CURVE ARRANGEMENTS, PENCILS, AND JACOBIAN SYZYGIES

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Abstract. Let $\mathcal{C} : f = 0$ be a curve arrangement in the complex projective plane. If $\mathcal{C}$ contains a curve subarrangement consisting of at least three members in a pencil, then one obtains an explicit syzygy among the partial derivatives of the homogeneous polynomial f. In many cases this observation reduces the question about the freeness or the nearly freeness of $\mathcal{C}$ to an easy computation of Tjurina numbers. Some consequences for Terao's conjecture in the case of line arrangements are also discussed.

1. Introduction

Let $S = \mathbb{C}[x, y, z]$ be the graded polynomial ring in the variables x, y, z with complex coefficients and let $\mathcal{C} : f = 0$ be a reduced curve of degree d in the complex projective plane $\mathbb{P}^2$. The minimal degree of a Jacobian syzygy for f is the integer $mdr(f)$ defined to be the smallest integer $r \geq 0$ such that there is a nontrivial relation

$$af_x + bf_y + cf_z = 0 \tag{1.1}$$

among the partial derivatives f_x, f_y and f_z of f with coefficients a, b, c in S_r, the vector space of homogeneous polynomials of degree r. The knowledge of the invariant $mdr(f)$ allows one to decide if the curve $\mathcal{C}$ is free or nearly free by a simple computation of the total Tjurina number $\tau(\mathcal{C})$, see [8], [4], and Corollary 4.7 and Theorem 1.12 below for nice geometric applications.

When $\mathcal{C}$ is a free (resp. nearly free) curve in the complex projective plane $\mathbb{P}^2$, such that $\mathcal{C}$ is not a union of lines passing through one point, then the exponents of $\mathcal{A}$ denoted by $d_1 \leq d_2$ satisfy $d_1 = mdr(f) \geq 1$ and one has

$$d_1 + d_2 = d - 1, \tag{1.2}$$

(resp. $d_1 + d_2 = d$). Moreover, all the pairs d_1, d_2 satisfying these conditions may occur as exponents, see [7]. For more on free hypersurfaces and free hyperplane arrangements see [16], [13], [20], [17]. A useful result is the following.

Theorem 1.1. *Let $\mathcal{C} : f = 0$ be a reduced curve of degree d. If $r_0 \leq mdr(f)$ for some integer $r_0 \geq 1$, then $\mathcal{C}$ is free with exponents $(r_0, d - r_0 - 1)$ if and only if*

$$\tau(\mathcal{C}) = (d - 1)^2 - r_0(d - r_0 - 1).$$

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In particular, the set $F(d,\tau)$ of free curves in the variety $C(d,\tau) \subset \mathbb{P}(S_d)$ of reduced plane curves of degree d with a fixed global Tjurina number τ is a Zariski open subset.

The interested reader may state and prove the completely similar result for nearly free curves. If the curve $\mathcal{C}$ is reducible, one calls it sometimes a curve arrangement. When the curve $\mathcal{C}$ can be written as the union of at least three members of one pencil of curves, we say that $\mathcal{C}$ is a curve arrangement of pencil type. Such arrangements play a key role in the theory of line arrangements, see for instance [11], [14] and the references therein.

In this note we show that the existence of a subarrangement $\mathcal{C}'$ in a curve arrangement $\mathcal{C} : f = 0$, with $\mathcal{C}'$ of pencil type, gives rise to an explicit Jacobian syzygy for f. We start with the simplest case, when $\mathcal{C} = \mathcal{A}$ is a line arrangement and the pencil type subarrangement $\mathcal{C}'$ comes from an intersection point having maximal multiplicity, say $m = m(\mathcal{A})$, in $\mathcal{A}$. This case was considered from a different point of view in the paper [10] by D. Faenzi and J. Vallès. However, the construction of interesting syzygies from points of high multiplicity in $\mathcal{A}$ is very explicit and elementary in our note, see the formula (2.4), while in [10] the approach involves a good amount of Algebraic Geometry. This explicit construction allows us to draw some additional conclusions for the nearly free line arrangements as well.

Our first main result is the following.

Theorem 1.2. *If $\mathcal{A} : f = 0$ is a line arrangement and $m = m(\mathcal{A})$ is the maximal multiplicity of its intersection points, then either $mdr(f) = d - m$, or $mdr(f) \leq d - m - 1$ and then one of the following two cases occurs.*

1. *$mdr(f) \leq m - 1$. Then equality holds, i.e. $mdr(f) = m - 1$, one has the inequality $2m < d + 1$ and the line arrangement $\mathcal{A}$ is free with exponents $d_1 = mdr(f) = m - 1$ and $d_2 = d - m$;*
2. *$m \leq mdr(f) \leq d - m - 1$, in particular $2m < d$.*

Theorem 1.1 can be used to identify the free curves in the case (2) above. We show by examples in the third section that all the cases listed in Theorem 1.2 can actually occur inside the class of free line arrangements. A number of corollaries of Theorem 1.2 on the multiplicity $m(\mathcal{A})$ of a free or nearly free line arrangement $\mathcal{A}$ are given in the second section.

On the other hand, as a special case of a result in [5] recalled in subsection 2.7 below, we have the following.

Proposition 1.3. *If $\mathcal{A} : f = 0$ is a line arrangement, then*

$$m \geq \frac{2d}{mdr(f) + 2}.$$

In particular, if $\mathcal{A}$ is free or nearly free with exponents $d_1 \leq d_2$, then

$$m \geq \frac{2d}{d_1 + 2}.$$

This inequality is sharp, i.e. an equality, for some arrangements, see Example 3.8.

We say that Terao's Conjecture holds for a free hyperplane arrangement $\mathcal{A}$ if any other hyperplane arrangement $\mathcal{B}$, having an isomorphic intersection lattice $L(\mathcal{B}) = L(\mathcal{A})$, is also free, see [13], [22]. This conjecture is open even in the case of line arrangements in the complex projective plane $\mathbb{P}^2$, in spite of a lot of efforts, see for instance [1], [2]. For line arrangements, since the total Tjurina number $\tau(\mathcal{A})$ is determined by the intersection lattice $L(\mathcal{A})$, it remains to check that $\mathcal{A} : f = 0$ and $\mathcal{B} : g = 0$ satisfy $mdr(f) = mdr(g)$ and then apply [8], [4]. Theorem 3.1 in [10] and our results above imply the following fact, to be proved in section 3.

Corollary 1.4. *Let $\mathcal{A}$ be a free line arrangement with exponents $d_1 \leq d_2$. If*

$$m = m(\mathcal{A}) \geq d_1,$$

then Terao's Conjecture holds for the line arrangement $\mathcal{A}$. In particular, this is the case when one of the following conditions hold.

(1) $d_1 = d - m$;
(2) $m \geq d/2$;
(3) $d_1 \leq \sqrt{2d+1} - 1$.

Remark 1.5. (i) The fact that the Terao's Conjecture holds for the line arrangement $\mathcal{A}$ when $m = m(\mathcal{A}) \geq d_1 + 2$ was established in [9] by an approach not involving Jacobian syzygies. The result for $m = m(\mathcal{A}) \geq d_1$ is implicit in [10], see Theorem 3.1 coupled with Remarks 3.2 and 3.3. Moreover, the case $m = m(\mathcal{A}) = d_1 - 1$ for some real line arrangements is discussed in Theorem 6.2 in [10].

(ii) The cases $d_1 = d - m$ and $m \geq d/2$ in Corollary 1.4 follow also from the methods described in [22], see in particular Proposition 1.23 (i) and Theorem 1.39. Corollary 1.4 follows also from [1], Theorem 1.1., the claims (1) and (3).

(iii) The case (3) in Corollary 1.4 improves Corollary 2.5 in [4] saying that Terao's conjecture holds for $\mathcal{A}$ if $d_1 \leq \sqrt{d-1}$.

It is known that Terao's Conjecture holds for the line arrangement $\mathcal{A}$ when $d = |\mathcal{A}| \leq 12$, see [10]. This result and the case (3) in Corollary 1.4 imply the following.

Corollary 1.6. *Let $\mathcal{A}$ be a free line arrangement with exponents $d_1 \leq d_2$. If*

$$d_1 \leq 4,$$

then Terao's Conjecture holds for the line arrangement $\mathcal{A}$.

In the case of nearly free line arrangements we have the following result, which can be proved by the interested reader using the analog of Theorem 1.1 for nearly free arrangements.

Corollary 1.7. *Let $\mathcal{A}$ be a nearly free line arrangement with exponents $d_1 \leq d_2$. If*

$$m = m(\mathcal{A}) \geq d_1,$$

then any other line arrangement $\mathcal{B}$, having an isomorphic intersection lattice $L(\mathcal{B}) = L(\mathcal{A})$, is also nearly free.

Now we present our results for curve arrangements. First we assume that $\mathcal{C}$ is itself an arrangement of pencil type.

Theorem 1.8. *Let $\mathcal{C} : f = 0$ be a curve arrangement in $\mathbb{P}^2$ such that the defining equation has the form*

$$f = q_1 q_2 \cdots q_m,$$

for some $m \geq 3$, where $\deg q_1 = \cdots = \deg q_m = k \geq 2$ and the curves $\mathcal{C}_i : q_i = 0$ for $i = 1, ..., m$ are members of the pencil $u\mathcal{C}_1 + v\mathcal{C}_2$, assumed to contain only reduced curves. Then either $mdr(f) = 2k - 2$, or $m = 3$, $mdr(f) \leq 2k - 3$ and in addition one of the following two cases occurs.

1. *$k \geq 4$ and $mdr(f) \leq k + 1$. Then equality holds, i.e. $mdr(f) = k + 1$, and the curve arrangement $\mathcal{C}$ is free with exponents $d_1 = k + 1$ and $d_2 = 2k - 2$;*
2. *$k \geq 5$ and $k + 2 \leq mdr(f) \leq 2k - 3$.*

Using [8], [4] we get the following consequence.

Corollary 1.9. *Let $\mathcal{C}$ be a curve arrangement of pencil type such that the corresponding pencil $u\mathcal{C}_1 + v\mathcal{C}_2$ contains only reduced curves. If the number m of pencil members, which are curves of degree k, is at least 4, then $mdr(f) = 2k - 2$. In particular, in this case the curve $\mathcal{C}$ is free if and only if*

$$\tau(\mathcal{C}) = (d-1)^2 - 2(k-1)(d-2k+1),$$

resp. $\mathcal{C}$ is nearly free if and only if

$$\tau(\mathcal{C}) = (d-1)^2 - 2(k-1)(d-2k+1) - 1.$$

And again Theorem 1.1 can be used to identify the free curves in the case (2) above. Now we discuss the case of a curve arrangement containing a subarrangement of pencil type.

Theorem 1.10. *Let $\mathcal{C} : f = 0$ be a curve arrangement in $\mathbb{P}^2$ such that the defining equation has the form*

$$f = q_1 q_2 \cdots q_m h,$$

for some $m \geq 2$, where $\deg q_1 = \cdots = \deg q_m = k$ and the curves $\mathcal{C}_i : q_i = 0$ for $i = 1, ..., m$ are reduced members of the pencil $u\mathcal{C}_1 + v\mathcal{C}_2$. Assume that the curves $\mathcal{C}_1 : q_1 = 0$, $\mathcal{C}_2 : q_2 = 0$ and $\mathcal{H} : h = 0$ have no intersection points and that the curve $\mathcal{H}$ is irreducible. Then either $mdr(f) = 2k - 2 + \deg(h) = d - (m-2)k - 2$, or $mdr(f) \leq d - (m-2)k - 3$ and then one of the following two cases occurs.

1. *$mdr(f) \leq (m-2)k + 1$. Then equality holds, i.e. $mdr(f) = (m-2)k + 1$, and the curve arrangement $\mathcal{C}$ is free with exponents $d_1 = (m-2)k + 1$ and $d_2 = d - (m-2)k - 2$;*
2. *$(m-2)k + 2 \leq mdr(f) \leq d - (m-2)k - 3$.*

In fact, this result holds also when $k = 1$ and $\mathcal{H}$ is just reduced, see Remark 2.3.

Remark 1.11. Note that when $\mathcal{C}$ is a line arrangement, containing strictly the pencil type arrangement $\mathcal{C}'$ and such that $\deg h > 1$, (i.e. $\mathcal{C}$ contains at least two lines not in $\mathcal{C}'$), then it is not clear whether the Jacobian syzygy constructed in (4.3) is primitive. Due to this fact, Theorem 1.2 cannot be regarded as a special case of Theorem 1.10.

Exactly as in Corollary 1.9, when $mdr(f) = d - (m-2)k - 2$, the freeness or nearly freeness of $\mathcal{C}$ is determined by the global Tjurina number $\tau(\mathcal{C})$. This can be seen in the examples given in the fourth section as well as in Corollary 4.7 and in the following result.

Theorem 1.12. *Let $\mathcal{C} : f = 0$ be a curve arrangement in $\mathbb{P}^2$ such that the defining equation has the form*

$$f = q_1 q_2 \cdots q_m,$$

for some $m \geq 3$, where $\deg q_1 = \cdots = \deg q_m = k \geq 2$ and the curves $\mathcal{C}_i : q_i = 0$ for $i = 1, ..., m$ are members of the pencil spanned by $\mathcal{C}_1$ and $\mathcal{C}_2$. Assume that the pencil $u\mathcal{C}_1 + v\mathcal{C}_2$ is generic, i.e. the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ meet transversely in exactly k^2 points. Then the following properties are equivalent.

1. *Any singularity of any singular member C_j^s of the pencil $u\mathcal{C}_1 + v\mathcal{C}_2$ is weighted homogeneous and all these singular members $\mathcal{C}_j^s$ are among the m curves $\mathcal{C}_i$ in the curve arrangement $\mathcal{C}$;*
2. *The curve $\mathcal{C}$ is free with exponents $(2k-2, mk-2k+1)$.*

This result was essentially stated and partially proved in [21], see also the Erratum to that paper.

2. Multiple points and Jacobian syzygies

2.1. **Proof of Theorem 1.1.** If $\mathcal{C}$ is free with exponents $(r_0, d - r_0 - 1)$, then the formula for $\tau(\mathcal{C})$ is well known, see for instance [6]. Suppose now that $\tau(\mathcal{C})$ is given by the formula in Theorem 1.1. Suppose first that $r_0 < r = mdr(f) \leq (d-1)/2$. Then one has

$$\tau(\mathcal{C}) = (d-1)^2 - r_0(d-1-r_0) > \phi_1(r) := (d-1)^2 - r(d-1-r), \tag{2.1}$$

since the function $\phi_1(r)$ is stricly deceasing on $[0, (d-1)/2]$, which contradicts Theorem 3.2 in [8]. Next suppose that $r_0 < r$ and $(d-1)/2 < r \leq d - r_0 - 1$. It follows then from Theorem 3.2 in [8] that one has

$$\tau(\mathcal{C}) \leq \phi_2(r) := (d-1)^2 - r(d-r-1) - \binom{2r+2-d}{2}. \tag{2.2}$$

The function $\phi_2(r)$ is strictly decreasing on $((d-4)/2, +\infty)$ and moreover

$$\phi_1\left(\frac{d-1}{2}\right) = \phi_2\left(\frac{d-1}{2}\right).$$

This yields a contradiction with (2.1), thus showing that the strict inequality $r > r_0$ is impossible. It follows that $r = r_0$ and one may use [8], [4] to complete the proof of the first claim.

To prove the second claim, consider the closed subvariety X_r in $\mathbb{P}(S_r^3) \times \mathbb{P}(S_d)$ given by

$$X_r = \{((a, b, c), f) \quad : \quad af_x + bf_y + cf_z = 0\}.$$

Note that a polynomial $f \in S_d$ satisfies $mdr(f) \leq r$ if and only if $[f] \in \mathbb{P}(S_d)$ is in the image Z_r of X_r under the second projection. If there is $0 < r_0 \leq (d-1)/2$ such that $\tau = \phi_1(r_0)$, then by the above discussion, $F(d,\tau)$ is exactly the complement of $Z_{r_0-1} \cap C(d,\tau)$ in $C(d,\tau)$. If such an r_0 does not exist, then $F(d,\tau) = \emptyset$, which completes the proof.

2.2. **Proof of Theorem 1.2.** We show first that an intersection point p of multiplicity m gives rise to a syzygy

$$R_p : a_p f_x + b_p f_y + c_p f_z = 0 \tag{2.3}$$

where $\deg a_p = \deg b_p = \deg c_p = d - m$ and such that the polynomials a_p, b_p, c_p have no common factor in S. Let $f = gh$, where g (resp. h) is the product of linear factors in f corresponding to lines in $\mathcal{A}$ passing (resp. not passing) through the point p. If we choose the coordinates on $\mathbb{P}^2$ such that $p = (1 : 0 : 0)$, then g is a homogeneous polynomial in y, z of degree m, while each linear factor L in h contains the term in x with a non zero coefficient a_L. Moreover, $\deg h = d - m$. It follows that

$$f_x = gh_x = gh \sum_L \frac{a_L}{L} = f\frac{P}{h},$$

where P is a polynomial of degree $d - m - 1$ such that P and h have no common factors. This implies that

$$dhf_x = dPf = xPf_x + yPf_y + zPf_z, \tag{2.4}$$

i.e. we get the required syzygy R_p by setting $a_p = xP - dh$, $b_p = yP$ and $c_p = zP$.

Now, by the definition of $mdr(f)$, we get $mdr(f) \leq d-m$ and it remains to analyse the case $mdr(f) < d - m$. Let R_1 be the syzygy of degree $mdr(f)$ among f_x, f_y, f_z. It follows that R_p is not a multiple of R_1, and hence when

$$\deg R_1 + \deg R_p = mdr(f) + d - m \leq d - 1$$

we can use Lemma 1.1 in [18] and get the case (1). The case (2) is just the situation when the case (1) does not hold, so there is nothing to prove.

Remark 2.3. The method of proof of Theorem 1.2 gives a proof of Theorem 1.10 when $k = 1$ and $\mathcal{H}$ is a reduced curve, not necessarily irreducible. Indeed, $\mathcal{H}$ reduced implies that h and h_x cannot have any common factor. Any such irreducible common fact would be a line passing through the point $p = (1 : 0 : 0)$, and h does not have such factors by assumption.

Theorem 1.2 clearly implies the following Corollary, saying that the highest multiplicity of a point of a (nearly) free line arrangement cannot take arbitrary values with respect to the exponents.

Corollary 2.4. *(i) If $\mathcal{A}$ is a free line arrangement with exponents $d_1 \leq d_2$, then either $m = d_2 + 1$ or $m \leq d_1 + 1$.*
(ii) If $\mathcal{A}$ is a nearly free line arrangement with exponents $d_1 \leq d_2$, then either $m = d_2$ or $m \leq d_1$.

The first claim (i) in Corollary 2.4 should be compared with the final claim in Corollary 4.5 in [10] and looks like a dual result to Corollary 1.2 in [1]. As a special case of Corollary 2.4 we get the following.

Corollary 2.5. *(i) If $\mathcal{A}$ is a free line arrangement with exponents $d_1 \leq d_2$ and $m > d/2$, then $m = d_2 + 1$.*
(ii) If $\mathcal{A}$ is a nearly free line arrangement with exponents $d_1 \leq d_2$ and $m \geq d/2$, then $m = d_2$.

The following consequence of Theorem 1.2 is also obvious.

Corollary 2.6. *If $\mathcal{A}$ is a line arrangement such that $2m = d$, then either $mdr(f) = m$ and $\mathcal{A}$ is not free, or $mdr(f) = m - 1$ and $\mathcal{A}$ is free with exponents $m - 1, m$.*

2.7. **Proof of Proposition 1.3.** For the reader's convenience, we recall some facts from [5], see also [6]. Let C be a reduced plane curve in $\mathbb{P}^2$ defined by $f = 0$. Let α_C be the minimum of the Arnold exponents α_p of the singular points p of C. The plane curve singularity (C, p) is weighted homogeneous of type $(w_1, w_2; 1)$ with $0 < w_j \leq 1/2$, if there are local analytic coordinates y_1, y_2 centered at p and a polynomial $g(y_1, y_2) = \sum_{u,v} c_{u,v} y_1^u y_2^v$, with $c_{u,v} \in \mathbb{C}$ and where the sum is over all pairs $(u, v) \in \mathbb{N}^2$ with $uw_1 + vw_2 = 1$. In this case one has

$$\alpha_p = w_1 + w_2, \tag{2.5}$$

see for instance [5]. With this notation, Corollary 5.5 in [5] can be restated as follows.

Theorem 2.8. *Let $C : f = 0$ be a degree d reduced curve in $\mathbb{P}^2$ having only weighted homogeneous singularities. Then $AR(f)_r = 0$ for all $r < \alpha_C d - 2$.*

In the case of a line arrangement $C = \mathcal{A}$, a point p of multiplicity k has by the above discussion the Arnold exponent $\alpha_p = 2/k$. It follows that, for $m = m(\mathcal{A})$, one has

$$\alpha_C = \frac{2}{m}, \tag{2.6}$$

and hence Theorem 2.8 implies

$$mdr(f) \geq \frac{2}{m} d - 2. \tag{2.7}$$

In other words

$$m \geq \frac{2d}{mdr(f) + 2}, \tag{2.8}$$

i.e. exactly what is claimed in Proposition 1.3.

2.9. **Proof of Corollary 1.7.** Let $\mathcal{B}$ be defined by $g = 0$. Then Corollary 2.5 (ii) applied to $\mathcal{A}$ implies that $d_1 = d - m$, and hence in particular

$$\tau(\mathcal{A}) = (d-1)^2 - (d-m)(m-1) - 1,$$

see [4]. Note that $\tau(\mathcal{A}) = \tau(\mathcal{B})$ as for a line arrangement the total Tjurina number is determined by the intersection lattice, see for instance [7], section (2.2). If $mdr(g) = d - m$, then our characterisation of nearly free arrangements in [4] via the total Tjurina number implies that $\mathcal{B}$ is also nearly free.

On the other hand, if $mdr(g) < d - m$, Theorem 1.2 applied to the arrangement $\mathcal{B}$ implies that the only possibility given the assumption $m \geq d/2$ is that $\mathcal{B}$ is free with exponents $m-1, d-m$, in particular

$$\tau(\mathcal{B}) = (d-1)^2 - (d-m)(m-1).$$

This is a contradiction with the above formula for $\tau(\mathcal{A})$, so this case is impossible.

3. On free line arrangements

3.1. **Proof of Corollary 1.4.** The proof of Corollary 1.4 is based on Theorem 3.1 in [10], which we recall now in a slightly modified form, see also [7], section (2.2).

Theorem 3.2. *Let $\mathcal{B}$ be an arrangement of d lines in $\mathbb{P}^2$ and suppose that there are two integers $k \geq 1$ and $\ell \geq 0$ such that $d = 2k + \ell + 1$ and there is an intersection point in $\mathcal{B}$ of multiplicity e such that*

$$k \leq e \leq k + \ell + 1. \tag{3.1}$$

Then the arrangement $\mathcal{B}$ is free with exponents $(k, k+\ell)$ if and only if the total Tjurina number of $\mathcal{B}$ satisfies the equality

$$\tau(\mathcal{B}) = (d-1)^2 - \ell(k+\ell). \tag{3.2}$$

Remark 3.3. A new proof of Theorem 3.2 can be given using our Theorem 1.1, Theorem 1.2 and Theorem 3.2 in [8].

To prove the first claim of Corollary 1.4, we apply Theorem 2 in [10] to the arrangement $\mathcal{B}$. Corollary 2.5 implies that $m = m(\mathcal{A}) = m(\mathcal{B}) \leq d_2 + 1$. Then we can set $k = d_1$, $\ell = d_2 - d_1$ and $e = m$ and we get

$$k = d_1 \leq e = m \leq k + \ell + 1 = d_2 + 1.$$

It follows that $\tau(\mathcal{B}) = \tau(\mathcal{A}) = (d-1)^2 - d_1 d_2$ and hence the line arrangement $\mathcal{B}$ is free with exponents d_1, d_2.

The last claim of Corollary 1.4 follows since $m < d_1$ implies via Proposition 1.3 that

$$\frac{2d}{d_1 + 2} < d_1.$$

But this quadratic inequality in d_1 holds if and only if $d_1 > \sqrt{2d+1} - 1$.

3.4. **Some examples of free line arrangements.** Now we consider some examples of line arrangements. First we show by examples that all the cases listed in Theorem 1.2 and Corollary 2.6 can actually occur inside the class of free line arrangements.

Example 3.5. The line arrangement

$$\mathcal{A} : f = xyz(x-z)(x+z)(x-y) = 0$$

is free with exponents $(2, 3)$ and has $m = 4 > d/2 = 3$. Hence we are in the situation $d_1 = mdr(f) = 2 = d - m$.

Example 3.6. The line arrangement

$$\mathcal{A} : f = xyz(x-z)(x+z)(x-y)(x+y)(y-z) = 0$$

is free with exponents $(3,4)$ and has $m = 4 = d/2$. Hence we are in the situation $d_1 = mdr(f) < 4 = d-m$ and $d_1 = mdr(f) = m-1$, as in Corollary 2.6.

Similarly, the line arrangement

$$\mathcal{A} : f = xyz(x-z)(x+z)(x-y)(x+y)(y-z)(y+z) = 0$$

is free with exponents $(3,5)$ and has $m = 4 < d/2$. Hence we are in the situation $d_1 = mdr(f) < 5 = d-m$ and $d_1 = mdr(f) = m-1$.

Example 3.7. The line arrangement

$$\mathcal{A} : f = xyz(x-z)(x+z)(x-y)(x+y)(y-z)(y+z)(x+2y)(x-2y)$$

$$(x+2z)(x-2z)(y-2z)(y+2z)(x+y-z)(x-y+z)(-x+y+z)(x+y+z) = 0$$

is free with exponents $(9,9)$ and has $m = 6 < 19/2$. Hence we are in the situation $m = 6 \leq mrd(f) = d_1 = 9 < d-m = 13$.

Finally we give an example showing that the inequality in Proposition 1.3 is sharp.

Example 3.8. The line arrangement

$$\mathcal{A} : f = (x^3-y^3)(y^3-z^3)(x^3-z^3) = 0$$

is free with exponents $(4,4)$ and has

$$m = 3 = \frac{2d}{d_1+2}.$$

4. Pencils and Jacobian syzygies

Let $\mathcal{C} : f = 0$ be a curve arrangement in $\mathbb{P}^2$ such that the defining equation has the form

$$f = q_1 q_2 \cdots q_m h = gh,$$

for some $m \geq 2$, where $\deg q_1 = \cdots = \deg q_m = k$ and the curves $\mathcal{C}_i : q_i = 0$ for $i = 1, ..., m$ are reduced members of the pencil $u\mathcal{C}_1 + v\mathcal{C}_2$, which contains only reduced curves. In terms of equations, one can write

$$q_i = q_1 + t_i q_2, \tag{4.1}$$

for $i = 3, ..., m$ and some $t_i \in \mathbb{C}^*$ distinct complex numbers. In other words, the curve subarrangement $\mathcal{C}' : g = 0$ of $\mathcal{C}$ consists of $m \geq 2$ reduced members of a pencil.

To find a Jacobian syzygy for f as in (1.1) is equivalent to finding a homogeneous 2-differential form ω on $\mathbb{C}^3$ with polynomial coefficients such that

$$\omega \wedge df = 0. \tag{4.2}$$

4.1. **The case when $\mathcal{C}$ is a pencil.** When $h = 1$, i.e. when $\mathcal{C} = \mathcal{C}'$ is a pencil itself, then one can clearly take

$$\omega = dq_1 \wedge dq_2, \tag{4.3}$$

see also Lemma 2.1 in [21]. This form yields a primitive syzygy of degree $2k-2$ if we show that
(i) $\omega \neq 0$, and
(ii) ω is primitive, i.e. ω cannot be written as $e\eta$, for $e \in S$ with $\deg e > 0$ and η a 2-differential form on $\mathbb{C}^3$ with polynomial coefficients. Such a polynomial e is called a *divisor* of ω.

The first claim follows from Lemma 3.3 in [4], since $q_1 = 0$ is a reduced curve and q_1 and q_2 are not proportional. The claim (ii) is a consequence of Lemma 2.5 in [21], or can easily be proven directly by the interested reader.

4.2. **The case when $\mathcal{C}$ is a not pencil.** If $\deg h = d - km > 0$, then we set

$$\omega = adq_1 \wedge dq_2 + bdq_1 \wedge dh + cdq_2 \wedge dh, \tag{4.4}$$

with $a, b, c \in S$ to be determined. The condition (4.2) becomes

$$g\left[a - bh\left(\frac{1}{q_2} + \frac{t_3}{q_3} + \cdots + \frac{t_m}{q_m}\right) + ch\left(\frac{1}{q_1} + \frac{1}{q_3} + \cdots + \frac{1}{q_m}\right)\right] dq_1 \wedge dq_2 \wedge dh = 0.$$

We have the following result.

Lemma 4.3. *Assume that the curves $\mathcal{C}_1 : q_1 = 0$, $\mathcal{C}_2 : q_2 = 0$ and $\mathcal{H} : h = 0$ have no common point. Then the 2-form $\omega = adq_1 \wedge dq_2 + bdq_1 \wedge dh + cdq_2 \wedge dh$ with $a = -mh$, $b = -q_2$ and $c = q_1$ is non-zero and satisfies $\omega \wedge df = 0$. Moreover, any divisor of ω is a divisor of the Jacobian determinant $J(q_1, q_2, h)$ of the polynomials q_1, q_2, h and of h. In particular, if h is irreducible, then ω is primitive.*

Proof. Since the ideal (q_1, q_2, h) is $\mathbf{m}$-primary, where $\mathbf{m} = (x, y, z)$, it follows that

$$dq_1 \wedge dq_2 \wedge dh = J(q_1, q_2, h) dx \wedge dy \wedge dz \neq 0,$$

see [12], p. 665. This shows in particular that $\omega \neq 0$. Indeed, one has

$$\omega \wedge dq_1 = q_1 J(q_1, q_2, h) dx \wedge dy \wedge dz$$

and

$$\omega \wedge dq_2 = q_2 J(q_1, q_2, h) dx \wedge dy \wedge dz.$$

Since q_1 and q_2 have no common factor, these equalities show that any divisor e of ω divides $J(q_1, q_2, h)$.

Let Δ be the contraction of differential forms with the Euler vector field, see Chapter 6 in [3] for more details if needed. Then one has

$$J(q_1, q_2, h)\Delta(dx \wedge dy \wedge dz) = \Delta(dq_1 \wedge dq_2 \wedge dh) =$$
$$= kq_1 dq_2 \wedge dh - kq_2 dq_1 \wedge dh + (d - mk) h dq_1 \wedge dq_2 =$$
$$= k\omega + d \cdot h dq_1 \wedge dq_2.$$

This implies that any divisor e of ω and of $J(q_1, q_2, h)$ divides h as well. Since h does not divide $J(q_1, q_2, h)$, see [12], p. 659, the last claim follows. □

The main results based on the above facts are Theorems 1.8 and 1.10, stated in the Introduction. Their proofs are exactly the same as the proof of Theorem 1.2 using the discussion above.

Remark 4.4. The case $r = d - m$ in Theorem 1.2, the case $r = 2k - 2$ in Theorem 1.8, or the case $r = 2k - 2 + \deg(h)$ in Theorem 1.10 can sometimes be discarded if $r = mdr(f) > (d-1)/2$ using the inequality (2.2).

Now we illustrate these results by some examples.

Example 4.5. (i) The line arrangement

$$\mathcal{A} : f = (x^k - y^k)(y^k - z^k)(x^k - z^k) = 0$$

for $k \geq 2$ is seen to be free with exponents $(k+1, 2k-2)$ using Theorem 1.12. This arrangement has $m(\mathcal{A}) = k$ for $k \geq 3$, hence the Jacobian syzygy constructed in the proof of Theorem 1.2 has degree $d - m(\mathcal{A}) = 2k$. The Jacobian syzygy constructed in (4.3) has degree $d_2 = 2k - 2$, hence we are in the case (1) of Theorem 1.8 when $k \geq 4$. Theorems 1.1 and 1.2 give an alternative proof for the freeness of $\mathcal{A}$. The same method shows that the arrangement

$$\mathcal{A}' : f = xyz(x^k - y^k)(y^k - z^k)(x^k - z^k) = 0$$

for $k \geq 2$ is free with exponents $(k+1, 2k+1)$.

(ii) The curve arrangement

$$\mathcal{C} : f = xyz(x^3 + y^3 + z^3)[(x^3 + y^3 + z^3)^3 - 27x^3y^3z^3] = 0$$

is just the Hesse arrangement from [21] with one more smooth member of the pencil added. One has $k = 3$ and $m = 5$, hence $r = mdr(f) = 4$ follows from Theorem 1.8. Moreover, the Jacobian syzygy constructed in (4.3) has minimal degree $r = mdr(f)$, and this is always the case by Theorem 1.8 when $k = 3$ or when $m > 3$. To compute the total Tjurina number $\tau(\mathcal{C})$ of $\mathcal{C}$, note that the 9 base points of the pencil are ordinary 5-fold points, hence each contributes with 16 to $\tau(\mathcal{C})$. There are four singular members of the pencil in $\mathcal{C}$, each a triangle, hence we should add 12 to $\tau(\mathcal{C})$ for these 12 nodes which are the vertices of the four triangles. It follows that

$$\tau(\mathcal{C}) = 9 \times 16 + 12 = 156 = (d-1)^2 - r(d-r-1) = 14^2 - 4 \times 10,$$

which shows that $\mathcal{C}$ is free with exponents $(4, 10)$ using [8], [4].

Example 4.6. (i) The line arrangement

$$\mathcal{A} : f = (x^k - y^k)(y^k - z^k)(x^k - z^k)x = 0$$

is seen by a direct computation to be free with exponents $(k+1, 2k-1)$. This arrangement has $m(\mathcal{A}) = k+1$ for $k \geq 2$, hence the Jacobian syzygy constructed in the proof of Theorem 1.2 has degree $d - m(\mathcal{A}) = 2k$. The Jacobian syzygy constructed in (4.3) has degree $d_2 = 2k - 1$, hence we are in the case (1) of Theorem 1.10.

(ii) Consider the curve arrangement $\mathcal{C} : f = x(x^{m-1} - y^{m-1})(xy + z^2)$ for $m \geq 3$. Here $k = 1$ and $d = m + 2$. Theorem 1.10 implies that $r = mdr(f) = \deg(h) = 2$. To compute the total Tjurina number $\tau(\mathcal{C})$ of $\mathcal{C}$, note that $(0 : 0 : 1)$ is an ordinary

m-fold point, hence it contributes to $\tau(\mathcal{C})$ by $(m-1)^2$. Each of the $(m-1)$ lines in $x^{m-1} - y^{m-1}$ meets the smooth conic $\mathcal{H} : xy + z^2 = 0$ in two points, so has a contribution to $\tau(\mathcal{C})$ equal to 2. The line $x = 0$ is tangent to $\mathcal{H}$ at the point $p = (0 : 1 : 0)$ and hence at p the curve $\mathcal{C}$ has an A_3 singularity. It follows that

$$\tau(\mathcal{C}) = (m-1)^2 + 2(m-1) + 3 = m^2 + 2 = (d-1)^2 - r(d-r-1) - 1.$$

Using [4], we infer that the curve $\mathcal{C}$ is nearly free with exponents $(2, m)$. The same method shows that the curve arrangement $\mathcal{C}' : f = xy(x^{m-2} - y^{m-2})(xy + z^2)$ for $m \geq 3$ is free with exponents $(2, m-1)$.

We end this section with a more general and geometric example.

Corollary 4.7. *Let $\mathcal{H} : h = 0$ be a smooth curve of degree $e \geq 2$ and let p be a generic point in $\mathbb{P}^2$, such that there are exactly $m = e(e-1)$ simple tangent lines to $\mathcal{H}$, say $L_1, ..., L_m$, passing through p. Then the curve $\mathcal{C} = \mathcal{H} \cup L_1 \cup ... \cup L_m$ is free with exponents $(e, e^2 - e - 1)$.*

Proof. It is known that the degree of the dual curve $\mathcal{H}^*$ is given by $e^* = e(e-1)$, see [12], p. 282, hence the existence of points p as claimed is clear. We apply Theorem 1.10 to the curve $\mathcal{C}$, with $k = 1$, $d = m + e = e^2$. When $e = 2$, we are in the case (1), hence $\mathcal{C}$ is free with exponents $(1, 2)$. For $e \geq 3$, we get $r = mdr(f) = e$. Then a computation of the global Tjurina number as in Example 4.6 (ii) (each line L_j contains $e - 2$ nodes of $\mathcal{C}$ and an A_3 singularity corresponding to the point where L_j is tangent to $\mathcal{H}$) shows that

$$\tau(\mathcal{C}) = e^4 - e^3 - e^2 + e + 1 = (d-1)^2 - r(d-r-1).$$

Hence $\mathcal{C}$ is free with exponents $(r, d-r-1) = (e, e^2 - e - 1)$ using [8], [4]. □

5. The case of generic pencils

Let $\mathcal{C} : f = 0$ be a curve arrangement in $\mathbb{P}^2$ such that the defining equation has the form

$$f = q_1 q_2 \cdots q_m,$$

for some $m \geq 2$, where $\deg q_1 = \cdots = \deg q_m = k$ and the curves $\mathcal{C}_i : q_i = 0$ for $i = 1, ..., m$ are members of the pencil spanned by $\mathcal{C}_1$ and $\mathcal{C}_2$. We say that this pencil is generic if the following condition is satisfied: the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ meet transversely in exactly k^2 points. If this holds, then the generic member of this pencil is smooth, and any member of the pencil $u\mathcal{C}_1 + v\mathcal{C}_2$ is smooth at any of the k^2 base points. Let us denote by $\mathcal{C}_j^s$ for $j = 1, ..., p$ all the singular members in this pencil. One has the following result.

Proposition 5.1. *If the pencil $u\mathcal{C}_1 + v\mathcal{C}_2$ is generic, then the sum of the total Milnor numbers of the singular members $\mathcal{C}_j^s$ in the pencil satisfies*

$$\sum_{j=1,p} \mu(\mathcal{C}_j^s) = 3(k-1)^2.$$

Proof. First recall that $\mu(\mathcal{C}_j^s)$ is the sum of the Milnor numbers of all the singularities of the curve $\mathcal{C}_j^s$. Then we consider two smooth members $D_1 : g_1' = 0$ and $D_2 : g_2' = 0$ in the pencil and consider the rational map $\phi : X \to \mathbb{C}$, where $X = \mathbb{P}^2 \setminus D_1$ and

$$\phi(x : y : z) = \frac{g_2'(x, y, z)}{g_1'(x, y, z)}.$$

Then it follows that ϕ is a tame regular function, see [19], whose singular points are exactly the union of the singular points of the curves $\mathcal{C}_j^s$ for $j = 1, ..., p$. From the general properties of tame functions it follows that

$$\sum_{j=1,p} \mu(\mathcal{C}_j^s) = \sum_{a \in X} \mu(\phi, a) = \chi(X, X \cap D_2).$$

Since the Euler characteristic of complex constructible sets is additive we get

$$\chi(X, X \cap D_2) = \chi(\mathbb{P}^2) - \chi(D_1) - \chi(D_2) + \chi(D_1 \cap D_2) =$$
$$= 3 + 2k(k-3) + k^2 = 3(k-1)^2.$$

□

5.2. **Proof of Theorem 1.12.** First we assume (1) and prove (2). For this, we compute the total Tjurina number $\tau(\mathcal{C})$, taking into account the fact that the singularities of $\mathcal{C}$ are of two types: the ones coming from the singularities of the singular members $\mathcal{C}_j^s$ and the k^2 base points, each of which is an ordinary m-fold point. It follows that

$$\tau(\mathcal{C}) = \sum_{j=1,p} \tau(\mathcal{C}_j^s) + k^2(m-1)^2 = 3(k-1)^2 + k^2(m-1)^2, \tag{5.1}$$

since $\tau(\mathcal{C}_j^s) = \mu(\mathcal{C}_j^s)$, all the singularities being weighted homogeneous.

Assume first that $m \geq 4$. Then Corollary 1.9 implies that $r = mdr(f) = 2k - 2$ and the equation (5.1) yields $\tau(\mathcal{C}) = (d-1)^2 - r(d-1-r)$, i.e. $\mathcal{C}$ is free.

Consider now the case $m = 3$. If $r = mdr(f) = 2k - 2$, the same proof as above works. Moreover, if we are in the case (1) of Theorem 1.8, i.e. $mdr(f) = k + 1 = r_0$, then again we get

$$\tau(\mathcal{C}) = (d-1)^2 - r_0(d-1-r_0),$$

and hence $\mathcal{C}$ is free in this case as well. It remains to discuss the case (2) in Theorem 1.8. This can be done using Theorem 1.1, thus completing the proof of the implication (1) ⇒ (2). The implication (2) ⇒ (1) is obvious using [15].

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