\gamma_p(e_i)+(1-\delta_{1,i})\rho_{e_i(pe_i)^*}(e_j)=0, & p\in\Omega,\\
\gamma_{e_jp}(e_i^*)+\rho_{e_ip^*}(e_j)=0, & p\in\Omega,\ p_z\neq e_i,\\
\alpha_v(e_j)+\gamma_{e_je_i}(e_i^*)+(1-\delta_{1,i})\rho_{e_ie_i^*}(e_j)=0, & \\
\rho_{w(e_jh)^*}(e_i^*)+\rho_{e_iwh^*}(e_j)=0, & wh^*\in\mathfrak{M},
\end{array}$$

for any $1\leq i,j\leq \ell$.

Proof. We have to proof only first and second equations, because another equations have been obtained in Theorem 2.1.

1) For the special edge e_1 we have

$$\begin{aligned}
A(e_1) \;=\;& \sum_{p\in\Omega}\nu_p\Big(pe_1-e_1p\Big)+\sum_{p\in\Omega}\nu'_p\Big(\delta_{p_0,e_1}(p/p_0)^*-e_1p^*\Big)+\\
&+\sum_{wh^*\in\mathfrak{M}}\nu''_{wh^*}\Big(\delta_{h_0,e_1}w(h/h_0)^*-e_1wh^*\Big)=\\
=\;& \sum_{p\in\Omega}\nu_p\Big(pe_1-e_1p\Big)+\nu'_{e_1}\Big(v-e_1e_1^*\Big)-\sum_{k=2}^{\ell}\nu'_{e_k}e_1e_k^*+\nu_{e_1e_1}\Big(e_1^*-e_1e_1^*e_1^*\Big)-\\
&-\sum_{k=2}^{\ell}\nu'_{e_ke_1}e_1e_1^*e_k^*+\sum_{k=2}^{\ell}\nu'_{e_1e_k}\Big(e_k^*-e_1(e_1e_k)^*\Big)-\sum_{k=2}^{\ell}\sum_{r=2}^{\ell}\nu'_{e_ke_r}e_1(e_ke_r)^*+\\
&+\sum_{p\in\Omega}\nu'_{e_1pe_1}\Big((pe_1)^*-e_1e_1^*(e_1p)^*\Big)-\sum_{k=2}^{\ell}\sum_{p\in\Omega}\nu'_{e_kpe_1}e_1e_1^*(e_kp)^*+\\
&+\sum_{k=2}^{\ell}\sum_{p\in\Omega}\nu'_{e_1pe_k}\Big((pe_k)^*-e_1(e_1pe_k)^*\Big)-\sum_{k=2}^{\ell}\sum_{r=2}^{\ell}\sum_{p\in\Omega}\nu'_{e_kpe_r}e_1(e_kpe_r)^*+\\
&+\sum_{we_1^*\in\mathfrak{M}}\nu''_{we_1^*}\Big(w-e_1we_1^*\Big)-\sum_{k=2}^{\ell}\sum_{w\in\Omega}\nu''_{we_k^*}e_1we_k^*+\\
&+\sum_{wh^*\in\mathfrak{M}}\nu''_{w(e_1h)^*}\Big(wh^*-e_1w(e_1h)^*\Big)-\sum_{k=2}^{\ell}\sum_{wh^*\in\mathfrak{M}}\nu''_{w(e_kh)^*}e_1w(e_kh)^*.
\end{aligned}$$

Using the equation $e_1e_1^*=v-\sum\limits_{k=2}^{\ell}e_ke_k^*$ we get,

$$\begin{aligned}
A(e_1) \;=\;& \sum_{p\in\Omega}\nu_p\Big(pe_1-e_1p\Big)+\sum_{k=2}^{\ell}\nu'_{e_1}e_ke_k^*-\sum_{k=2}^{\ell}\nu'_{e_k}e_1e_k^*+\sum_{k=2}^{\ell}\nu_{e_1e_1}e_ke_k^*e_1^*-\\
&-\sum_{k=2}^{\ell}\nu'_{e_ke_1}e_k^*+\sum_{r=2}^{\ell}\sum_{k=2}^{\ell}\nu'_{e_ke_1}e_r(e_ke_r)^*+\sum_{k=2}^{\ell}\nu'_{e_1e_k}\Big(e_k^*-e_1(e_1e_k)^*\Big)-\sum_{k=2}^{\ell}\sum_{r=2}^{\ell}\nu'_{e_ke_r}e_1(e_ke_r)^*+\\
&+\sum_{p\in\Omega}\nu'_{e_1pe_1}\Big((pe_1)^*-(e_1p)^*\Big)+\sum_{k=2}^{\ell}\sum_{p\in\Omega}\nu'_{e_1pe_1}e_k(e_1pe_k)^*-\sum_{k=2}^{\ell}\sum_{p\in\Omega}\nu'_{e_kpe_1}(e_kp)^*+\\
&+\sum_{k=2}^{\ell}\sum_{r=2}^{\ell}\sum_{p\in\Omega}\nu'_{e_kpe_1}e_r(e_kpe_r)^*+\sum_{k=2}^{\ell}\sum_{p\in\Omega}\nu'_{e_1pe_k}\Big((pe_k)^*-e_1(e_1pe_k)^*\Big)-\\
&-\sum_{k=2}^{\ell}\sum_{r=2}^{\ell}\sum_{p\in\Omega}\nu'_{e_kpe_r}e_1(e_kpe_r)^*+\sum_{we_1^*\in\mathfrak{M}}\nu''_{we_1^*}\Big(w-e_1we_1^*\Big)-\sum_{k=2}^{\ell}\sum_{w\in\Omega}\nu''_{we_k^*}e_1we_k^*+\\
&+\sum_{wh^*\in\mathfrak{M}}\nu''_{w(e_1h)^*}\Big(wh^*-e_1w(e_1h)^*\Big)-\sum_{k=2}^{\ell}\sum_{wh^*\in\mathfrak{M}}\nu''_{w(e_kh)^*}e_1w(e_kh)^*.
\end{aligned}$$

We have

$$\begin{aligned} A(e_1)\Big|_V &= 0,\\ A(e_1)\Big|_\Omega &= \sum_{p\in\Omega}\nu_p\Big(pe_1-e_1p\Big)+\sum_{we_1^*\in\mathfrak{M}}\nu''_{we_1^*}w, \end{aligned}$$

it follows that

$$\beta\underbrace{e_1\cdots e_1}_{t}(e_1)=0,\qquad t\geq 1,$$

Further, we have

$$\begin{aligned} A(e_1)\Big|_{\Omega^*} &= \sum_{p\in\Omega}\nu'_{e_1pe_1}\Big((pe_1)^*-(e_1p)^*\Big)+\sum_{k=2}^{\ell}\nu'_{e_1e_k}e_k^*-\sum_{k=2}^{\ell}\nu'_{e_ke_1}e_k^*+\\ &+\sum_{k=2}^{\ell}\sum_{p\in\Omega}\nu'_{e_1pe_k}(pe_k)^*-\sum_{k=2}^{\ell}\sum_{p\in\Omega}\nu'_{e_kpe_1}(e_kp)^*=\\ &= \sum_{k=2}^{\ell}\Big(\nu'_{e_1e_k}-\nu'_{e_ke_1}\Big)e_k^*+\sum_{k=2}^{\ell}\Big(\nu'_{e_1e_1e_k}-\nu'_{e_1e_ke_1}\Big)(e_1e_k)^*+\sum_{k=2}^{\ell}\Big(\nu'_{e_1e_ke_1}-\nu'_{e_ke_1e_1}\Big)(e_ke_1)^*+\\ &+\sum_{r=2}^{\ell}\sum_{k=2}^{\ell}\Big(\nu'_{e_1e_re_k}-\nu'_{e_re_ke_1}\Big)(e_re_k)^*+\sum_{p\in\Omega}\Big(\nu'_{e_1e_1pe_1}-\nu'_{e_1pe_1e_1}\Big)(e_1pe_1)^*+\\ &+\sum_{k=2}^{\ell}\sum_{p\in\Omega}\Big(\nu'_{e_1e_1pe_k}-\nu'_{e_1pe_ke_1}\Big)(e_1pe_k)^*+\sum_{k=2}^{\ell}\sum_{p\in\Omega}\Big(\nu'_{e_1e_kpe_1}-\nu'_{e_kpe_1e_1}\Big)(e_kpe_1)^*+\\ &+\sum_{r=2}^{\ell}\sum_{k=2}^{\ell}\sum_{p\in\Omega}\Big(\nu'_{e_1e_rpe_k}-\nu'_{e_rpe_ke_1}\Big)(e_rpe_k)^*=\\ &= \sum_{k=2}^{\ell}\Big(\nu'_{e_1e_k}-\nu'_{e_ke_1}\Big)e_k^*+\sum_{k=1}^{\ell}\sum_{r=1}^{\ell}\Big(\nu'_{e_1e_re_k}-\nu'_{e_re_ke_1}\Big)(e_re_k)^*+\\ &+\sum_{k=1}^{\ell}\sum_{r=1}^{\ell}\sum_{p\in\Omega}\Big(\nu'_{e_1e_rpe_k}-\nu'_{e_rpe_ke_1}\Big)(e_rpe_k)^*, \end{aligned}$$

it follows that

$$\gamma\underbrace{e_1\cdots e_1}_{t}(e_1)=0,\qquad t\geq 1$$

Further, we have

$$\begin{aligned} A(e_1)\Big|_{\mathfrak{M}} &= \sum_{k=2}^{\ell}\nu'_{e_1}e_ke_k^*-\sum_{k=2}^{\ell}\nu'_{e_k}e_1e_k^*+\sum_{r=2}^{\ell}\nu'_{e_1e_1}e_r(e_1e_r)^*+\\ &+\sum_{r=2}^{\ell}\sum_{k=2}^{\ell}\nu'_{e_ke_1}e_r(e_ke_r)^*-\sum_{r=2}^{\ell}\nu'_{e_1e_r}e_1(e_1e_r)^*-\\ &-\sum_{r=2}^{\ell}\sum_{k=2}^{\ell}\nu'_{e_ke_r}e_1(e_ke_r)^*+\sum_{r=2}^{\ell}\sum_{p\in\Omega}\nu'_{e_1pe_1}e_r(e_1pe_r)^*+\\ &+\sum_{r=2}^{\ell}\sum_{k=2}^{\ell}\sum_{p\in\Omega}\nu'_{e_kpe_1}e_r(e_kpe_r)^*-\sum_{r=2}^{\ell}\sum_{p\in\Omega}\nu'_{e_1pe_r}e_1(e_1pe_r)^*-\\ &-\sum_{r=2}^{\ell}\sum_{k=2}^{\ell}\sum_{p\in\Omega}\nu'_{e_kpe_r}e_1(e_kpe_r)^*-\sum_{we_1^*\in\mathfrak{M}}\nu''_{we_1^*}e_1we_1^*-\\ &-\sum_{k=2}^{\ell}\sum_{w\in\Omega}\nu''_{we_k^*}e_1we_k^*+\sum_{wh^*\in\mathfrak{M}}\nu''_{w(e_1h)^*}\Big(wh^*-e_1w(e_1h)^*\Big)-\\ &-\sum_{k=2}^{\ell}\sum_{wh^*\in\mathfrak{M}}\nu''_{w(e_kh)^*}e_1w(e_kh)^*. \end{aligned}$$

Let us add up similar terms,

$$\begin{aligned}
A(e_1)\Big|_{\mathfrak{M}} &= \sum_{k=2}^{\ell}\Big(\nu'_{e_1}+\nu''_{e_k(e_1e_k)^*}\Big)e_ke_k^* + \sum_{k=2}^{\ell}\Big(\nu''_{e_1(e_1e_k)^*}-\nu'_{e_k}\Big)e_1e_k^* + \\
&+\sum_{k=1}^{\ell}\sum_{r=2}^{\ell}\Big(\nu'_{e_ke_1}+\nu''_{e_r(e_1e_ke_r)^*}\Big)e_r(e_ke_r)^* + \sum_{k=1}^{\ell}\sum_{r=2}^{\ell}\Big(\nu''_{e_1(e_1e_ke_r)^*}-\nu'_{e_ke_r}\Big)e_1(e_ke_r)^* + \\
&+\sum_{k=1}^{\ell}\sum_{r=2}^{\ell}\sum_{p\in\Omega}\Big(\nu'_{e_kpe_1}+\nu''_{e_r(e_1e_kpe_r)^*}\Big)e_r(e_kpe_r)^* + \sum_{k=1}^{\ell}\sum_{r=2}^{\ell}\sum_{p\in\Omega}\Big(\nu''_{e_1(e_1e_kpe_r)^*}-\nu'_{e_kpe_r}\Big)e_1(e_kpe_r)^* + \\
&+\sum_{k=1}^{\ell}\sum_{we_k^*\in\mathfrak{M}}\Big(\nu''_{e_1w(e_1e_k)^*}-\nu''_{we_k^*}\Big)e_1we_k^* + \sum_{k=1}^{\ell}\sum_{wh^*\in\mathfrak{M}}\Big(\nu''_{e_1w(e_1e_kh)^*}-\nu''_{w(e_kh)^*}\Big)e_1w(e_kh)^* + \\
&+\sum_{wh^*\in\mathfrak{M},w_0\neq e_1,\,w\notin E}\nu''_{w(e_1h)^*}wh^*,
\end{aligned}$$

we see there are no zero terms.

2) Let us consider the edges $e_r\in E$, $2\le r\le\ell$. For fixed e_r we have

$$\begin{aligned}
A(e_r) &= \sum_{p\in\Omega}\nu_p\Big(pe_r-e_rp\Big)+\sum_{p\in\Omega}\nu'_p\Big(\delta_{p_0,e_r}(p/p_0)^*-e_rp^*\Big)+ \\
&+\sum_{wh^*\in\mathfrak{M}}\nu''_{wh^*}\Big(\delta_{h_0,e_r}w(h/h_0)^*-e_rwh^*\Big)= \\
&= \sum_{p\in\Omega}\nu_p\Big(pe_r-e_rp\Big)+\nu'_{e_r}\Big(v-e_re_r^*\Big)-\sum_{k=1,k\neq r}^{\ell}\nu'_{e_k}e_re_k^*+ \\
&+\sum_{p\in\Omega}\nu'_{e_rp}\Big(p^*-e_r(e_rp)^*\Big)-\sum_{k=1,k\neq r}^{\ell}\sum_{p\in\Omega}\nu'_{e_kp}e_r(e_kp)^*+ \\
&+\sum_{w\in\Omega}\nu''_{we_r^*}\Big(w-e_rwe_r^*\Big)+\sum_{wh^*\in\mathfrak{M}}\nu''_{w(e_rh)^*}\Big(wh^*-e_rw(e_rh)^*\Big)- \\
&-\sum_{k=1,k\neq r}^{\ell}\sum_{we_k^*\in\mathfrak{M}}\nu''_{we_k^*}e_rwe_k^*-\sum_{k=1,k\neq r}^{\ell}\sum_{wh^*\in\mathfrak{M}}\nu''_{w(e_kw)^*}e_rw(e_kh)^*.
\end{aligned}$$

Let us add up similar summands,

$$\begin{aligned}
A(e_r)\Big|_V &= \nu'_{e_r}, \\
A(e_r)\Big|_\Omega &= \sum_{p\in\Omega}\nu_p\Big(pe_r-e_rp\Big)+\sum_{p\in\Omega}\nu''_{pe_r^*}p, \\
A(e_r)\Big|_{\Omega^*} &= \sum_{p\in\Omega}\nu'_{e_rp}p^*, \\
A(e_r)\Big|_{\mathfrak{M}} &= -\sum_{k=1}^{\ell}\nu'_{e_k}e_re_k^*-\sum_{k=1}^{\ell}\sum_{p\in\Omega}\nu'_{e_kp}e_r(e_kp)^*-\sum_{k=1}^{\ell}\sum_{we_k^*\in\mathfrak{M}}\nu''_{we_k^*}e_rwe_k^*- \\
&-\sum_{k=1}^{\ell}\sum_{wh^*\in\mathfrak{M}}\nu''_{w(e_kh)^*}e_rw(e_kh)^*+\sum_{wh^*\in\mathfrak{M}}\nu''_{w(e_rh)^*}wh^*= \\
&= \sum_{k=1}^{\ell}\sum_{wh^*\in\mathfrak{M}}\Big(\nu''_{e_r(e_re_k)^*}-\nu'_{e_k}\Big)e_re_k^*+\sum_{k=1}^{\ell}\sum_{p\in\Omega}\Big(\nu''_{e_r(e_re_kp)^*}-\nu'_{e_kp}\Big)e_r(e_kp)^*+ \\
&+\sum_{k=1}^{\ell}\sum_{we_k^*\in\mathfrak{M}}\Big(\nu''_{e_rw(e_re_k)^*}-\nu''_{we_k^*}\Big)e_rwe_k^*+\sum_{k=1}^{\ell}\sum_{wh^*\in\mathfrak{M}}\Big(\nu''_{e_rw(e_re_kh)^*}-\nu''_{w(e_kh)^*}\Big)e_rw(e_kh)^*+ \\
&+\sum_{wh^*\in\mathfrak{M},w_0\neq e_k}\nu''_{w(e_rh)^*}wh^*,
\end{aligned}$$

we see there are no zero terms.

4) For the e_1^* we have

$$\begin{aligned} A(e_1^*) &= \sum_{p\in\Omega}\nu_p\Big(pe_1^*-\delta_{p_0,e_1}(p/p_0)\Big)+\sum_{p\in\Omega^*}\nu'_p\Big((e_1p)^*-(pe_1)^*\Big)+\\ &+\sum_{wh^*\in\mathfrak{M}}\nu''_{wh^*}\Big(w(e_1h)^*-\delta_{w_0,e_1}(w/w_0)h^*\Big)=\\ &= \nu_{e_1}\Big(e_1e_1^*-v\Big)+\sum_{k=2}^{\ell}\nu_{e_k}e_ke_1^*+\nu_{e_1e_1}\Big(e_1e_1e_1^*-e_1\Big)+\\ &+\sum_{k=2}^{\ell}\nu_{e_ke_1}e_ke_1e_1^*+\sum_{k=2}^{\ell}\nu_{e_1e_k}\Big(e_1e_ke_1^*-e_k\Big)+\sum_{k=2}^{\ell}\sum_{r=2}^{\ell}\nu_{e_ke_r}e_ke_re_1^*+\\ &+\sum_{p\in\Omega}\nu_{e_1pe_1}\Big(e_1pe_1e_1^*-pe_1\Big)+\sum_{k=2}^{\ell}\sum_{p\in\Omega}\nu_{e_kpe_1}e_kpe_1e_1^*+\\ &+\sum_{k=2}^{\ell}\sum_{p\in\Omega}\nu_{e_1pe_k}\Big(e_1pe_ke_1^*-pe_k\Big)+\sum_{k=2}^{\ell}\sum_{r=2}^{\ell}\sum_{p\in\Omega}\nu_{e_kpe_r}e_kpe_re_1^*+\\ &+\sum_{p\in\Omega}\nu'_p\Big((e_1p)^*-(pe_1)^*\Big)+\sum_{e_1h^*\in\mathfrak{M}}\nu''_{e_1h^*}\Big(e_1(e_1h)^*-h^*\Big)+\sum_{k=2}^{\ell}\sum_{h\in\Omega}\nu''_{e_kh^*}e_k(e_1h)^*+\\ &+\sum_{wh^*\in\mathfrak{M}}\nu''_{e_1wh^*}\Big(e_1w(e_1h)^*-wh^*\Big)+\sum_{k=2}^{\ell}\sum_{wh^*\in\mathfrak{M}}\nu''_{e_kwh^*}e_kw(e_1h)^*. \end{aligned}$$

Using the equation $e_1e_1^*=v-\sum\limits_{k=2}^{\ell}e_ke_k^*$, we get

$$\begin{aligned} A(e_1^*) &= -\sum_{k=2}^{\ell}\nu_{e_1}e_ke_k^*+\sum_{k=2}^{\ell}\nu_{e_k}e_ke_1^*-\sum_{k=2}^{\ell}\nu_{e_1e_1}e_1e_ke_k^*+\sum_{k=2}^{\ell}\nu_{e_ke_1}e_k-\sum_{k=2}^{\ell}\sum_{r=2}^{\ell}\nu_{e_ke_1}e_ke_re_r^*+\\ &+\sum_{k=2}^{\ell}\nu_{e_1e_k}e_1e_ke_1^*-\sum_{k=2}^{\ell}\nu_{e_1e_k}e_k+\sum_{k=2}^{\ell}\sum_{r=2}^{\ell}\nu_{e_ke_r}e_ke_re_1^*+\sum_{p\in\Omega}\nu_{e_1pe_1}\Big(e_1p-pe_1\Big)-\\ &-\sum_{k=2}^{\ell}\sum_{p\in\Omega}\nu_{e_1pe_1}e_1pe_ke_k^*+\sum_{k=2}^{\ell}\sum_{p\in\Omega}\nu_{e_kpe_1}e_kp-\sum_{k=2}^{\ell}\sum_{r=2}^{\ell}\sum_{p\in\Omega}\nu_{e_kpe_1}e_kpe_re_r^*+\\ &+\sum_{k=2}^{\ell}\sum_{p\in\Omega}\nu_{e_1pe_k}\Big(e_1pe_ke_1^*-pe_k\Big)+\sum_{k=2}^{\ell}\sum_{r=2}^{\ell}\sum_{p\in\Omega}\nu_{e_kpe_r}e_kpe_re_1^*+\\ &+\sum_{p\in\Omega}\nu'_p\Big((e_1p)^*-(pe_1)^*\Big)+\sum_{e_1h^*\in\mathfrak{M}}\nu''_{e_1h^*}\Big(e_1(e_1h)^*-h^*\Big)+\sum_{k=2}^{\ell}\sum_{h\in\Omega}\nu''_{e_kh^*}e_k(e_1h)^*+\\ &+\sum_{wh^*\in\mathfrak{M}}\nu''_{e_1wh^*}\Big(e_1w(e_1h)^*-wh^*\Big)+\sum_{k=2}^{\ell}\sum_{wh^*\in\mathfrak{M}}\nu''_{e_kwh^*}e_kw(e_1h)^*. \end{aligned}$$

Let us add up similar terms,

$$
\begin{aligned}
A(e_1^*)\Big|_{\Omega} \quad = \quad & \sum_{k=2}^{\ell} \nu_{e_k e_1} e_k - \sum_{k=2}^{\ell} \nu_{e_1 e_k} e_k + \sum_{p\in\Omega} \nu_{e_1 p e_1} \Big(e_1 p - p e_1 \Big) + \sum_{k=2}^{\ell} \sum_{p\in\Omega} \nu_{e_k p e_1} e_k p - \\
& - \sum_{k=2}^{\ell} \sum_{p\in\Omega} \nu_{e_1 p e_k} p e_k = \\
= \quad & \sum_{k=2}^{\ell} \Big(\nu_{e_k e_1} - \nu_{e_1 e_k} \Big) e_k + \sum_{k=2}^{\ell} \Big(\nu_{e_1 e_k e_1} - \nu_{e_1 e_1 e_k} \Big) e_1 e_k + \sum_{k=2}^{\ell} \Big(\nu_{e_k e_1 e_1} - \nu_{e_1 e_k e_1} \Big) e_k e_1 + \\
& + \sum_{k=2}^{\ell} \sum_{r=2}^{\ell} \Big(\nu_{e_k e_r e_1} - \nu_{e_1 e_k e_r} \Big) e_k e_r + \sum_{k=2}^{\ell} \sum_{p\in\Omega} \Big(\nu_{e_1 p e_k e_1} - \nu_{e_1 e_1 p e_k} \Big) e_1 p e_k + \\
& + \sum_{k=2}^{\ell} \sum_{p\in\Omega} \Big(\nu_{e_k p e_1 e_1} - \nu_{e_1 e_k p e_1} \Big) e_k p e_1 + \sum_{k=2}^{\ell} \sum_{r=2}^{\ell} \sum_{p\in\Omega} \Big(\nu_{e_k p e_r e_1} - \nu_{e_1 e_k p e_r} \Big) e_k p e_r = \\
= \quad & \sum_{k=2}^{\ell} \Big(\nu_{e_k e_1} - \nu_{e_1 e_k} \Big) e_k + \sum_{k=1}^{\ell} \sum_{r=1}^{\ell} \Big(\nu_{e_k e_r e_1} - \nu_{e_1 e_k e_r} \Big) e_k e_r + \\
& + \sum_{k=1}^{\ell} \sum_{r=1}^{\ell} \sum_{p\in\Omega} \Big(\nu_{e_k p e_r e_1} - \nu_{e_1 e_k p e_r} \Big) e_k p e_r .
\end{aligned}
$$

We get

$$
\beta \underbrace{e_1 \cdots e_1}_{t} (e_1^*) = 0, \qquad t \geq 1,
$$

Further,

$$
\begin{aligned}
A(e_1^*)\Big|_{\Omega^*} \quad = \quad & \sum_{p\in\Omega} \nu'_p \Big((e_1 p)^* - (p e_1)^* \Big) - \sum_{e_1 h^* \in \mathfrak{M}} \nu''_{e_1 h^*} h^* = \\
= \quad & - \sum_{k=2}^{\ell} \nu''_{e_1 e_k^*} e_k^* - \sum_{k=2}^{\ell} \nu'_{e_k} (e_k e_1)^* + \sum_{k=2}^{\ell} \Big(\nu'_{e_k} - \nu''_{e_1 (e_1 e_k)^*} \Big) (e_1 e_k)^* - \sum_{k=2}^{\ell} \sum_{r=2}^{\ell} \nu''_{e_1 (e_k e_r)^*} (e_k e_r)^* + \\
& + \sum_{p\in\Omega} \Big(\nu'_{p e_1} - \nu'_{e_1 p} \Big) (e_1 p e_1)^* - \sum_{k=2}^{\ell} \sum_{p\in\Omega} \nu'_{e_k p} (e_k p e_1)^* + \Big(\nu'_{p e_k} - \nu''_{e_1 (e_1 p e_k)^*} \Big) (e_1 p e_k)^* - \\
& - \sum_{k=2}^{\ell} \sum_{r=2}^{\ell} \sum_{p\in\Omega} \nu''_{e_1 (e_r p e_k)^*} (e_r p e_k)^*,
\end{aligned}
$$

it follows that

$$
\gamma \underbrace{e_1 \cdots e_1}_{t} (e_1^*) = 0, \qquad t \geq 1.
$$

Further,

$$
\begin{aligned}
A(e_1^*)\Big|_{\mathfrak{M}} \quad = \quad & - \sum_{k=2}^{\ell} \nu_{e_1} e_k e_k^* + \sum_{k=2}^{\ell} \nu_{e_k} e_k e_1^* - \sum_{k=2}^{\ell} \nu_{e_1 e_1} e_1 e_k e_k^* - \sum_{k=2}^{\ell} \sum_{r=2}^{\ell} \nu_{e_k e_1} e_k e_r e_r^* + \\
& + \sum_{k=2}^{\ell} \nu_{e_1 e_k} e_1 e_k e_1^* + \sum_{k=2}^{\ell} \sum_{r=2}^{\ell} \nu_{e_k e_r} e_k e_r e_1^* - \sum_{k=2}^{\ell} \sum_{p\in\Omega} \nu_{e_1 p e_1} e_1 p e_k e_k^* - \\
& - \sum_{k=2}^{\ell} \sum_{r=2}^{\ell} \sum_{p\in\Omega} \nu_{e_k p e_1} e_k p e_r e_r^* + \sum_{k=2}^{\ell} \sum_{p\in\Omega} \nu_{e_1 p e_k} e_1 p e_k e_1^* + \sum_{k=2}^{\ell} \sum_{r=2}^{\ell} \sum_{p\in\Omega} \nu_{e_k p e_r} e_k p e_r e_1^* + \\
& + \sum_{e_1 h^* \in \mathfrak{M}} \nu''_{e_1 h^*} e_1 (e_1 h)^* + \sum_{k=2}^{\ell} \sum_{h\in\Omega} \nu''_{e_k h^*} e_k (e_1 h)^* + \\
& + \sum_{w h^* \in \mathfrak{M}} \nu''_{e_1 w h^*} \Big(e_1 w (e_1 h)^* - w h^* \Big) + \sum_{k=2}^{\ell} \sum_{w h^* \in \mathfrak{M}} \nu''_{e_k w h^*} e_k w (e_1 h)^*,
\end{aligned}
$$

let us add up similar terms,

$$\begin{aligned}
A(e_1^*)\Big|_{\mathfrak{M}} \quad = \quad & \sum_{k=2}^{\ell}\Big(-\nu_{e_1}-\nu''_{e_1e_ke_k^*}\Big)e_ke_k^* + \sum_{k=2}^{\ell}\Big(\nu_{e_k}-\nu''_{e_1e_ke_1^*}\Big)e_ke_1^* + \\
& + \sum_{k=1}^{\ell}\sum_{r=2}^{\ell}\Big(-\nu_{e_ke_1}-\nu''_{e_1e_ke_re_r^*}\Big)e_ke_re_r^* + \sum_{k=1}^{\ell}\sum_{r=2}^{\ell}\Big(\nu_{e_ke_r}-\nu''_{e_1e_ke_re_1^*}\Big)e_ke_re_1^* + \\
& + \sum_{k=1}^{\ell}\sum_{r=2}^{\ell}\sum_{p\in\Omega}\Big(-\nu_{e_kpe_1}-\nu''_{e_1e_kpe_re_r^*}\Big)e_kpe_re_r^* + \sum_{k=1}^{\ell}\sum_{r=2}^{\ell}\sum_{p\in\Omega}\Big(\nu_{e_kpe_r}-\nu''_{e_1e_kpe_re_1^*}\Big)e_kpe_re_1^* + \\
& + \sum_{k=1}^{\ell}\sum_{e_1h^*\in\mathfrak{M}}\Big(\nu''_{e_kh^*}-\nu''_{e_1e_k(e_1h)^*}\Big)e_k(e_1h)^* + \sum_{k=1}^{\ell}\sum_{wh^*\in\mathfrak{M}}\Big(\nu''_{e_kwh^*}-\nu''_{e_1e_kw(e_1h)^*}\Big)e_kw(e_1h)^* - \\
& - \sum_{wh^*\in\mathfrak{M},h_0\neq e_1}\nu''_{e_1wh^*}wh^*,
\end{aligned}$$

we see there are no zero terms.

4) For $e_r^*\in E^*$, $2\le r\le\ell$, we have

$$\begin{aligned}
A(e_r^*) \quad = \quad & \sum_{p\in\Omega}\nu_p\Big(pe_r^*-\delta_{p_0,e_r}(p/p_0)\Big) + \sum_{p\in\Omega}\nu'_p\Big((e_rp)^*-(pe_r)^*\Big) + \\
& + \sum_{wh^*\in\mathfrak{M}}\nu''_{wh^*}\Big(w(e_rh)^*-\delta_{w_0,e_r}(w/w_0)h^*\Big) = \\
= \quad & \nu_{e_r}\Big(e_re_r^*-v\Big) + \sum_{k=1,k\neq r}^{\ell}\nu_{e_k}e_ke_r^* + \\
& + \sum_{p\in\Omega}\nu_{e_rp}\Big(e_rpe_r^*-p\Big) + \sum_{k=1,k\neq r}^{\ell}\sum_{p\in\Omega}\nu_{e_kp}e_kpe_r^* + \\
& + \sum_{p\in\Omega}\nu'_p\Big((e_rp)^*-(pe_r)^*\Big) + \sum_{h\in\Omega}\nu''_{e_rh^*}\Big(e_r(e_rh)^*-h^*\Big) + \\
& + \sum_{k=1,k\neq r}^{\ell}\nu''_{e_kh^*}e_k(e_rh)^* + \sum_{wh^*\in\mathfrak{M}}\nu''_{e_rwh^*}\Big(e_rw(e_rh)^*-wh^*\Big) + \\
& + \sum_{k=1,k\neq r}^{\ell}\sum_{wh^*\in\mathfrak{M}}\nu''_{e_kwh^*}e_kw(e_rh)^*.
\end{aligned}$$

Let us add up similar terms,

$$\begin{aligned}
A(e_r^*)|_V \quad = \quad & -\nu_{e_r}, \\
A(e_r^*)|_\Omega \quad = \quad & -\sum_{p\in\Omega}\nu_{e_rp}p, \\
A(e_r^*)|_{\Omega^*} \quad = \quad & \sum_{p\in\Omega}\nu'_p\Big((e_rp)^*-(pe_r)^*\Big) - \sum_{h\in\Omega}\nu''_{e_rh^*}h^* = \\
= \quad & -\sum_{k=1}^{\ell}\nu''_{e_re_k^*}e_k^* + \sum_{k=1,k\neq r}^{\ell}\Big(-\nu'_{e_k}-\nu''_{e_r(e_ke_r)^*}\Big)(e_ke_r)^* + \sum_{k=1,k\neq r}^{\ell}\Big(\nu'_{e_k}-\nu''_{e_r(e_re_k)^*}\Big)(e_re_k)^* - \\
& -\nu''_{e_r(e_re_r)^*}(e_re_r)^* - \sum_{k=1,k\neq r}^{\ell}\sum_{t=1,t\neq r}^{\ell}\nu''_{e_r(e_ke_t)^*}(e_ke_t)^* + \sum_{p\in\Omega}\Big(\nu'_{pe_r}-\nu'_{e_rp}-\nu''_{e_r(e_rpe_r)^*}\Big)(e_rpe_r)^* + \\
& + \sum_{k=1,k\neq r}^{\ell}\sum_{p\in\Omega}\Big(-\nu'_{e_kp}-\nu''_{e_r(e_kpe_r)^*}\Big)(e_kpe_r)^* + \sum_{k=1,k\neq r}^{\ell}\sum_{p\in\Omega}\Big(\nu'_{pe_k}-\nu''_{e_r(e_rpe_k)^*}\Big)(e_rpe_k)^* - \\
& - \sum_{k=1,k\neq r}^{\ell}\sum_{t=1,t\neq r}^{\ell}\sum_{p\in\Omega}\nu''_{e_r(e_kpe_t)^*},
\end{aligned}$$

and

$$
\begin{aligned}
A(e_r^*)|_{\mathfrak{M}} &= \sum_{k=1}^{\ell}\nu_{e_k}e_ke_r^* + \sum_{k=1}^{\ell}\sum_{p\in\Omega}\nu_{e_kp}e_kpe_r^* + \\
&\quad + \sum_{k=1}^{\ell}\sum_{h\in\Omega}\nu''_{e_kh^*}e_k(e_rh)^* + \sum_{k=1}^{\ell}\sum_{wh^*\in\mathfrak{M}}\nu''_{e_kwh^*}e_kw(e_rh)^* - \\
&\quad - \sum_{wh^*\in\mathfrak{M}}\nu''_{e_rwh^*}wh^* = \\
&= \sum_{k=1}^{\ell}\Big(\nu_{e_k} - \nu''_{e_re_ke_r^*}\Big)e_ke_r^* + \sum_{k=1}^{\ell}\sum_{p\in\Omega}\Big(\nu_{e_kp} - \nu'_{e_re_kpe_r^*}\Big)e_kpe_r^* + \\
&\quad + \sum_{k=1}^{\ell}\sum_{p\in\Omega}\Big(\nu''_{e_kp^*} - \nu''_{e_re_k(e_rp)^*}\Big)e_k(e_rp)^* + \sum_{k=1}^{\ell}\sum_{wh^*\in\mathfrak{M}}\Big(\nu''_{e_kwh^*} - \nu''_{e_re_kw(e_rh)^*}\Big)e_kw(e_rh)^* - \\
&\quad - \sum_{k=1,k\neq r}^{\ell}\sum_{wh^*\in\mathfrak{M}}\nu''_{e_rw(e_kh)^*}w(e_kh)^*,
\end{aligned}
$$

we see there are no zero terms. It completes the proof. □

As corollary of this Theorem follows the full description of all outer derivations of the Leavitt path algebra $W(\ell)$.

Theorem 3.2. *Any outer derivation $\mathscr{D}$ of the Leavitt path algebra $W(\ell)$ can be described as follows,*

$$
\mathscr{D}(x) = \begin{cases} 0, & \text{if } x = v, \\ \alpha_v(x)v + \sum_{p\in\Omega}\Big(\beta_p(x)p + \gamma_p(x)p^*\Big) + \sum_{wh^*\in\mathfrak{M}}\rho_{wh^*}(x)wh^*, & \text{if } x \in E\cup E^*, \end{cases}
$$

where almost all coefficients $\alpha(x),\beta(x),\gamma(x),\rho(x)\in R$ are zero and they satisfy the following equations,

$$
\begin{aligned}
&\gamma_{e_j}(e_i^*) + \beta_{e_i}(e_j) = 0, && \\
&\beta_p(e_i^*) + (1-\delta_{1,j})\rho_{pe_je_j^*}(e_i^*) + \beta_{e_ipe_j}(e_i) = 0, && p\in\Omega, \\
&\rho_{pe_j^*}(e_i^*) + \beta_{e_ip}(e_j) = 0, && p\in\Omega,\ p_z\neq e_j, \\
&\alpha_v(e_i^*) + (1-\delta_{1,j})\rho_{e_je_j^*}(e_i^*) + \beta_{e_ie_j}(e_j) = 0, && \\
&\gamma_{e_jpe_i}(e_i^*) + \gamma_p(e_i) + (1-\delta_{1,i})\rho_{e_i(pe_i)^*}(e_j) = 0, && p\in\Omega, \\
&\gamma_{e_jp}(e_i^*) + \rho_{e_ip^*}(e_j) = 0, && p\in\Omega,\ p_z\neq e_i, \\
&\alpha_v(e_j) + \gamma_{e_je_i}(e_i^*) + (1-\delta_{1,i})\rho_{e_ie_i^*}(e_j) = 0, && \\
&\rho_{w(e_jh)^*}(e_i^*) + \rho_{e_iwh^*}(e_j) = 0, && wh^*\in\mathfrak{M},
\end{aligned}
$$

for any $1\le i,j\le\ell$, $t\ge 1$, and at least one of the following scalers $\beta_{\underbrace{e_1\cdots e_1}_{t}}(e_1)$, $\beta_{\underbrace{e_1\cdots e_1}_{t}}(e_1^*)$, $\gamma_{\underbrace{e_1\cdots e_1}_{t}}(e_1)$, $\gamma_{\underbrace{e_1\cdots e_1}_{t}}(e_1^*)$ are not zero.

Proof. It immediately follows from Theorem 2.1 and Theorem 3.1. □

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