A NEW TEST FOR ASPHERICITY AND DIAGRAMMATIC REDUCIBILITY OF GROUP PRESENTATIONS

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ABSTRACT. We present a new test for studying asphericity and diagrammatic reducibility of group presentations. We use this criterion to generalize results of J. Howie and S.M. Gersten on asphericity of LOTs and of Adian presentations. Our test can be applied to prove diagrammatic reducibility in cases where the classical weight test fails. We also use our methods to investigate a conjecture of S.V. Ivanov related to Kaplansky problem on zero divisors: we strengthen Ivanov's result for locally indicable groups and prove a weak version of the conjecture.

1. INTRODUCTION

The notion of asphericity is central to topology, geometry and algebra. Recall that a path-connected space X is called aspherical if its homotopy groups $\pi_n(X)$ are trivial for $n \geq 2$. A longstanding open problem in low dimensional topology is the Whitehead asphericity conjecture, which asserts that a path-connected subcomplex of an aspherical 2-complex is itself aspherical [3, 27, 29]. There have been considerable advances in this question which include works of Cockcroft [7], Adams [1] and Howie [15, 16]. A closely related unsolved problem is whether ribbon disc complements are necessarily aspherical [17, 18]. This question was shown to be equivalent to whether complexes associated to *labeled oriented trees* (LOTs) are aspherical. LOT-complexes are combinatorially encoded 2-complexes which naturally appear as spines of ribbon disc complements [17]. A proof of the asphericity of ribbon disc complements would generalize, in some sense, Papakyriakopoulos' analogous result for knot complements. It is easy to see that every LOT-complex embeds in a contractible 2-complex. Therefore LOT-complexes are considered test cases for the Whitehead conjecture. The concept of asphericity plays a key role in combinatorial group theory (see for example [5, 9, 12, 15, 17, 20, 23]). In this context, a group presentation is called aspherical if its standard 2-complex is aspherical. In geometry, asphericity is fundamental in the study of manifolds and it is related to the theory of hyperbolic groups (see [2, 13, 22]).

One of the best-known asphericity tests for 2-complexes (or group presentations) is Gersten's weight test [9, 10] which is a generalization of Sieradski's coloring test [26]. Gersten's weight test was further developed by Pride [24] and generalized by Huck and Rosebrock [19]. All these variants of the weight test are based on the combinatorial Gauss-Bonnet theorem. Howie attacked asphericity problems with a different approach, using the notion of local indicability. Recall that a group G is *indicable* if it admits a nontrivial homomorphism to the infinite cyclic group, and it is *locally indicable* if all its nontrivial finitely generated subgroups are indicable. Howie showed that a connected 2-complex

with locally indicable fundamental group and trivial second homology group is aspherical [15]. He also used the concept of local indicability to deduce that *reducible* presentations with no proper powers are aspherical and to prove asphericity of certain classes of LOT-complexes [17]. In [18] he showed that for $n \geq 3$, ribbon n-knots in S^{n+2} admit minimal Seifert manifolds provided their associated LOTs have diameter at most 3.

In this paper we present a new test for studying asphericity of group presentations. The I-test provides a criterion for deciding when a presentation $\mathcal{P}$ of an indicable group G is aspherical. Our test is based on a description of the second homotopy group of the 2-complex $K_{\mathcal{P}}$ associated to $\mathcal{P}$ as the kernel of the boundary map $C_2(\widetilde{K}_{\mathcal{P}}) \to C_1(\widetilde{K}_{\mathcal{P}})$ in the cellular chain complex of the universal cover of $K_{\mathcal{P}}$, and it uses basic linear algebra methods. We compare our test with the different variations of the weight test and use our methods to prove asphericity in cases where known tests fail.

Similarly as in the case of Gersten's weight test, our I-test guarantees *diagrammatic reducibility* of the presentation. This concept, first studied by Sieradski [26] and Gersten [9, 10], is stronger than asphericity and is intimately related to the existence of solutions of equations over groups. The Kervaire-Laudenbach-Howie conjecture asserts that any independent system of equations over a group H has a solution over H [14, 10, 25]. Howie showed that the conjecture is true for locally indicable groups and Gersten proved that if a presentation $\mathcal{P}$ is diagrammatically reducible then all systems of equations modeled on $\mathcal{P}$ over any group H are solvable in an overgroup of H [10] (see also [4, 21] for more results in these directions). Gersten also used the concept of diagrammatic reducibility to study subgroups of hyperbolic groups [11].

We use our test to obtain generalizations of results of Gersten and Howie on Adian presentations and LOTs [9, 17]. We introduce the notions of *deforestable* and *weakly deforestable* labeled oriented graph (LOG) and show that their associated presentations are diagrammatically reducible. These classes strictly contain those LOGs Γ such that either the initial graph $I(\Gamma)$ or the terminal graph $T(\Gamma)$ has no cycles (cf. [17]).

We also apply our methods to investigate a conjeture of Ivanov related to Kaplansky problem on zero divisors [20]. Suppose $\mathcal{P}$ is a finite presentation and $\mathcal{Q}$ is obtained from $\mathcal{P}$ by adding one generator x and one relator r. Ivanov conjectures that, under certain hypotheses, the asphericity of $\mathcal{P}$ implies that of $\mathcal{Q}$. Ivanov showed that a counterexample to this conjecture would provide a torsion-free group G whose integral group ring $\mathbb{Z}G$ contains zero divisors. He proved that the conjecture holds in certain cases, including the case where the group H presented by $\mathcal{P}$ is locally indicable. This follows from results of Howie. In fact under the hypotheses of the conjecture it is immediate that the locally indicability of H implies that of G. We prove that the conjecture remains valid if one only requires that some specific (finitely many) subgroups of G, determined by the relator r, be indicable (see Theorem 5.4 below). Finally in Theorem 5.5 we show that the conjecture is true if we are allowed to *perturb* the relation r.

2. The I-test

Let $\mathcal{P} = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$ be a presentation of a group G, where $\mathcal{A} = \{x_1, x_2, \ldots, x_n\}$ is an alphabet and the relators r_j are non-necessarily reduced words in $\mathcal{A} \cup \mathcal{A}^{-1}$. If w is a word in $\mathcal{A} \cup \mathcal{A}^{-1}$, w will also denote the corresponding element in the free group $F(x_1, x_2, \ldots, x_n)$ with basis $\mathcal{A}$ and the element $p(w) \in G$, where $p : F(x_1, x_2, \ldots, x_n) \to G$ is the quotient map. In particular if $g \in G$, gw denotes the element $g.p(w) \in G$. The second homotopy group $\pi_2(K_{\mathcal{P}})$ of the associated 2-complex

is isomorphic to the second homology group $H_2(\widetilde{K}_{\mathcal{P}})$ of the universal cover of $K_{\mathcal{P}}$. The complex $\widetilde{K}_{\mathcal{P}}$ has one 0-cell e_g^0 for each element $g \in G$. For each $1 \leq i \leq n$, there is an oriented 1-cell $e_{i,g}^1$ from e_g^0 to $e_{gx_i}^0$. For each $g \in G$ and $1 \leq j \leq m$ there is an oriented 2-cell $e_{j,g}^2$. If x_i is a letter of r_j, say $r_j = wx_i w'$ for certain words w, w', then $e_{i,gw}^1$ is a face of $e_{j,g}^2$ with incidence 1. If $r_j = wx_i^{-1}w'$, then $e_{i,gwx_i^{-1}}^1$ is a face of $e_{j,g}^2$ with incidence -1. This yields a description of $\pi_2(K_{\mathcal{P}})$ that goes back to Reidemeister and Whitehead (see [27, pp. 84]), as the kernel of the boundary map $\partial : C_2(\widetilde{K}_{\mathcal{P}}) \to C_1(\widetilde{K}_{\mathcal{P}})$ in the cellular chain complex. Theorem 2.1 below summarizes these ideas giving equations for π_2 which can be read off from the presentation. If $w = x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \ldots x_{i_l}^{\epsilon_l}$ is a (non-necessarily reduced) word in $\mathcal{A} \cup \mathcal{A}^{-1}$, where $\epsilon_{k,w} = \epsilon_k = \pm 1$ for each $1 \leq k \leq l$, the length $\mathrm{l}(w) = l$ of w is the number of letters in w. Given $1 \leq k \leq \mathrm{l}(w)$, denote by $w^{(k)}$ the subword of w obtained by removing the first $k-1$ letters of w. That is, $w^{(k)} = x_{i_k}^{\epsilon_k} x_{i_{k+1}}^{\epsilon_{k+1}} \ldots x_{i_l}^{\epsilon_l}$. For $1 \leq k \leq \mathrm{l}(w)$ define $s(k, w) = w^{(k)}$ if $\epsilon_k = 1$, and $s(k, w) = w^{(k+1)}$ if $\epsilon_k = -1$. Let $1 \leq i \leq n$. Denote by $\mathrm{occ}(x_i, w) = \{1 \leq k \leq l \mid x_{i_k} = x_i\}$ the set of positions in which the letter x_i (or its inverse x_i^{-1}) occurs in w.

Theorem 2.1. *Let* $\mathcal{P} = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_m \rangle$ *be a presentation of a group* G. *For each* $1 \leq i \leq n$ *and each* $g \in G$ *consider the integral linear equation* $E_{i,g}$:

$$\sum_{1 \leq j \leq m} \sum_{k \in \mathrm{occ}(x_i, r_j)} \epsilon_{k,r_j} n_{gs(k,r_j)}^j = 0$$

in the unknowns n_h^j, *for* $1 \leq j \leq m$, $h \in G$.

Then $K_{\mathcal{P}}$ *is aspherical if and only if the unique solution of the system* $\{E_{i,g}\}_{i,g}$ *with finitely many nontrivial* n_g^j *is the trivial solution,* $n_g^j = 0$ *for each* $1 \leq j \leq m, g \in G$.

Remark 2.2. A term n_h^j appears in the equation $E_{i,g}$ if and only if $e_{i,g}^1$ is a face of $e_{j,h}^2$.

Example 2.3. Let $\mathcal{P} = \langle x, y, z, w | x^2 y^2 z^2, xyx^{-1}zyz^{-1}, w^2 x^{-1} w^{-1} z \rangle$. The equations associated to the generators x, y, z, w are respectively

$$n_{gx^2y^2z^2}^1 + n_{gxy^2z^2}^1 + n_{gxyx^{-1}zyz^{-1}}^2 - n_{gzyz^{-1}}^2 - n_{gw^{-1}z}^3 = 0,$$
$$n_{gy^2z^2}^1 + n_{gyz^2}^1 + n_{gyx^{-1}zyz^{-1}}^2 + n_{gyz^{-1}}^2 = 0,$$
$$n_{gz^2}^1 + n_{gz}^1 + n_{gzyz^{-1}}^2 - n_g^2 + n_{gz}^3 = 0,$$
$$n_{gw^2x^{-1}w^{-1}z}^3 + n_{gwx^{-1}w^{-1}z}^3 - n_{gz}^3 = 0.$$

Let G be a finitely presented group. Recall that G is said to be indicable if there exists an epimorphism from G onto the infinite cyclic group $\mathbb{Z}$. We denote by $G^{ab} = G/[G, G]$ the abelianization of G. Let $\mathcal{P} = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$ be a presentation of G. Denote by $q : F(x_1, x_2, \ldots, x_n) \to F(x_1, x_2, \ldots, x_n)^{ab} = \mathbb{Z}^n$ the quotient map. We identify $q(x_i) \in F(x_1, x_2, \ldots, x_n)^{ab}$ with the i-th vector of the standard basis of $\mathbb{Z}^n$.

Lemma 2.4. G *is indicable if and only if there is a nonzero vector* $v \in \mathbb{R}^n$ *orthogonal to each* $q(r_j)$.

Proof. An epimorphism $G \to \mathbb{Z}$ induces an epimorphism $G^{ab} \to \mathbb{Z}$. Since G^{ab} is isomorphic to $\mathbb{Z}^n/\langle q(r_j)\rangle_j$, there exists an epimorphism $\mathbb{Z}^n \to \mathbb{Z}$ which is zero in each $q(r_j)$. A homomorphism $\mathbb{Z}^n \to \mathbb{Z}$ is the multiplication by a vector $v \in \mathbb{Z}^n$. Conversely, a nonzero vector $v \in \mathbb{R}^n$ orthogonal to each $q(r_j)$ induces a nontrivial homomorphism $f : G^{ab} \to \mathbb{R}$.

Since $f(G^{ab})$ is a finitely generated torsion free abelian group, there is a nontrivial map $G^{ab} \to \mathbb{Z}^r$ and then G is indicable. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Assume that G is indicable an let $v \in \mathbb{R}^n$ be a vector orthogonal to each $q(r_j)$. Given $w \in F(x_1, x_2, \ldots, x_n)$, its *weight* (relative to v) is $\overline{w} = \langle q(w), v \rangle \in \mathbb{R}$, where $\langle, \rangle$ denotes the standard inner product of $\mathbb{R}^n$. If $g \in G$ is represented by an element $w \in F(x_1, x_2, \ldots, x_m)$, we define its weight as $\overline{g} = \overline{w}$. Note that the definition does not depend on the representative. We define the *weight matrix M* of $\mathcal{P}$ as follows. It is an $n \times m$ matrix whose entries are families of real numbers. The family $M_{i,j}$ is

$$M_{i,j} = \{\overline{s(k, r_j)}\}_{k \in \mathrm{occ}(x_i, r_j)}$$

In other words, $M_{i,j}$ contains the weights of the subindices g of the unknowns n_g^j which appear in the equation $E_{i,1}$ (corresponding to the generator x_i and the trivial element $1 \in G$) in the statement of Theorem 2.1. Of course, M depends on the vector v. It is important to note that a number can appear in $M_{i,j}$ with multiplicity and that $M_{i,j}$ is empty if x_i does not occur in r_j.

Definition 2.5. An $n \times m$ matrix M whose entries are (possibly empty) families of real numbers will be called *good* if $n \geq m$ and there exists an ordering $j_1, j_2, \ldots, j_m$ of the columns of M and an ordering $i_1, i_2, \ldots, i_m$ of m of the rows, such that for each $1 \leq k \leq m$

(1) M_{i_k, j_k} is non-empty,

(2) the maximum λ_k of M_{i_k, j_k} is the maximum of the whole row $\bigcup_{j=1}^{m} M_{i_k, j}$ and

(3) the multiplicity of λ_k in $\bigcup_{l=k}^{m} M_{i_k, j_l}$ is one.

A presentation $\mathcal{P} = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$ satisfies the *I-test* if there exists a vector $v \in \mathbb{R}^n$, orthogonal to each $q(r_j)$ such that the corresponding weight matrix is good. In this case we say that $\mathcal{P}$ satisfies the I-test for v, and for the orderings $j_1, j_2, \ldots, j_m$ and $i_1, i_2, \ldots, i_m$ that make M good.

In Example 2.3 above, the vector $v = (1, 0, -1, 2)$ is orthogonal to $q(r_1) = (2, 2, 2, 0)$, $q(r_2) = (0, 2, 0, 0)$ and $q(r_3) = (-1, 0, 1, 1)$. The weight matrix M is

$$\begin{array}{c} \\ x \\ y \\ z \\ w \end{array} \begin{pmatrix} \begin{array}{ccc} r_1 & r_2 & r_3 \\ 0, -1 & 0, 0 & -3 \\ -2, -2 & -1, 1 & \emptyset \\ -2, -1 & 0, 0 & -1 \\ \emptyset & \emptyset & 0, -2, -1 \end{array} \end{pmatrix}$$

$\mathcal{P}$ satisfies the I-test with respect to the orders $2, 3, 1$ of the columns and $2, 4, 1$ of the rows: the maximum of the second row is $1 \in M_{2,2}$ and has multiplicity one in $M_{2,2} \cup M_{2,3} \cup M_{2,1} = \{-2, -2, -1, 1\}$, the maximum of the fourth row is $0 \in M_{4,3}$ and has multiplicity one in $M_{4,3} \cup M_{4,1} = \{-2, -1, 0\}$, the maximum of the first row is $0 \in M_{1,1}$ and has multiplicity one in $M_{1,1} = \{-1, 0\}$.

Theorem 2.6. *If $\mathcal{P}$ satisfies the I-test, it is aspherical.*

Proof. Suppose $\mathcal{P}$ satisfies the I-test and let v, $j_1, j_2, \ldots, j_m$ and $i_1, i_2, \ldots, i_m$ be as in the definition. If $\mathcal{P}$ is not aspherical, there is a nontrivial family of integer numbers $\{n_g^j\}_{j,g}$

with finite support, which is a solution to the equations $E_{i,g}$. Let $\alpha = \min\{\overline{h} \mid h \in G$ and there exists $1 \leq j \leq m$ such that $n_h^j \neq 0\}$. Let $t = \min\{1 \leq l \leq m \mid$ there exists $h \in G$ with $n_h^{j_l} \neq 0$ and $\overline{h} = \alpha\}$ and let $h \in G$ be such that $n_h^{j_t} \neq 0$ and $\overline{h} = \alpha$. The equation $E_{i_t,g}$ is

$$\sum_{1 \leq j \leq m} \sum_{k \in \mathrm{occ}(x_{i_t}, r_j)} \epsilon_{k,r_j} n_{gs(k,r_j)}^j = 0.$$

The maximum of M_{i_t,j_t} is $\lambda = \overline{s(k, r_{j_t})}$ for certain $k \in \mathrm{occ}(x_{i_t}, r_{j_t})$. Let $g = hs(k, r_{j_t})^{-1}$. Then, the term $n_h^{j_t}$ appears in the equation $E_{i_t,g}$. Since $n_h^{j_t} \neq 0$, there must be a second nonzero term in the equation. However, $\alpha = \overline{h} = \overline{g} + \lambda$ is the maximum among the weights of all the subindices in $E_{i_t,g}$. Therefore, any nonzero term must have subindex of weight α. This cannot happen for terms with superindex j_l and $l < t$ by definition of t and this cannot happen for the other terms either, by the hypothesis on the multiplicity of λ in $\bigcup_{l=t}^m M_{i_t,j_l}$. $\qquad\square$

In the proof of Theorem 2.6 we have only used part of the information given by the equations $E_{i,g}$. Namely, that if the sum of the terms in each equation is zero, then they are all zero or there are at least two nonzero terms. Therefore we have proved that the I-test guarantees that $\mathcal{P}$ satisfies the following property (p): If $\{n_g^j\}_{j,g}$ is a family of integer numbers with finite support, and in each equation $E_{i,g}$ all the terms are zero or there are at least two nonzero terms (we do not actually need the family to satisfy the equations), then all the n_g^j must be trivial.

We will see that property (p) is equivalent to the notion of diagrammatic reducibility. Recall that a combinatorial 2-complex K (for instance, the complex associated with a presentation) is said to be diagrammatically reducible (DR) if for each cell structure C on S^2 and each combinatorial map $f : C \to K$, there exist distinct 2-cells $e^2, \widetilde{e}^2$ of C with a common 1-face e^1 and a homeomorphism $h : e^2 \to \widetilde{e}^2$ fixing e^1 pointwise such that $f|_{\widetilde{e}^2}h = f|_{e^2}$. We will say that the presentation $\mathcal{P}$ is DR if its standard complex is DR. This notion is stronger than asphericity. For basic definitions and applications of diagrammatic reducibility see [9, 10]. In any case, for our purposes only the following characterization will be needed. This result, due to Corson and Trace, appeared in [8, Theorem 2.4].

Theorem 2.7 (Corson-Trace). *A combinatorial 2-complex K is DR if and only if every finite subcomplex of the universal cover $\widetilde{K}$ collapses to a 1-dimensional complex.*

To prove that $\mathcal{P}$ satisfies property (p) if and only if $K_\mathcal{P}$ is DR, we argue as follows. Suppose $K_\mathcal{P}$ is not DR. Then there exists a finite subcomplex L of $\widetilde{K}_\mathcal{P}$ of dimension 2 without free faces. We define a family $\{n_g^j\}_{j,g}$ associated to L. For each $1 \leq j \leq m$ and $g \in G$ define $n_g^j = 1$ if $e_{j,g}^2 \in L$ and $n_g^j = 0$ otherwise. Then $\{n_g^j\}_{j,g}$ is a nontrivial family with finite support which is not a solution to the system $\{E_{i,g}\}_{i,g}$, in general. However, since each 1-cell $e_{i,g}^1$ is a proper face of 0 or ≥ 2 cells, by Remark 2.2, each equation $E_{i,g}$ has zero or at least two nonzero terms. Conversely, if $\{n_g^j\}_{j,g}$ is a nontrivial family with finite support and each equation $E_{i,g}$ has zero or at least two nonzero terms, then the subcomplex of $\widetilde{K}_\mathcal{P}$ generated by the cells $e_{j,g}^2$ such that $n_g^j \neq 0$, is finite, 2-dimensional, and has no free faces. Then $K_\mathcal{P}$ is not DR. Therefore, we have proved

Theorem 2.8. *If $\mathcal{P}$ satisfies the I-test, $K_{\mathcal{P}}$ is DR.*

Note that the I-test can only be applied to presentations $\mathcal{P}$ such that the presented group G is indicable and the deficiency of $\mathcal{P}$ is non-negative, i.e. $n \geq m$. The positivity of the deficiency automatically implies indicability.

Remark 2.9. Given a presentation $\mathcal{P}$, the set $V \subseteq \mathbb{R}^n$ of vectors v orthogonal to every $q(r_j)$ is naturally described as the solution of a system of m linear equations in n variables. Given $v \in V$, it is easy to decide algorithmically whether $\mathcal{P}$ satisfies the I-test with respect to v. Moreover, if $\{v_1, v_2, \ldots, v_k\}$ is a basis of V, then $\mathcal{P}$ satisfies the I-test with respect to a vector $v = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_k v_k$ and orderings $j_1, j_2, \ldots, j_m$ and $i_1, i_2, \ldots, i_m$ of columns and rows if and only if certain linear inequalities on the λ_i are satisfied. A system of equations of the form $\sum \lambda_i c_i \geq 0$ or $\sum \lambda_i c_i > 0$ has a solution if and only if the system obtained by replacing each $\sum \lambda_i c_i > 0$ by $\sum \lambda_i c_i \geq 1$ has a solution. The feasibility can be then decided with linear programming. Therefore it is algorithmically decidable whether a presentation satisfies the I-test for some vector v.

If $\mathcal{P}$ satisfies the I-test for v and orderings $j_1, j_2, \ldots, j_m$ and $i_1, i_2, \ldots, i_m$ of columns and rows, then any partial orderings $j_1', j_2', \ldots, j_l'$ and $i_1', i_2', \ldots, i_l'$ $(l \leq m)$ satisfying Definition 2.5 for $1 \leq k \leq l$ can be extended to orderings $j_1', j_2', \ldots, j_m'$ and $i_1', i_2', \ldots, i_m'$ such that $\mathcal{P}$ satisfies the I-test for those orderings.

Remark 2.10. If $\mathcal{P}$ satisfies the I-test, then any subpresentation of $\mathcal{P}$ does. In particular the Whitehead conjecture is true for presentations satisfying the I-test. The later follows also from Theorem 2.8.

The following example shows a class of presentations which satisfy our test.

Example 2.11. Let $n \in \mathbb{N}$ and let
$$\mathcal{P} = \langle x_1, x_2, \ldots, x_n, x | x_i x \omega_i = x \tau_i, 1 \leq i \leq n \rangle$$
where ω_i, τ_i are words in which x occurs only with positive exponent. Suppose further that for each i the total exponent $exp(x, \omega_i)$ of x in ω_i coincides with $exp(x, \tau_i)$. We show that $\mathcal{P}$ satisfies our test and therefore it is DR.

The vector $v = (0, 0, \ldots, 0, 1)$ is orthogonal to each $q(r_i)$, where $r_i = x_i x \omega_i (\tau_i)^{-1} x^{-1}$. The weight of $s(1, r_i) = r_i$ is 0. For any other occurrence $k \in \mathrm{occ}(x_i, r_i)$ of x_i in r_i, $\overline{s(k, r_i)} = exp(x, s(k, r_i)) < 0$. At the same time, all the elements in $M_{i,j}$ are negative for $j \neq i$. Then $\mathcal{P}$ satisfies the I-test with respect to v and the natural order $1, 2, \ldots, n$ of the columns and rows.

3. Relationship with previous weight tests

Gersten's weight test provides a useful tool for proving diagrammatic reducibility (and hence asphericity) of group presentations [9]. In [19] Huck and Rosebrock introduced a more general weight test which guarantees asphericity. We recall these criteria and exhibit examples of presentations which satisfy the I-test but not the previous weight tests.

As explained in [19], given a presentation $\mathcal{P} = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$, the *Whitehead graph* $W_{\mathcal{P}}$ is the boundary of a regular nighborhood of the unique vertex of $K_{\mathcal{P}}$. In other words, $W_{\mathcal{P}}$ is the undirected graph (possibly with parallel edges) which can be described as follows. For each word $w = x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \ldots x_{i_l}^{\epsilon_l}$, with $\epsilon_i = \pm 1$, consider the graph W_w whose vertex set is $\{x_1, -x_1, x_2, -x_2, \ldots, x_n, -x_n\}$ and with an edge $(\epsilon_k x_{i_k}, -\epsilon_{k+1} x_{i_{k+1}})$

for each $1 \leq k \leq l$ (subindices considered modulo l). Then $W_{\mathcal{P}}$ is the union of all the graphs W_{r_j}, $1 \leq j \leq m$.

Suppose that no relator r_j is a proper power. A *weight function* on $W_{\mathcal{P}}$ is a real valued function g on the edges of $W_{\mathcal{P}}$ (i.e. the *korners* of $K_{\mathcal{P}}$). A weight function g satisfies Huck and Rosebrock's weight test if

(1) For each $1 \leq j \leq m$, $\sum\limits_{e \in E(W_{r_j})} g(e) \leq \mathrm{l}(r_j) - 2$ and

(2) For each simple cycle z in $W_{\mathcal{P}}$, $\sum\limits_{e \in z} g(e) \geq 2$.

Theorem 2.2 in [19] claims that the existence of a weight function satisfying the test implies vertex asphericity, a concept which in turn implies asphericity of $\mathcal{P}$. Gersten's original weight test requires a stronger hypothesis on the weight function g and implies diagrammatic reducibility (see [9, Theorem 4.7]).

The following example shows a presentation which satisfies the I-test while no weight function g satisfies the weight test.

Example 3.1. Let $\mathcal{P} = \langle x, y | x^3 y x y \rangle$. Then $\mathcal{P}$ satisfies the I-test but it does not satisfy the weight test. The Whitehead graph of $\mathcal{P}$ appears in Figure 1.
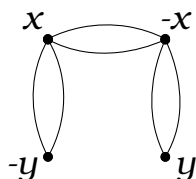

FIGURE 1. The Whitehead graph of $\mathcal{P} = \langle x, y | x^3 y x y \rangle$.

Suppose a weight function g satisfies the weight test. Condition (2) implies that the sum of the weights of the two edges from x to $-y$ is at least 2. The same holds for the two edges from x to $-x$ and from $-x$ to y. Then condition (1) gives $6 \leq \sum\limits_{e \in W_{\mathcal{P}}} g(e) \leq \mathrm{l}(x^3 y x y) - 2 = 4$, a contradiction.

The vector $v = (1, -2)$ is orthogonal to $q(x^3 y x y) = (4, 2)$. The coefficient of the second row in the weight matrix is $M_{2,1} = \{-3, -2\}$, so $\mathcal{P}$ satisfies the I-test.

Similarly, the presentation $\mathcal{P} = \langle x, y | x^3 y^3 x y \rangle$ provides another example of a presentation which satisfies the I-test but not the weight test.

The presentations of Example 2.11, which are proved to be DR by the I-test, do not satisfy in general the weight tests. Take for instance $\omega_1 = x_1^{-1} x$ and $\tau_1 = x_1 x x_1$. Then the weight test fails independently of the choices of ω_i, τ_i for $2 \leq i \leq n$.

4. ADIAN PRESENTATIONS AND LOGS

We now apply our test to generalize some results of Howie and Gersten on LOTs and Adian presentations. This will produce extensive families of DR presentations.

Recall that a labeled oriented graph (LOG) Γ consists of two finite sets V, E, and three maps $\iota, \tau, \lambda : E \to V$. The elements of V are called vertices and the elements of E, edges. If $e \in E$, $\iota(e)$ is called the initial vertex of e, $\tau(e)$ its terminal vertex, and $\lambda(e)$ its label. A LOG Γ is called a labeled oriented tree (LOT) if the underlying graph is a tree. Associated

with a LOG Γ there is a presentation. An edge e of Γ with initial vertex x, terminal vertex y and label z, has an associated relator $r_e = xzy^{-1}z^{-1}$. The presentation corresponding to a LOG Γ with vertex set $V = \{x_1, x_2, \ldots, x_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$ is $\mathcal{P}_\Gamma = \langle x_1, x_2, \ldots, x_n \mid r_{e_1}, r_{e_2}, \ldots, r_{e_m} \rangle$.

Given a LOG Γ, consider the undirected graph $I(\Gamma)$ with the same vertex set as Γ and with an edge (x, y) for each edge of Γ with label x and terminal vertex y. Similarly, define $T(\Gamma)$ with the same vertex set and an edge (x, y) for each edge of Γ with initial vertex x and label y. The following result by Howie ([17, Corollary 10.2]) says that these graphs can be used to prove asphericity of Γ.

Theorem 4.1 (Howie). *If either $I(\Gamma)$ or $T(\Gamma)$ is a tree, $\mathcal{P}_\Gamma$ is aspherical.*

We will prove a generalization of this result.

Definition 4.2. Let Γ be a LOG and let Γ' be a sub-LOG (i.e. an *admissible* subgraph, in Howie's terminology [17]). We say that there is a *deforestation of type IL/T* from Γ to Γ' if the number of edges in $\Gamma \smallsetminus \Gamma'$ equals the number of vertices in $\Gamma \smallsetminus \Gamma'$ and there exists an ordering $e_1, e_2, \ldots, e_m$ of the edges of $\Gamma \smallsetminus \Gamma'$ and an ordering $x_1, x_2, \ldots, x_m$ of the vertices of $\Gamma \smallsetminus \Gamma'$ such that for each $1 \leq j \leq m$ one of the following holds:

- (IL) The number of edges in $\Gamma_{j-1} = \Gamma \smallsetminus \{e_1, e_2, \ldots, e_{j-1}\}$ which have initial vertex x_j plus the number of edges in Γ_{j-1} having label x_j is one. Moreover, this unique edge with initial vertex or label x_j is e_j.
- (T) x_j is the terminal vertex of a unique edge in Γ_{j-1}, this edge being e_j. Moreover, there is no edge in Γ whose initial vertex or label is x_j.

A *deforestation of type TL/I* is defined similarly by interchanging the words "initial" and "terminal". A LOG Γ is *deforestable* if there is a deforestation of type IL/T to a discrete subgraph or a deforestation of type TL/I to a discrete subgraph. Of course, a LOG Γ is deforestable if and only if its opposite Γ^{op} is deforestable.

Proposition 4.3. *If Γ is deforestable LOG, $\mathcal{P}_\Gamma$ satisfies the I-test.*

Proof. Suppose there is a deforestation of type IL/T from Γ to a discrete sub-LOG Γ'. Let $e_1, e_2, \ldots, e_m$ and $x_1, x_2, \ldots, x_m$ be as in Definition 4.2. Let $x_{m+1}, x_{m+2}, \ldots, x_n$ be the vertices of Γ'. Then $\mathcal{P}_\Gamma = \langle x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n \mid r_1, r_2, \ldots, r_m \rangle$, where r_j is the relation associated to e_j.

Suppose e_j is an edge of Γ with initial vertex x, terminal vertex y and label z (we do not require them to be different). Then $r_j = xzy^{-1}z^{-1}$. In particular $v = (1, 1, \ldots, 1)$ is orthogonal to each $q(r_j)$. We will prove that $\mathcal{P}_\Gamma$ satisfies the I-test for v and the natural order of the columns and rows of M. The coefficient $M_{i,j}$ consists of a (possibly empty) family which contains the elements -1 and 0 with certain multiplicity. If x_i is the terminal vertex of e_j, then -1 is an element of this family, if x_i is the initial vertex, then $0 \in M_{i,j}$, and if x_i is the label, $-1, 0 \in M_{i,j}$.

Let $1 \leq j \leq m$. If condition (IL) holds, then 0 is an element of $M_{j,j}$ of multiplicity one and $0 \notin M_{j,k}$ for each $k > j$. If condition (T) holds, then $M_{j,j} = \{-1\}$, $M_{j,k} = \emptyset$ for each $k > j$ and $M_{j,k} = \emptyset$ or $\{-1\}$ for each $1 \leq k \leq m$. This proves that $\mathcal{P}_\Gamma$ satisfies our test.

Finally, if there is a deforestation of type TL/I from Γ to a discrete sub-LOG, $\mathcal{P}_\Gamma$ satisfies the I-test for $v = (-1, -1, \ldots, -1)$. $\square$

Proposition 4.4. *Let Γ be a LOG. If $T(\Gamma)$ or $I(\Gamma)$ has no cycles, Γ is deforestable.*

Proof. Suppose $T(\Gamma)$ has no cycles and that x_1 is a leaf of $T(\Gamma)$. Then there is a unique edge e_1 of Γ whose initial vertex or label is x_1, but not both simultaneously.

Let $\Gamma_1 = \Gamma \smallsetminus \{e_1\}$ and let T_1 be the subgraph of $T(\Gamma)$ induced by the vertices different from x_1. By induction suppose defined $x_1, x_2, \ldots, x_j$ and $e_1, e_2, \ldots, e_j$. Let $\Gamma_j = \Gamma \smallsetminus \{e_1, e_2, \ldots, e_j\}$ and let T_j be the subgraph of $T(\Gamma)$ induced by the vertices different from $x_1, x_2, \ldots, x_j$. Let x_{j+1} be a leaf of T_j. Then there is a unique edge e_{j+1} of Γ_j whose initial vertex or label is x_{j+1}. This process ends when T_j is discrete for some j, say $j = m$, and, then, Γ_m is also discrete. The orders $e_1, e_2, \ldots, e_m$ and $x_1, x_2, \ldots, x_m$ constructed show that there is a deforestation of type IL/T from Γ to $\Gamma' = \Gamma_m \smallsetminus \{x_1, x_2, \ldots, x_m\}$. Therefore, Γ is deforestable. Note that condition (T) is never used. If $I(\Gamma) = T(\Gamma^{op})$ has no cycles, Γ^{op} is deforestable, and then so is Γ. $\square$

Example 4.5. The following LOT is deforestable but both $I(\Gamma)$ and $T(\Gamma)$ have cycles.

There is a deforestation of type IL/T from Γ to a one-vertex LOT. An order of the vertices which satisfies Definition 4.2 is $5, 3, 2, 1$.

In Section 6 we will extend Proposition 4.3 to a wider class of LOGs.

An *Adian* presentation is a presentation $\mathcal{P} = \langle x_1, x_2, \ldots, x_n | U_j = V_j, j \in J \rangle$ where U_j and V_j are non-trivial positive words. Gersten defines the *left graph* $L(\mathcal{P})$ of $\mathcal{P}$ as the undirected graph whose vertices are the first letters of the words U_j and the first letters of the words V_j. For each $j \in J$ there is an edge (the j-th edge) from the first letter of U_j to the first letter of V_j. The *right graph* $R(\mathcal{P})$ is defined similarly considering the last letters of the words U_j, V_j. Gersten proved the following result [9, Proposition 4.15].

Proposition 4.6 (Gersten). *Let $\mathcal{P} = \langle x_1, x_2, \ldots, x_n | U_j = V_j, j \in J \rangle$ be an Adian presentation such that $l(U_j) = l(V_j)$ for each $j \in J$. If either $L(\mathcal{P})$ or $R(\mathcal{P})$ has no cycles, $K_\mathcal{P}$ is DR.*

Note that the presentation $\mathcal{P}_\Gamma$ associated to a LOG Γ is Adian, and in this case $L(\mathcal{P}) = T(\Gamma)$ and $R(\mathcal{P}) = I(\Gamma)$. Hence, Proposition 4.6 generalizes Theorem 4.1. We will now use our methods to prove a generalization of Proposition 4.6.

Let $\mathcal{P} = \langle x_1, x_2, \ldots, x_n | U_j = V_j, j \in J \rangle$ be an Adian presentation. We label each edge of $L(\mathcal{P})$ with a family of generators of $\mathcal{P}$ in the following way. Given a generator $x = x_i$, consider those words U_j, V_j containing x, in which x occurs in the left-most position among all the words U_k, V_k. Then label the corresponding j-th edges of $L(\mathcal{P})$ with x. An edge may have various labels. In particular, if x occurs simultaneously in U_j and V_j in the left-most position among all the U_k and V_k, then the j-th edge will have the label x twice. Note that every generator which appears in some relator will be label of at least one edge of $L(\mathcal{P})$. Note also that the vertices adjacent to an edge are always labels of that edge. These labels are called *trivial* and they are not written in the graphical representation of $L(\mathcal{P})$. Similarly, we label the edges of the graph $R(\mathcal{P})$ with the generators that occur at the right-most positions among all the words.

Example 4.7. Let $\mathcal{P} = \langle x, y, z, t, u, v | xuz^2 = yutv, yu = zv, y^3 = tx^2, tv^2 = zxu, tx = zy \rangle$. Note that both $L(\mathcal{P})$ and $R(\mathcal{P})$ have cycles. Figure 2 shows the labeled left and right graphs of $\mathcal{P}$.
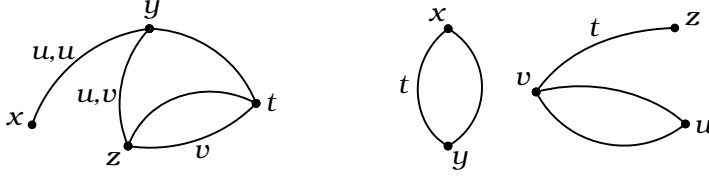
FIGURE 2. The labeled left and right graphs of $\mathcal{P}$.

Definition 4.8. Let G be an undirected graph in which each edge is labeled with a family of elements in a set S. If e is an edge of G which is labeled with a family that contains an element $s \in S$, which appears only once in total in all the labels of G, then we can remove the edge e from G. This removal is called a *deletion*. The graph G is called *discretizable* if a graph with no edges can be obtained from G by a sequence of deletions.

Remark 4.9. If $\mathcal{P}$ is an Adian presentation and $L(\mathcal{P})$ has no cycles, then $L(\mathcal{P})$ is discretizable. Note that if $x = x_i$ is a leaf of $L(\mathcal{P})$, then the edge incident to x is the unique edge labeled with x. Thus we can delete leaf edges one at the time. Analogously, if $R(\mathcal{P})$ has no cycles, it is discretizable. Example 4.7 shows a presentation $\mathcal{P}$ such that $L(\mathcal{P})$ is discretizable, although it has cycles. Since x is a leaf of $L(\mathcal{P})$, the edge xy can be deleted. Now, the generator u appears in a unique label, so edge yz can be removed. Then we can delete yt, then the edge containing v in its label, and finally the last edge. By Theorem 4.10 below, this presentation is DR. Note that the right graph is not discretizable: although the edge zv and both edges between x and y can be deleted, the edges between u and v cannot be removed.

Theorem 4.10. *Let $\mathcal{P} = \langle x_1, x_2, \ldots, x_n | U_j = V_j, j \in J \rangle$ be an Adian presentation such that $l(U_j) = l(V_j)$ for each $j \in J$. If either $L(\mathcal{P})$ or $R(\mathcal{P})$ is discretizable, $\mathcal{P}$ satisfies the I-test.*

Proof. Suppose $L(\mathcal{P})$ is discretizable. We prove that $\mathcal{P}$ satisfies the I-test for the vector $v = (1, 1, \ldots, 1)$. If x_i is the k-th letter of U_j, the weight $\overline{s(k, U_j V_j^{-1})} = 1 - k$ is an element of $M_{i,j}$. If x_i is the k-th letter of V_j, then $\overline{s(2l(U_j) - k + 1, U_j V_j^{-1})} = 1 - k \in M_{i,j}$. Thus, if x_i appears only once in the labels of $L(\mathcal{P})$, in the j-th edge, then x_i is the k-th letter of U_j or of V_j for some k (but not of both), the first $k - 1$ letters of U_j and V_j are different from x_i, and the first k letters of $U_{j'}$ and $V_{j'}$ are different from x_i for every $j' \neq j$. Then $1 - k \in M_{i,j}$ is the maximum of the whole i-th row in the weight matrix M and it appears with multiplicity one. The order of deletion of the edges of $L(\mathcal{P})$ induces an order in the columns and $|J|$ of the rows which proves that $\mathcal{P}$ satisfies the I-test. If $R(\mathcal{P})$ is discretizable, $\mathcal{P}$ satisfies our test for $v = (-1, -1, \ldots, -1)$. $\square$

Remark 4.11. If Γ is a LOG, then $L(\mathcal{P}_\Gamma)$ or $R(\mathcal{P}_\Gamma)$ is discretizable if and only if Γ is deforestable.

Remark 4.12. These ideas can be easily extended to study diagrammatic reducibility of more general presentations $\mathcal{P} = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$ such that the total exponent of each relator r_j is 0 and each proper final segment $(r_j)^{(k)}$ has negative total exponent. In this case, define $L(\mathcal{P})$ to be the undirected graph whose vertices are the first and last letters of the relators r_j and with an edge (the j-th edge) from the first to the last letter of r_j for each $1 \leq j \leq m$. The labels of the edges in $L(\mathcal{P})$ are the following. For each

generator x_i consider all the total exponents of the words $s(k, r_j)$ with $1 \leq j \leq m$ and $k \in \mathrm{occ}(x_i, r_j)$ and label the j-th edge of $L(\mathcal{P})$ with x_i if the maximum of those exponents is attained at r_j, once for each time this number is attained at r_j. Then, if $L(\mathcal{P})$ is discretizable, $\mathcal{P}$ is aspherical.

5. IVANOV'S ASPHERICITY CONJECTURE

In [20] Ivanov proposed the following conjecture.

Conjecture 5.1 (Ivanov). *Let $\mathcal{P} = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$ be an aspherical presentation and let $\mathcal{Q} = \langle x_1, x_2, \ldots, x_n, x | r_1, r_2, \ldots, r_m, r \rangle$ be such that:*

- *The total exponent of x in r is non-zero,*
- *The group H presented by $\mathcal{P}$ naturally embeds in the group G presented by $\mathcal{Q}$, and*
- *G is torsion-free.*

Then $\mathcal{Q}$ is aspherical.

It is unknown whether the first hypothesis implies the second one (this is in fact the statement of the Kervaire Conjecture). Ivanov's conjecture is related to the Kaplansky conjecture on zero divisors of a group ring. If Conjecture 5.1 is false, there is a torsion free group G such that $\mathbb{Z}G$ has zero divisors ([20, Theorem 1]). In this section we will prove two results related to Conjecture 5.1.

Lemma 5.2. *Let $\mathcal{P} = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$ be an aspherical presentation and let $\mathcal{Q} = \langle x_1, x_2, \ldots, x_n, x | r_1, r_2, \ldots, r_m, r \rangle$ be such that the group H presented by $\mathcal{P}$ naturally embeds in the group G presented by $\mathcal{Q}$. Moreover, suppose that for any solution $\{n_g^j\}_{1 \leq j \leq m, g \in G} \cup \{n_g\}_{g \in G}$ of the system $\{E_{i,g}\}_{1 \leq i \leq n, g \in G} \cup \{E_g\}_{g \in G}$ with finite support, we have that $n_g = 0$ for every $g \in G$. Here n_g denotes the coefficients corresponding to the relator r and E_g the equations associated to the generator x. Then $\mathcal{Q}$ is aspherical.*

Proof. Let $\{n_g^j\}_{1 \leq j \leq m, g \in G} \cup \{n_g\}_{g \in G}$ be a solution of the system $\{E_{i,g}\}_{1 \leq i \leq n, g \in G} \cup \{E_g\}_{g \in G}$ with finite support. We want to show that all the n_g^j and n_g are zero. By hypothesis $n_g = 0$ for every $g \in G$. Let $k \in G$. Let $\iota : H \to G$ be the natural embedding. For each $1 \leq j \leq m$ and each $h \in H$, define $\widetilde{n}_h^j = n_{k\iota(h)}^j$. Since $n_g = 0$ for every $g \in G$, the equation $\widetilde{E}_{i,h}$ associated to the presentation $\mathcal{P}$, reduces to equation $E_{i,k\iota(h)}$ associated to $\mathcal{Q}$. Therefore, $\{\widetilde{n}_h^j\}_{1 \leq j \leq m, h \in H}$ is a solution of the system $\{\widetilde{E}_{i,h}\}_{1 \leq i \leq n, h \in H}$. Moreover, since $n_g^j = 0$ except for finitely many (g, j) and the kernel of ι is finite (trivial, in fact), only finitely many $\widetilde{n}_h^j$ are non-zero. Theorem 2.1 implies then that $\widetilde{n}_h^j = 0$ for every $h \in H$ and $1 \leq j \leq m$. In particular, $n_k^j = \widetilde{n}_1^j = 0$ for every j. Since $k \in G$ is arbitrary, by Theorem 2.1, $\mathcal{Q}$ is aspherical. $\square$

Recall that a group G is said to have the *unique product property (upp)* if for any two non-empty finite subsets $A, B \subseteq G$, there exists $g \in G$ such that $gA \cap B$ has exactly one element. It is easy to see that if G has the upp, then it is torsion-free and the group ring $\mathbb{Z}G$ has no zero divisors (see [28] for examples of torsion-free groups without the upp). The result of Ivanov [20, Theorem 1] mentioned above implies then that his conjecture is true if G has the upp. We give next a direct proof of this result using Lemma 5.2.

Proposition 5.3. *Ivanov's conjecture is true if the group G presented by $\mathcal{Q}$ has the unique product property.*

Proof. Let $\{n_g^j\}_{1 \le j \le m, g \in G} \cup \{n_g\}_{g \in G}$ be a solution with finite support of the system $\{E_{i,g}\}_{1 \le i \le n, g \in G} \cup \{E_g\}_{g \in G}$ associated to $\mathcal{Q}$. Since the generator x does not appear in the relators r_j, the equation E_g is

$$\sum_{k \in \mathrm{occ}(x,r)} \epsilon_{k,r} n_{gs(k,r)} = 0,$$

where $\mathrm{occ}(x,r)$ denotes the set of occurrences of x in r. Since the total exponent $\sum \epsilon_{k,r}$ of x in r is non-zero, the equation above is not trivial, and it can be rewritten as

$$(1) \qquad \sum_{i=1}^{p} a_i n_{gs_i} = 0,$$

where $p \ge 1$, $0 \ne a_i \in \mathbb{Z}$ for every $1 \le i \le p$, and $s_1, s_2, \ldots, s_p$ are different elements of G. Let $A = \{s_1, s_2, \ldots, s_p\}$ and $B = \{g \in G | n_g \ne 0\}$. Suppose B is non-empty. By hypothesis, there exists $g \in G$ such that $gA \cap B$ has a unique element, say $h \in G$. Then equation (1) above says that $n_h = 0$, a contradiction. Thus, $B = \emptyset$ and Lemma 5.2 implies that $\mathcal{Q}$ is aspherical. $\qquad \square$

As explained in [20], Howie's results show that Conjecture 5.1 is true when G (equivalently H) is locally indicable, that is, if every nontrivial finitely generated subgroup of G is indicable (this follows also from Proposition 5.3). The next result improves this. Concretely, if x occurs k times in r, then we only have to check that certain $2^k - \frac{k(k+1)}{2} - 1$ subgroups of G are indicable. In the case of 4 occurrences, for instance, the asphericity of $\mathcal{Q}$ would follow from the indicability of a subgroup of G generated by 3 elements and four subgroups generated by 2 elements each. Recall that a group G is said to be left-orderable if there exists a total order $\le$ in the underlying set of G such that $h \le h'$ implies $gh \le gh'$ for every $h, h', g \in G$.

Theorem 5.4. *Under the hypothesis of Conjecture 5.1, suppose that for every subset $S \subseteq \{s(k,r) \in G | k \in occ(x,r)\}$ with more than two elements, the subgroup G_S of G generated by the elements $s^{-1}t$ for $s, t \in S$, is indicable, or, more generally, there exists a nontrivial homomorphism $G_S \to L_S$ to a left-orderable group L_S. Then $\mathcal{Q}$ is aspherical.*

Proof. For simplicity we assume that each subgroup G_S in the statement is indicable. The proof of the general case is identical. Suppose $\{n_g^j\}_{1 \le j \le m, g \in G} \cup \{n_g\}_{g \in G}$ is a solution with finite support of the system $\{E_{i,g}\}_{1 \le i \le n, g \in G} \cup \{E_g\}_{g \in G}$. Assume there exists $g_0 \in G$ such that $n_{g_0} \ne 0$. Similarly as in the proof of Proposition 5.3, let

$$(1) \qquad \sum_{i=1}^{p} a_i n_{gs_i} = 0,$$

be the equation E_g associated to x and $g \in G$, where $p \ge 1$, a_i are nonzero integers and $s_1, s_2, \ldots, s_p$ are pairwise different elements of G. Then $\{s_1, s_2, \ldots, s_p\} \subseteq \{s(k,r) \in G | k \in \mathrm{occ}(x,r)\}$. Let $G_1 = \langle s_i^{-1} s_j | 1 \le i, j \le p \rangle$. If $p_1 = p > 1$, by hypothesis there exists a nontrivial homomorphism $\varphi_1 : G_1 \to \mathbb{Z}$ (for $p = 2$ this follows from the fact that G is torsion-free). Let $m_1 = \min\{\varphi_1(g) | g \in G_1$ and $n_{g_0 g} \ne 0\}$. Note that m_1 is well-defined since $n_{g_0} \ne 0$ and only finitely many n_g are non-zero. Let $g_1 \in G_1$ be such that $n_{g_0 g_1} \ne 0$ and $\varphi_1(g_1) = m_1$. Let $M_1 = \max\{\varphi_1(s_1), \varphi_1(s_2), \ldots, \varphi_1(s_{p_1})\}$. By reordering the s_i if needed, there exists $1 \le p_2 \le p_1$ such that $\varphi_1(s_i) = M_1$ for $1 \le i \le p_2$ and $\varphi_1(s_i) < M_1$ for $p_2 < i \le p_1$. Since φ_1 is non-trivial, $p_2 < p_1$. Let $G_2 = \langle s_i^{-1} s_j | 1 \le i, j \le p_2 \rangle$.

If $p_2 > 1$, by hypothesis there exists a nontrivial homomorphism $\varphi_2 : G_2 \to \mathbb{Z}$. Let $m_2 = \min\{\varphi_2(g)|g \in G_2 \text{ and } n_{g_0g_1g} \neq 0\}$. Let $g_2 \in G_2$ be such that $n_{g_0g_1g_2} \neq 0$ and $\varphi_2(g_2) = m_2$. Let $M_2 = \max\{\varphi_2(s_1), \varphi_2(s_2), \ldots, \varphi_2(s_{p_2})\}$. By reordering the s_i, there exists $1 \leq p_3 \leq p_2$ such that $\varphi_2(s_i) = M_2$ for $1 \leq i \leq p_3$ and $\varphi_2(s_i) < M_2$ for $p_3 < i \leq p_2$. Since φ_2 is non-trivial, $p_3 < p_2$. This process ends when $p_{l+1} = 1$ for some l. Then $m_l = \min\{\varphi_l(g)|g \in G_l \text{ and } n_{g_0g_1\ldots g_{l-1}g} \neq 0\}$, the minimum attained by some $g_l \in G_l$ and the maximum $M_l = \max\{\varphi_l(s_1), \varphi_l(s_2), \ldots, \varphi_l(s_{p_l})\}$ is only attained by s_1. Note that $G_1 \supseteq G_2 \supseteq \ldots \supseteq G_l$.

Let $g = g_0g_1\ldots g_l s_1^{-1} \in G$. Equation (1) becomes then

$$\sum_{i=1}^{p} a_i n_{g_0g_1\ldots g_l s_1^{-1}s_i} = 0.$$

Let $i > p_2$. Since φ_1 is trivial in G_2, $\varphi_1(g_1g_2\ldots g_l s_1^{-1}s_i) = \varphi_1(g_1) - \varphi_1(s_1) + \varphi_1(s_i) < \varphi_1(g_1) = m_1$, thus $n_{g_0g_1\ldots g_l s_1^{-1}s_i} = 0$. Then

$$\sum_{i=1}^{p_2} a_i n_{g_0g_1\ldots g_l s_1^{-1}s_i} = 0.$$

If $i > p_3$, then $\varphi_2(g_2g_3\ldots g_l s_1^{-1}s_i) = \varphi_2(g_2) - \varphi_2(s_1) + \varphi_2(s_i) < \varphi_2(g_2) = m_2$, thus $n_{g_0g_1\ldots g_l s_1^{-1}s_i} = 0$ and

$$\sum_{i=1}^{p_3} a_i n_{g_0g_1\ldots g_l s_1^{-1}s_i} = 0.$$

Following this reasoning we conclude $a_1 n_{g_0g_1\ldots g_l} = 0$, which contradicts the definition of g_l. The contradiction arose from the faulty assumption that there exists $g_0 \in G$ with $n_{g_0} \neq 0$. By Lemma 5.2, $\mathcal{Q}$ is aspherical.

$\square$

The following result is a weak version of the conjecture, where we are allowed to perturb the relation r.

Theorem 5.5. *Let* $\mathcal{P} = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$ *be an aspherical presentation of an indicable group* H *and let* $\mathcal{Q} = \langle x_1, x_2, \ldots, x_n, x | r_1, r_2, \ldots, r_m, r \rangle$ *be such that the total exponent of* x *in* r *is non-zero. Then there exists a cyclic permutation* r' *of* r, $1 \leq i \leq n$ *and* $M_0 \in \mathbb{N}$ *such that, for every integer* M *with* $|M| \geq M_0$, *the perturbed presentation*

$$\mathcal{Q}' = \langle x_1, x_2, \ldots, x_n, x | r_1, r_2, \ldots, r_m, x_i^M r' \rangle$$

is aspherical if H *naturally embeds in the group* G' *presented by* $\mathcal{Q}'$.

Proof. Let β be the total exponent of x in r. Replacing r by r^{-1} if needed, we can assume $\beta > 0$. Let $\text{occ}(x, r) = \{k_1, k_2, \ldots, k_p\}$ be the set of occurrences of x in r, with $1 \leq k_1(r) < k_2(r) < \ldots < k_p(r) \leq l(r)$. Consider the sequence

$$S(r) : a_1(r), a_2(r), \ldots, a_p(r)$$

where a_i is the total exponent of x in the subword $s(k_i, r)$. We will prove the following

Claim: there exists a cyclic permutation r' of r such that $k_1(r') = 1$, $\epsilon_{1,r'} = +1$ (that is, that the first letter of r' is x^{+1}), and $a_i(r') < a_1(r') = \beta$ for every $i > 1$.

If r' is a cyclic permutation of r which begins with x^{+1}, then it is easy to see that

(i) The sequence $S(r')$ begins with $a_1(r') = \beta$,

(ii) $|a_{i+1}(r') - a_i(r')| \leq 1$ for every $1 \leq i < p$,

(iii) $a_p(r')$ is 0 or 1 (it is 0 if $\epsilon_{k_p(r'),r'} = -1$ and 1 if $\epsilon_{k_p(r'),r'} = 1$).

Suppose that for a given cyclic permutation $r' = r'_1$ of r which begins with x^{+1}, there exists $1 < i \leq p$ such that $a_i(r_1) \geq \beta$. Assume $i = i_1$ is maximum with that property. Then by (ii) and (iii), $a_i(r') = \beta$. We will show that $\epsilon_{k_i(r'),r'} = 1$. Assume on the contrary that $\epsilon_{k_i(r'),r'} = -1$, then $i \neq p$ by (iii), since $\beta \neq 0$. Now there are two cases: $\epsilon_{k_{i+1}(r'),r'} = -1$ or $\epsilon_{k_{i+1}(r'),r'} = 1$. In the first case, $a_{i+1}(r') = a_i(r') + 1 = \beta + 1$, and in the second, $a_{i+1}(r') = a_i(r') = \beta$. Any of these contradicts the maximality of i. We conclude then that $\epsilon_{k_i(r'),r'} = 1$. Note that the total exponent of x in the subword $w = w_1$ given by the first $k_i(r') - 1$ letters of r' is 0. We can consider then the cyclic permutation r'_2 of $r' = r'_1$ which begins with the $k_{i_1}(r'_1)$-th letter of r'_1 and repeat this argument. If r'_2 does not satisfy the property of the Claim, then there exists $1 < i_2 \leq p$ such that the total exponent of x in the subword w_2 given by the first $k_{i_2}(r'_2) - 1$ letters of r_2 is 0. If the Claim is false, there is a sequence $w_1, w_2, \ldots$ of subwords of cyclic permutations of r, all of which have zero total exponent for x. Moreover, the infinite word $w_1 w_2 w_3 \ldots$ coincides with the infinite word $r' r' r' \ldots$. There exist then $k, l \in \mathbb{N}$ with $k < l$ such that $w_k w_{k+1} \ldots w_l$ is a nontrivial power $(r'_0)^N$ of a cyclic permutation r'_0 of r. But this is absurd since the total exponent of x in $w_k w_{k+1} \ldots w_l$ is 0, while the exponent of x in $(r'_0)^N$ is $N\beta > 0$. This finishes the proof of the Claim.

Let r' be a cyclic permutation of r which satisfies the properties stated in the Claim. Let $q : F(x_1, x_2, \ldots, x_n) \to \mathbb{Z}^n$ and $q' : F(x_1, x_2, \ldots, x_n, x) \to \mathbb{Z}^{n+1}$ denote the abelianization maps. Since H is indicable, there exists a nonzero vector $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ orthogonal to each $q(r_j)$. Let $1 \leq i \leq n$ be such that $v_i \neq 0$ and let $M \in \mathbb{Z}$. Define $\tilde{r} = x_i^M r'$, let $\mathcal{Q}'$ be the presentation defined in the statement of the theorem and let G' be the group presented by $\mathcal{Q}'$. Let $q'(r) = (\beta_1, \beta_2, \ldots, \beta_i, \ldots, \beta_n, \beta)$. Then $q'(\tilde{r}) = (\beta_1, \beta_2, \ldots, \beta_i + M, \ldots, \beta_n, \beta)$. Define $\alpha_M = -\frac{\sum v_l \beta_l + M v_i}{\beta}$. Then $v'_M = (v_1, v_2, \ldots, v_n, \alpha_M) = (v_1, v_2, \ldots, v_n, \alpha_0) - (0, 0, \ldots, 0, \frac{M v_i}{\beta})$ is orthogonal to each $q'(r_j)$ and to $q'(\tilde{r})$.

Let $g \in G'$ and let E_g

$$0 = \sum_{k \in \text{occ}(x, \tilde{r})} \epsilon_{k, \tilde{r}} n_{gs(k, \tilde{r})} = \sum_{k \in \text{occ}(x, r')} \epsilon_{k, r'} n_{gs(k, r')} = \sum_{j=1}^{p} \epsilon_{k_j(r'), r'} n_{gs(k_j(r'), r')}$$

be the equation associated to x and g in $\mathcal{Q}'$.

The weight of $s(k_j(r'), r')$ is $\overline{s(k_j(r'), r')} = \langle q'(s(k_j(r'), r')), v'_M \rangle = \langle q'(s(k_j(r'), r')), v'_0 \rangle - \langle q'(s(k_j(r'), r')), (0, 0, \ldots, 0, \frac{M v_i}{\beta}) \rangle = c_j - \frac{M v_i}{\beta} a_j(r')$, where c_j is a constant which does not depend on M, and $a_j(r')$ is the total exponent of x in $s(k_j(r'), r')$, which appears in the Claim above. By the Claim $a_j(r') < a_1(r') = \beta$ for every $j > 1$. Then, if $|M|$ is big enough, the family of weights

$$\{\overline{s(k_j(r'), r')} \mid 1 \leq j \leq p\}$$

has a maximum or a minimum with multiplicity one. Then, the ideas of Section 2 apply to show that $n_g = 0$ for every $g \in G'$. The theorem follows then from Lemma 5.2.

$\square$

Remark 5.6. There is a DR analogue of Theorem 5.5 obtained by replacing the words "aspherical" by "DR" in the statement. The proof is identical, considering a finite sub-complex L of $\check{K}_{Q'}$ without free faces, then proving that it cannot have cells of type $e^2_{m+1,g}$ and then using the hypothesis on $\mathcal{P}$ to conclude that L is 1-dimensional.

The DR versions of Proposition 5.3 and Theorem 5.4 do not hold. In fact $\mathcal{P} = \langle | \rangle$, $\mathcal{Q} = \langle x | x^2 x^{-1} \rangle$ show that even when $\mathcal{P}$ is DR, $\mathcal{Q}$ can be non-DR.

6. An extension of the test

The ideas of the previous sections can be applied to a obtain a more general method. We use this extended test to prove diagrammatic reducibility of a larger class of LOTs and to study asphericity of presentations when our original I-test fails.

Definition 6.1. A function $\varphi : G \to P$ from a group G to a poset P is called *order preserving* if $\varphi(h) \leq \varphi(h')$ implies $\varphi(gh) \leq \varphi(gh')$ for each $g \in G$ (in particular $\varphi(h) < \varphi(h')$ implies $\varphi(gh) < \varphi(gh')$). More generally, a function $\varphi : \coprod_{j=1}^m G = \bigcup_{j=1}^m G \times \{j\} \to P$ is *order preserving* if $\varphi(h, j) \leq \varphi(h', j')$ implies $\varphi(gh, j) \leq \varphi(gh', j')$ for every $g \in G$.

We say that a presentation $\mathcal{P} = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$ of a group G satisfies the *extended test* if there exists a poset P and an order preserving map $\varphi : \coprod_{j=1}^m G \to P$ such that for every $1 \leq j \leq m$ there is some $1 \leq i \leq n$ with the following property: there exists $k \in \mathrm{occ}(x_i, r_j)$ such that $\varphi(s(k, r_j), j) > \varphi(s(k', r_{j'}), j')$ for every $1 \leq j' \leq m$ and $k' \in \mathrm{occ}(x_i, r_{j'})$ with $(k', j') \neq (k, j)$.

Proposition 6.2. *If $\mathcal{P}$ satisfies the extended test, it is DR.*

Proof. Let $\{n_g^j\}_{j,g}$ be a nontrivial family of integer numbers with finite support such that each equation $E_{i,g}$ has zero or at least two nonzero terms. Let P and φ be as in the definition. The finite subposet $\{\varphi(g, j) | n_g^j \neq 0\} \subseteq P$ has at least one minimal element, say $\varphi(h, j)$ for $h \in G, 1 \leq j \leq m$. Let $1 \leq i \leq n$ and $k \in \mathrm{occ}(x_i, r_j)$ be as in Definition 6.1. Let $g = hs(k, r_j)^{-1}$. Since $\varphi(gs(k', r_{j'}), j') < \varphi(gs(k, r_j), j) = \varphi(h, j)$ for each $k' \in \mathrm{occ}(x_i, r_{j'})$ with $(k', j') \neq (k, j)$, the equation $E_{i,g}$ has a unique nonzero term, which is a contradiction. $\square$

Example 6.3. The I-test is a very particular case of the extended test. Suppose $\mathcal{P}$ satisfies the I-test for $v \in \mathbb{R}^n$ and the orders $j_1, j_2, \ldots, j_m$ and $i_1, i_2, \ldots, i_m$ of columns and rows. Consider $P = \mathbb{R} \times \{j_1 < j_2 < \ldots < j_m\}$ with the lexicographic order and let $\varphi : \coprod G \to P$ be defined by $\varphi(g, j) = (\overline{g}, j)$. Then φ is order preserving and the fact that the weight matrix M is good implies that φ satisfies the required property.

Suppose now that $\mathcal{P} = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$ is a presentation and let $1 \leq n_1 < n_2 < \ldots < n_k = n$, $1 \leq m_1 < m_2 < \ldots < m_k = m$ be such that for each $1 \leq l \leq k$, the relators $r_1, r_2, \ldots, r_{m_l}$ are words in the first n_l generators. In particular, if $v \in \mathbb{R}^n$ is orthogonal to each $q(r_j)$, the weight matrix $M(v)$ is a block matrix

$$
\begin{pmatrix}
\begin{array}{c|c|c|c}
M_1(v) & * & \cdots & * \\
\hline
\emptyset & M_2(v) & & * \\
\hline
\vdots & & \ddots & \vdots \\
\hline
\emptyset & \emptyset & \cdots & M_k(v)
\end{array}
\end{pmatrix}
$$

Here $M_l(v)$ is the submatrix of $M(v)$ corresponding to rows $n_{l-1}+1, n_{l-1}+2, \ldots, n_l$ and columns $m_{l-1}+1, m_{l-1}+2, \ldots, m_l$.

Theorem 6.4. *Let $\mathcal{P} = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$ and let $1 \le n_1 < n_2 < \ldots < n_k = n$, $1 \le m_1 < m_2 < \ldots < m_k = m$ be such that for each $1 \le l \le k$, the relators $r_1, r_2, \ldots, r_{m_l}$ are words in the first n_l generators. Let $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$ such that each v_l is orthogonal to every $q(r_j)$ and for each $1 \le l \le k$, $M_l(v_l)$ is good (see Definition 2.5). Then $\mathcal{P}$ is DR.*

Proof. We show that $\mathcal{P}$ satisfies the extended test. Let $j_{m_{l-1}+1}, j_{m_{l-1}+2}, \ldots, j_{m_l}$ be an order of the columns $m_{l-1}+1, m_{l-1}+2, \ldots, m_l$ that makes $M_l(v_l)$ good. Recall that for posets P, Q, the join $P * Q$ is the order in the disjoint union of P and Q obtained by preserving the ordering in each copy and setting $p < q$ for each $p \in P$ and $q \in Q$. Define then

$$P = (\mathbb{R} \times \{j_{m_{k-1}+1} < j_{m_{k-1}+2} < \ldots < j_{m_k}\}) * (\mathbb{R} \times \{j_{m_{k-2}+1} < j_{m_{k-2}+2} < \ldots < j_{m_{k-1}}\}) * \ldots$$
$$\ldots * (\mathbb{R} \times \{j_1 < j_2 < \ldots < j_{m_1}\}),$$

in which each factor $\mathbb{R} \times \{j_{m_{l-1}+1}, j_{m_{l-1}+2}, \ldots, j_{m_l}\}$ is considered with the lexicographic order. Define $\varphi : \coprod_{j=1}^{m} G \to P$ by $\varphi(g, j) = (\overline{g}, j)$, where the weight $\overline{g}$ is taken with respect to the vector v_l if $m_{l-1}+1 \le j \le m_l$. Then φ is order preserving and $\mathcal{P}$ satisfies the extended test. $\square$

Remark 6.5. Note that even if all the vectors v_l in the previous theorem are equal, it may happen that the blocks $M_l(v_l)$ are good while the weight matrix M is not.

Theorem 6.4 can be used to generalize the notion of deforestability for LOGs.

Definition 6.6. A LOG Γ is *weakly deforestable* if there is a sequence $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_k$ of sub-LOGs of Γ with Γ_k discrete, such that for each $1 \le i \le k$ there is a deforestation of type IL/T (see Definition 4.2) or a deforestation of type TL/I from Γ_{i-1} to Γ_i.

Example 6.7. Figure 3 shows a non-deforestable LOT Γ of diameter 4. There is a deforestation of type IL/T from Γ to the sub-LOT Γ' given by the horizontal edges. This sub-LOT is deforestable. Hence Γ is weakly deforestable.

Theorem 6.8. *If a LOG Γ is weakly deforestable, the associated presentation $\mathcal{P}$ satisfies the hypothesis of Theorem 6.4. In particular $K_{\mathcal{P}}$ is DR.*

Proof. If $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_k$ are as in Definition 6.6, there exist $1 \le n_1 < n_2 < \ldots < n_k = n$, $1 \le m_1 < m_2 < \ldots < m_k = m$ such that $\mathcal{P} = \mathcal{P}_\Gamma = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$ and for each $1 \le l \le k$, the relators $r_1, r_2, \ldots, r_{m_l}$ are words in the first n_l generators. Moreover, $n_l - n_{l-1} = m_l - m_{l-1}$ is the number of vertices (and edges) in $\Gamma_{k-l} \smallsetminus \Gamma_{k-l+1}$ for each $l \ge 2$. There exist $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$, each v_l equal to $(1, 1, \ldots, 1)$ or to $(-1, -1, \ldots, -1)$, such that $M_1(v_1)$ and the square matrices $M_2(v_2), \ldots, M_k(v_k)$ are good. Then Theorem 6.4 applies. $\square$

A particular case of the extended test which gives a useful generalization of the I-test is the following. Suppose L is a left-orderable group and $\psi : G \to L$ is a group homomorphism. Then for an ordering $j_1, j_2, \ldots, j_m$ of $\{1, 2, \ldots, m\}$, consider $P = L \times \{j_1 < j_2 < \ldots < j_m\}$, again with the lexicographic order. The map $\varphi : \coprod_{j=1}^{m} G \to P$
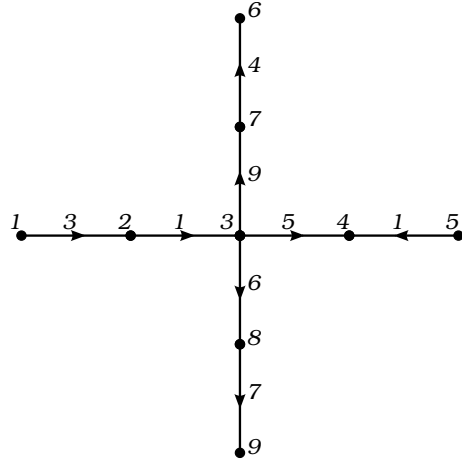
FIGURE 3. A weakly deforestable LOT of diameter 4.

given by $\varphi(g, j) = (\psi(g), j)$ is order preserving and can be used to test the diagrammatic reducibility of $\mathcal{P}$.

Example 6.9. Let $\mathcal{P} = \langle x, y, z | xzx^{-1}yzy^{-1}z^{-1}y^{-1}, xy^{-1}x^{-1}y^{-1}xy \rangle$ be a presentation of a group G. The subspace of vectors orthogonal to $q(r_1)$ and $q(r_2)$ is $\langle (1, 1, 1) \rangle \subseteq \mathbb{R}^3$. It is easy to check that for any $v \in \langle (1, 1, 1) \rangle$, the weight matrix $M(v)$ is not good. However we will see that the extended test can be applied to prove diagrammatic reducibility of $\mathcal{P}$.

Let $B_4 = \langle x, y, z | xzx^{-1}z^{-1}, xyxy^{-1}x^{-1}y^{-1}, yzyz^{-1}y^{-1}z^{-1} \rangle$ be the braid group of 4-braids. The (opposite of the) Dehornoy ordering is a left-ordering in B_4 which satisfies the following: if an element $g \in B_4$ is represented by a word w in the generators which contains the letter x and no occurrence of x in w has positive exponent, then $g > 1$ in B_4 (see [6, Section 7.2] for more details). Let $\varphi : G \coprod G \to B_4$ be the map whose restriction to each copy of G is the natural homomorphism $G \to B_4$.

The 3×2 matrix M defined by $M_{i,j} = \{\varphi(s(k, r_j), j)\}_{k \in \mathrm{occ}(x_i, r_j)}$ is

$$
\begin{array}{c}
\\
x \\
y \\
z
\end{array}
\begin{array}{c}
r_1 \qquad\qquad\qquad r_2 \\
\left(
\begin{array}{cc}
1, z^{-1} & 1, y^{-1}xy, xy \\
z^{-1}, z^{-1}y^{-1}, 1 & yx^{-1}, xy, y \\
x^{-1}, y^{-1}z^{-1}, y^{-1} & \emptyset
\end{array}
\right)
\end{array}
$$

Then x^{-1} is the greatest element of the third row while yx^{-1} is the greatest of the second one. Therefore $\mathcal{P}$ satisfies the extended test.

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