SMALL DRIFT LIMIT THEOREMS FOR RANDOM WALKS

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Abstract. We show analogs of the classical arcsine theorem for the occupation time of a random walk in $(-\infty, 0)$ in the case of a small positive drift. To study the asymptotic behavior of the total time spent in $(-\infty, 0)$ we consider some parametrized classes of random walks, where the convergence of the parameter to zero implies the convergence of the drift to zero. We begin with the simple random walk with ± 1 steps (for which we also consider the asymptotic distribution for the number of visits to a finite set) and generalize this to steps of size -1 and $m \geq 1$. Then we study families of associated distributions and shift families (generated by a centered random walk by adding to each step a shift constant $a > 0$ and then letting a tend to zero). In all cases we arrive at the same limiting distribution, which is the distribution of the time spent below zero of a standard Brownian motion with drift 1. For shift families this is explained by a functional limit theorem. In the course we also give a new form of the first arcsine law for the Brownian motion with drift.

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1. Introduction

For the classical symmetric random walk it is well known that the three random variables "time spent on the positive axis", "position of the first maximum" and "last exit from zero" are identically distributed and (suitably normalized) asymptotically arcsine-distributed. Here the norming factor is the length of the time interval the random walk has been observed, so that the limiting statements refer to "relative" times.

Consider now a classical random walk with drift $\delta \neq 0$. Clearly the same "relative" variables can be studied. The asymptotic distribution of the random variable "(fraction of) time spent in $(-\infty, \alpha]$ has been determined by Takács [22], by applying a functional limit theorem.

But if $\delta \neq 0$ there is also another, "absolute" perspective. If for example $\delta > 0$ for a general random walk, it is clear that $Z(\delta) =$ "number of visits in $(-\infty, 0)$ " is almost surely finite, and that $Z(\delta) \longrightarrow \infty$ in probability as $\delta \searrow 0$. One may ask if $Z(\delta)$, after multiplication with some deterministic function $a(\delta)$, has a non-degenerate limit distribution. This paper aims to answer these and related questions for some parametrized families of random walks, where the convergence of the parameter to zero implies the convergence of the drift to zero. In all cases the limiting distribution for the occupation time in $(-\infty, 0)$ turns out to have the density

$$p(t) = 2\,\frac{\varphi(\sqrt{2t})}{\sqrt{2t}} - 2\,\Phi(-\sqrt{2t}), \quad t > 0$$

where φ and Φ are the density and the distribution function of $N(0,1)$, respectively.

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The arcsine law and its ramifications are a classical topic but there are always recent contributions, for example some new explicit distributions [16], new proofs [11], or asymptotic considerations [17]. Interesting results on the number of visits to one point by skipfree random walks and related questions can be found in [4]. The problem considered in this paper is also connected to the heavy traffic approximation problem in queueing theory, in which the growth of the all-time maximum of $S_n - na$ (where S_n is the nth partial sum of iid random variables with mean zero) is studied as $a \searrow 0$. In the queueing context this is equivalent to the growth of the steady-state waiting time in a $GI/G/1$ system when the traffic load tends to 1. This question was first posed by Kingman (see [14]) and was investigated by many authors (e.g. [3, 15, 18, 19, 21]).

2. Some fluctuation theory

The topics investigated here belong to the fluctuation theory of random walks. We recall some basic facts, which which will be used in the sequel and can e.g. be found in Section XII.7 of [8].

For a random walk $(S_n)_{n\geq 1}$, i.e., a sequence of partial sums of iid random variables, let $R = \inf\{n \geq 1 \;:\; S_n < 0\}$ and $W = \inf\{n \geq 1 \;:\; S_n \geq 0\}$ be the length of the first strictly descending and weakly ascending ladder epochs of the random walk, respectively, and let $r(z)$ and $a(z)$ denote the corresponding probability generating functions.

Theorem 2.1. *(Sparre Andersen) For $|z| < 1$*

$$\frac{1}{1-r(z)} = \exp\left\{\sum_{n=1}^{\infty} \frac{z^n}{n}\mathbf{P}(S_n < 0)\right\}$$

$$\frac{1}{1-a(z)} = \exp\left\{\sum_{n=1}^{\infty} \frac{z^n}{n}\mathbf{P}(S_n \geq 0)\right\}$$

An immediate consequence is the factorization theorem.

Theorem 2.2. *(Duality) For $|z| < 1$*

$$(1-r(z))(1-a(z)) = 1-z.$$

It follows from the factorization theorem is that W (R) has a finite expected value if and only if R (W) is defective, and that the relations $\mathbf{E}(R)\mathbf{P}(W=\infty) = 1$ $\mathbf{E}(W)\mathbf{P}(R=\infty) = 1$ hold.

At the combinatorial heart of fluctuation theory is the "Sparre Andersen transformation" (made explicit by Feller and refined by Bizley and Joseph) given in Lemma 3 of XII.8 of [8]:

Lemma 2.3. *Let $x_1, \ldots, x_n$ be real numbers with exactly $k \geq 0$ negative partial sums $s_{i_1}, \ldots, s_{i_k}$, where $i_1 > \ldots > i_k$. Write down $x_{i_1}, \ldots, x_{i_k}$ followed by the remaining x_i in their original* *order. (If $k = 0$, the sequence remains unchanged). The transformation thus defined is invertible, and the first (absolute) minimum of the partial sums of the new arrangement occurs at the k-th place.*

Clearly this extends to *infinite sequences* with exactly k negative partial sums: just apply the bijection above to an initial section large enough to contain all the negative partial sums, and leave the rest unchanged.

3. The transient simple random walk

In this section we use the following notation. For $k \in \mathbf{Z}$ let $(S_n^{(k)})_{n\in\mathbf{Z}_+}$ be the simple upward biased random walk starting at k, i.e., $S_0^{(k)} = k$ and

$$S_n^{(k)} = k + X_1 + \cdots + X_n, \quad n \in \mathbf{N}$$

where the X_n are iid and

$$\mathbf{P}(X_n = 1) = p > 1/2, \ \mathbf{P}(X_n = -1) = q = 1 - p < 1/2.$$

Let $Z_k = Z_k(p)$ be the number of visits of $(S_n^{(k)})_{n\in\mathbf{Z}_+}$ to $(-\infty, 0)$, formally defined by

$$Z_k = \sum_{n=0}^{\infty} 1_{(-\infty,0)}(S_n^{(k)}).$$

Due to the positive drift of the random walk, $Z_k(p)$ is an almost surely finite random variable. We are interested in the asymptotic distribution of $Z_0 = Z_0(p)$ as $p \searrow 1/2$. Clearly, $Z_0(p) \to \infty$ in probability as p decreases to $1/2$, so we are looking for constants $c_p \to 0$ as $p \searrow 1/2$ for which $c_p Z_0(p)$ converges in distribution to a finite random variable. It turns out that $c_p = (p - \frac{1}{2})^2$ is the right choice.

Introduce the generating functions $f_k(z) = \mathbf{E}z^{Z_k}$, $0 \le z \le 1$. By a standard first-step argument they satisfy the difference equations

$$f_k(z) = pf_{k+1}(z) + qf_{k-1}(z), \quad k \ge 0$$
$$f_k(z) = pzf_{k+1}(z) + qzf_{k-1}(z), \quad k < 0.$$

The solution is of the form

$$f_k(z) = a(z) + b(z)(q/p)^k, \quad k \ge -1$$
$$f_k(z) = c(z)\Big(\frac{1 - \sqrt{1 - 4pqz^2}}{2pz}\Big)^k + d(s)\Big(\frac{1 + \sqrt{1 - 4pqz^2}}{2pz}\Big)^k, \quad k \le 0$$

with functions $a(z), b(z), c(z), d(z)$ still to be determined. It is obvious that

$$0 \le f_k(z) \le 1, \ \lim_{k\to\infty} f_k(z) = 1, \ \lim_{k\to-\infty} f_k(z) = 0.$$

Hence, $a(z) = 1$ and $c(z) = 0$ so that

$$f_k(z) = 1 + b(z)(q/p)^k, \quad k \ge -1$$
$$f_k(z) = d(z)\left(\frac{1 + \sqrt{1 - 4pqz^2}}{2pz}\right)^k, \quad k \le 0.$$

For $k = 0$ and $k = -1$ both equations hold. A quick calculation yields

$$f_0(z) = d(z) = 1 + b(z) = \frac{(p-q)(1+\sqrt{1-4pqz^2})}{p(1-2z^2+\sqrt{1-4pqz^2})}.$$

An alternative equivalent formula is

$$f_0(z) = \frac{2z(p-q)}{2pz - 1 + \sqrt{1-4pqz^2}}. \tag{3.1}$$

Now let $g(s) = \mathbf{E}(\exp\{-sZ_0\}) = f_0(e^{-s})$, $s > 0$, be the Laplace transform of Z_0. In order to obtain a limit for $g(s)$ as $p \searrow 1/2$ we replace s by $(p-q)^2 s$. Let $\delta = p - q$. Then

$$p = (1+\delta)/2,\ q = (1-\delta)/2,\ pq = (1-\delta^2)/4.$$

We find that, as $\delta \searrow 0$,

$$\begin{aligned}
g((p-q)^2 s) &= \frac{(p-q)\sqrt{1-4pqe^{-2(p-q)^2 s}}}{1-2qe^{-(p-q)^2 s} + \sqrt{1-4pqe^{-2(p-q)^2 s}}} \\
&= \frac{2\delta(1+o(1))}{1-(1-\delta)e^{-\delta^2 s} + \sqrt{1-(1-\delta^2)e^{-2\delta^2 s})}} \\
&= \frac{2\delta(1+o(1))}{1-(1-\delta)[1+O(\delta^2)] + \sqrt{1-(1-\delta^2)[1-2\delta^2 s + O(\delta^4)]}} \\
&\to \frac{2}{1+\sqrt{1+2s}}.
\end{aligned}$$

To identify the limit law, let A be a random variable with Laplace transform $\ell_A(s) = 2/(1+\sqrt{1+s})$. We show that the density of A is given by

$$p_A(t) = \frac{2}{\sqrt{\pi}}\left(t^{-1/2}e^{-t} - \int_t^\infty x^{-1/2}e^{-x}\,dx\right) = 2\,\frac{\varphi(\sqrt{2t})}{\sqrt{2t}} - 2\,\Phi(-\sqrt{2t}),\quad t > 0. \tag{3.2}$$

To see this, note that $1/\sqrt{1+s}$ is the Laplace transform of the gamma distribution $\Gamma_{1,\frac{1}{2}}$, which has density

$$\gamma_{1,\frac{1}{2}}(t) = 1_{(0,\infty)}(t)\,\frac{e^{-t}}{\sqrt{\pi t}}.$$

Therefore $1/[1-(1/\sqrt{1+s})]$ is the Laplace transform of $1 - \Gamma_{1,\frac{1}{2}}(t) = \int_t^\infty \gamma_{1,\frac{1}{2}}(x)\,dx$. The equality

$$\frac{1}{1+\sqrt{1+s}} = \frac{1}{\sqrt{1+s}} - \frac{1}{s}\Big(1 - \frac{1}{\sqrt{1+s}}\Big)$$

now yields density (3.2). We have proved

Theorem 3.1. *For $p > 1/2$ the occupation time $Z_0(p)$ has generating function (3.1) and $2(p-\frac{1}{2})^2 Z_0(p)$ converges in distribution as $p \searrow 1/2$ to a random variable A, which has Laplace transform $2/(1+\sqrt{1+s})$ and the density given by (3.2).*

Remark 3.1. Let $T_0(p) = \sup\{n \geq 0 : S_n^{(0)} = 0\}$ the time of the last return to the origin. In the symmetric case $p = 1/2$ the walk is persistent and $T_0(1/2) = \infty$ a.s. In the transient case $p > 1/2$, $T_0(p)$ has generating function

$$h(z) = \frac{p-q}{\sqrt{1-4pqz^2}}.$$

In the same way as above one sees that $\frac{1}{2}(p-q)^2 T_0(p)$ converges in distribution as $p \searrow 1/2$, the limiting distribution having Laplace transform $\frac{1}{\sqrt{1+s}}$, i.e., being the $\Gamma_{1,\frac{1}{2}}$ distribution with density $\gamma_{1,\frac{1}{2}}(t)$ as above. ■

Remark 3.2. Let $N_0(p)$ denote the number of zeros of the random walk. Then

$$\mathbf{P}(N_0(p) = r,\, T_0(p) = 2n) = \frac{r}{n-r}\binom{2n-r}{n} 2\,(pq)^n$$

and $(\delta N_0(p), \frac{1}{2}\delta^2 T_0(p))$ converges weakly to the distribution with density

$$f(y,t) = 1_{(0,\infty)}(y)\, 1_{(0,\infty)}(t)\, \frac{y}{2t}\frac{1}{\sqrt{2\pi t}}\, e^{-(y^2/4t)-t} \quad .$$

In particular, $\delta N_0(p)$ is asymptotically $\exp(1)$. For the symmetric random walk let $N_{0,2n}$ denote the number of zeros up to time $2n$. A classical theorem of Chung-Hunt [5] states that $\sqrt{2/n} N_{0,2n}$ is asymptotically distributed as $|N(0,1)|$. All these results show that deviations from the symmetric random walk become clearly visible after $n \approx \delta^{-2}$ steps. while characteristics like the positive sojourn time and the last exit time from zero are in both cases of approximately the same size their distributions differ. For the last exit from zero a precise description is given in 7.3 below. ■

Remark 3.3. It is well-known that

$$r(z) = \frac{1-\sqrt{1-4pqz^2}}{2pz}$$

is the generating function of the length of the first strictly descending ladder epoch, so (3.1) shows that

$$f_0(z) = \frac{p-q}{p}\frac{1}{1-r(z)}.$$

By Theorem 4 in Section XII.7 of [8] this means that for each $n \geq 1$

$$P(Z_0 = n) = \frac{p-q}{p}\, \mathbf{P}(S_1 < 0, \ldots, S_n < 0). \tag{3.3}$$

■

Relation (3.3) is no coincidence. We will see in the next section that it can be generalized to arbitrary random walks. But first let us consider a few other occupation times for the transient simple random walk.

Let U be the total number of visits to a *fixed single point*, say zero. First assume the simple random walk starts from zero. The probability to eventually return to zero is q/p when the first move is to the right and 1 otherwise, so that it is given by $p(q/p)+q = 1-(p-q) = 1-\delta$. Hence,

$$\mathbf{P}(U = n) = \delta\,(1-\delta)^{n-1}, \quad n \in \mathbf{N}.$$

Therefore, as $p \searrow 1/2$,

$$\begin{aligned}\mathbf{P}((p-q)U < x) &= \sum_{n < x/(p-q)} \delta\,(1-\delta)^{n-1} \\ &= 1-(1-\delta)^{x/\delta}(1+o(1)) \\ &= 1-e^{-x}+o(1).\end{aligned}$$

So we obtain an exponential distribution in the limit.

If the random walk starts at an arbitrary point k, then it reaches zero almost surely if $k \leq 0$ and with probability $(q/p)^k$ if $k > 0$. Thus the limiting distribution of $(p-q)Z$ remains the same.

Now let us solve the *two-point case* and derive the distribution of the total number of visits to $\{0,1\}$, which we call again U. We present two methods to derive its generating function when starting from k, again called $f_k(z)$.

The *first method* is first-step analysis, which yields difference equations leading to

$$f_k(z) = 1 + b(z)(q/p)^k, \quad k \geq 1$$

$$f_k(z) = f_0(z), \quad k \leq 0$$

For $k = 0, 1$ we obtain two linear equations for $b(z)$ and $f_0(z)$. After tedious algebra we arrive at

$$f_0(z) = \frac{z^2(p^2 - pq)}{1 - 2qz + z^2(q^2 - pq)}.$$

As above it can be shown that the Laplace transform of $(p-q)U$ converges to $1/(1+2s)$, so that the asymptotic distribution is $\exp(1/2)$.

The *second method* gives more insight into the structure of U. Suppose the random walk starts at 0. The total number of visits can be decomposed by splitting the random walk in independent excursions as follows:

(a) First an excursion in $(-\infty, 0]$, starting at 0, until the random walk reaches 1 for the first time. The number V_1 of visits to 0 during this excursion has a geometric distribution with parameter p: $\mathbf{P}(V_1 = n) = q^{n-1}p,\ n \in \mathbf{N}$.

(b) This first excursion is followed by one in $[1, \infty)$ which either ends by reaching 0 or goes on indefinitely. During this second excursion a return to 1 takes place when making one step from 1 to 2 and then eventually getting back from 2 to 1; since this has probability $p \times (q/p) = q$, the total number T_1 of visits to 1 during such an excursion is also geometrically distributed: $\mathbf{P}(T_1 = n) = q^{n-1}p,\ n \in \mathbf{N}$.

If the random walk has returned to 0 after an excursion of type (b), a new excursion of type (a) starts. This way we obtain a sequence of iid random variables $V_1, T_1, V_2, T_2, \ldots$ such that

$$U = V_1 + T_1 + \cdots + V_N + T_N,$$

where N is the number of the first excursion of type (b) that does not end at state 0. The probability for one excursion of type (b) to go on forever is

$$p(1 - \frac{q}{p}) + p\frac{q}{p}p(1 - \frac{q}{p}) + p\frac{q}{p}p\frac{q}{p}p(1 - \frac{q}{p}) + \cdots = 1 - \frac{q}{p}.$$

Hence,

$$\mathbf{P}(N = n) = (\frac{q}{p})^{n-1}(1 - \frac{q}{p}), \quad n \in \mathbf{N}.$$

It is clear that N is independent of the V_i and a little reflection shows that also N and the T_i are independent. The generating functions of V_1, T_1 and N are given by $g_V(z) = g_T(z) =$

$pz/(1-qz)$ and $g_N(z) = (p-q)z/(p-qz)$. Hence we obtain

$$\mathbf{E}z^U = g_N(g_V(z)g_T(z)) = \frac{(p-q)p^2z^2}{p(1-qz)^2 - qp^2z^2}.$$

After a few manipulations this is the same as our previous formula.

Remark 3.4. The case of an *arbitrary finite number of points*, say the total number of visits to $\{0, 1, \ldots, N\}$, can, in principle, also be tackled by first-step analysis. For the corresponding generating functions, again called $f_k(z)$ for simplicity, it is easily seen that G700G they are of the form

$$f_k(z) = 1 + b(z)(q/p)^k, \quad k \geq N$$
$$f_k(z) = f_0(z), \quad k \leq 0$$

and satisfy the difference equation

$$f_k(z) = z^{\varepsilon_k}[pf_{k+1}(z) + qf_{k-1}(z)],$$

where $\varepsilon_k = 1$ if $0 \leq k \leq N$ and $\varepsilon_k = 0$ otherwise. For $k = 0, 1, \ldots, N$ we obtain two $N+1$ linear equations for the unknowns $f_0(z), \ldots, f_N(z), b(z)$ from which these functions can be determined. ■

4. More general random walks

For an arbitrary random walk S_n with positive drift starting at 0, define Z_0 as above and let $r(z)$ denote the generating function of the length of the random walk's first strictly descending ladder epoch. Let $a(z)$ denote the generating function of the length of the first weakly ascending ladder epoch and μ be its expected value. The following formulas express the generating function of Z_0 in terms of $r(z)$ or of $a(z)$, respectively.

Theorem 4.1.

$$\mathbf{E}z^{Z_0} = \frac{1-r(1)}{1-r(z)} = \frac{1}{\mu}\frac{1-a(z)}{1-z}. \tag{4.1}$$

Proof. According to Lemma 2.3, each sequence $x_1, x_2 \ldots$ with exactly k negative partial sums there corresponds (by a finite reordering) a unique sequence with first (absolute) minimum at the kth place. The partial sums $s_0 = 0, s_1, s_2, \ldots$ of the rearranged sequence consist of a first part $s_0, s_1, \ldots, s_k$ and a second part $s_{k+1}, s_{k+2}, \ldots$ such that the partials sums satisfy $s_i > s_k$ for $i \leq k$ and $s_i - s_k \geq 0$ for $i > k$. For a random walk the joint distribution of the X_i is invariant under finite permutations, and the two parts are independent. The first part has probability

$$\mathbf{P}(0 > S_k, S_1 > S_k, \ldots, S_{k-1} > S_k) = \mathbf{P}(S_1 < 0, \ldots, S_k < 0)$$

(by reversing the order of the variables), the second part has probability

$$\mathbf{P}(S_{k+1} - S_k \geq 0, S_{k+2} - S_k \geq 0, \ldots) = \mathbf{P}(S_1 \geq 0, S_2 \geq 0, \ldots) = 1 - r(1).$$

This yields the first equation of (4.1). The second one follows immediately from the factorization identity $(1-a(z))(1-r(z)) = 1-z$ (recall Theorem 2.2). ■

We use Theorem 4.1 to study further examples.

Theorem 4.2. *Let $t > 1$ be a positive integer, and consider the random walk generated by X_n with $\mathbf{P}(X_n = t-1) = p$, $\mathbf{P}(X_n = -1) = 1-p = q$. Let $p > 1/t$ and set $\delta = \mathbf{E}X_n = tp-1$. Then the corresponding occupation time in $(-\infty, 0)$, say $Z_0(p,t)$, satisfies*

$$\frac{\delta^2}{2(t-1)}\, Z_0(p,t) \longrightarrow A \quad \text{in distribution as} \quad p \searrow 1/t.$$

Proof. First-step analysis yields

$$r(z) = qz + pz\,(r(z))^t.$$

Thus

$$r(z) = qz\, T_t(pq^{t-1}z^t)$$

where T_t denotes the so-called t-ary tree function, which is explicitly given by its power series expansion

$$T_t(z) = \sum_{n=0}^{\infty} \binom{nt}{n} \frac{z^n}{n(t-1)+1}.$$

Define $p_0 = 1/t, q_0 = 1-(1/t)$. The function $\beta(y) = (y-1)/y^t$ has a maximum for $y_0 = 1/q_0$ and T_t is its inverse on the interval $[1, 1/q_0]$. Singular expansion of T_t around $\beta(y_0) = p_0 q_0^{t-1}$ yields

$$q_0 T_t(x) = 1 - w + O(w^2)$$

around y_0, where

$$w = \sqrt{\frac{2}{t(t-1)}\left(1 - \frac{x}{p_0 q_0^{t-1}}\right)}.$$

Let $z = z(s) = e^{-\delta^2 s}$. Expanding $p \mapsto pq^{t-1}e^{-t(tp-1)^2 s}$ around p_0 gives

$$p_0 q_0^{t-1}(1 - t(\frac{1}{2(t-1)} + s)\delta^2 + O(\delta^3).$$

Furthermore, $q = q_0 - (\delta/t)$. Thus,

$$\begin{aligned}
1 - r(z) &= 1 - qz\, T_t(pq^{t-1}z^t) \\
&= 1 - (q_0 - \frac{\delta}{t})\, T_t\left(p_0 q_0^{t-1}(1 + t(\frac{1}{2(t-1)} + s)\delta^2 + O(\delta^3))\right) \\
&= 1 - (q_0 - \frac{\delta}{t})\left(\frac{1}{q_0} + \sqrt{\frac{2}{t(t-1)}\left(1 - (1 - t(\frac{1}{2(t-1)} + s)\delta^2 + O(\delta^3))\right)}\right) \\
&= \delta\left(\frac{1}{t-1} + \frac{1}{t-1}\sqrt{1 + 2(t-1)s}\right) + o(\delta)
\end{aligned}$$

and as $p \searrow 1/t$ we obtain

$$\frac{1 - r(1)}{1 - r(z)} \longrightarrow \frac{2}{1 + \sqrt{1 + 2(t-1)s}}.$$

∎

In the same way one can show

Theorem 4.3. *(a) Let $a > 1$ and $X_n = a - Y_n$ with Y_n being iid and* $\exp(1)$*-distributed, and let $\delta = a - 1$. Then $\frac{1}{2}\delta^2 Z_0$ converges in distribution to A as $\delta \searrow 0$.*
(b) Let $a < 1$ and $X_n = Y_n - a$ with Y_n being iid and $\exp(1)$*-distributed, and let $\delta = 1 - a$. Then $\frac{1}{2}\delta^2 Z_0$ converges in distribution to A as $\delta \searrow 0$.*

5. Occupation times for associated distributions

We observe that the distribution of A occurs in a natural way as limit of *stationary delay distributions*. Recall that in renewal theory a distribution G on $[0, \infty)$ is called stationary delay if it has a density of the form $(1/\mu)(1 - F(x))$, where F is a distribution function on $[0, \infty)$ with finite mean $\mu > 0$ (the reason being that the renewal process generated by F becomes stationary when started with G).

Lemma 5.1. *For $\alpha \in (0, 1)$ let $\ell_\alpha(s) = e^{-s^\alpha}$ be the Laplace transform of a standard positive stable random variable S_α. For $p > 0$ let $\ell_{\alpha,p}(s)$ be the Laplace transform*

$$\ell_{\alpha,p}(s) = \frac{\ell_\alpha(p+s)}{\ell_\alpha(p)}$$

of the associated distribution, which has expected value $\mu(p) = -\ell_\alpha(p)'/\ell_\alpha(p)$. Then we have, for every $s > 0$,

$$\frac{1 - \ell_{\alpha,p}(ps)}{\mu(p)\, ps} \longrightarrow \frac{(1+s)^\alpha - 1}{\alpha s} =: g_\alpha(s) \quad \text{as } p \searrow 0.$$

Proof. Straightforward computation. ∎

Note that

$$g_\alpha(s) = \frac{1}{(1+s)^{1-\alpha}} - \frac{1}{s}\left(1 - \frac{1}{(1+s)^{1-\alpha}}\right).$$

In particular, $g_{1/2}(s)$ is the Laplace transform of A.

Next we study a *generic example*: a large class of families of associated distributions generated by a distribution of mean zero. The generated random walk is skipfree to the left. The corresponding random walk turns out to have the same limit law for its occupation time in $(-\infty, 0)$ as the simple random walk considered in Theorem 3.1.

Let $f(z) = \mathbf{E}z^X$ be the generating function of a nonnegative integer-valued random variable X such that

$$p_0 = \mathbf{P}(X = 0) > 0, \ \mathbf{E}X = 1.$$

We assume that the moment-generating function $\mathbf{E}e^{uX}$ exists in a neighborhood U of 0. Let $m = f''(1) = \mathbf{E}X(X-1)$. Here $u \in U$ will serve as a parameter describing the drift: to parametrize the expected value we consider random variables X_u with generating functions

$$f_u(z) = f(e^u z)/f(e^u).$$

We consider the random walk generated by iid random variables Y_i^u where each Y_i^u is distributed as $X_u - 1$, and the corresponding length of the strictly descending ladder epoch with generating function $r_u(z)$. Our aim are assertions for the case $\delta = \delta(u) = \mathbf{E}Y_1^u - 1 \longrightarrow 0$.

Let us first consider the case $u = 0$. First-step analysis for $r(z)$ leads to the equation

$$r(z) = p_0\, z + z\,(f(r(z)) - p_0) = z\, f(r(z)).$$

Under the stated conditions the singular inversion theorem is applicable (see e.g. Theorem VI.6 in [9]), so $r(z)$ has a singular expansion

around its singular radius ρ of the form

$$r(z) = \tau - \sqrt{\frac{2f(\tau)}{f''(\tau)}\left(1 - \frac{z}{\rho}\right)} + O(1 - \frac{z}{\rho}).$$

Here τ is the unique positive solution of the critical equation $f(z) - z\,f'(z) = 0$, and $\rho = \tau/f(\tau)$ is the radius of convergence of $r(z)$ around zero.

The same is true for the corresponding generating function $r_u(z)$ with u in the neighborhood U of zero, and we want to investigate how these expansions evolve as u varies. We write $m(u)$, $\tau(u)$, $\rho(u)$ in this case. We find for $u = 0$: $\tau(0) = 1, m(0) = m, \rho(0) = 1$ and

$$r_0(z) = 1 - \sqrt{\frac{2}{m}\,(1 - z)} + O(1 - z).$$

For $u \neq 0$ we find $\tau(u) = e^{-u}$, $m(u) = e^{2u}\,m$, $\rho(u) = e^{-u}f(e^u)$ and

$$r_u(z) = p_0 e^u\,z + z\,(f_u(r_u(z)) - p_0) = z\,f_u(r_u(z)).$$

It follows that

$$r_u(z) = e^{-u}\left(1 - \sqrt{\frac{2}{m}\left(1 - \frac{z}{\rho(u)}\right)}\right) + O(1 - \frac{z}{\rho(u)}).$$

Recall that $\delta(u) = \mathbf{E}X_u - 1 = (e^u f'(e^u)/f(e^u)) - 1$ and set $z = e^{-s\delta^2(u)}$ for a fixed $s > 0$. Taylor expansion shows that

$$\rho(u) = 1 + m\frac{u^2}{2} + O(u^3)$$
$$\delta(u) = mu + O(u^2)$$
$$e^{-s\delta^2(u)} = 1 - m^2u^2s + O(u^3).$$

Inserting these expansions we finally arrive at

$$1 - r_u(e^{-s\delta^2(u)}) = u + |u|\sqrt{1 + 2ms + O(u)} + o(u).$$

As a consequence we have:

Theorem 5.2. *For the random walks described above, and under the conditions given there,*

$$\frac{\delta^2}{2m}Z_0 \longrightarrow A \quad as \quad \delta \searrow 0.$$

Proof. $\delta \searrow 0$ corresponds to $u \searrow 0$, and by the expansion above we find that, as $u \searrow 0$,

$$\frac{1 - r_u(1)}{1 - r_u(e^{-s\delta^2(u)})} \longrightarrow \frac{2}{1 + \sqrt{1 + 2ms}}.$$

∎

The latter result covers the case $u > 0$. For $u < 0$ we get

Theorem 5.3. *For the random walks described above, and under the conditions given there,*

$$(r(e^{-s\delta^2(u)}))^{1/|u|} \longrightarrow e^{-\sqrt{1-2ms}+1} \quad as \quad u \nearrow 0.$$

As $|u| \approx m/|\delta(u)|$, this may be rephrased as

$$(r(e^{-s\delta^2}))^{1/|\delta|} \longrightarrow e^{-\frac{1}{m}(\sqrt{1-ms}+1)} \quad \text{as} \quad \delta \nearrow 0.$$

In other words, if we denote the length of the strictly descending ladder epoch with expected value $1/\delta$ by R_δ, the sum of $1/|\delta|$ independent copies of the random variables $\delta^2 R_\delta$ will converge in distribution to a random variable with Laplace transform $\exp\{-(1/m)(\sqrt{1-ms}+1)\}$ as $\delta \nearrow 0$, which belongs to an associated distribution of the standard stable distribution $S_{1/2}$

For the weakly ascending ladder epoch we find:

Theorem 5.4. *For the random walks described above, and under the conditions given there,*

$$\frac{1-a_u(1)}{1-a_u(e^{-s\delta^2(u)})} \Longrightarrow \frac{2}{1+\sqrt{1+2ms}} \quad as \quad u \nearrow 0.$$

Proof. We have

$$r'_u(1) = \frac{1}{1-f'_u(1)} = \frac{1}{|\delta(u)|}$$

and

$$\frac{1-a_u(1)}{1-a_u(e^{-s\delta^2(u)})} = \frac{1}{r'_u(1)} \frac{1-r(e^{-s\delta^2(u)})}{1-e^{-s\delta^2(u)}}.$$

From the expansion above we obtain

$$\frac{1}{r'_u(1)} \frac{1-r(e^{-s\delta^2(u)})}{1-e^{-s\delta^2(u)}} = \frac{(|u|+O(u^2))\big(\big(|u|(\sqrt{1+2ms}-1)+o(u)\big)}{u^2m^2s+O(u^3)}$$
$$\longrightarrow \frac{2}{1+\sqrt{1+2ms}}.$$

■

Note that Theorems 5.3 and 5.4 hold under the sole assumption that X is a nonnegative random variable with finite variance, since the moment-generating function certainly exists for $u \leq 0$. Further, since the lengths of the weak ascending ladder epochs corresponding to $X-1$ have the same distribution as those of the weak descending ladder epochs corresponding to $1-X$, the limit A appears on "both sides", for $\delta \searrow 1$ as the limit in distribution of the number of visits to the negative reals, for $\delta \nearrow 1$ as the limit in distribution of the number of visits to the nonnegative reals.

Remark 5.5 Note that in all cases where we used the singular inversion theorem above, it provides (in principle) a complete asymptotic expansion. ■

Remark 5.6. The results can be interpreted in terms of branching processes. It is known (see [7], p. 298) that the unique positive solution $r(z)$ of $r(z) = z\,f(r(z))$ is the generating function of the total progeny of a Galton-Watson process with reproduction function $f(z)$, and in the situation above we describe the transition from the supercritical case ($u > 0$) via the critical case ($u = 0$) to the subcritical case ($u < 0$) for a certain parametrized class of distributions. In subcritical and critical cases extinction occurs almost surely, in the "slightly" supercritical case we have from the derivation above that the survival probability is given by

$$1 - r_u(1) = 2u + o(u) = \frac{2(\delta(u))}{m} + o(\delta(u)).$$

This reaffirms the following result (see e.g. Theorem 5.5. in [10]): for a sequence of reproduction functions f_ε with expected values $1+\varepsilon$ and variances σ_ε^2 we have $1-r_\varepsilon(1) = 2\sigma_0^{-2}\varepsilon + o(\varepsilon)$ whenever $\sigma_\varepsilon^2 \longrightarrow \sigma_0^2$ as $\varepsilon \searrow 0$, if $\sup_{\varepsilon\le\varepsilon_0} \mathbf{E}X_\varepsilon^{2+\delta} < \infty$ for some $\varepsilon_0 > 0, \delta > 0$. ∎

6. The normal random walk

In this section we consider the asymptotic behavior of Z_0 for the random walk with iid steps having the normal distribution with expected value δ and variance σ^2, and let $d = |\delta| \searrow 0$. In this case we find for $r(z)$, directly from Sparre Andersen's theorem,

$$\begin{aligned}\log\left(\frac{1}{1-r(z)}\right) &= \sum_{n=1}^{\infty} \frac{z^n}{n}\mathbf{P}(S_n < 0)\\ &= \sum_{n=1}^{\infty} \frac{z^n}{n}\int_{-\infty}^{-n\delta} \frac{1}{\sqrt{2n\pi\sigma^2}}\, e^{-x^2/2n\sigma^2}\, dx\\ &= \sum_{n=1}^{\infty} \frac{z^n}{n}\left(\frac{1}{2} - \operatorname{sign}(\delta)\int_0^{nd} \frac{1}{\sqrt{2n\pi\sigma^2}}\, e^{-x^2/2n\sigma^2}\, dx\right).\end{aligned}$$

Hence,

$$1 - r(z) = (1-z)^{\frac{1}{2}} \exp(\operatorname{sign}(\delta)G(z)), \tag{6.1}$$

where

$$G(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}\int_0^{nd} \frac{1}{\sqrt{2n\pi\sigma^2}}\, e^{-x^2/2n\sigma^2}\, dx.$$

We have

$$\begin{aligned}\int_0^{nd} \frac{1}{\sqrt{2n\pi\sigma^2}}\, e^{-x^2/2n\sigma^2}\, dx &= \int_0^{d} \sqrt{\frac{n}{2\pi\sigma^2}}\, e^{-ny^2/2\sigma^2}\, dy\\ &= \frac{n}{\pi\sigma^2}\int_0^{d}\int_0^{\infty} e^{-n(y^2+x^2)/2\sigma^2}\, dy\, dx\\ &= \frac{nd^2}{\pi\sigma^2}\int_0^{1}\int_0^{\infty} e^{-nd^2(y^2+x^2)/2\sigma^2}\, dy\, dx\end{aligned}$$

and therefore

$$G(z) = \frac{d^2}{\pi\sigma^2}\int_0^{1}\int_0^{\infty} \frac{e^{-d^2(y^2+x^2)/2\sigma^2}}{1-ze^{-d^2(y^2+x^2)/2\sigma^2}}\, dy\, dx.$$

Thus $G(z)$ depends only on the ratio $q = d^2/2\sigma^2$. Fix $s > 0$. Setting $z = e^{-qs}$ we obtain for $q \searrow 0$ (by dominated convergence):

$$\begin{aligned}G(e^{-qs}) &= \frac{2}{\pi}\int_0^{1}\int_0^{\infty} \frac{qe^{-q(y^2+x^2)}}{1-e^{-q(s+y^2+x^2)}}\, dy\, dx\\ &\longrightarrow \frac{2}{\pi}\int_0^{1}\int_0^{\infty} \frac{1}{s+y^2+x^2}\, dy\, dx\\ &= \log\left(\frac{1+\sqrt{1+s}}{\sqrt{s}}\right).\end{aligned}$$

Inserting this into (6.1) we find in a similar way as above

Theorem 6.1.

(a) $qZ_0 \longrightarrow A$ *in distribution as* $q \searrow 0, \delta \searrow 0$.

(b) $r(e^{-qs})^{1/\sqrt{q}} \longrightarrow e^{-(\sqrt{1+s}-1)}$ *as* $q \searrow 0, \delta \nearrow 0$.

Note that here σ^2 may vary with δ, it is only essential that $q \longrightarrow 0$.

7. Some auxiliary results for Brownian motion with drift

We present some related results on Brownian motion with positive drift $\delta > 0$ and variance σ^2. Let B_t be a standard Brownian motion and $X_t = \sigma B_t + \delta t$.

Lemma 7.1. *(1) Let* $z > 0$ *and* $T_z = \inf\{t \geq 0 \,:\, X_t \geq z\}$ *be the first time when* X_t *reaches level* z. *Then* T_z *has Laplace transform*

$$\ell_{T_z}(s) = \mathbf{E}(e^{-sT_z}) = \exp\left(-\frac{z}{\sigma^2}(\sqrt{\delta^2 + 2\sigma^2 s} - \delta)\right).$$

(2) Let $V_0 = V_0(\delta) = \int_0^\infty 1_{(-\infty,0)}(X_t)\, dt$ *be the total time that* X_t *spends below zero. Then* V_0 *has Laplace transform*

$$\ell_{V_0}(s) = \mathbf{E}(e^{-sV_0}) = \frac{2\delta}{\delta + \sqrt{\delta^2 + 2\sigma^2 s}}$$

Proofs for (1) resp. (2) (for $\sigma^2 = 1$) can be found in [13] resp. [12]. Note that the distribution of $(\delta^2/2\sigma^{2^2})V_0(\delta)$ is equal to that of A for all $\delta > 0$.

For $z \geq 0$ let $V_z : \int_0^\infty 1_{(-\infty,z)}(X_t)\, dt$ the total time the process spends below z. Then the obvious decomposition $V_z = T_z + V_0$ (obtained by conditioning on T_z) yields

Lemma 7.2. V_z *has Laplace transform*

$$\ell_{V_z}(s) = \mathbf{E}(e^{-sV_z}) = \ell_{T_z}(s)\, \ell_{V_0}(s).$$

The density and distribution function are given in [12].

Now let $\delta \in \mathbf{R} \setminus \{0\}, \sigma^2 = 1$, so that $X_t = B_t + \delta t$, and consider $W = \sup\{t \in [0,1] \,:\, X_t = 0\}$, the last time X_t visits 0 in $[0,1]$.

Theorem 7.3. $W \stackrel{d}{=} C \cdot \min\{[1, D_\delta\}$ *where* C *and* D_δ *are independent,* C *is arcsine-distributed, and* D_δ *is* $\exp(\delta^2/2)$*-distributed. The moments of* W *are given by*

$$\mathbf{E}W^k = \binom{2k}{k} \frac{1}{2^{2k}} \int_0^1 k y^{k-1} e^{-\delta^2 y/2}\, dy, \quad k \geq 1.$$

Proof. We use a random walk approximation in the style of Takács [22]. Let $Y_1, Y_2, \dots$ be iid with

$$\mathbf{P}(Y_i = 1) = p = \frac{1}{2} + \frac{\delta}{2\sqrt{n}}, \quad \mathbf{P}(Y_i = -1) = q = 1 - p$$

(p and q depend on n, but this is suppressed in the notation) and partial sums $S_0 = 0$, $S_k = \sum_{i=1}^k Y_i$.

It it easy to see that the processes $X^{(n)}$ defined by

$$X^{(n)}(t) = \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}, \quad 0 \le t \le 1$$

converge in distribution to $X = (X_t)_{t\in[0,1]}$ in $D[0,1]$.

Furthermore, the last-exit time from 0 is continuous in the Skorohod topology on $D[0,1]$ on a set of P_X-measure 1, and

$$T_n = \sup\{t \in [0,1] \ : \ X^{(n)}(t) = 0\} = \frac{1}{n} \max\{0 \le k \le n \ : \ S_k = 0\}.$$

Then it suffices to show that $T_N/N \longrightarrow C \cdot \min\{1, D_\delta\}$ as $N \longrightarrow \infty$.

Since $\frac{1}{\sqrt{1-4pqz^2}}$ and $\frac{\sqrt{1-4pqz^2}}{1-z}$ are the generating functions for pq-paths which start and end at 0 and for those which start at 0 and never return to 0, respectively, the generating function of T_N is

$$\begin{aligned}\mathbf{E}(t^{T_N}) &= [z^N] \frac{1}{\sqrt{1-4pqt^2z^2}} \frac{\sqrt{1-4pqz^2}}{1-z} \\ &= [z^N] \frac{1}{\sqrt{1-4pqt^2z^2}} \frac{\sqrt{1-4pqz^2}}{1-z^2}(1+z).\end{aligned}$$

(Here $[z^N]f(z)$ denotes the coefficient of $[z^N]$ in the Taylor expansion of the function $f(z)$ around zero.) Thus the generating functions for $N = 2n+1$ and $N = 2n$ are identical and it is enough to consider even N. Let $N = 2n$ be even (and $n > m^2$) and $U_n = T_N/2$. Then the generating function of U_n is

$$\begin{aligned}\mathbf{E}(t^{U_n}) &= [z^{2n}] \frac{1}{\sqrt{1-4pqtz^2}} \frac{\sqrt{1-4pqz^2}}{1-z^2} \\ &= [z^n] \frac{1}{\sqrt{1-4pqtz}} \frac{\sqrt{1-4pqz}}{1-z}\end{aligned}$$

so that the $k-$th factorial moment $u_{k,n} = \mathbf{E}\left(U_n(U_n-1)\cdots(U_n-k+1)\right)$ of U_n is given by

$$\begin{aligned}u_{k,n} &= k!(-1)^k \binom{-\frac{1}{2}}{k} (4pq)^k \, [z^{n-k}] \frac{1}{(1-4pqz)^k \, (1-z)} \\ &= k(-1)^k \binom{-\frac{1}{2}}{k} (4pq)^k \, [z^{n-k}] \frac{1}{(1-z)} \int_0^\infty x^{k-1} e^{-(1-4pqz)x} \, dx \\ &= (-1)^k \binom{-\frac{1}{2}}{k} (4pq)^k \int_0^\infty k x^{k-1} e^{-x} \left(\sum_{j=0}^{n-k} \frac{(4pqx)^j}{j!} \right) dx.\end{aligned}$$

Now denote by $Poiss(\lambda)$ a random variable having the Poisson distribution with parameter λ. As $4pq = 1 - (\delta^2/2n)$, we obtain

$$\begin{aligned}&\int_0^\infty k x^{k-1} e^{-x} \left(\sum_{j=0}^{n-k} \frac{(4pqx)^j}{j!} \right) dx \\ &= \int_0^\infty k x^{k-1} e^{-x(1-4pq)} \, \mathbf{P}\left(Poiss(4pqx) \le n-k\right) dx\end{aligned}$$

$$= n^k \int_0^\infty k y^{k-1} e^{-\delta^2 y/2} \, \mathbf{P}\left(Poiss((n - \frac{\delta^2}{2})y) \le n - k \right) dy.$$

By the central limit theorem,

$$\mathbf{P}\left(Poiss((n - \frac{\delta^2}{2})y) \le n - k \right) \longrightarrow \begin{cases} 1 & \text{for } 0 \le y < 1 \\ \frac{1}{2} & \text{for } y = 1 \\ 0 & \text{for } y > 1 \end{cases}$$

so that for every k we have

$$\frac{u_{n,k}}{n^k} \longrightarrow (-1)^k \binom{-\frac{1}{2}}{k} \int_0^1 k y^{k-1} e^{-y\delta^2/2} \, dy.$$

Hence $\mathbf{E}T_N^k / N^k$ tends to the same limit. This shows the second assertion. Finally,

$$\mathbf{E}C^k = (-1)^k \binom{-\frac{1}{2}}{k} = \binom{2k}{k} \frac{1}{2^{2k}}$$

and integration by parts shows that $\int_0^1 k y^{k-1} e^{-y\delta^2/2} \, dy = \mathbf{E} \min\{1, D_\delta^k\}$. Thus all moments of T_N/N converge to the corresponding moments of $C \cdot \min\{1, D_\delta\}$. Since the distribution of $C \cdot \min\{1, D_\delta\}$ is clearly determined by its moments the first assertion follows. ∎

As an immediate consequence of the scaling properties of Brownian motion we see that the distribution of

$$W(T) = \sup\{t \le T \, : \, \sigma B_t + \delta t = 0\}$$

is the same as that of $C \cdot \min\{T, D_{\delta/\sigma}\}$. The time of the last zero of $\sigma B_t + \delta t$ in the interval $[0, \infty)$ is thus distributed as $C \cdot D_{\delta/\sigma}$, which is the gamma distribution $\gamma(\frac{\delta^2}{2\sigma^2}, \frac{1}{2})$.

Remark 7.4. Last-exit times of Brownian motion from moving boundaries have been studied intensively, and a more complicated expression for the density of the last-exit time from a linear boundary was derived in [20]. The representation in 7.3 appears to be new, as it is not mentioned in the encyclopaedic monograph [2]. For the density of the sojourn time found by Takács by a random walk limit two "purely Brownian" explanations have been given by in [6]. It is natural to ask for such an explanation for the representation in 7.3. ∎

8. Occupation time for a shifted random walk

In this last section we consider a shifted random walk. Specifically, let $(X_1, X_2, \ldots)$ be a sequence of iid random variables with $\mathbf{E}(X_i) = 0$, $\mathrm{Var}(X_i) = \sigma^2 \in (0, \infty)$. Let $\delta > 0$ and $Y_i^\delta = X_i + \delta$, $S_n^\delta = \sum_{i=1}^n Y_i^\delta$. We are interested in $Z_0^\delta = \sum_{i=1}^\infty 1_{(-\infty,0)}(S_n^\delta)$.

We use the same ideas as Prohorov [18], who proved

Theorem 8.1. *(Prohorov) In the situation above let* $M^\delta = \min\{S_n^\delta \, : \, n \ge 0\}$. *Then*

$$\mathbf{P}(\delta M^\delta > x) \longrightarrow e^{-2x/\sigma^2} \quad \textit{for all } x > 0.$$

In [18] the maximum in the case of negative drift was considered instead of M^δ. The result had been proved earlier by Kingman under the assumption of the existence of an exponential moment.

Theorem 8.2.

$$\frac{\delta^2}{2\sigma^2} Z_0^\delta \longrightarrow A \quad \textit{in distribution as } \delta \searrow 0.$$

Proof. Let $T > 0$ and consider the sequence of processes

$$U^\delta(t) = \delta \sum_{i=1}^{\lfloor T/\delta^2 \rfloor} Y_i^\delta, \quad 0 \le t \le T.$$

By Donsker's limit theorem, the sequence $U^\delta \longrightarrow \sigma B + id$ in distribution in $D[0,T]$, where $\sigma B + id$ denotes the Brownian motion with variance σ^2 and drift 1, i.e. with coordinate variables $\sigma B_t + t$. For any bounded Borel function v on $[0,T]$ the functional $x \mapsto \int_0^T v(x_t)\, dt$ on $D[0,T]$ is Skorohod-measurable and continuous except on a set of B-measure 0 (see e.g. [1], p. 247). Thus,

$$\begin{aligned} \delta^2 \mathrm{card}(\{n \ : \ S_n^\delta < 0, 1 \le n \le T/\delta^2\}) &= \int_0^{\delta^2 \lfloor T/\delta^2 \rfloor} 1_{(-\infty,0)}(U^\delta(t))\, dt \\ &\longrightarrow \int_0^T 1_{(-\infty,0)}(X_t)\, dt \text{ as } \delta \searrow 0 \end{aligned}$$

in distribution and we will be done if we can justify the interchange of the limits $T \longrightarrow \infty, \delta \searrow 0$. Let $\delta_k > 0$ be a sequence decreasing to zero. Prohorov showed in essence (by an application of Kolmogorov's inequality) that for every $\varepsilon > 0$ we can find an N such that $\mathbf{P}(\min_{n \ge N/\delta_k^2} S_n^{\delta_k} \le 0) < \varepsilon$ for all k. Thus

$$\lim_{T \longrightarrow \infty} \sup_{k \ge 1} \mathbf{P}(\min_{n \ge T/\delta_k^2} S_n^{\delta_k} \le 0) = 0 \tag{8.1}$$

and the assertion follows, since (by the monotone convergence theorem)

$$\lim_{T \longrightarrow \infty} \int_0^T 1_{(-\infty,0)}(X_t)\, dt = \int_0^\infty 1_{(-\infty,0)}(X_t)\, dt.$$

∎

Remark 8.5. A similar discussion can be found in [21]. In that paper, Shneer and Wachtel derived an extension of Kolmogrov's inequality and treated the maximum of random walks with negative drift and step size distributions attracted to a stable law of index $\alpha \in (1,2]$. In the case of finite variance ($\alpha = 2$) they remark that their results (including in particular the crucial relation (8.1)) remain valid if the step size distribution depends on δ in such a way that $\sigma^2(\delta) \longrightarrow \sigma^2 > 0$ and if additionally the following Lindeberg-type condition holds: for every $\varepsilon > 0$,

$$\lim_{\delta \longrightarrow 0} \int_{|\delta X^\delta| > \varepsilon} (X_1^\delta)^2 \, d\mathbf{P} = 0.$$

It is then not hard to show that theorem 8.2 holds under these more general conditions. ∎

Remark 8.6. Assume the X_i are independent with $\mathbf{E}(X_i) = 0$ and variances $\mathrm{Var}(X_i) = \sigma^2{}_i$ and satisfy Lindeberg's condition. Let $s_i^2 = \sum_{k=1}^i \sigma^2{}_k$. Then the step process $X_n(t)$ which jumps to the value S_i/s_n at time s_i^2/s_n^2 converges weakly to a standard Brownian in $D[0,1]$ (by Prohorov's extension of Donsker's theorem). One may thus expect that they exhibit similar limiting behavior. ∎

Finally it is now a routine matter to show

Theorem 8.3. *In the situation above let* $z > 0$ *and* $Z_z = \sum_{n=1}^{\infty} 1_{(-\infty,z)}(S_n)$. *Then* $\delta^2 Z_{z/\delta} \longrightarrow V_z$ *in distribution, where the Laplace transform of* V_z *is given in Lemma 7.2. with* $\mu = 1$.

Apparently the distribution of A occurs naturally as a limit of occupation times for random walks with drift. It is well-known (see e.g. Section XIV.3 in [8]) that the deeper reason for the frequent occurrence of the (generalized) arcsine distributions lies in their intimate connection to distribution functions with regularly varying tails. The same explanation applies here. In the case of drift zero the distribution functions of the ladder epochs are attracted to the standard positive stable distribution $S_{1/2}$, and the positive sojourn times are asymptotically arcsine-distributed. In the cases with small drift (and finite variance) the ladder epochs are attracted to an associated distribution of $S_{1/2}$, and therefore the positive sojourn times have the distribution of A given above.

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