KAM FOR THE KLEIN GORDON EQUATION ON $\mathbb{S}^d$.

by

Benoît Grébert & Eric Paturel

Abstract. — Recently the KAM theory has been extended to multidimensional PDEs. Nevertheless all these recent results concern PDEs on the torus, essentially because in that case the corresponding linear PDE is diagonalized in the Fourier basis and the structure of the resonant sets is quite simple. In the present paper, we consider an important physical example that do not fit in this context: the Klein Gordon equation on $\mathbb{S}^d$. Our abstract KAM theorem also allow to prove the reducibility of the corresponding linear operator with time quasiperiodic potentials.

Contents

1. Introduction.

If the KAM theorem is now well documented for nonlinear Hamiltonian PDEs in 1-dimensional context (see [**22, 23, 25**]) only few results exist for multidimensional PDEs.

Existence of quasi-periodic solutions of space-multidimensional PDE were first proved in [**8**] (see also [**9**]) but with a technique based on the Nash-Moser thorem that does not allow to analyze the linear stability of the obtained solutions. Some KAM-theorems for small-amplitude solutions of multidimensional beam equations (see (3.6) above) with typical m were obtained in [**16, 17**]. Both works treat equations with a constant-coefficient nonlinearity $g(x, u) = g(u)$, which is significantly easier than the general case. The first complete KAM theorem for space-multidimensional PDE was obtained in [**15**]. Also see [**4, 5**].
The techniques developed by Eliasson-Kuksin have been improved in [**13, 12**] to allow a KAM result without external parameters. In these two papers the authors prove the existence of small amplitude quasi-periodic solutions of the beam equation on the d-dimensional torus. They further investigate the stability of these solutions and give explicit examples where the solution is linearly unstable and thus exhibits hyperbolic features (a sort of whiskered torus).

2000 ***Mathematics Subject Classification.*** — .
Key words and phrases. — KAM theory, Regularizing PDE, Klein Gordon equation.

All these examples concern PDEs on the torus, essentially because in that case the corresponding linear PDE is diagonalized in the Fourier basis and the structure of the resonant sets is the same for NLS, NLW or beam equation. In the present paper, adapting the technics in [**15**], we consider an important example that do not fit in the Fourier analysis: the Klein Gordon equation on the sphere $\mathbb{S}^d$.
Notice that existence of quasi-periodic solutions for NLW and NLS on compact Lie groups via Nash Moser technics (and without linear stability) has been proved recently in [**7, 6**].

To understand the new difficulties, let us start with a brief overview of the method developed in [**15**]. Consider the nonlinear Schrödinger equation on $\mathbb{T}^d$

$$iu_t = -\Delta u + \text{nonlinear terms}, \quad x \in \mathbb{T}^d,\ t \in \mathbb{R}.$$

In Fourier variables it reads [1]

$$i\,\dot u_k = |k|^2 u_k + \text{nonlinear terms}, \quad k \in \mathbb{Z}^d.$$

So two Fourier modes indexed by $k, j \in \mathbb{Z}^d$ are (linearly) resonant when $|k|^2 = |j|^2$. For the beam equation on the torus, the resonance relation is the same. The resonant sets $\mathcal{E}_k = \{j \in \mathbb{Z}^d \mid |j|^2 = |k|^2\}$ define a natural clustering of $\mathbb{Z}^d$. All the modes in the block $\mathcal{E}_k$ have the same energy, and we can expect that the interactions between different blocks are small, but the interactions inside a block could be of order one. With this idea in mind, the principal step of the KAM technique, i.e. the resolution of the so called homological equation, leads to the inversion of an infinite matrix which is block-diagonal with respect to this clustering. It turns out that these blocks have cardinality growing with $|k|$ making harder the control of the inverse of this matrix. As a consequence we lose regularity each time we solve the homological equation. Of course, this is not acceptable for an infinite induction. The very nice idea in [**15**] consists in considering a sub-clustering constructed as the equivalence classes of the equivalence relation on $\mathbb{Z}^d$ generated by the pre-equivalence relation

$$a \sim b \Longleftrightarrow \begin{cases} |a| = |b| \\ |a-b| \le \Delta \end{cases}$$

Let $[a]_\Delta$ denote the equivalence class of a. The crucial fact (proved in [**15**]) is that the blocks are *finite* with a maximal "diameter"

$$\max_{[a]_\Delta = [b]_\Delta} |a-b| \le C_d \Delta^{\frac{(d+1)!}{2}}$$

depending only on Δ. With such a clustering, we do not lose regularity when we solve the homological equation. Furthermore, working in a phase space of analytic functions u or equivalently, exponentially decreasing Fourier coefficients u_k, it turns out that the homological equation is "almost" block diagonal relatively to this clustering. Then we let the parameter Δ grow at each step of the KAM iteration.

Unfortunately, this estimate of the diameter of a block $[a]_\Delta$ by a constant independent of $|a|$ is a sort of miracle that does not persist in other cases. For instance if we consider the nonlinear Klein Gordon equation on the sphere $\mathbb{S}^2$,

$$(\partial_t^2 - \Delta + m)u = \text{nonlinear terms}, \quad t \in \mathbb{R},\ x \in \mathbb{S}^2$$

then the linear part diagonalizes in the harmonic basis $\Psi_{j,\ell}$ (see Section 3) and the natural clustering is given by the resonant sets $\{(j,\ell) \in \mathbb{N}^2 \mid \ell = -j, \cdots, j\}$. We can easily convince ourself that there is no simple construction of a sub-clustering compatible with the equation, in such a way that the size of the blocks does no more depend on the energy.
So we have to invent a new way to proceed. First we consider a phase space Y_s with polynomial decay on the Fourier coefficient (corresponding to Sobolev regularity for u) instead of

1. The space $\mathbb{Z}^d$ is equipped with standard euclidian norm: $|k|^2 = k_1^2 + \cdots k_d^2$.

exponential decay and we use a different norm on the Hessian matrix that takes into account the polynomial decrease of the off-diagonal blocks:

$$|M|_{\beta,s} = \sup_{j,k\in\mathbb{N}} \|M_{[k]}^{[j]}\|(kj)^{\beta}\left(\frac{\min(j,k)+|j^2-k^2|}{\min(j,k)}\right)^{s/2} \tag{1.1}$$

where $[j] = \{(n,m)\in\mathbb{N}^2 \mid n+m=j\}$ is the block of energy j, $M_{[k]}^{[j]}$ is the interaction matrix M reduced to the eigenspace of energy j and of energy k, and $\|\cdot\|$ is the operator norm in ℓ^2. This norm was suggested by our study of the Birkhoff normal form in [**3**] and [**18**].
This technical changes make disappear the loss of regularity in the resolution of the homological equation. Nevertheless this is not the end of the story, since this Sobolev structure of the phase space $\mathcal{T}^{s,\beta}$ (see Section 2) is not stable by Poisson bracket and thus is not adapted to an iterative scheme. So the second ingredient consists in a trick previously used in [**20**]: we take advantage of the regularizing effect of the homological equation to obtain a solution in a slightly more regular space $\mathcal{T}^{s,\beta+}$ and then we verify that $\{\mathcal{T}^{s,\beta},\mathcal{T}^{s,\beta+}\}\in\mathcal{T}^{s,\beta}$ (see Section 4) which enables an iterative procedure. The last problem is to check that the non linear term, say P, belongs to the class $\mathcal{T}^{s,\beta}$ which imposes a decreasing condition on the operator norm of the blocks of the Hessian of P. It turns out that this condition is satisfied for the Klein Gordon equation on spheres (and also on Zoll manifold, see Remark 3.3). A similar condition is also satisfied for the quantum harmonic oscillator on $\mathbb{R}^d$

$$i\,u_t = -\Delta u + |x|^2 u + \text{nonlinear terms}, \quad x\in\mathbb{R}^d.$$

But unfortunately, in order to belong in the class $\mathcal{T}^{s,\beta}$, the gradient of the nonlinear term has to be regularizing, a fact that is not true for the quantum harmonic oscillator, and thus our KAM theorem does not apply in this case. Nevertheless, this last condition is not required when P is quadratic and thus this method allows to obtain a reducibility result for the quantum harmonic oscillator with time quasi periodic potential. This is detailed in our forthcoming paper [**19**].
In this paper we only consider PDEs with external parameters (similar to a convolution potential in the case of NLS on the torus). Following [**12**] we could expect to remove these external parameters (and to use only internal parameters) but the technical cost would be very high.

We now state our result for the Klein Gordon equation on the sphere. Denote by Δ the Laplace-Beltrami operator on the sphere $\mathbb{S}^d$, $m>0$ and let $\Lambda_0 = (-\Delta+m)^{1/2}$. The spectrum of Λ_0 equals $\{\sqrt{j(j+d-1)+m} \mid j\geq 0\}$. For each $j\geq 1$ let E_j be the associated eigenspace, its dimension is $d_j = O(j^{d-1})$. We denote by $\Psi_{j,l}$ the harmonic function of degree j and order ℓ so that we have

$$E_j = \operatorname{Span}\{\Psi_{j,l},\ l=1,\cdots,d_j\}.$$

We denote

$$\mathcal{E} := \{(j,\ell)\in\mathbb{N}\times\mathbb{Z} \mid j\geq 0 \text{ and } \ell = 1,\cdots,d_j\}$$

in such a way that $\{\Psi_a,\, a\in\mathcal{E}\}$ is a basis of $L^2(\mathbb{S}^d,\mathbb{C})$.
We introduce the harmonic multiplier M_ρ defined on the basis $(\Psi_a)_{a\in\mathcal{E}}$ of $L^2(\mathbb{S}^d)$ by

$$M_\rho\Psi_a = \rho_a\Psi_a \quad \text{for } a\in\mathcal{E} \tag{1.2}$$

where $(\rho_a)_{a\in\mathcal{E}}$ is a bounded sequence of nonnegative real numbers.

Let g be a real analytic function on $\mathbb{S}^d\times\mathbb{R}$ such that g vanishes at least at order 2 in the second variable at the origin. We consider the following nonlinear Klein Gordon equation

$$(\partial_t^2 - \Delta + m + \delta M_\rho)u + \varepsilon g(x,u) = 0, \quad t\in\mathbb{R},\ x\in\mathbb{S}^d \tag{1.3}$$

where $\delta>0$ and $\varepsilon>0$ are small parameters.
Introducing $\Lambda = (-\Delta+m+\delta M_\rho)^{1/2}$ and $v = -u_t \equiv -\dot{u}$, (1.3) reads

$$\begin{cases} \dot{u} &= -v, \\ \dot{v} &= \Lambda^2 u + \varepsilon g(x,u). \end{cases}$$

Defining $\psi = \frac{1}{\sqrt{2}}(\Lambda^{1/2}u + i\Lambda^{-1/2}v)$ we get

$$\frac{1}{i}\dot{\psi} = \Lambda\psi + \frac{\varepsilon}{\sqrt{2}}\Lambda^{-1/2} g\left(x, \Lambda^{-1/2}\left(\frac{\psi+\bar{\psi}}{\sqrt{2}}\right)\right).$$

Thus, if we endow the space $L^2(\mathbb{S}^d, \mathbb{C})$ with the standard real symplectic structure given by the two-form $-id\psi \wedge d\bar{\psi}$ then equation (1.3) becomes a Hamiltonian system

$$\dot{\psi} = i\frac{\partial H}{\partial \bar{\psi}}$$

with the hamiltonian function

$$H(\psi, \bar{\psi}) = \int_{\mathbb{S}^d} (\Lambda\psi)\bar{\psi}\mathrm{d}x + \varepsilon \int_{\mathbb{S}^d} G\left(x, \Lambda^{-1/2}\left(\frac{\psi+\bar{\psi}}{\sqrt{2}}\right)\right)\mathrm{d}x.$$

where G is a primitive of g with respect to the variable u: $g = \partial_u G$.
The linear operator Λ is diagonal in the basis $\{\Psi_a,\, a \in \mathcal{E}\}$:

$$\Lambda\Psi_a = \lambda_a \Psi_a, \quad \lambda_a = \sqrt{w_a(w_a + d - 1) + m + \delta\rho_a}, \qquad \forall\, a \in \mathcal{E}$$

where we set

$$w_{(j,\ell)} = j \quad \forall\,(j,\ell) \in \mathcal{E}.$$

Let us decompose ψ and $\bar{\psi}$ in the basis $\{\Psi_a,\, a \in \mathcal{E}\}$:

$$\psi = \sum_{a\in\mathcal{E}} \xi_a \Psi_a, \quad \bar{\psi} = \sum_{a\in\mathcal{E}} \eta_a \Psi_a\,.$$

On $\mathcal{P}_{\mathbb{C}} := \ell^2(\mathcal{E}, \mathbb{C}) \times \ell^2(\mathcal{E}, \mathbb{C})$ endowed with the complex symplectic structure $-i\sum_s \mathrm{d}\xi_s \wedge \mathrm{d}\eta_s$ we consider the Hamiltonian system

$$\begin{cases} \dot{\xi}_a &= i\frac{\partial H}{\partial \eta_a} \\ \dot{\eta}_a &= -i\frac{\partial H}{\partial \xi_a} \end{cases} \qquad a \in \mathcal{E} \tag{1.4}$$

where the Hamiltonian function H is given by

$$H = \sum_{a\in\mathcal{E}} \lambda_a \xi_a \eta_a + \varepsilon \int_{\mathbb{S}^d} G\left(x, \sum_{a\in\mathcal{E}} \frac{(\xi_a + \eta_a)\Psi_a}{\sqrt{2}\ \lambda_a^{1/2}}\right)\mathrm{d}x. \tag{1.5}$$

The Klein Gordon equation (1.3) is then equivalent to the Hamiltonian system (1.4) restricted to the real subspace

$$\mathcal{P}_{\mathbb{R}} := \{(\xi, \eta) \in \ell^2(\mathcal{E}, \mathbb{C}) \times \ell^2(\mathcal{E}, \mathbb{C}) \mid \eta_a = \bar{\xi}_a,\ a \in \mathcal{E}\}.$$

Definition 1.1. — *Let $\mathcal{A} \subset \mathcal{E}$ a finite subset of cardinal n. This set is* admissible *if and only if*

$$\mathcal{A} \ni (j_1, \ell_1) \neq (j_2, \ell_2) \in \mathcal{A} \Rightarrow j_1 \neq j_2. \tag{1.6}$$

We fix $I_a \in [1, 2]$ for $a \in \mathcal{A}$, the initial n actions, and we write the modes $\mathcal{A}$ in action-angle variables:

$$\xi_a = \sqrt{I_a + r_a}e^{i\theta_a}, \quad \eta_a = \sqrt{I_a + r_a}e^{-i\theta_a}.$$

We define $\mathcal{L} = \mathcal{E} \setminus \mathcal{A}$ and, to simplify the presentation, we assume that

$$\rho_{j,l} = \rho_j \text{ for } (j,\ell) \in \mathcal{A}\ ;\ \rho_{j,l} = 0 \text{ for } (j,\ell) \in \mathcal{L}.$$

Set

$$\begin{aligned} w_{j,\ell} &= j \quad \text{ for } (j,\ell) \in \mathcal{E}, \\ \lambda_{j,\ell} &= \sqrt{j(j+d-1)+m} \text{ for } (j,\ell) \in \mathcal{L}, \\ (\omega_0)_{j,\ell}(\rho) &= \sqrt{j(j+d-1)+m+\delta\rho_j} \text{ for } (j,\ell) \in \mathcal{A}, \\ \zeta &= (\xi_a, \eta_a)_{a\in\mathcal{L}}. \end{aligned} \tag{1.7}$$

With this notation H reads (up to a constant)

$$H(r,\theta,\zeta)=\langle\omega_0(\rho),r\rangle+\sum_{a\in\mathcal{L}}\lambda_a\xi_a\eta_a+\varepsilon f(r,\theta,\zeta)$$

where

$$f(r,\theta,\zeta)=\int_{\mathbb{S}^d}G\left(x,\hat{u}(r,\theta,\zeta)(x)\right)dx$$

and

$$\hat{u}(r,\theta,\zeta)(x)=\sum_{a\in\mathcal{A}}\frac{\sqrt{2(I_a+r_a)}\cos\theta_a}{\lambda_a^{1/2}}\Psi_a(x)+\sum_{a\in\mathcal{L}}\frac{(\xi_a+\eta_a)}{\sqrt{2}\,\lambda_a^{1/2}}\Psi_a(x). \tag{1.8}$$

Let us set $u_1(\theta,x)=\hat{u}(0,\theta;0)(x)$. Then for any $I\in[1,2]^n$ and $\theta_0\in\mathbb{T}^n$ the function $(t,x)\mapsto u_1(\theta_0+t\omega,x)$ is a quasi-periodic solution of (1.3) with $\varepsilon=0$. Our main theorem states that for most external parameter ρ this quasi-periodic solution persists (but is slightly deformed) when we turn on the nonlinearity:

Theorem 1.2. — *Fix n the cardinality of an admissible set $\mathcal{A}$, $s>1$ the Sobolev regularity and g the nonlinearity. There exists an exponent $\upsilon(d)>0$ such that, for ε sufficiently small (depending on n, s and g) and satisfying*

$$\varepsilon\leq\delta^{\upsilon(d)},$$

there exists a Borel subset $\mathcal{D}'$, positive constants α and C with

$$\mathcal{D}'\subset[1,2]^n,\quad \operatorname{meas}([1,2]^n\setminus\mathcal{D}')\leq C\varepsilon^{\alpha},$$

such that for $\rho\in\mathcal{D}'$, there is a function $u(\theta,x)$, analytic in $\theta\in\mathbb{T}^n_{\frac{\sigma}{2}}$ and smooth in $x\in\mathbb{S}^d$, satisfying

$$\sup_{|\Im\theta|<\frac{\sigma}{2}}\|u(\theta,\cdot)-u_1(\theta,\cdot)\|_{H^s(\mathbb{S}^d)}\leq\varepsilon^{11/12},$$

and there is a mapping

$$\omega':\mathcal{D}'\to\mathbb{R}^n,\quad \|\omega'-\omega\|_{C^1(\mathcal{D}')}\leq\varepsilon,$$

such that for any $\rho\in\mathcal{D}'$ the function

$$u(t,x)=u(\theta+t\omega'(\rho),x)$$

is a solution of the Klein Gordon equation (1.3)*. Furthermore this solution is linearly stable. The positive constant α depends only on n while C also depends on g and s.*

Notice that in this work we did not try to optimize the exponents. In particular 11/12 could be replaced by any number strictly less than 1 and the choice of $\upsilon(d)$ obtained by inserting (3.1) in (6.6) is far from optimal. Actually we could expect that $\varepsilon\ll\delta$ is sufficient but the technical cost would be very high. This effort is justified when we try to prove a KAM result without external parameters (see [**24**] where the authors obtained a condition of the form $\varepsilon\ll\delta$ in the context of the NLS equation; see also [**13**], [**12**] for the beam equation and [**10**] for the 1d wave equation where the authors obtained a condition of the form $\varepsilon\ll\delta^{1+\alpha}$ for suitable $\alpha>0$).

We will deduce Theorem 1.2 from an abstract KAM result stated in Section 2 and proved in Section 6. The application to the Klein Gordon equation is detailed in Section 3. Roughly speaking, our abstract theorem applies to any multidimensional PDE with regularizing nonlinearity and which satisfies the second Melnikov condition (see Hypothesis A3). For instance, it doesn't apply to nonlinear Schrödinger on any compact manifold since we have no regularizing effect in that case. On the contrary, it applies to the beam equation on the torus $\mathbb{T}^d$ (see Remark 3.4). Unfortunately it doesn't apply to the nonlinear wave equation on $\mathbb{T}^d$ (see Remark 3.5), since in that case the second Melnikov condition is not satisfied.

In Section 4 we study the Hamiltonian flows generated by Hamiltonian functions in $\mathcal{T}^{s,\beta}$. In Section 5 we detail the resolution of the homological equation. In both Sections 4 and 5 we use techniques and proofs that were developed in [**15**] and [**13**]. The novelty lies in the use of different norms (see (1.1)) and the use of two different classes of Hamiltonians: $\mathcal{T}^{s,\beta}$ and $\mathcal{T}^{s,\beta+}$ which, of course, complicate the technical arguments. For convenience of the reader we repeat most of the proofs. We point out that, for the resolution of the homological equation (Section 5), we use a variant of a Lemma due to Delort-Szeftel [**11**], whose proof is given in Appendix A.

Acknowledgement: The authors acknowledge the support from the project ANAE (ANR-13-BS01-0010-03) of the Agence Nationale de la Recherche.

2. Setting and abstract KAM theorem.

Notations. In this section we state a KAM result for a Hamiltonian $H = h + f$ of the following form

$$H = \langle \omega(\rho), r \rangle + \frac{1}{2}\langle \zeta, A(\rho)\zeta \rangle + f(r, \theta, \zeta; \rho)$$

where

- $\omega \in \mathbb{R}^n$ is the frequencies vector corresponding to the internal modes in action-angle variables $(r, \theta) \in \mathbb{R}^n_+ \times \mathbb{T}^n$.
- $\zeta = (\zeta_s)_{s\in\mathcal{L}}$ are the external modes: $\mathcal{L}$ is an infinite set of indices, $\zeta_s = (p_s, q_s) \in \mathbb{R}^2$ and $\mathbb{R}^2$ is endowed with the standard symplectic structure $dq \wedge dp$.
- A is a linear operator acting on the external modes, typically A is diagonal.
- f is a perturbative Hamiltonian depending on all the modes and is of order ε where ε is a small parameter.
- ρ is an external parameter in $\mathcal{D}$ a compact subset of $\mathbb{R}^p$ with $p \geq n$.

We now detail the structures behind these objects and the hypothesis needed for the KAM result.

Cluster structure on $\mathcal{L}$. Let $\mathcal{L}$ be a set of indices and $w : \mathcal{L} \to \mathbb{N} \setminus \{0\}$ be an "energy" function [2] on $\mathcal{L}$. We consider the clustering of $\mathcal{L}$ given by $\mathcal{L} = \cup_{a\in\mathcal{L}}[a]$ associated to equivalence relation

$$b \sim a \Longleftrightarrow w_a = w_b.$$

We denote $\hat{\mathcal{L}} = \mathcal{L}/\sim$. We assume that the cardinal of each energy level is finite and that there exist $C_b > 0$ and $d^* > 0$ two constants such that the cardinality of $[a]$ is controlled by $C_b w_a^d$:

$$d_a = d_{[a]} = \operatorname{card}\{b \in \mathcal{L} \mid w_b = w_a\} \leq C_b w_a^{d^*}. \tag{2.1}$$

Linear space. Let $s \geq 0$, we consider the complex weighted ℓ_2-space

$$Y_s = \{\zeta = (\zeta_a \in \mathbb{C}^2,\ a \in \mathcal{L}) \mid \|\zeta\|_s < \infty\}$$

where [3]

$$\|\zeta\|_s^2 = \sum_{a\in\mathcal{L}} |\zeta_a|^2 w_a^{2s}.$$

In the spaces Y_s acts the linear operator J,

$$J\ :\ \{\zeta_a\} \mapsto \{\sigma_2\zeta_a\}, \quad \text{with } \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It provides the spaces Y_s, $s \geq 0$, with the symplectic structure $J\mathrm{d}\zeta \wedge \mathrm{d}\zeta$. To any C^1-smooth function defined on a domain $\mathcal{O} \subset Y_s$, corresponds the Hamiltonian equation

$$\dot{\zeta} = J\nabla f(\zeta),$$

where ∇f is the gradient with respect to the scalar product in Y.

2. We could replace the assumption that w takes integer values by $\{w_a - w_b \mid a, b \in \mathcal{L}\}$ accumulates on a discrete set.

3. We provide $\mathbb{C}^2$ with the hermitian norm, $|\zeta_a| = |(p_a, q_a)| = \sqrt{|p_a|^2 + |q_a|^2}$.

Infinite matrices. We denote by $\mathcal{M}_{s,\beta}$ the set of infinite matrices $A : \mathcal{L} \times \mathcal{L} \to \mathcal{M}_{2\times 2}(\mathbb{R})$ with value in the space of real 2×2 matrices that are symmetric

$$A_a^b = {}^tA_b^a, \quad \forall a,\ b \in \mathcal{L}$$

and satisfy

$$|A|_{s,\beta} := \sup_{a,b\in\mathcal{L}} (w_a w_b)^\beta \left\| A_{[a]}^{[b]} \right\| \left(\frac{w(a,b) + |w_a^2 - w_b^2|}{w(a,b)} \right)^{s/2} < \infty$$

where $A_{[a]}^{[b]}$ denotes the restriction of A to the block $[a] \times [b]$, $w(a,b) = \min(w_a, w_b)$ and $\| \cdot \|$ denotes the operator norm induced by the Y_0-norm.

A class of regularizing Hamiltonian functions. Let us fix any $n \in \mathbb{N}$. On the space

$$\mathbb{C}^n \times \mathbb{C}^n \times Y_s$$

we define the norm

$$\|(z, r, \zeta)\|_s = \max(|z|, |r|, \|\zeta\|_s).$$

For $\sigma > 0$ we denote

$$\mathbb{T}_\sigma^n = \{z \in \mathbb{C}^n : |\Im z| < \sigma\}/2\pi\mathbb{Z}^n.$$

For $\sigma, \mu \in (0, 1]$ and $s \geq 0$ we set

$$\mathcal{O}^s(\sigma, \mu) = \mathbb{T}_\sigma^n \times \{r \in \mathbb{C}^n : |r| < \mu^2\} \times \{\zeta \in Y_s : \|\zeta\|_s < \mu\}$$

We will denote points in $\mathcal{O}^s(\sigma, \mu)$ as $x = (\theta, r, \zeta)$. A function defined on a domain $\mathcal{O}^s(\sigma, \mu)$, is called *real* if it gives real values to real arguments.
Let

$$\mathcal{D} = \{\rho\} \subset \mathbb{R}^p$$

be a compact set of positive Lebesgue measure. This is the set of parameters upon which will depend our objects. Differentiability of functions on $\mathcal{D}$ is understood in the sense of Whitney. So $f \in C^1(\mathcal{D})$ if it may be extended to a C^1-smooth function $\tilde{f}$ on $\mathbb{R}^p$, and $|f|_{C^1(\mathcal{D})}$ is the infimum of $|\tilde{f}|_{C^1(\mathbb{R}^p)}$, taken over all C^1-extensions $\tilde{f}$ of f.
If (z, r, ζ) are C^1 functions on $\mathcal{D}$, then we define

$$\|(z, r, \zeta)\|_{s,\mathcal{D}} = \max_{j=0,1}(|\partial_\rho^j z|, |\partial_\rho^j r|, \|\partial_\rho^j \zeta\|_s).$$

Let $f : \mathcal{O}^0(\sigma, \mu) \times \mathcal{D} \to \mathbb{C}$ be a C^1-function, real holomorphic in the first variable x, such that for all $\rho \in \mathcal{D}$

$$\mathcal{O}^s(\sigma, \mu) \ni x \mapsto \nabla_\zeta f(x, \rho) \in Y_{s+\beta}$$

and

$$\mathcal{O}^s(\sigma, \mu) \ni x \mapsto \nabla_\zeta^2 f(x, \rho) \in \mathcal{M}_{s,\beta}$$

are real holomorphic functions. We denote this set of functions by $\mathcal{T}^{s,\beta}(\sigma, \mu, \mathcal{D})$. We notice that for $\beta > 0$, both the gradient and the hessian of $f \in \mathcal{T}^{s,\beta}(\sigma, \mu, \mathcal{D})$ have a regularizing effect.
For a function $f \in \mathcal{T}^{s,\beta}(\sigma, \mu, \mathcal{D})$ we define the norm

$$[f]_{\sigma,\mu,\mathcal{D}}^{s,\beta}$$

through

$$\sup \max(|\partial_\rho^j f(x, \rho)|, \mu\|\partial_\rho^j \nabla_\zeta f(x, \rho)\|_{s+\beta}, \mu^2|\partial_\rho^j \nabla_\zeta^2 f(x, \rho)|_{s,\beta}),$$

where the supremum is taken over all

$$j = 0, 1,\ x \in \mathcal{O}^s(\sigma, \mu),\ \rho \in \mathcal{D}.$$

In the case $\beta = 0$ we denote $\mathcal{T}^s(\sigma, \mu, \mathcal{D}) = \mathcal{T}^{s,0}(\sigma, \mu, \mathcal{D})$ and

$$[f]_{\sigma,\mu,\mathcal{D}}^s = [f]_{\sigma,\mu,\mathcal{D}}^{s,0}.$$

Normal form: We introduce the orthogonal projection Π defined on the 2×2 complex matrices

$$\Pi : \mathcal{M}_{2\times 2}(\mathbb{C}) \to \mathbb{C}I + \mathbb{C}J$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Definition 2.1. — *A matrix $A : \mathcal{L} \times \mathcal{L} \to \mathcal{M}_{2\times 2}(\mathbb{C})$ is on normal form and we denote $A \in \mathcal{NF}$ if*

(i) A is real valued,
(ii) A is symmetric, i.e. $A_b^a = {}^t A_a^b$,
(iii) A satisfies $\Pi A = A$,
(iii) A is block diagonal, i.e. $A_b^a = 0$ for all $w_a \neq w_b$.

To a real symmetric matrix $A = (A_a^b) \in \mathcal{M}$ we associate in a unique way a real quadratic form on $Y_s \ni (\zeta_a)_{a\in\mathcal{L}} = (p_a, q_a)_{a\in\mathcal{L}}$

$$q(\zeta) = \frac{1}{2} \sum_{a,b\in\mathcal{L}} \langle \zeta_a,\ A_a^b \zeta_b \rangle.$$

In the complex variables, $z_a = (\xi_a, \eta_a),\ a \in \mathcal{L}$, where

$$\xi_a = \frac{1}{\sqrt{2}}(p_a + iq_a), \quad \eta_a = \frac{1}{\sqrt{2}}(p_a - iq_a),$$

we have

$$q(\zeta) = \frac{1}{2}\langle \xi, \nabla_\xi^2 q\ \xi \rangle + \frac{1}{2}\langle \eta, \nabla_\eta^2 q\ \eta \rangle + \langle \xi, \nabla_\xi \nabla_\eta q\ \eta \rangle.$$

The matrices $\nabla_\xi^2 q$ and $\nabla_\eta^2 q$ are symmetric and complex conjugate of each other while $\nabla_\xi \nabla_\eta q$ is Hermitian. If $A \in \mathcal{M}_{s,\beta}$ then

$$\sup_{a,b} \left\| (\nabla_\xi \nabla_\eta q)_{[a]}^{[b]} \right\| \leq \frac{|A|_{s,\beta}}{(w_a w_b)^\beta \,(1 + |w_a - w_b|)^s}. \tag{2.2}$$

We note that if A is on normal form, then the associated quadratic form $q(\zeta) = \frac{1}{2}\langle \zeta, A\zeta \rangle$ reads in complex variables

$$q(\zeta) = \langle \xi, Q\eta \rangle \tag{2.3}$$

where $Q : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ is

(i) Hermitian, i.e. $Q_b^a = \overline{Q_a^b}$,
(ii) Block-diagonal.

In other words, when A is on normal form, the associated quadatic form reads

$$q(\zeta) = \frac{1}{2}\langle p, A_1 p \rangle + \langle p, A_2 q \rangle + \frac{1}{2}\langle p, A_1 q \rangle$$

with $Q = A_1 + iA_2$ Hermitian.
By extension we will say that a Hamiltonian is on normal form if it reads

$$h = \langle \omega(\rho), r \rangle + \frac{1}{2}\langle \zeta, A(\rho)\zeta \rangle \tag{2.4}$$

with $\omega(\rho) \in \mathbb{R}^n$ a frequency vector and $A(\rho)$ on normal form for all ρ.

2.1. Hypothesis on the spectrum of A_0.— We assume that A_0 is a real diagonal matrix whose diagonal elements $\lambda_a > 0,\ a \in \mathcal{L}$ are C^1. Our hypothesis depend on two constants $1 > \delta_0 > 0$ and $c_0 > 0$ fixed once for all.

Hypothesis A1 – Asymptotics. We assume that there exist $\gamma \geq 1$ such that

$$c_0\, w_a^\gamma \leq \lambda_a \leq \frac{1}{c_0} w_a^\gamma \quad \text{for } \rho \in \mathcal{D} \text{ and } a \in \mathcal{L} \tag{2.5}$$

and

$$|\lambda_a - \lambda_b| \geq c_0 |w_a - w_b| \quad \text{for } a, b \in \mathcal{L}\,. \tag{2.6}$$

Hypothesis A2 – Non resonances. There exists a $\delta_0 > 0$ such that for all $\mathcal{C}^1$-functions

$$\omega : \mathcal{D} \to \mathbb{R}^n, \quad |\omega - \omega_0|_{\mathcal{C}^1(\mathcal{D})} < \delta_0,$$

the following holds for each $k \in \mathbb{Z}^n \setminus 0$: either we have the following properties :

$$\begin{cases} |\langle k, \omega(\rho)\rangle| \geq \delta_0 & \text{for all } \rho \in \mathcal{D}, \\ |\langle k, \omega(\rho)\rangle + \lambda_a| \geq \delta_0 w_a & \text{for all } \rho \in \mathcal{D} \text{ and } a \in \mathcal{L}, \\ |\langle k, \omega(\rho)\rangle + \lambda_a + \lambda_b| \geq \delta_0(w_a + w_b) & \text{for all } \rho \in \mathcal{D} \text{ and} a,\ b \in \mathcal{L}, \\ |\langle k, \omega(\rho)\rangle + \lambda_a - \lambda_b| \geq \delta_0(1 + |w_a - w_b|) & \text{for all } \rho \in \mathcal{D} \ \text{ and } a,\ b \in \mathcal{L}, \end{cases}$$

or there exists a unit vector $\mathfrak{z} \in \mathbb{R}^p$ such that

$$(\nabla_\rho \cdot \mathfrak{z})(\langle k, \omega\rangle) \geq \delta_0$$

for all $\rho \in \mathcal{D}$. The first term of the alternative will be used in order to control the small divisors for large k, and the second one is featured to control them for small k.

The last assumption above will be used to bound from below divisors $|\langle k, \omega(\rho)\rangle + \lambda_a(\rho) - \lambda_b(\rho)|$ with $w_a,\ w_b \sim 1$. To control the (infinitely many) divisors with $\max(w_a, w_b) \gg 1$ we need another assumption:

Hypothesis A3 – Second Melnikov condition in measure. There exist absolute constants $\alpha_1 > 0$, $\alpha_2 > 0$ and $C > 0$ such that for all $\mathcal{C}^1$-functions

$$\omega : \mathcal{D} \to \mathbb{R}^n, \quad |\omega - \omega_0|_{\mathcal{C}^1(\mathcal{D})} < \delta_0,$$

the following holds:
for each $\kappa > 0$ and $N \geq 1$ there exists a closed subset $\mathcal{D}' = \mathcal{D}'(\omega_0, \kappa, N) \subset \mathcal{D}$ satisfying

$$\text{meas}(\mathcal{D} \setminus \mathcal{D}') \leq C N^{\alpha_1} (\frac{\kappa}{\delta_0})^{\alpha_2} \quad (\alpha_1, \alpha_2 \geq 0) \tag{2.7}$$

such that for all $\rho \in \mathcal{D}'$, all $0 < |k| \leq N$ and all $a, b \in \mathcal{L}$ we have

$$|\langle k, \omega(\rho)\rangle + \lambda_a - \lambda_b| \geq \kappa(1 + |w_a - w_b|). \tag{2.8}$$

2.2. The abstract KAM Theorem.

— We are now in position to state our abstract KAM result.

Theorem 2.2. — *Assume that*

$$h_0 = \langle \omega_0(\rho), r\rangle + \frac{1}{2}\langle \zeta, A_0 \zeta\rangle \tag{2.9}$$

with the spectrum of A_0 satisfying Hypothesis A1, A2, A3 and let $f \in \mathcal{T}^{s,\beta}(\mathcal{D}, \sigma, \mu)$ with $\beta > 0$, $s > 0$. There exists $\varepsilon_0 > 0$ (depending on $n, d, s, \beta, \sigma, \mu$, on $\mathcal{A}$, c_0 and $\sup|\nabla_\rho \omega|$), $\alpha > 0$ (depending on n, d^*, s, β, α_1, α_2) and $\upsilon(\beta, d^*) > 0$ such that[4] *if*

$$[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}} = \varepsilon < \min\left(\varepsilon_0, \delta_0^{\upsilon(\beta,d^*)}\right)$$

there is a $\mathcal{D}' \subset \mathcal{D}$ with meas$(\mathcal{D} \setminus \mathcal{D}') \leq \varepsilon^\alpha$ such that for all $\rho \in \mathcal{D}'$ the following holds: There are a real analytic symplectic diffeomorphism

$$\Phi : \mathcal{O}^s(\sigma/2, \mu/2) \to \mathcal{O}^s(\sigma, \mu)$$

and a vector $\omega = \omega(\rho)$ such that

$$(h_0 + f) \circ \Phi = \langle \omega(\rho), r\rangle + \frac{1}{2}\langle \zeta, A(\rho)\zeta\rangle + \tilde{f}(r, \theta, \zeta; \rho)$$

where $\partial_\zeta \tilde{f} = \partial_r \tilde{f} = \partial^2_{\zeta\zeta} \tilde{f} = 0$ for $\zeta = r = 0$ and $A : \mathcal{L} \times \mathcal{L} \to \mathcal{M}_{2\times 2}(\mathbb{R})$ is on normal form, i.e. A is real symmetric and block diagonal: $A_a^b = 0$ for all $w_a \neq w_b$.
Moreover Φ satisfies

$$\|\Phi - \text{Id}\|_s \leq \varepsilon^{11/12}\ ,$$

4. An explicit choice of υ is given in (6.6) but is surely far from optimality.

for all $(r,\theta,\zeta)\in\mathcal{O}^s(\sigma/2,\mu/2)$, *and*

$$|A(\rho)-A_0|_\beta\le\varepsilon,$$
$$|\omega(\rho)-\omega_0(\rho)|_{C^1}\le\varepsilon$$

for all $\rho\in\mathcal{D}'$.

This normal form result has dynamical consequences. For $\rho\in\mathcal{D}'$, the torus $\{0\}\times\mathbb{T}^n\times\{0\}$ is invariant by the flow of $(h_0+f)\circ\Phi$ and the dynamics of the Hamiltonian vector field of h_0+f on the $\Phi(\{0\}\times\mathbb{T}^n\times\{0\})$ is the same as that of

$$\langle\omega(\rho),r\rangle+\frac{1}{2}\langle\zeta,A(\rho)\zeta\rangle.$$

The Hamiltonian vector field on the torus $\{\zeta=r=0\}$ is

$$\begin{cases}\dot\zeta=0\\ \dot\theta=\omega\\ \dot r=0,\end{cases}$$

and the flow on the torus is linear: $t\mapsto\theta(t)=\theta_0+t\omega$.
Moreover, the linearized equation on this torus reads

$$\begin{cases}\dot\zeta=JA\zeta+J\partial^2_{r\zeta}f(0,\theta_0+\omega t,0)\cdot r\\ \dot\theta=\partial^2_{r\zeta}f(0,\theta_0+\omega t,0)\cdot\zeta+\partial^2_{rr}f(0,\theta_0+\omega t,0)\cdot r\\ \dot r=0.\end{cases}$$

Since A is on normal form (and in particular real symmetric and block diagonal) the eigenvalues of the ζ-linear part are purely imaginary: $\pm i\tilde\lambda_a,\ a\in\mathcal{L}$. Therefore the invariant torus is linearly stable in the classical sense (all the eigenvalues of the linearized system are purely imaginary). Furthermore if the $\tilde\lambda_a$ are non-resonant with respect to the frequency vector ω (a property which can be guaranteed restricting the set $\mathcal{D}'$ arbitrarily little) then the linearized equation is reducible to constant coefficients. Then the ζ-component (and of course also the r-component) will have only quasi-periodic (in particular bounded) solutions.

3. Applications to Klein Gordon on $\mathbb{S}^d$

In this section we prove Theorem 1.2 as a corollary of Theorem 2.2. We use notations introduced in the introduction (see in particular (1.7)). Then the Klein Gordon Hamiltonian H reads (up to a constant)

$$H(r,\theta,\zeta)=\langle\omega_0(\rho),r\rangle+\sum_{a\in\mathcal{L}}\lambda_a\xi_a\eta_a+\varepsilon f(r,\theta,\zeta)$$

where

$$f(r,\theta,\zeta)=\int_{\mathbb{S}^d}G\left(x,\hat u(r,\theta,\zeta)(x)\right)dx.$$

Lemma 3.1. — *Hypothesis A1, A2 and A3 hold true with* $\mathcal{D}=[1,2]^n$ *and*

$$\delta_0=\Big(\frac{\delta}{2\sqrt{2+d+m}\max(w_a,\ a\in\mathcal{A})}\Big)^3. \tag{3.1}$$

Proof. — Hypothesis A1 is clearly satisfied with $c_0=1/2$ and $\gamma=1$. The control of the cardinality of the clusters (2.1) is given with $d^*=d-1$.
On the other hand choosing $\mathfrak{z}\equiv z_k=\frac{k}{|k|}$ we have

(3.2)
$$(\nabla_\rho\cdot\mathfrak{z})(\langle k,\omega\rangle)\ge\frac{\delta}{2\max((\omega_0)_a,\ a\in\mathcal{A})}|k|\ge\frac{\delta}{\sqrt{2+d+m}\max(w_a,\ a\in\mathcal{A})}|k|\quad\text{for all }k\neq0$$

while

$$(\nabla_\rho\cdot\mathfrak{z})\lambda_a=0\quad\text{for all }a\in\mathcal{L}. \tag{3.3}$$

Then for all $k \neq 0$ the second part of the alternative in Hypothesis A2 is satisfied choosing

$$\delta_0 \leq \delta_* := \frac{\delta}{2\max(w_a,\ a \in \mathcal{A})\sqrt{2+d+m}}.$$

It remains to verify A3. Without loss of generality we can assume $w_a \leq w_b$.
First denoting

$$F_\kappa(k,a,b) := \{\rho \in \mathcal{D} \mid |\langle \omega, k\rangle + \lambda_a - \lambda_b| \leq \kappa\},$$

we have using (3.2) that

$$\operatorname{meas} F_\kappa(k,a,b) \leq C(k,a,b)\frac{\kappa}{\delta_*}.$$

On the other hand, defining

$$G_\nu(k,e) := \{\rho \in \mathcal{D} \mid |\langle \omega, k\rangle + e| \leq 2\nu\},$$

we have, using again (3.2) that

$$\operatorname{meas} G_\nu(k,e) \leq C\frac{\nu}{\delta_*}.$$

Further $|\langle \omega, k\rangle + e| \leq 1$ can occur only if $|e| \leq C|k|$ and thus

$$G_\nu = \bigcup_{\substack{0<|k|\leq N\\ e\in\mathbb{Z}}} G_\nu(k,e)$$

has a Lebesgue measure less than $CN^{n+1}\frac{\nu}{\delta_*}$.
Now we remark that

$$|j + \frac{d-1}{2} - \sqrt{j(j+d-1)+m}| \leq \frac{C'_{m,d}}{j}$$

where $C'_{m,d}$ only depends on m and d, from which we deduce

$$|\lambda_a - \lambda_b - (w_a - w_b)| \leq \frac{2C'_{m,d}}{w_a}.$$

Therefore for $\rho \in \mathcal{D} \setminus G_\nu$ and $w_a \geq \frac{2C'_{m,d}}{\nu}$ we have for all $0 < |k| \leq N$

$$|\langle \omega, k\rangle + \lambda_a - \lambda_b| \geq \nu.$$

Finally $w_a \leq \frac{2C'_{m,d}}{\nu}$ and $|\langle \omega, k\rangle + \lambda_a - \lambda_b| \leq 1$ leads to $w_b \leq \frac{2C'_{m,d}}{\nu} + CN$ and thus, if we restrict ρ to

$$\mathcal{D}' = \mathcal{D} \setminus \Big[G_\nu \cup \Big(\bigcup_{\substack{0<|k|\leq N\\ w_a,w_b\leq \frac{2C'_{m,d}}{\nu}+CN}} F_\kappa(k,a,b)\Big)\Big]$$

we get

$$|\langle \omega, k\rangle + \lambda_a - \lambda_b| \geq \min(\kappa,\nu), \quad 0<|k|\leq N,\ a,b \in \mathcal{L}.$$

Further

$$\operatorname{meas} \mathcal{D}\setminus\mathcal{D}' \leq CN^{n+1}\frac{\nu}{\delta_*} + \left(\frac{2C'_{m,d}}{\nu} + CN\right)^2 N^n\frac{\kappa}{\delta_*}.$$

Then choosing $\nu = \kappa^{1/3}$ and $\delta_0 = \delta_*^3$, this measure is controlled by

$$CN^{n+2}\big(\frac{\kappa}{\delta_0}\big)^{1/3}$$

and we have

$$|\langle \omega, k\rangle + \lambda_a - \lambda_b| \geq \kappa, \quad \text{for } \rho \in \mathcal{D}',\ 0<|k|\leq N \text{ and } a,b \in \mathcal{L}.$$

Now we remark that for $|\lambda_a - \lambda_b| \geq 2|\langle \omega, k\rangle|$,

$$|\langle \omega, k\rangle + \lambda_a - \lambda_b| \geq \frac{1}{2}|\lambda_a - \lambda_b| \geq \frac{1}{4}(1+|w_a - w_b|) \geq \kappa(1+|w_a - w_b|)$$

if we assume $\kappa \leq \frac{1}{4}$.
On the other hand, when $|\lambda_a - \lambda_b| \leq 2|\langle\omega, k\rangle| \leq CN$,

$$|\langle\omega, k\rangle + \lambda_a - \lambda_b| \geq \tilde{\kappa}(1 + |w_a - w_b|)$$

where $\tilde{\kappa} = \frac{\kappa}{1+CN}$. Thus we get

$$|\langle\omega, k\rangle + \lambda_a - \lambda_b| \geq \tilde{\kappa}(1 + |w_a - w_b|), \quad \text{for } \rho \in \mathcal{D}',\ 0 < |k| \leq N \text{ and } a, b \in \mathcal{L}$$

with

$$\text{meas}\left(\mathcal{D} \setminus \mathcal{D}'\right) \leq CN^{n+3}\big(\frac{\tilde{\kappa}}{\delta_0}\big)^{1/3}.$$

□

Lemma 3.2. — *Assume that $(x, u) \mapsto g(x, u)$ is real analytic on $\mathbb{S}^d \times \mathbb{R}$ and $s > 1$ then there exist $\sigma > 0$, $\mu > 0$ such that the maping*

$$\mathcal{O}^s(\sigma, \mu) \times \mathcal{D} \ni (r, \theta, \zeta; \rho) \mapsto f(r, \theta, \zeta; \rho) := \int_{S^d} G(x, \hat{u}(r, \theta, \zeta)(x))dx,$$

where $\hat{u}$ is defined in (1.8), belongs to $\mathcal{T}^{s,1/2}(\sigma, \mu, \mathcal{D})$ for any s of the form $2N - \frac{1}{2}$ with $N \in \mathbb{N}$ and $N > d$.

Proof. — First we notice that f does not depend on the parameter ρ. Due to the analyticity of g and the fact that [5] $s > d/2$, there exist positive σ and μ such that $f : \mathcal{O}(\sigma, \mu) \times \mathcal{D} \to \mathbb{C}$ is a C^1-function, analytic in the first variables (r, θ, ζ), whose gradient in ζ analytically maps Y_s to Y_{-s}. Further we have

$$\frac{\partial f}{\partial \xi_a} = \frac{\partial f}{\partial \eta_a} = \frac{1}{2\lambda_a^{1/2}} \int_{\mathbb{S}^d} G(x, \hat{u}(x))\Psi_a(x)\, dx.$$

Since $x \mapsto G(x, \hat{u}(x)) \in H^s(\mathbb{S}^d)$, we deduce that $\nabla_\zeta f \in Y_{s+1/2}$.
It remains to verify that $\nabla^2_\zeta f(r, \theta, \zeta; \rho) \in \mathcal{M}_{s,1/2}$.
We have

$$\frac{\partial^2 f}{\partial \xi_a \xi_b} = \frac{\partial^2 f}{\partial \eta_a \eta_b} = \frac{\partial^2 f}{\partial \xi_a \eta_b} = \frac{1}{2\lambda_a^{1/2}\lambda_b^{1/2}} \int_{\mathbb{S}^d} G(x, \hat{u}(x))\Psi_a \Psi_b\, dx. \tag{3.4}$$

We note that for $s > d/2$ and $(r, \theta, \zeta) \in \mathcal{O}^s(\sigma, \mu)$, $x \mapsto \hat{u}(x)$ is bounded on $\mathbb{S}^d$.
It remains to prove that the infinite matrix M defined by

$$M_a^b = \frac{1}{\lambda_a^{1/2}\lambda_b^{1/2}} \int_{\mathbb{S}^d} G(x, \hat{u})\Psi_a \Psi_b\, dx$$

belongs to $\mathcal{M}_{s,1/2}$, i.e.

$$\sup_{a,b \in \mathcal{L}} w_a^{1/2} w_b^{1/2} \left\| M_{[a]}^{[b]} \right\| \Big(\frac{w(a,b) + |w_a^2 - w_b^2|}{w(a,b)}\Big)^{s/2+1/4} < \infty$$

where we recall that $w(a, b) = \min(w_a, w_b)$. The case $w_a = w_b$ is straightforward, since $\lambda_a \sim w_a$, $x \mapsto \hat{u}(x)$ is bounded on $\mathbb{S}^d$, and Φ_a, Φ_b are normalized in $L^2(\mathbb{S}^d)$.
If $w_a \neq w_b$, first we notice that

$$\left\| M_{[a]}^{[b]} \right\| = \sup_{\|u\|, \|v\| = 1} |\langle M_{[a]}^{[b]} u, v\rangle| = \frac{1}{\lambda_a^{1/2}\lambda_b^{1/2}} \sup_{\substack{\Phi_a \in E_{[a]},\ \|\Phi_a\| = 1 \\ \Phi_b \in E_{[b]},\ \|\Phi_b\| = 1}} \Big| \int_{\mathbb{S}^d} G(x, \hat{u})\Phi_a \Phi_b\, dx \Big|,$$

where $E_{[a]}$ (resp. $E_{[b]}$) is the eigenspace of $-\Delta$ associated to the cluster $[a]$ (resp. $[b]$). Then we follow arguments developed in [**2**, Proposition 2]. The basic idea lies in the following commutator lemma: Let A be a linear operator which maps $H^s(S^d)$ into itself and define the sequence of operators

$$A_N := [-\Delta, A_{N-1}], \quad A_0 := A$$

5. $s > d/2$ is needed to insure that Y_s is an algebra.

where Δ denotes the Laplace Beltrami operator on $\mathbb{S}^d$, then with [**2**, Lemma 7], we have for any $a, b \in \mathcal{L}$ with $w_a \neq w_b$ and any $N \geq 0$

$$|\langle A\Phi_a, \Phi_b\rangle| \leq \frac{1}{|w_a^2 - w_b^2|^N} |\langle A_N \Phi_a, \Phi_b\rangle|.$$

Let A be the operator given by the multiplication by the function

$$\Phi(x) = G(x, \hat{u}(r, \theta, \zeta)(x)).$$

We note that $\Phi \in H^{s+1/2}$ for $(r, \theta, \zeta) \in \mathcal{O}^s(\sigma, \mu)$. Then, by an induction argument,

$$A_N = \sum_{0 \leq |\alpha| \leq N} C_{\alpha,N} D^\alpha$$

where

$$C_{\alpha,N} = \sum_{0 \leq |\beta| \leq 2N - |\alpha|} V_{\alpha,\beta,N}(x) D^\beta \Phi$$

and $V_{\alpha,\beta,N}$ are C^∞ functions (cf. [**2**, Lemma 8]). Therefore one gets

$$\begin{aligned}
|\int_{S^d} \Phi_a \Phi_b \Phi dx| &\leq \frac{1}{|w_a^2 - w_b^2|^N} \|A_N \Phi_a\|_{L^2} \\
&\leq C \frac{1}{|w_a^2 - w_b^2|^N} \sum_{0 \leq |\alpha| \leq N} \sum_{0 \leq |\beta| \leq 2N - |\alpha|} ||D^\beta \Phi D^\alpha \Psi_a||_{L^2} \\
&\leq C \frac{1}{|w_a^2 - w_b^2|^N} \Big(\sum_{0 \leq |\alpha| \leq N/2} \sum_{0 \leq |\beta| \leq 2N - |\alpha|} \|\Phi_a\|_{|\alpha| + \nu_0} ||\Phi||_{|\beta|} \\
&+ \sum_{N/2 < |\alpha| \leq N} \sum_{0 \leq |\beta| \leq 2N - |\alpha|} \|\Phi_a\|_{|\alpha|} ||\Phi||_{|\beta| + \nu_0} \Big) \\
&\leq C \frac{1}{|w_a^2 - w_b^2|^N} \|\Phi_a\|_N ||\Phi||_{2N}
\end{aligned}$$

where we used

$$\forall \nu_0 > d/2 \qquad \|fg\|_{L^2} \leq C \|f\|_{\nu_0} \|g\|_{L^2}.$$

On the other hand since $-\Delta \Phi_a = w_a(w_a + d - 1)\Phi_a$

$$\|\Phi_a\|_N \leq C w_a^N. \tag{3.5}$$

Therefore choosing $N = \frac{1}{2}(s + \frac{1}{2})$

$$\begin{aligned}
|\int_{S^d} \Phi_a \Phi_b \Phi dx| &\leq C \Big(\frac{w_a}{|w_a^2 - w_b^2|} \Big)^{s/2 + 1/4} \\
&\leq 2^{s/2 + 1/4} C \Big(\frac{w_a}{w(a,b) + |w_a^2 - w_b^2|} \Big)^{s/2 + 1/4}.
\end{aligned}$$

Clearly the same estimate remains true when interchanging a and b. □

So Main Theorem applies (for any choice of vector $I \in [1,2]^{\mathcal{A}}$) and Theorem 1.2 is proved.

Remark 3.3. — Theorem 1.2 still holds true when we consider the Klein Gordon equation on a Zoll manifold. This technical extension follows from results and computations in [**11**] and [**3**]. We prefer to focus on the sphere in order to simplify the presentation.

Remark 3.4. — We can also consider the Beam equation on the torus $\mathbb{T}^d$ with convolution potential in a Sobolev-like phase space:

$$u_{tt} + \Delta^2 u + mu + V \star u + \varepsilon \partial_u G(x, u) = 0, \quad x \in \mathbb{T}^d. \tag{3.6}$$

Here m is the mass, G is a real analytic function on $\mathbb{T}^d \times \mathbb{R}$ vanishing at least of order 3 at the origin. The convolution potential $V : \mathbb{T}^d \to \mathbb{R}$ is supposed to be analytic with real positive Fourier coefficients $\hat{V}(a)$, $a \in \mathbb{Z}^d$. The same equation, but in an analytic phase space, were considered in [**13, 12**]. Actually following [**13**] and the proof of Lemma 3.2, in order to apply our abstract KAM theorem, it remains to control the $|\cdot|_{s,1/2}$-norm of the infinite matrix [(6)]

$$M_a^b = \frac{1}{\lambda_a^{1/2}\lambda_b^{1/2}}\int_{\mathbb{T}^d}\partial_u^2 G(x,u)\Psi_a\Psi_b\,dx$$

restricted to the block defined by $[a]=\{b\in\mathbb{Z}^d \mid |a|=|b|\}$. This is achieved in the same way as in Lemma 3.2.

Remark 3.5. — Notice that our theorem does not apply to the nonlinear wave equation:

$$u_{tt}+\Delta u+mu+V\star u+\varepsilon\partial_u G(x,u)=0,\quad x\in\mathbb{T}^d \tag{3.7}$$

since in that case the second Melnikov condition is not satisfied.

4. Poisson brackets and Hamiltonian flows.

It turns out that the space $\mathcal{T}^{s,\beta}(\sigma,\mu,\mathcal{D})$ is not stable by Poisson brackets. Therefore, in this section, we first define a new space $\mathcal{T}^{s,\beta+}(\sigma,\mu,\mathcal{D})\subset\mathcal{T}^{s,\beta}(\sigma,\mu,\mathcal{D})$ and then we prove a structural stability which is essentially contained in the claim

$$\{\mathcal{T}^{s,\beta+}(\sigma,\mu,\mathcal{D})\ ,\ \mathcal{T}^{s,\beta}(\sigma,\mu,\mathcal{D})\}\in\mathcal{T}^{s,\beta}(\sigma,\mu,\mathcal{D}).$$

We will also study the hamiltonian flows generated by hamiltonian functions in $\mathcal{T}^{s,\beta+}(\sigma,\mu,\mathcal{D})$. In this section, all constants C will depend only on s, β and n.

4.1. New Hamiltonian space. — We introduce $\mathcal{T}^{s,\beta+}(\sigma,\mu,\mathcal{D})$ defined by

$$\mathcal{T}^{s,\beta+}(\sigma,\mu,\mathcal{D})=\{f\in\mathcal{T}^{s,\beta}(\sigma,\mu,\mathcal{D})\mid\ \partial_\rho^j\nabla_\zeta^2 f\in\mathcal{M}_{s,\beta}^+,\ j=0,1\}$$

where

$$\mathcal{M}_{s,\beta}^+=\{M\in\mathcal{M}_{s,\beta}\mid |M|_{s,\beta+}<\infty\}$$

and

$$|M|_{s,\beta+}=\sup_{a,b\in\mathcal{L}}(1+|w_a-w_b|)\Big(\frac{w(a,b)+|w_a^2-w_b^2|}{w(a,b)}\Big)^{\frac{s}{2}}(w_aw_b)^\beta\left\|M_{[a]}^{[b]}\right\|.$$

We endow $\mathcal{T}^{s,\beta+}(\sigma,\mu,\mathcal{D})$ with the norm

$$[f]_{\sigma,\mu,\mathcal{D}}^{s,\beta+}=[f]_{\sigma,\mu,\mathcal{D}}^{s,\beta}+\sup_{j=0,1}\Big(\mu^2|\partial_\rho^j\nabla_\zeta^2 f|_{s,\beta+}\Big).$$

Lemma 4.1. — *Let $0<\beta\le 1$ and $s>d/2$ there exists a constant $C\equiv C(\beta,s)>0$ such that*

(i) Let $A\in\mathcal{M}_{s,\beta}$ and $B\in\mathcal{M}_{s,\beta}^+$ then AB and BA belong to $\mathcal{M}_{s,\beta}$ and

$$|AB|_{s,\beta},\ |BA|_{s,\beta}\le C|A|_{s,\beta}|B|_{s,\beta+}.$$

(ii) Let $A,B\in\mathcal{M}_{s,\beta}^+$ then AB and BA belong to $\mathcal{M}_{s,\beta}^+$ and

$$|AB|_{s,\beta+},\ |BA|_{s,\beta+}\le C|A|_{s,\beta+}|B|_{s,\beta+}.$$

(iii) Let $A\in\mathcal{M}_{s,\beta}^+$ then $A\in\mathcal{L}(Y_s,Y_{s+\beta})$ and

$$\|A\zeta\|_{s+\beta}\le C|A|_{s,\beta+}\|\zeta\|_s.$$

(iv) Let $X\in Y_s$ and $Y\in Y_s$ and denote $A=X\otimes Y$ then A and ${}^tA$ belong to $\mathcal{L}(Y_s)$ and

$$\|A\|_{\mathcal{L}(Y_s)},\|{}^tA\|_{\mathcal{L}(Y_s)}\le C\|X\|_s\|Y\|_s.$$

(v) Let $X\in Y_{s+\beta}$ and $Y\in Y_{s+\beta}$ then $A=X\otimes Y\in\mathcal{M}_{s,\beta}$ and

$$\|A\|_{s,\beta}\le C\|X\|_{s+\beta}\|Y\|_{s+\beta}.$$

6. Here $\lambda_a=\sqrt{|a|^4+m}$ and $\Psi_a(x)=e^{ia\cdot x}$, $a\in\mathbb{Z}^d$.

Proof. — (i) Let $a,b\in\mathcal{L}$

$$\begin{aligned}\left\|(AB)_{[a]}^{[b]}\right\| &\leq \sum_{c\in\hat{\mathcal{L}}}\left\|A_{[a]}^{[c]}\right\|\left\|B_{[c]}^{[b]}\right\|\\ &\leq \frac{|A|_{\beta+}|B|_{\beta}}{(w_aw_b)^{\beta}}\Big(\frac{w(a,b)}{w(a,b)+|w_a^2-w_b^2|}\Big)^{\frac{s}{2}}\sum_{c\in\hat{\mathcal{L}}}\frac{1}{w_c^{2\beta}(1+|w_a-w_c|)}\\ &\leq C\frac{|A|_{\beta+}|B|_{\beta}}{(w_aw_b)^{\beta}}\Big(\frac{w(a,b)}{w(a,b)+|w_a^2-w_b^2|}\Big)^{\frac{s}{2}}\end{aligned}$$

where we used that by Lemma A.1

$$\frac{w(a,b)}{w(a,b)+|w_a^2-w_b^2|}\geq\frac{w(a,c)}{w(a,c)+|w_a^2-w_c^2|}\frac{w(c,b)}{w(c,b)+|w_c^2-w_b^2|}$$

and that by Lemma A.2, $\sum_{c\in\hat{\mathcal{L}}}\frac{1}{w_c^{2\beta}(1+|w_a-w_c|)}\leq C$ where C only depends on β.

(ii) Similarly let $a,b\in\mathcal{L}$ and assume without loss of generality that $w_a\leq w_b$

$$\begin{aligned}\left\|(AB)_{[a]}^{[b]}\right\| &\leq \sum_{c\in\hat{\mathcal{L}}}\left\|A_{[a]}^{[c]}\right\|\left\|B_{[c]}^{[b]}\right\|\\ &\leq \frac{|A|_{\beta+}|B|_{\beta+}}{(w_aw_b)^{\beta}}\Big(\frac{w(a,b)}{w(a,b)+|w_a^2-w_b^2|}\Big)^{\frac{s}{2}}\sum_{c\in\hat{\mathcal{L}}}\frac{1}{w_c^{2\beta}(1+|w_a-w_c|)(1+|w_b-w_c|)}\\ &\leq \frac{2|A|_{\beta+}|B|_{\beta+}}{(w_aw_b)^{\beta}(1+|w_a-w_b|)}\Big(\frac{w(a,b)}{w(a,b)+|w_a^2-w_b^2|}\Big)^{\frac{s}{2}}\\ &\Big(\sum_{\substack{c\in\hat{\mathcal{L}}\\ w_c\leq\frac{1}{2}(w_a+w_b)}}\frac{1}{w_c^{2\beta}(1+|w_a-w_c|)}+\sum_{\substack{c\in\hat{\mathcal{L}}\\ w_c\geq\frac{1}{2}(w_a+w_b)}}\frac{1}{w_c^{2\beta}(1+|w_b-w_c|)}\Big)\\ &\leq C\frac{|A|_{\beta+}|B|_{\beta+}}{(w_aw_b)^{\beta}(1+|w_a-w_b|)}\Big(\frac{w(a,b)}{w(a,b)+|w_a^2-w_b^2|}\Big)^{\frac{s}{2}}.\end{aligned}$$

(iii) Let $\zeta\in Y_s$ we have

$$\begin{aligned}\|A\zeta\|_{s+\beta}^2 &\leq \sum_{a\in\hat{\mathcal{L}}}w_a^{2s+2\beta}\big(\sum_{b\in\hat{\mathcal{L}}}\|A_{[a]}^{[b]}\|\|\zeta_{[b]}\|\big)^2\\ &\leq |A|_{s,\beta+}^2\sum_{a\in\hat{\mathcal{L}}}\Big(\sum_{b\in\hat{\mathcal{L}}}\frac{w_a^s\|w_b^s\zeta_{[b]}\|}{w_b^{s+\beta}(1+|w_a-w_b|)}\Big(\frac{w(a,b)}{w(a,b)+|w_a^2-w_b^2|}\Big)^{\frac{s}{2}}\Big)^2\\ &\leq 2^{2s+1}|A|_{s,\beta+}^2\sum_{a\in\hat{\mathcal{L}}}\Big(\sum_{\substack{b\in\hat{\mathcal{L}}\\ w_a\leq 2w_b)}}\frac{\|w_b^s\zeta_{[b]}\|}{w_b^{\beta}(1+|w_a-w_b|)}\\ &\quad+\sum_{\substack{b\in\hat{\mathcal{L}}\\ w_a\geq 2w_b}}\frac{\|w_b^s\zeta_{[b]}\|w(a,b)^{\frac{s}{2}}}{w_b^{s+\beta}(1+|w_a-w_b|)}\Big)^2\\ &\leq 2^{2s+1}|A|_{s,\beta+}^2\sum_{a\in\hat{\mathcal{L}}}\big(\sum_{b\in\hat{\mathcal{L}}}\frac{\|w_b^s\zeta_{[b]}\|}{w_b^{\beta}(1+|w_a-w_b|)}\big)^2\\ &\leq C|A|_{s,\beta+}^2\|\zeta\|_s^2\end{aligned}$$

where we used that the convolution between the ℓ^p sequence, $p<2$, $\|w_b^{s-\beta}\zeta_{[b]}\|$ and the ℓ^q sequence, $q=\frac{2p}{3p-2}>1$, $\frac{1}{(1+|w_b|)}$ is a ℓ^2 sequence in a whose norm is bounded by $C\|\zeta\|_s$.

(iv) Let $u\in Y_s$, we have

$$\|Au\|_s=|\langle Y,u\rangle|\|X\|_s\leq\|X\|_s\|Y\|_s\|u\|_s.$$

(v) Let $a, b \in \mathcal{L}$

$$\begin{aligned}\left\|A_{[a]}^{[b]}\right\| &= \|X_{[a]}\|\|Y_{[b]}\| \leq (w_a w_b)^{-s-\beta}\|X\|_{s+\beta}\|Y\|_{s+\beta} \\ &\leq (w_a w_b)^{-\beta}\frac{1}{(1+|w_a^2-w_b^2|)^{s/2}}\|X\|_{s+\beta}\|Y\|_{s+\beta} \\ &\leq (w_a w_b)^{-\beta}\Big(\frac{w(a,b)}{(w(a,b)+|w_a^2-w_b^2|)}\Big)^{s/2}\|X\|_{s+\beta}\|Y\|_{s+\beta}.\end{aligned}$$

□

4.2. Jets of functions.— For any function $h \in \mathcal{T}^s(\sigma,\mu,\mathcal{D})$ we define its jet $h^T = h^T(x,\rho)$ as the following Taylor polynomial of h at $r = 0$ and $\zeta = 0$:

$$\begin{aligned}h^T =& h_\theta + \langle h_r, r\rangle + \langle h_\zeta, \zeta\rangle + \frac{1}{2}\langle h_{\zeta\zeta}\zeta, \zeta\rangle \\ =& h(\theta,0,\rho) + \langle \nabla_r h(\theta,0,\rho), r\rangle + \langle \nabla_\zeta h(\theta,0,\rho), \zeta\rangle + \frac{1}{2}\langle \nabla^2_{\zeta\zeta} h(\theta,0,\rho)\zeta, \zeta\rangle\end{aligned} \tag{4.1}$$

Functions of the form h^T will be called *jet-functions.*
Directly from the definition of the norm $[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}$ we get that

$$\begin{aligned}&|h_\theta(\theta,\rho)| \leq [h]^s_{\sigma,\mu,\mathcal{D}}, \quad |h_r(\theta,\rho)| \leq \mu^{-2}[h]^s_{\sigma,\mu,\mathcal{D}}, \\ &\|h_\zeta(\theta,\rho)\|_{s+\beta} \leq \mu^{-1}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}, \quad |h_{\zeta\zeta}(\theta,\rho)|_{s,\beta} \leq \mu^{-2}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}},\end{aligned} \tag{4.2}$$

for any $\theta \in \mathbb{T}^n_\sigma$ and any $\rho \in \mathcal{D}$. Moreover, the first derivative with respect to ρ will satisfy the same estimates.
We also notice that by Cauchy estimates we have that for $x \in \mathcal{O}(\sigma,\mu')$

$$\|\nabla^2_\zeta h(x)\|_{\mathcal{L}(Y_s,Y_{s+\beta})} \leq \frac{\sup_{y\in\mathcal{O}(\sigma,\mu)}\|\nabla_\zeta h(y)\|_s}{\mu-\mu'}. \tag{4.3}$$

Thus $h_{\zeta\zeta}$ is a linear continuous operator from Y_s to $Y_{s+\beta}$ and

$$\|h_{\zeta\zeta}(\theta,\rho)\|_{\mathcal{L}(Y_s,Y_{s+\beta})} \leq \mu^{-2}[h]^s_{\sigma,\mu,\mathcal{D}} \tag{4.4}$$

for any $\theta \in \mathbb{T}^n_\sigma$ and any $\rho \in \mathcal{D}$.

Proposition 4.2. — *For any $h \in \mathcal{T}^{s,\beta}(\sigma,\mu,\mathcal{D})$ we have $h^T \in \mathcal{T}^{s,\beta}(\sigma,\mu,\mathcal{D})$,*

$$[h^T]^{s,\beta}_{\sigma,\mu,\mathcal{D}} \leq C[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}\,,$$

and, for any $0 < \mu' < \mu$,

$$[h-h^T]^{s,\beta}_{\sigma,\mu',\mathcal{D}} \leq C\left(\frac{\mu'}{\mu}\right)^3[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}\,,$$

where C is an absolute constant.

Proof. — We start with the second statement. Consider first the hessian $\nabla^2_{\zeta\zeta}(h-h^T)(x)$ for $x = (\theta, r, \zeta) \in \mathcal{O}^s(\sigma,\mu')$. Let us denote $m = \mu'/\mu$. Then for $z \in \overline{D}_1 = \{z \in \mathbb{C} : |z| \leq 1\}$ we have $(\theta, (z/m)^2 r, (z/m)\zeta) \in \mathcal{O}^s(\sigma,\mu)$. Consider the function

$$\begin{aligned}&f : D_1 \times \mathcal{O}^s(\sigma,\mu') \to \mathcal{M}_\beta\,, \\ &(z,x) \mapsto \nabla^2_{\zeta\zeta}h(\theta,(z/m)^2 r,(z/m)\zeta) = h_0(x) + h_1(x)z + \dots\,.\end{aligned}$$

It is holomorphic and its norm is bounded by $\mu^{-2}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}$. So, by the Cauchy estimate, $|h_j(x)|_{s,\beta} \leq \mu^{-2}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}$ for $j = 1, 2, \dots$ and $x \in \mathcal{O}^s(\sigma,\mu')$. Since $\nabla^2_{\zeta\zeta}(h-h^T)(x) = h_1(x)m + h_2(x)m^2 + \cdots$, then $\nabla^2_{\zeta\zeta}(h-h^T)$ is holomorphic in $x \in \mathcal{O}^s(\sigma,\mu)$, and

$$|\nabla^2_{\zeta\zeta}(h-h^T)(x)|_{s,\beta} \leq \mu^{-2}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}(m+m^2+\dots) \leq \mu^{-2}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}\frac{m}{1-m}.$$

So $\nabla^2_{\zeta\zeta}(h-h^T)$ satisfies the required estimate with $C=2$, if $\mu'\le\mu/2$.

Same argument applies to bound the norms of $\partial_\rho\nabla^2_{\zeta\zeta}(h-h^T)$, $h-h^T$ and $\nabla_\zeta(h-h^T)$ if $\mu'\le\mu/2$, and to prove the analyticity of these mappings.

Now we turn to the first statement and write h^T as $h-(h-h^T)$. This implies that h^T, $\nabla_\zeta h^T$ and $\nabla^2_{\zeta\zeta}h^T$ are analytic on $\mathcal{O}^s(\sigma,\frac12\mu)$ and that

$$[h^T]^{s,\beta}_{\sigma,\frac12\mu,\mathcal{D}}\le C_1[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}\,.$$

Since h^T is a quadratic polynomial, then the mappings h^T, $\nabla_\zeta h^T$ and $\nabla^2_{\zeta\zeta}h^T$ are as well analytic on $\mathcal{O}^s(\sigma,\ \mu)$, and the norm $[h^T]^{s,\beta}_{\sigma,\mu,\mathcal{D}}$ satisfies the same estimate, modulo another constant factor, for any $0<\mu'\le\mu$.

Finally, the estimate for $[h-h^T]^{s,\beta}_{\sigma,\mu',\mathcal{D}}$ when $\mu/2\le\mu'\le\mu$, with a suitable constant C, follows from the estimate for $[h^T]^{s,\beta}_{\sigma,\mu,\mathcal{D}}$ since $[h-h^T]^{s,\beta}_{\sigma,\mu',\mathcal{D}}\le[h^T]^{s,\beta}_{\sigma,\mu,\mathcal{D}}+[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}$. □

4.3. Poisson brackets and flows. — The Poisson brackets of functions is defined by

$$\{f,g\}=\nabla_r f\cdot\nabla_\theta g-\nabla_\theta f\cdot\nabla_r g+\langle J\nabla_\zeta f,\nabla_\zeta g\rangle\,. \tag{4.5}$$

Lemma 4.3. — *Let $s\ge1$. Let $f\in\mathcal{T}^{s,\beta+}(\sigma,\mu,\mathcal{D})$ and $g\in\mathcal{T}^{s,\beta}(\sigma,\mu,\mathcal{D})$ be two jet functions then for any $0<\sigma'<\sigma$ we have $\{f,g\}\in\mathcal{T}^{s,\beta}(\sigma',\mu,\mathcal{D})$ and*

$$[\{f,g\}]^{s,\beta}_{\sigma',\mu,\mathcal{D}}\le C(\sigma-\sigma')^{-1}\mu^{-2}[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}}[g]^{s,\beta}_{\sigma,\mu,\mathcal{D}}.$$

Proof. — Let denote by h_1, h_2, h_3 the three terms on the right hand side of (4.5). Since $\nabla_r f(\theta,r,\zeta,\rho)=f_r(\theta,\rho)$ and $\nabla_r g(\theta,r,\zeta,\rho)=g_r(\theta,\rho)$ are independent of r and ζ, the control of h_1 and h_2 is straightforward by Cauchy estimates and (4.2).
We focus on the third term in formula: $h_3=\langle J\nabla_\zeta f,\nabla_\zeta g\rangle$. As, from (4.1), we have $\nabla_\zeta f=f_\zeta+f_{\zeta\zeta}\zeta$ and similarly for $\nabla_\zeta g$, we obtain

$$h_3=\langle Jf_\zeta,g_\zeta\rangle-\langle\zeta,f_{\zeta\zeta}Jg_\zeta\rangle+\langle g_{\zeta\zeta}Jf_\zeta,\zeta\rangle+\langle g_{\zeta\zeta}Jf_{\zeta\zeta}\zeta,\zeta\rangle.$$

Using (4.2), (4.4) and $\|\zeta\|_s\le\mu$, we get

$$|h_3(x,\cdot)|\le C\mu^{-2}[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}}[g]^{s,\beta}_{\sigma,\mu,\mathcal{D}}\,,$$

for any $x\in\mathcal{O}(\sigma,\mu)$ and $\rho\in\mathcal{D}$.
Since

$$\nabla_\zeta h_3=-f_{\zeta\zeta}Jg_\zeta+g_{\zeta\zeta}Jf_\zeta+g_{\zeta\zeta}Jf_{\zeta\zeta}\zeta-f_{\zeta\zeta}Jg_{\zeta\zeta}\zeta,$$

then, using (4.4) and Lemma 4.1, we get that for $x\in\mathcal{O}^s(\sigma,\mu)$ and $\rho\in\mathcal{D}$

$$\|\nabla_\zeta h_3(x,\cdot)\|_{s+\beta}\le C\mu^{-3}[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}}[g]^{s,\beta}_{\sigma,\mu,\mathcal{D}}\,.$$

Finally, as $\nabla^2h_3=g_{\zeta\zeta}Jf_{\zeta\zeta}-f_{\zeta\zeta}Jg_{\zeta\zeta}$, then, using again Lemma 4.1 we get that for $x\in\mathcal{O}^s(\sigma,\mu)$ and $\rho\in\mathcal{D}$

$$|\nabla^2h_3(x,\cdot)|_{s,\beta}\le C\mu^{-4}[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}}[g]^{s,\beta}_{\sigma,\mu,\mathcal{D}}\,.$$

□

4.4. Hamiltonian flows. — To any C^1-function f on a domain $\mathcal{O}^s(\sigma,\mu)\times\mathcal{D}$ we associate the Hamilton equations

$$\begin{cases}\dot r= & \nabla_\theta f(r,\theta,\zeta;\rho),\\ \dot\theta= & -\nabla_r f(r,\theta,\zeta;\rho),\\ \dot\zeta= & J\nabla_\zeta f(r,\theta,\zeta;\rho).\end{cases} \tag{4.6}$$

and denote by $\Phi^t_f\equiv\Phi^t$, $t\in\mathbb{R}$, the corresponding flow map (if it exists). Now let $f\equiv f^T$ be a jet-function

$$f=f_\theta(\theta;\rho)+f_r(\theta;\rho)\cdot r+\langle f_\zeta(\theta;\rho),\zeta\rangle+\frac12\langle f_{\zeta\zeta}(\theta;\rho)\zeta,\zeta\rangle. \tag{4.7}$$

Then Hamilton equations (4.6) take the form [7]

$$\begin{cases} \dot r = & -\nabla_\theta f(r,\theta,\zeta), \\ \dot\theta = & f_r(\theta), \\ \dot\zeta = & J\left(f_\zeta(\theta) + f_{\zeta\zeta}(\theta)\zeta\right). \end{cases} \tag{4.8}$$

Denote by $V_f = (V_f^r, V_f^\theta, V_f^\zeta)$ the corresponding vector field. It is analytic on any domain $\mathcal{O}^s(\sigma-2\eta,\mu-2\nu) =: \mathcal{O}_{2\eta,2\nu}$, where $0<2\eta<\sigma$, $0<2\nu<\mu$. The flow maps Φ_f^t of V_f on $\mathcal{O}_{2\eta,2\nu}$ are analytic as long as they exist. We will study them as long as they map $\mathcal{O}_{2\eta,2\nu}$ to $\mathcal{O}_{\eta,\nu}$.
Assume that

$$[f]^s_{\sigma,\mu,\mathcal{D}} \le \frac{1}{2}\nu^2\eta. \tag{4.9}$$

Then for $x=(r,\theta,\zeta)\in\mathcal{O}_{2\eta,2\nu}$ by the Cauchy estimate [8] and (4.4) we have

$$\begin{aligned} |V_f^r|_{\mathbb{C}^n} &\le (2\eta)^{-1}[f]^s_{\sigma,\mu,\mathcal{D}} \le \nu^2, \\ |V_f^\theta|_{\mathbb{C}^n} &\le (4\nu^2)^{-1}[f]^s_{\sigma,\mu,\mathcal{D}} \le \eta, \\ \|V_f^\zeta\|_s &\le \left(\mu^{-1}+\mu^{-2}\mu\right)[f]^s_{\sigma,\mu,\mathcal{D}} \le \nu. \end{aligned}$$

Noting that the distance from $\mathcal{O}_{2\eta,2\nu}$ to $\partial\mathcal{O}_{\eta,\nu}$ in the r-direction is $2\nu\mu-3\nu^2>\nu^2$, in the θ-direction is η and in the ζ-direction is ν, we see that the flow maps

$$\Phi_f^t : \mathcal{O}^s(\sigma-2\eta,\mu-2\nu)\to\mathcal{O}^s(\sigma-\eta,\mu-\nu), \qquad 0\le t\le 1, \tag{4.10}$$

are well defined and analytic.

For $x\in\mathcal{O}_{2s,2\nu}$ denote $\Phi_f^t(x)=(r(t),\theta(t),\zeta(t))$. Since V_f^θ is independent from r and ζ, then $\theta(t)=K(\theta;t)$, where K is analytic in both arguments. As $V_f^\zeta = Jf_\zeta + Jf_{\zeta\zeta}\zeta$, where the non autonomous linear operator $Jf_{\zeta\zeta}(\theta(t))$ is bounded in the space Y_s and both the operator and the curve $Jf_\zeta(\theta(t))$ analytically depend on θ (through $\theta(t)=K(\theta;t)$), then $\zeta(t)=T(\theta,t)+U(\theta;t)\zeta$, where $U(\theta;t)$ is a bounded linear operator, both U and T analytic in θ. Similar since V_f^ζ is a quadratic polynomial in ζ and an affine function of r, then $r(t)=L(\theta,\zeta;t)+S(\theta;t)r$, where S is an $n\times n$ matrix and L is a quadratic polynomial in ζ, both analytic in θ.

The vector field V_f is real for real arguments, and so behaves its flow map. Since the vector field is hamiltonian, then the flow maps are symplectic (e.g., see [**23**]). We have proven

Lemma 4.4. — *Let $0<2\eta<\sigma$, $0<2\nu<\mu$ and $f=f^T\in\mathcal{T}^s(\sigma,\mu,\mathcal{D})$ satisfy* (4.9). *Then for $0\le t\le 1$ the flow maps Φ_f^t of equation* (4.8) *define analytic mappings* (4.10) *and define symplectomorphisms from $\mathcal{O}^s(\sigma-2\eta,\mu-2\nu)$ to $\mathcal{O}^s(\sigma-\eta,\mu-\nu)$. They have the form*

$$\Phi_f^t : \begin{pmatrix} r \\ \theta \\ \zeta \end{pmatrix} \to \begin{pmatrix} L(\theta,\zeta;t)+S(\theta;t)r \\ K(\theta;t) \\ T(\theta;t)+U(\theta;t)\zeta \end{pmatrix}, \tag{4.11}$$

where $L(\theta,\zeta;t)$ is quadratic in ζ, while $U(\theta;t)$ and $S(\theta;t)$ are bounded linear operators in corresponding spaces.

Our next result specifies the flow maps Φ_f^t and their representation (4.11) when $f\in\mathcal{T}^{s,\beta+}(\sigma,\mu,\mathcal{D})$:

Lemma 4.5. — *Let $0<2\eta<\sigma\le 1$, $0<2\nu<\mu\le 1$ and $f=f^T\in\mathcal{T}^{s,\beta+}(\sigma,\mu,\mathcal{D})$ satisfy*

$$[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}} \le \frac{1}{2}\nu^2\eta \tag{4.12}$$

7. Here and below we often suppress the argument ρ.
8. Notice that the distance from $\mathcal{O}^s(\sigma-2\eta,\mu-2\nu)$ to $\partial\mathcal{O}^s(\sigma,\mu)$ in the r-direction is $4\nu\mu-4\nu^2>4\nu^2$.

Then:
1) Mapping L is analytic in $(\theta,\zeta)\in\mathbb{T}^{\sigma-2\eta}\times\mathcal{O}_\mu(Y_s)$. Mappings K,T and operators S and U analytically depend on $\theta\in\mathbb{T}^{\sigma-2\eta}$; their norms and operator-norms satisfy

$$\begin{aligned}\|S(\theta;t)\|_{\mathcal{L}(\mathbb{C}^n,\mathbb{C}^n)},\ \|{}^tU(\theta;t)-I\|_{\mathcal{L}(Y_s,Y_{s+\beta})},\\ \|U(\theta;t)-I\|_{\mathcal{L}(Y_s,Y_{s+\beta})},\ |U(\theta;t)-I|_{s,\beta+}\le 2,\end{aligned}\tag{4.13}$$

while for any component L^j of L and any $(\theta,r,\zeta)\in\mathcal{O}^s(\sigma-2\eta,\mu-2\nu)$ we have

$$\begin{aligned}\|\nabla_\zeta L^j(\theta,\zeta;t)\|_{s+\beta}&\le C\eta^{-1}\mu^{-1}[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}},\\ |\nabla^2_\zeta L^j(\theta,\zeta;t)|_{s,\beta+}&\le C\eta^{-1}\mu^{-2}[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}}.\end{aligned}\tag{4.14}$$

2) The flow maps Φ^t_f analytically extend to mappings
$\mathbb{C}^n\times\mathbb{T}^n_{\sigma-2\eta}\times Y_s\ni x^0=(r^0,\theta^0,\zeta^0)\mapsto x(t)\in\mathbb{C}^n\times\mathbb{T}^n_\sigma\times Y_s$,
$x(t)=(r(t),\theta(t),\zeta(t))$, *which satisfy*

$$\begin{aligned}&|r(t)-r^0|\le 4\eta^{-1}\big(1+\mu^{-1}\|\zeta^0\|_s+\mu^{-2}|r^0|+\mu^{-2}||\zeta^0||^2_s\big)[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}},\\ &|\theta(t)-\theta^0|\le\mu^{-2}[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}},\\ &\|\zeta(t)-\zeta^0\|_{s+\beta}\le\big(\mu^{-2}\|\zeta^0\|_s+\mu^{-1}\big)\,[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}},\end{aligned}\tag{4.15}$$

Moreover, the ρ-derivative of the mapping $x^0\mapsto x(t)$ satisfies the same estimates as the increments $x(t)-x^0$.

Proof. — Consider the equation for $\zeta(t)$ in (4.8):

$$\dot\zeta(t)=a(t)+B(t)\zeta(t),\quad \zeta(0)=\zeta^0\in\mathcal{O}_{\mu-2\nu}(Y_s),\tag{4.16}$$

where $a(t)=Jf_\zeta(\theta(t))$ is an analytic curve $[0,1]\to Y_\gamma$ and $B(t)=Jf_{\zeta\zeta}(\theta(t))$ is an analytic curve $[0,1]\to\mathcal{M}$. Both analytically depend on θ^0. By the hypotheses and using (4.3)

$$||a(t)||_s\le\mu^{-1}[f]^s_{\sigma,\mu,\mathcal{D}},\quad \|B\|_{\mathcal{L}(Y_s,Y_s)}\le\mu^{-2}[f]^s_{\sigma,\mu,\mathcal{D}}\le\frac12\nu\le\frac12.\tag{4.17}$$

On the other hand by Lemma 4.1 (iii), $B\in\mathcal{L}(Y_s,Y_{s+\beta})$ and

$$\|B\|_{\mathcal{L}(Y_s,Y_{s+\beta})}\le\mu^{-2}[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}}.\tag{4.18}$$

By re-writing (4.16) in the integral form $\zeta(t)=\zeta^0+\int_0^t(a(t')+B(t')\zeta(t'))\mathrm{d}t'$ and iterating this relation, we get that

$$\zeta(t)=a^\infty(t)+(I+B^\infty(t))\zeta^0,\tag{4.19}$$

where

$$a^\infty(t)=\int_0^t a(t_1)\mathrm{d}t_1+\sum_{k\ge2}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}}\prod_{j=1}^{k-1}B(t_j)a(t_k)\mathrm{d}t_k\cdots\mathrm{d}t_2\,\mathrm{d}t_1,$$

and

$$B^\infty(t)=\sum_{k\ge1}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}}\prod_{j=1}^{k}B(t_j)\mathrm{d}t_k\cdots\mathrm{d}t_2\,\mathrm{d}t_1.$$

Due to (4.12), (4.17) and (4.18), for each k and for $0\le t_k\le\ldots t_1\le1$ we have that

$$\|B(t_1)\ldots B(t_k)\|_{\mathcal{L}(Y_s,Y_{s+\beta})}\le(\frac12)^{k-1}\mu^{-2}[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}}.$$

By this relation and (4.17) we get that a^∞ and B^∞ are well defined for $t\in[0,1]$ and satisfy

$$\begin{aligned}\|B^\infty(t)\|_{\mathcal{L}(Y_s,Y_{s+\beta})}&\le\mu^{-2}[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}},\\ \|a^\infty(t)\|_{s+\beta}&\le\mu^{-3}([f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}})^2\le\mu^{-1}[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}}.\end{aligned}\tag{4.20}$$

Again, the curves a^∞ and B^∞ analytically depend on θ^0. Inserting (4.20) in (4.19) we get that $\zeta=\zeta(t)$ satisfies the third estimate of (4.15).
On the other hand for all $t\in[0,1]$, $B\in\mathcal{M}^+_{s,\beta}$ and

$$|B(t)|_{s,\beta+}\le \mu^{-2}[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}}.$$

Therefore using Lemma 4.1 we get

$$|B^\infty(t)|_{s,\beta+}\le \mu^{-2}[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}}. \tag{4.21}$$

Since in (4.11) $U(\theta;t)=I+B^\infty(t)$, then the estimates on U in (4.13) follow from (4.20) and (4.21).

Now consider equation for $r(t)$:

$$\dot r(t)=-\alpha(t)-\Lambda(t)r(t),\quad r(0)=r^0\in\mathcal{O}_{(\mu-2\nu)^2}(\mathbb{C}^n)$$

where $\Lambda(t)=\nabla_\theta f_r(\theta(t))$ and

$$\alpha(t)=\nabla_\theta f_\theta(\theta(t))+\langle\nabla_\theta f_\zeta(\theta(t)),\zeta(t)\rangle+\frac12\langle\nabla_\theta f_{\zeta\zeta}(\theta(t))\zeta(t),\zeta(t)\rangle. \tag{4.22}$$

The curve of matrices $\Lambda(t)$ and the curve of vectors $\alpha(t)$ analytically depend on $\theta^0\in\mathbb{T}^n_{\sigma-2\eta}$. Besides, $\alpha(t)$ analytically depends on $\zeta^0\in Y_s$, while Λ is ζ^0-independent.

By the Cauchy estimate and (4.12), for any $\theta(t)\in\mathbb{T}^n_{\sigma-\eta}$ we have

$$\begin{aligned}&|\Lambda(t)|_{\mathcal{L}(\mathbb{C}^n,\mathbb{C}^n)}\le\eta^{-1}\mu^{-2}[f]^s_{\sigma,\mu,\mathcal{D}}\le\frac12,\\ &|\alpha(t)|\le 2\eta^{-1}[f]^s_{\sigma,\mu,\mathcal{D}}(1+\mu^{-1}\|\zeta^0\|_s+\mu^{-2}\|\zeta^0\|_s^2)\end{aligned} \tag{4.23}$$

where for the second estimate we used that $\|\zeta(t)-\zeta^0\|_s\le 1+\|\zeta^0\|_s$.
Since $\nabla_{\zeta(t)}\alpha(t)=\nabla_\theta f_\zeta(\theta(t))+\nabla_\theta f_{\zeta\zeta}(\theta(t))\zeta(t)$ and $\nabla_{\zeta_0}={}^tU(\theta;t)\nabla_\zeta$, then using (4.13) and Lemma 4.1 we obtain

$$\|\nabla_{\zeta^0}\alpha(t)\|_{s+\beta}\le 4\eta^{-1}\mu^{-1}[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}}(1+\mu^{-1}\|\zeta^0\|_s). \tag{4.24}$$

Since $\nabla^2_{\zeta^0}\alpha(t)={}^tU\nabla^2_{\zeta(t)}\alpha(t)U={}^tU\nabla_\theta f_{\zeta\zeta}(\theta(t))U$, then due to (4.13) and Lemma 4.1

$$|\nabla^2_{\zeta^0}\alpha(t)|_{s,\beta+}\le 4\eta^{-1}\mu^{-2}[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}}. \tag{4.25}$$

We proceed as for the ζ-equation to derive

$$r(t)=-\alpha^\infty(t)+(1-\Lambda^\infty(t))r^0, \tag{4.26}$$

where

$$\alpha^\infty(t)=\int_0^t\alpha(t_1)\mathrm{d}t_1+\sum_{k\ge2}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}}\prod_{j=1}^{k-1}\Lambda(t_j)\alpha(t_k)\mathrm{d}t_k\cdots\mathrm{d}t_2\,\mathrm{d}t_1, \tag{4.27}$$

and

$$\Lambda^\infty(t)=\sum_{k\ge1}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}}\prod_{j=1}^{k}\Lambda(t_j)\mathrm{d}t_k\cdots\mathrm{d}t_2\,\mathrm{d}t_1. \tag{4.28}$$

Using (4.23) we get that

$$\begin{aligned}&|\Lambda^\infty(t)|_{\mathcal{L}(\mathbb{C}^n\times\mathbb{C}^n)}\le\frac12,\\ &|\alpha^\infty(t)|_{\mathbb{C}^n}\le 2\eta^{-1}\big(1+\mu^{-1}\|\zeta^0\|_s+\mu^{-2}\|\zeta^0\|_s^2\big)[f]^s_{\sigma,\mu,\mathcal{D}}.\end{aligned}$$

Since in (4.11) $S(\theta;1)=I-\Lambda^\infty(t)$, then the first estimate in (4.13) follows. Since $\Lambda^\infty(t)$ in (4.26) is ζ^0-independent, then $L(\theta,\zeta;t)=-\alpha^\infty(t)$. This is a quadratic in ζ^0 expression, and the estimates (4.14) follow from (4.24)–(4.25) and in view of the estimate for Λ^∞ above.
Finally using the estimates for Λ^∞ and α^∞ we get from (4.26) that $r=r(t)$ satisfies $(4.15)_1$, as $(4.15)_2$ directly comes from (4.8) and (4.2). □

Next we study how the flow maps Φ_f^t transform functions from $\mathcal{T}^{s,\beta}(\sigma,\mu,\mathcal{D})$.

Lemma 4.6. — *Let $0<2\eta<\sigma\le 1$, $0<2\nu<\mu\le 1$. Assume that $f=f^T\in\mathcal{T}^{s,\beta+}(\sigma,\mu,\mathcal{D})$ satisfies* (4.12). *Let $h\in\mathcal{T}^{s,\beta}(\sigma,\mu,\mathcal{D})$ and denote for $0\le t\le 1$*

$$h_t(x;\rho)=h(\Phi_f^t(x;\rho);\rho).$$

Then $h_t\in\mathcal{T}^{s,\beta}(\sigma-2\eta,\mu-2\nu,\mathcal{D})$ and

$$[h_t]^{s,\beta}_{\sigma-2\eta,\mu-2\nu,\mathcal{D}}\le C\frac{\mu}{\nu}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}$$

where C is an absolute constant.

Proof. — Let us write the flow map Φ_f^t as

$$x^0=(r^0,\theta^0,\zeta^0)\mapsto x(t)=(r(t),\theta(t),\zeta(t)).$$

By Lemma 4.5, $h_t(x^0)$ is analytic in $x^0\in\mathcal{O}(\sigma-2\eta,\mu-2\nu)$. Clearly $|h_t(x^0,\cdot)|\le[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}$ for $x^0\in\mathcal{O}(\sigma-2s,\mu-2\nu)$ and $\rho\in\mathcal{D}$. So it remains to estimate the gradient and hessian of $h(x^0)$.

1) *Estimating the gradient.* Since $\theta(t)$ does not depend on ζ^0, we have

$$\frac{\partial h_t}{\partial\zeta^0}=\sum_{k=1}^{n}\frac{\partial h(x(t))}{\partial r_k}\,\frac{\partial r_k(t)}{\partial\zeta^0}+\sum_{b\in\mathcal{L}}\frac{\partial h(x(t))}{\partial\zeta_b(t)}\,\frac{\partial\zeta_b(t)}{\partial\zeta^0}=\Sigma_1+\Sigma_2.$$

i) Since $x(t)\in\mathcal{O}(\sigma-\eta,\mu-\nu)$, we get by the Cauchy estimate that

$$\left|\frac{\partial h(x(t))}{\partial r_k}\right|\le\frac{1}{3\nu^2}[h]^s_{\sigma,\mu,\mathcal{D}}.$$

As $\nabla_{\zeta^0}r_k(t)$ was estimated in (4.14), then using (4.12) we get

$$\|\Sigma_1\|_{s+\beta}\le C\nu^{-2}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}\,\eta^{-1}\mu^{-1}[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}}\le C\mu^{-1}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}\,.$$

ii) Noting that $\Sigma_2(r,\theta,\zeta)={}^tU(\theta;t)\nabla_\zeta h$, we get using (4.13):

$$\|\Sigma_2\|_{s+\beta}\le 4\mu^{-1}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}.$$

Estimating similarly $\frac{\partial}{\partial\rho}\frac{\partial h_t}{\partial\zeta}$ we see that for $x\in\mathcal{O}(\sigma-2\eta,\mu-2\nu)$

$$\|\partial_\rho\nabla_{\zeta^0}h_t\|_{s+\beta}\le C\mu^{-1}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}.$$

2) *Estimating the hessian.* Since $\theta(t)$ does not depend on ζ^0 and since $\zeta(t)$ is affine in ζ^0, then

$$\begin{aligned}\frac{\partial^2 h_t}{\partial\zeta_a^0\partial\zeta_b^0}(x)&=\frac{\partial^2 h(x(t))}{\partial\zeta\partial\zeta}\frac{\partial\zeta(t)}{\partial\zeta_a^0}\frac{\partial\zeta(t)}{\partial\zeta_b^0}+\frac{\partial^2 h(x(t))}{\partial r^2}\frac{\partial r(t)}{\partial\zeta_a^0}\frac{\partial r(t)}{\partial\zeta_b^0}\\&\quad+\frac{\partial^2 h(x(t))}{\partial r\partial\zeta}\frac{\partial r(t)}{\partial\zeta_a^0}\frac{\partial\zeta(t)}{\partial\zeta_b^0}+\frac{\partial h(x(t))}{\partial r}\frac{\partial^2 r(t)}{\partial\zeta_a^0\partial\zeta_b^0}\\&=:\Delta_1+\Delta_2+\Delta_3+\Delta_4.\end{aligned}\tag{4.29}$$

i) We have $|\partial^2 h/\partial\zeta_a\partial\zeta_b|_\beta\le C\mu^{-2}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}$. Using this estimate jointly with (4.13) and Lemma 4.1 we see that

$$|\Delta_1|_\beta\le C\mu^{-2}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}.$$

ii) Since for $x^0\in\mathcal{O}^s(\sigma-2s,\mu-2\nu)$ by (4.14) we have

$$\|\nabla_\zeta r_j\|_{s+\beta}\le C\eta^{-1}\mu^{-1}[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}}\,,$$

and since by Cauchy estimate $|d_r^2h|\le C\nu^{-4}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}$, we get using Lemma 4.1(v) and (4.12)

$$|\Delta_2|_\beta\le C\nu^{-4}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}\eta^{-2}\mu^{-2}([f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}})^2\le C\mu^{-2}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}\,.$$

iii) For any j we have by the Cauchy estimate that $\|\frac{\partial}{\partial r_j}\nabla_\zeta h\|_{s+\beta} \le C\nu^{-3}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}$. Therefore by (4.13)

$$\Big\| \sum_{a'} \frac{\partial^2 h}{\partial r_j \partial \zeta_{a'}} \frac{\partial \zeta_{a'}}{\partial \zeta_a^0} \Big\|_{s+\beta} \le C\nu^{-3}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}\,.$$

Since

$$\|\nabla_{\zeta^0} r_j\|_{s+\beta} \le C\eta^{-1}\mu^{-1}[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}} \le C\nu^2\mu^{-1}$$

by (4.14), then using Lemma 4.1 (v) we find that

$$|\Delta_3|_\beta \le C\nu^{-1}\mu^{-1}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}\,.$$

iv) As $|\partial h/\partial r(x(t))| \le \nu^{-2}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}$ and

$$\left| \frac{\partial^2 r}{\partial \zeta_a^0 \partial \zeta_b^0} \right|_\beta \le C\eta^{-1}\mu^{-2}[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}}$$

by (4.14), then

$$|\Delta_4|_\beta \le C\mu^{-2}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}.$$

The ρ-gradient of the hessian leads to estimates similar to the above. So the lemma is proven.

□

We summarize the results of this section into a proposition.

Proposition 4.7. — *Let $0 < \sigma' < \sigma \le 1$, $0 < \mu' < \mu \le 1$. There exists an absolute constant $C \ge 1$ such that*
(i) if $f = f^T \in \mathcal{T}^{s,\beta}(\sigma,\mu,\mathcal{D})$ and

$$[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}} \le \frac{1}{2}(\mu-\mu')^2(\sigma-\sigma'), \tag{4.30}$$

then for all $0 \le t \le 1$, the Hamiltonian flow map Φ^t_f is a $\mathcal{C}^1$-map

$$\mathcal{O}^s(\sigma',\mu') \times \mathcal{D} \to \mathcal{O}^s(\sigma,\mu);$$

real holomorphic and symplectic for any fixed $\rho \in \mathcal{D}$. Moreover,

$$||\Phi^t_f(x,\cdot) - x||_{s,\mathcal{D}} \le C\left(\frac{1}{\sigma-\sigma'} + \frac{1}{\mu^2}\right)[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}}$$

for any $x \in \mathcal{O}^s(\sigma',\mu')$.
(ii) if $f = f^T \in \mathcal{T}^{s,\beta+}(\sigma,\mu,\mathcal{D})$ and

$$[f]^{s,\beta+}_{\sigma,\mu,\mathcal{D}} \le \frac{1}{2}(\mu-\mu')^2(\sigma-\sigma'), \tag{4.31}$$

then for all $0 \le t \le 1$ and all $h \in \mathcal{T}^{s,\beta}(\sigma,\mu,\mathcal{D})$, the function $h_t(x;\rho) = h(\Phi^t_f(x,\rho);\rho)$ belongs to $\mathcal{T}^{s,\beta}(\sigma',\mu',\mathcal{D})$ and

$$[h_t]^{s,\beta}_{\sigma',\mu',\mathcal{D}} \le C\frac{\mu}{(\mu-\mu')}[h]^{s,\beta}_{\sigma,\mu,\mathcal{D}}.$$

Proof. — Take $\sigma' = \sigma - 2s$ and $\mu' = \mu - 2\nu$ and apply Lemmas 4.5 and 4.6. □

5. Homological equation

Let us first recall the KAM strategy. Let h_0 be the normal form Hamiltonian given by (2.9)

$$h_0(r,\zeta,\rho)=\langle\omega_0(\rho),r\rangle+\frac{1}{2}\langle\zeta,A_0\zeta\rangle$$

satisfying Hypotheses A1-A3. Let f be a perturbation and

$$f^T=f_\theta+\langle f_r,r\rangle+\langle f_\zeta,\zeta\rangle+\frac{1}{2}\langle f_{\zeta\zeta}\zeta,\zeta\rangle$$

be its jet (see (4.1)). If f^T were zero, then $\{\zeta=r=0\}$ would be an invariant n-dimensional torus for the Hamiltonian h_0+f. In general we only know that f is small, say $f=\mathcal{O}(\varepsilon)$, and thus $f^T=\mathcal{O}(\varepsilon)$. In order to decrease the error term we search for a hamiltonian jet $S=S^T=\mathcal{O}(\varepsilon)$ such that its time-one flow map $\Phi_S=\Phi_S^1$ transforms the Hamiltonian h_0+f to

$$(h_0+f)\circ\Phi_S=h+f^+,$$

where h is a new normal form, ε-close to h_0, and the new perturbation f^+ is such that its jet is much smaller than f^T. More precisely,

$$h=h_0+\tilde{h},\qquad \tilde{h}=c(\rho)+\langle\chi(\rho),r\rangle+\frac{1}{2}\langle\zeta,B(\rho)\zeta\rangle=\mathcal{O}(\varepsilon),$$

and $(f^+)^T=\mathcal{O}(\varepsilon^2)$.

As a consequence of the Hamiltonian structure we have (at least formally) that

$$(h_0+f)\circ\Phi_S=h_0+\{h_0,S\}+f^T+\mathcal{O}(\varepsilon^2).$$

So to achieve the goal above we should solve the *homological equation*:

$$\{h_0,S\}=\tilde{h}-f^T+\mathcal{O}(\varepsilon^2). \tag{5.1}$$

Repeating iteratively the same procedure with h instead of h_0 etc., we will be forced to solve the homological equation, not only for the normal form Hamiltonian (2.9), but for more general normal form Hamiltonians (2.4) with ω close to ω_0 and A close to A_0 .

In this section we will consider a homological equation (5.1) with f in $\mathcal{T}^{s,\beta}(\sigma,\mu,\mathcal{D})$ and we will build a solution S in $\mathcal{T}^{s,\beta+}(\sigma,\mu,\mathcal{D})$. In this section, constants C may take different values, but will only depend on s, β, n, d^*, γ, c_0, α_1 and α_2 given in Hypothesis A1, A2 and A3.

5.1. Four components of the homological equation. — Let h be a normal form Hamiltonian (2.4),

$$h(r,\zeta,\rho)=\langle\omega(\rho),r\rangle+\frac{1}{2}\langle\zeta,A(\rho)\zeta\rangle\,,$$

and let us write a jet-function S as

$$S(\theta,r,\zeta)=S_\theta(\theta)+\langle S_r(\theta),r\rangle+\langle S_\zeta(\theta),\zeta\rangle+\frac{1}{2}\langle S_{\zeta\zeta}(\theta)\zeta,\zeta\rangle.$$

Therefore the Poisson bracket of h and S equals

$$\begin{aligned}\{h,S\}=(\nabla_\theta\cdot\omega)S_\theta&+\langle(\nabla_\theta\cdot\omega)S_r,r\rangle+\langle(\nabla_\theta\cdot\omega)S_\zeta,\zeta\rangle\\&+\frac{1}{2}\langle(\nabla_\theta\cdot\omega)S_{\zeta\zeta}\zeta,\zeta\rangle-\langle AJS_\zeta,\zeta\rangle+\langle S_{\zeta\zeta}JA\zeta,\zeta\rangle.\end{aligned}$$

Accordingly the homological equation (5.1) with h_0 replaced by h decomposes into four linear equations. The first two are

$$\langle\nabla_\theta S_\theta,\omega\rangle=-\,f_\theta+c+\mathcal{O}(\varepsilon^2), \tag{5.2}$$

$$\langle\nabla_\theta S_r,\omega\rangle=-\,f_r+\chi+\mathcal{O}(\varepsilon^2). \tag{5.3}$$

In these equations, we are forced to choose

$$c(\rho)=[\![f_\theta(\cdot,\rho)]\!]\quad\text{and}\quad\chi(\rho)=[\![f_r(\cdot,\rho)]\!]$$

where $[\![f]\!]$ denotes averaging of a function f in $\theta \in \mathbb{T}^n$, to get that the space mean-value of the r.h.s. vanishes. The other two equations are

$$\langle \nabla_\theta S_\zeta, \omega\rangle - AJS_\zeta = -f_\zeta + \mathcal{O}(\varepsilon^2), \tag{5.4}$$

$$\langle \nabla_\theta S_{\zeta\zeta}, \omega\rangle - AJS_{\zeta\zeta} + S_{\zeta\zeta}JA = -f_{\zeta\zeta} + B + \mathcal{O}(\varepsilon^2)\,, \tag{5.5}$$

where the operator B will be chosen later. The most delicate, involving the small divisors (see (2.8)), is the last equation.

5.2. The first two equations. — We begin with equations (5.2) and (5.3) which are both of the form

$$\langle \nabla_\theta \varphi(\theta,\rho), \omega(\rho)\rangle = \psi(\theta,\rho) \tag{5.6}$$

with $[\![\psi]\!] = 0$. Here $\omega : \mathcal{D} \to \mathbb{R}^n$ is $\mathcal{C}^1$ and verifies

$$|\omega - \omega_0|_{\mathcal{C}^1(\mathcal{D})} \le \delta_0.$$

Expanding φ and ψ in Fourier series,

$$\varphi = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \hat\varphi(k) e^{ik\cdot\theta}, \quad \psi = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \hat\psi(k) e^{ik\cdot\theta},$$

we solve eq. (5.6) by choosing

$$\hat\varphi(k) = -\frac{i}{\langle\omega, k\rangle}\hat\psi(k), \quad k\in\mathbb{Z}^n\setminus\{0\}; \qquad \hat\varphi(0) = 0.$$

Using Assumption A2 we have, for each $k \neq 0$, either that

$$|\langle\omega(\rho), k\rangle| \ge \delta_0 \quad \forall\rho$$

or that

$$(\nabla_\rho \cdot \mathfrak{z})(\langle k, \omega(\rho)\rangle) \ge \delta_0 \quad \forall\rho$$

for a suitable choice of a unit vector $\mathfrak{z}$. The second case implies that

$$|\langle\omega(\rho), k\rangle| \ge \kappa\,,$$

where $\kappa \le \delta_0$, for all ρ outside some open set $F_k \equiv F_k(\omega)$ of Lebesgue measure $\le \delta_0^{-1}\kappa$. Let

$$\mathcal{D}_1 = \mathcal{D} \setminus \bigcup_{0<|k|\le N} F_k.$$

Then the closed set $\mathcal{D}_1$ satisfies

$$\text{meas}(\mathcal{D}\setminus\mathcal{D}_1) \le N^n \frac{\kappa}{\delta_0}\,,$$

and $|\langle\omega(\rho), k\rangle| \ge \kappa$ for all $\rho \in \mathcal{D}_1$. Hence, for $\rho\in\mathcal{D}_1$ and all $0 < |k| \le N$ we have

$$|\hat\varphi(k)| \le \frac{1}{\kappa}|\hat\psi(k)|\,.$$

Setting $\varphi(\theta,\rho) = \sum_{0<|k|\le N} \hat\varphi(k,\rho)e^{ik\cdot\theta}$, we get that

$$\langle \nabla_\theta \varphi(\theta,\rho), \omega(\rho)\rangle = \psi(\theta,\rho) + R(\theta,\rho). \tag{5.7}$$

Hence φ is an approximate solution of eq. (5.6) with the error term $R(\theta,\rho) = -\sum_{|k|>N}\hat\psi(k,\rho)e^{ik\cdot\theta}$. We obtain by a classical argument that for $(\theta,\rho)\in\mathbb{T}^n_{\sigma'}\times\mathcal{D}_1$, $0<\sigma'<\sigma$, and $j = 0,1$

$$\begin{aligned} |\varphi(\theta,\rho)| &\le \frac{C}{\kappa(\sigma-\sigma')^n}\sup_{|\Im\theta|<\sigma}|\psi(\theta,\rho)|,\\ |\partial_\rho^j R(\theta,\rho)| &\le \frac{C\,e^{-\frac12(\sigma-\sigma')N}}{(\sigma-\sigma')^n}\sup_{|\Im\theta|<\sigma}|\partial_\rho^j\psi(\theta,\rho)|\,, \end{aligned} \tag{5.8}$$

where C only depends on n. If ψ is a real function, then so are φ and R.
Differentiating in ρ the definition of $\hat{\varphi}(k)$ gives [9]

$$\partial_\rho \hat{\varphi}(k) = \chi_{|k|\le N}(k)\Big(-\frac{i}{\langle \omega, k\rangle}\partial_\rho \hat{\psi}(k) + \frac{i}{\langle \omega, k\rangle^2}\langle \partial_\rho \omega, k\rangle \hat{\psi}(k)\Big).$$

From this we derive that

$$|\partial_\rho \varphi(\theta,\rho)| \le \frac{C(|\omega_0(\rho)|_{C^1}+1)N}{\kappa^2(\sigma-\sigma')^n}\big(\sup_{|\Im\theta|<\sigma}|\psi(\theta,\rho)| + \sup_{|\Im\theta|<\sigma}|\partial_\rho\psi(\theta,\rho)|\big),$$

where we estimated the derivative of ω by $|\omega_0(\rho)|_{C^1}+\delta_0 \le |\omega_0(\rho)|_{C^1}+1$.

Applying this construction to (5.2) and (5.3) we get

Proposition 5.1. — *Let $\omega : \mathcal{D} \to \mathbb{R}^n$ be $\mathcal{C}^1$ and verifying $|\omega-\omega_0|_{\mathcal{C}^1(\mathcal{D})} \le \delta_0$. Let $f \in \mathcal{T}^s(\sigma,\mu,\mathcal{D})$ and let $\delta_0 \ge \kappa > 0$, $N \ge 1$. Then there exists a closed set $\mathcal{D}_1 = \mathcal{D}_1(\omega,\kappa,N) \subset \mathcal{D}$, satisfying*

$$\operatorname{meas}(\mathcal{D}\setminus\mathcal{D}_1) \le CN^n\frac{\kappa}{\delta_0},$$

and

(i) there exist real $\mathcal{C}^1$-functions S_θ and R_θ on $\mathbb{T}^n_\sigma \times \mathcal{D}_1 \to \mathbb{C}$, analytic in θ, such that

$$\langle \nabla_\theta S_\theta(\theta,\rho), \omega(\rho)\rangle = -f_\theta(\theta,\rho) + [\![f_\theta(\cdot,\rho)]\!] + R_\theta(\theta,\rho)$$

and for all $(\theta,\rho) \in \mathbb{T}^n_{\sigma'} \times \mathcal{D}_1$, $\sigma' < \sigma$, and $j = 0,1$

$$|\partial^j_\rho S_\theta(\theta,\rho)| \le \frac{CN}{\kappa^2(\sigma-\sigma')^n}[f]^s_{\sigma,\mu,\mathcal{D}_1},$$

$$|\partial^j_\rho R_\theta(\theta,\rho)| \le \frac{Ce^{-\frac12(\sigma-\sigma')N}}{(\sigma-\sigma')^n}[f]^s_{\sigma,\mu,\mathcal{D}_1}.$$

(ii) there exist real $\mathcal{C}^1$ vector-functions S_r and R_r on $\mathbb{T}^n_\sigma \times \mathcal{D}_1$, analytic in θ, such that

$$\langle \nabla_\theta S_r(\theta,\rho), \omega(\rho)\rangle = -f_r(\theta,\rho) + [\![f_r(\cdot;\rho)]\!] + R_r(\theta,\rho),$$

and for all $(\theta,\rho) \in \mathbb{T}^n_{\sigma'} \times \mathcal{D}_1$, $\sigma' < \sigma$, and $j = 0,1$

$$|\partial^j_\rho S_r(\theta,\rho)| \le \frac{C}{\kappa^2(\sigma-\sigma')^n}[f]^s_{\sigma,\mu,\mathcal{D}_1},$$

$$|\partial^j_\rho R_r(\theta,\rho)| \le \frac{Ce^{-\frac12(\sigma-\sigma')N}}{(\sigma-\sigma')^n}[f]^s_{\sigma,\mu,\mathcal{D}_1}.$$

The constant C only depends on $|\omega_0|_{\mathcal{C}^1(\mathcal{D})}$.

5.3. The third equation. — To begin with, we recall a result proved in the appendix of [**15**].

Lemma 5.2. — *Let $A(t)$ be a real diagonal $N\times N$-matrix with diagonal components a_j which are $\mathcal{C}^1$ on $I =]-1,1[$, satisfying for all $j = 1,\dots,N$ and all $t \in I$*

$$a_j'(t) \ge \delta_0.$$

Let $B(t)$ be a Hermitian $N\times N$-matrix of class $\mathcal{C}^1$ on I such that [10]

$$\|B'(t)\| \le \delta_0/2,$$

for all $t \in I$. Then

$$\|(A(t)+B(t))^{-1}\| \le \frac{1}{\varepsilon}$$

outside a set of $t \in I$ of Lebesgue measure $\le CN\varepsilon\delta_0^{-1}$, where C is a numerical constant.

Concerning the third component (5.4) of the homological equation we have

9. Here and below $\chi_Q(k)$ stands for the characteristic function of a set $Q \subset \mathbb{Z}^n$.
10. Here $\|\cdot\|$ means the operator-norm of a matrix associated to the euclidean norm on $\mathbb{C}^N$.

Proposition 5.3. — *Let* $\omega : \mathcal{D} \to \mathbb{R}^n$ *be* $\mathcal{C}^1$ *and verifying* $|\omega - \omega_0|_{\mathcal{C}^1(\mathcal{D})} \le \delta_0$. *Let* $\mathcal{D} \ni \rho \mapsto A(\rho) \in \mathcal{NF} \cap \mathcal{M}_0$ *be* $\mathcal{C}^1$ *and verifying*

$$\|\partial_\rho^j (A(\rho) - A_0)_{[a]}\| \le \frac{1}{2}\delta_0 \tag{5.9}$$

for $j = 0, 1$, $a \in \mathcal{L}$ *and* $\rho \in \mathcal{D}$. *Let* $f \in \mathcal{T}^s(\sigma, \mu, \mathcal{D})$, $0 < \kappa \le \min(\frac{\delta_0}{2}, \frac{c_0}{2})$ *and* $N \ge 1$. *Then there exists a closed set* $\mathcal{D}_2 = \mathcal{D}_2(\omega, A, \kappa, N) \subset \mathcal{D}$, *satisfying*

$$\text{meas}(\mathcal{D} \setminus \mathcal{D}_2) \le C N^{\exp} \frac{\kappa}{\delta_0},$$

and there exist real $\mathcal{C}^1$*-functions* S_ζ *and* R_ζ *from* $\mathbb{T}^n \times \mathcal{D}_2$ *to* Y_s, *analytic in* θ, *such that*

$$\langle \nabla_\theta S_\zeta(\theta, \rho), \omega(\rho)\rangle - A(\rho) J S_\zeta(\theta, \rho) = -f_\zeta(\theta, \rho) + R_\zeta(\theta, \rho) \tag{5.10}$$

and for all $(\theta, \rho) \in \mathbb{T}^n_{\sigma'} \times \mathcal{D}_2$, $\sigma' < \sigma$, *and* $j = 0, 1$

$$\mu \|\partial_\rho^j S_\zeta(\theta, \rho)\|_{s+1} \le \frac{CN}{\kappa^2 (\sigma - \sigma')^{2n}} [f]^{s,\beta}_{\sigma,\mu,\mathcal{D}},$$

$$\mu \|\partial_\rho^j R_\zeta(\theta, \rho)\|_{s} \le \frac{C e^{-\frac{1}{2}(\sigma - \sigma')N}}{(\sigma - \sigma')^{n}} [f]^{s,\beta}_{\sigma,\mu,\mathcal{D}} \, .$$

The exponent exp *only depends on* d^*, n, γ *while the constant* C *also depends on* $|\omega_0|_{\mathcal{C}^1(\mathcal{D})}$.

Proof. — It is more convenient to deal with the hamiltonian operator JA than with operator AJ. Therefore we multiply eq. (5.10) by J and obtain for JS_ζ the equation

$$\big\langle \nabla_\theta (JS_\zeta)(\theta, \rho),\, \omega(\rho) \big\rangle - JA(\rho)(JS_\zeta)(\theta, \rho) = -Jf_\zeta(\theta, \rho) + JR_\zeta(\theta, \rho) \tag{5.11}$$

Let us re-write (5.11) in the complex variables ${}^t(\xi, \eta)$. For $a \in \mathcal{L}$

$$\zeta_a = \begin{pmatrix} p_a \\ q_a \end{pmatrix} = U_a \begin{pmatrix} \xi_a \\ \eta_a \end{pmatrix}, \quad U_a = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}. \tag{5.12}$$

The symplectic operator U_a transforms the quadratic form $(\lambda_a/2)\langle \zeta_a, \zeta_a\rangle$ to $i\lambda_a \xi_a \eta_a$. Therefore, if we denote by U the direct product of the operators diag $(U_a, a \in \mathcal{L})$ then it transforms $(1/2)\langle \zeta, A_0 \zeta\rangle$ to $\sum_{a \in \mathcal{L}} i\lambda_a \xi_a \eta_a$. So it transforms the hamiltonian matrix JA_0 to the diagonal hamiltonian matrix

$$\text{diag}\,\{i\lambda_a \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, a \in \mathcal{L}\}.$$

Then we make in (5.11) the substitution $JS_\zeta = US$, $JR_\zeta = UR$ and $-Jf_\zeta = UF_\zeta$, where $S = {}^t(S^\xi, S^\eta)$, etc. In this notation eq. (5.10) decouples into two equations

$$\begin{aligned} &\langle \nabla_\theta S^\xi, \omega\rangle - i\, {}^tQ S^\xi = F^\xi + R^\xi, \\ &\langle \nabla_\theta S^\eta, \omega\rangle + iQ S^\eta = F^\eta + R^\eta. \end{aligned} \tag{5.13}$$

Here $Q : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ is the scalar valued matrix associated to A via the formula (2.3), i.e.

$$Q = \text{diag}\{\lambda_a,\ a \in \mathcal{L}\} + B,$$

where B is Hermitian and block diagonal.

Written in the Fourier variables, eq. (5.13) becomes

$$\begin{aligned} i(\langle k, \omega\rangle - {}^tQ)\ \hat{S}^\xi(k) &= \hat{F}^\xi(k) + \hat{R}^\xi(k), \quad k \in \mathbb{Z}^n, \\ i(\langle k, \omega\rangle + Q)\ \hat{S}^\eta(k) &= \hat{F}^\eta(k) + \hat{R}^\eta(k), \quad k \in \mathbb{Z}^n. \end{aligned} \tag{5.14}$$

The two equations in (5.14) are similar, so let us consider (for example) the second one, and let us decompose it into its "components" over the blocks $[a]$:

$$i(\langle k, \omega(\rho)\rangle + Q(\rho)_{[a]})\hat{S}_{[a]}(k) = \hat{F}_{[a]}(k, \rho) + \hat{R}_{[a]}(k) \tag{5.15}$$

where the matrix $Q_{[a]}$ is the restriction of Q to $[a] \times [a]$ and the vector $\hat{F}_{[a]}(k, \rho)$ is the restriction of $\hat{F}(k, \rho)$ to $[a]$ – here we have suppressed the upper index η. Denoting by $L(k, [a], \rho)$ the Hermitian operator in the left hand side of equation (5.15), we want to estimate the operator norm of $L(k,[a],\rho)^{-1}$, i.e. we look for a lower bound of the modulus of the eigenvalues of $L(k,[a],\rho)$.

Let $\alpha(\rho)$ denote an eigenvalue of the matrix $Q_{[a]}(\rho)$, $a \in \mathcal{L}$. It follows from (5.9) that

$$|\alpha(\rho) - \lambda_a| \le \frac{\delta_0}{2} \le \frac{c_0}{2}$$

for some appropriate $a \in [a]$, which implies that

$$|\alpha(\rho)| \ge \frac{c_0}{2} w_a^\gamma \ge 2\kappa w_a$$

by (2.5). Hence,

$$\|L(0,[a],\rho)^{-1}\| \le \big(\kappa w_a\big)^{-1} \quad \forall \rho,\ \forall a.$$

Assume that $0 < |k| \le N$. Since $|\langle k, \omega(\rho)\rangle| \le CN$ it follows from (2.5) that

$$|\langle k, \omega(\rho)\rangle + \alpha(\rho)| \ge \frac{c_0}{4} w_a^\gamma \ge \kappa\ w_a$$

whenever $w_a \ge (\frac{4CN}{c_0})^{\frac{1}{\gamma}}$. Hence for these a's we get

$$\|L(k,[a],\rho)^{-1}\| \le \big(\kappa\ w_a\big)^{-1} \quad \forall \rho. \tag{5.16}$$

Now let $w_a \le (\frac{4CN}{c_0})^{\frac{1}{\gamma}}$. By Hypothesis A2 we have either

$$|\langle k, \omega(\rho)\rangle + \lambda_a| \ge \delta_0 w_a \quad \forall \rho, \forall a$$

or we have a unit vector $\mathfrak{z}$ such that

$$(\nabla_\rho \cdot \mathfrak{z})(\langle k, \omega(\rho)\rangle + \lambda_a) \ge \delta_0 \quad \forall \rho, \forall a.$$

The first case clearly implies (5.16), so let us consider the second case. By (5.9) it follows that

$$\|(\nabla_\rho \cdot \mathfrak{z}) H_{[a]}(\rho)\| \le \frac{\delta_0}{2}.$$

The Hermitian matrix $(\langle k, \omega(\rho)\rangle + Q(\rho)_{[a]})$ is of dimension $\lesssim w_a^{d^*}$ (see (2.1)) therefore, by Lemma 5.2, we conclude that (5.16) holds for all ρ outside a suitable set $F_{a,k}$ of measure $\lesssim w_a^{d^*+1}\kappa\delta_0^{-1}$. Let

$$\mathcal{D}_2 = \mathcal{D} \setminus \bigcup_{\substack{|k|\le N \\ w_a \le (\frac{4CN}{c_0})^{\frac{1}{\gamma}}}} F_{a,k}.$$

Then we get

$$\text{meas}(\mathcal{D} \setminus \mathcal{D}_2) \le CN^n \Big(\frac{N}{c_0}\Big)^{\frac{d^*+2}{\gamma}} \frac{\kappa}{\delta_0}$$

and (5.16) holds for all $\rho \in \mathcal{D}_2$, all $|k| \le N$ and all $[a]$.

Equation (5.15) is now solved by

$$\hat{S}_{[a]}(k,\rho) = \chi_{|k|\le N}(k) L(k,[a],\rho)^{-1} \hat{F}_{[a]}(k,\rho), \quad a \in \mathcal{L}\,, \tag{5.17}$$

and

$$\hat{R}_{[a]}(k,\rho) = \chi_{|k|>N}(k) \hat{F}_{[a]}(k,\rho), \quad a \in \mathcal{L}\,. \tag{5.18}$$

Using (5.16), we have for $\rho \in \mathcal{D}_2$

$$\|S_{[a]}(\theta,\rho)\| \le \frac{C}{\kappa\ w_a(\sigma-\sigma')^n} \sup_{|\Im\theta|<\sigma} \|F_{[a]}(\theta,\rho)\|,$$

$$\|R_{[a]}(\theta,\rho)\| \le \frac{Ce^{-\frac{1}{2}(\sigma-\sigma')N}}{(\sigma-\sigma')^n} \sup_{|\Im\theta|<\sigma} |F_{[a]}(\theta,\rho)|.$$

for $\theta \in \mathbb{T}^d_{\sigma'}$, see (5.8).
Since $\|S\|_s^2 = \sum_{a\in\mathcal{L}} w_a^{2s}|S_a|^2 = \sum_{a\in\hat{\mathcal{L}}} w_a^{2s}\|S_{[a]}\|^2$ these estimates imply that

$$\|S(\theta,\rho)\|_{s+1} \leq \frac{C}{\kappa(\sigma-\sigma')^n} \sup_{|\Im\theta|<\sigma} \|F(\theta,\rho)\|_s,$$
$$\|R(\theta,\rho)\|_s \leq \frac{Ce^{-\frac{1}{2}(\sigma-\sigma')N}}{(\sigma-\sigma')^n} \sup_{|\Im\theta|<\sigma} \|F(\theta,\rho)\|_s,$$

for any $\sigma' \leq \sigma$. The estimates of the derivatives with respect to ρ are obtained by differentiating (5.15) to obtain

$$L(k,[a],\rho)[\partial_\rho \hat{S}_{[a]}(k)] = -[\partial_\rho L(k,[a],\rho)]\hat{S}_{[a]}(k) + [\partial_\rho \hat{F}_{[a]}(k,\rho)] + [\partial_\rho \hat{R}_{[a]}(k)]$$

which is an equation of the same type as (5.15) for $\partial_\rho \hat{S}_{[a]}(k)$ and $\partial_\rho \hat{R}_{[a]}(k)$ where $-[\partial_\rho L(k,[a],\rho)]\hat{S}_{[a]}(k) + [\partial_\rho \hat{F}_{[a]}(k,\rho)] := B_{[a]}(k,\rho)$ plays the role of $\hat{F}_{[a]}(k,\rho)$. We solve this equation as in (5.17)-(5.18) and we note that

$$\chi_{|k|>N}(k)B_{[a]}(k,\rho) = \chi_{|k|>N}(k)[\partial_\rho \hat{F}_{[a]}(k,\rho)]$$

and thus

$$\|R(\theta,\rho)\|_s \leq \frac{Ce^{-\frac{1}{2}(\sigma-\sigma')N}}{(\sigma-\sigma')^n} \sup_{|\Im\theta|<\sigma} \|F(\theta,\rho)\|_s.$$

On the other hand

$$\|B_{[a]}(k,\rho)\|_s \leq \frac{CN}{\kappa(\sigma-\sigma')^n} \sup_{|\Im\theta|<\sigma} \| F(\theta,\rho)\|_s + \sup_{|\Im\theta|<\sigma} \|\partial_\rho F(\theta,\rho)\|_s$$

and therefore we get

$$\|\partial_\rho S(\theta,\rho)\|_{s+1} \leq \frac{CN\mu^{-1}}{\kappa^2(\sigma-\sigma')^{2n}}[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}}.$$

The functions F and R are complex, and the constructed solution S_ζ may also be complex. Instead of proving that it is real, we replace $S_\zeta, \theta \in \mathbb{T}^n$, by its real part and then analytically extend it to $\mathbb{T}^n_{\sigma'}$, using the relation $\Re S_\zeta(\theta,\rho) := \frac{1}{2}(S_\zeta(\theta,\rho) + \bar{S}_\zeta(\bar{\theta},\rho))$. Thus we obtain a real solution which obeys the same estimates. □

5.4. The last equation. — Concerning the fourth component of the homological equation, (5.5), we have the following result

Proposition 5.4. — *Let $\omega : \mathcal{D} \to \mathbb{R}^n$ be $\mathcal{C}^1$ and verifying $|\omega - \omega_0|_{\mathcal{C}^1(\mathcal{D})} \leq \delta_0$. Let $\mathcal{D} \ni \rho \mapsto A(\rho) \in \mathcal{NF} \cap \mathcal{M}_{s,\beta}$ be $\mathcal{C}^1$ and verifying*

$$\left|\partial_\rho^j (A(\rho) - A_0)\right|_{s,\beta} \leq \frac{\delta_0}{4} \tag{5.19}$$

for $j = 0,1$ and $\rho \in \mathcal{D}$. Let $f \in \mathcal{T}^{s,\beta}(\sigma,\mu,\mathcal{D})$, $0 < \kappa \leq \frac{\delta_0}{2}$ and $N \geq 1$.
Then there exists a subset $\mathcal{D}_3 = \mathcal{D}_3(h,\kappa,N) \subset \mathcal{D}$, satisfying

$$\text{meas}(\mathcal{D}\setminus\mathcal{D}_3) \leq C\Big(\frac{N}{c_0}\Big)^{\exp}\Big(\frac{\kappa}{\delta_0}\Big)^{\exp'},$$

and there exist real $\mathcal{C}^1$-functions $B : \mathcal{D}_3 \to \mathcal{M}_{s,\beta} \cap \mathcal{NF}$, $S_{\zeta\zeta}(\cdot;\rho) : \mathcal{D}_3 \to \mathcal{M}^+_{s,\beta}$ and $R_{\zeta\zeta}(\cdot;\rho) : \mathbb{T}^n_\sigma \times \mathcal{D}_3 \to \mathcal{M}_{s,\beta}$, analytic in θ, such that

$$\langle \nabla_\theta S_{\zeta\zeta}(\theta,\rho), \omega(\rho)\rangle - A(\rho)JS_{\zeta\zeta}(\theta,\rho) + S_{\zeta\zeta}(\theta,\rho)JA(\rho) = -f_{\zeta\zeta}(\theta,\rho) + B(\rho) + R_{\zeta\zeta}(\theta,\rho) \tag{5.20}$$

and for all $(\theta,\rho)\in\mathbb{T}^n_{\sigma'}\times\mathcal{D}_3$, $\sigma'<\sigma$, *and* $j=0,1$

$$\mu^2\left|\partial_\rho^j R_{\zeta\zeta}(\theta,\rho)\right|_{s,\beta}\le C\ \frac{e^{-\frac12(\sigma-\sigma')N}}{(\sigma-\sigma')^n}[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}}, \tag{5.21}$$

$$\mu^2\left|\partial_\rho^j S_{\zeta\zeta}(\theta,\rho)\right|_{s,\beta+}\le C\ \frac{N^{1+d^*/\gamma}}{\kappa^{2+d^*/2\beta}(\sigma-\sigma')^n}[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}}, \tag{5.22}$$

$$\mu^2\left|\partial_\rho^j B(\rho)\right|_{s,\beta}\le C\ [f]^{s,\beta}_{\sigma,\mu,\mathcal{D}}. \tag{5.23}$$

The two exponents exp *and* exp′ *are positive numbers depending on* n, γ, d^*, α_1, α_2, β. *The constant* C *also depends on* $|\omega_0|_{C^1(\mathcal{D})}$.

Proof. — As in the previous section, and using the same notation, we re-write (5.20) in complex variables. So we introduce $S={}^tUS_{\zeta,\zeta}U$, $R={}^tUR_{\zeta\zeta}U$ and $F={}^tUf_{\zeta\zeta}U$.
By construction, $S_a^b\in\mathcal{M}_{2\times 2}$ for all $a,b\in\mathcal{L}$. Let us denote

$$S_a^b=\begin{pmatrix}(S_a^b)^{\xi\xi} & (S_a^b)^{\xi\eta}\\ (S_a^b)^{\xi\eta} & (S_a^b)^{\eta\eta}\end{pmatrix}$$

and then

$$S^{\xi\xi}=((S_a^b)^{\xi\xi})_{a,b\in\mathcal{L}},\quad S^{\xi\eta}=((S_a^b)^{\xi\eta})_{a,b\in\mathcal{L}},\quad S^{\eta\eta}=((S_a^b)^{\eta\eta})_{a,b\in\mathcal{L}}.$$

We use similar notations for R, B and F.
In this notation (5.20) decouples into three equations [11]

$$\begin{aligned}
&\langle\nabla_\theta S^{\xi\xi},\omega\rangle+iQS^{\xi\xi}+iS^{\xi\xi}\,{}^tQ=B^{\xi\xi}-F^{\xi\xi}+R^{\xi\xi},\\
&\langle\nabla_\theta S^{\eta\eta},\omega\rangle-i\,{}^tQS^{\eta\eta}-iS^{\eta\eta}Q=B^{\eta\eta}-F^{\eta\eta}+R^{\eta\eta},\\
&\langle\nabla_\theta S^{\xi\eta},\omega\rangle+iQS^{\xi\eta}-iS^{\xi\eta}Q=B^{\xi\eta}-F^{\xi\eta}+R^{\xi\eta},
\end{aligned}$$

where we recall that Q is the scalar valued matrix associated to A via the formula (2.3). The first and the second equations are of the same type, so we focus on the resolution of the second and the third equations. Written in Fourier variables, they read

$$i(\langle k,\omega\rangle-{}^tQ)\hat S^{\eta\eta}(k)-i\hat S^{\eta\eta}(k)Q=\delta_{k,0}B^{\eta\eta}-\hat F^{\eta\eta}(k)+\hat R^{\eta\eta}(k),\quad k\in\mathbb{Z}^n, \tag{5.24}$$

$$i(\langle k,\omega\rangle+Q)\hat S^{\xi\eta}(k)-i\hat S^{\xi\eta}(k)Q=\delta_{k,0}B^{\xi\eta}-\hat F^{\xi\eta}(k)+\hat R^{\xi\eta}(k),\quad k\in\mathbb{Z}^n, \tag{5.25}$$

where $\delta_{k,j}$ denotes the Kronecker symbol.

Equation (5.24). We chose $B^{\eta\eta}=0$ and decompose the equation into "components" on each product block $[a]\times[b]$:

$$L\,\hat S_{[a]}^{[b]}(k)=i\hat F_{[a]}^{[b]}(k,\rho)-i\hat R_{[a]}^{[b]}(k) \tag{5.26}$$

where we have suppressed the upper index $\eta\eta$ and the operator $L:=L(k,[a],[b],\rho)$ is the linear Hermitian operator, acting in the space of complex $[a]\times[b]$-matrices defined by

$$L\,M=\big(\langle k,\omega(\rho)\rangle-{}^tQ_{[a]}(\rho)\big)M-MQ_{[b]}(\rho).$$

The matrix $Q_{[a]}$ can be diagonalized in an orthonormal basis:

$${}^tP_{[a]}Q_{[a]}P_{[a]}=D_{[a]}.$$

Therefore denoting $\hat S'^{[b]}_{[a]}={}^tP_{[a]}S^{[b]}_{[a]}P_{[b]}$, $\hat F'^{[b]}_{[a]}={}^tP_{[a]}F^{[b]}_{[a]}P_{[b]}$ and $\hat R'^{[b]}_{[a]}={}^tP_{[a]}R^{[b]}_{[a]}P_{[b]}$ the homological equation (5.26) reads

$$(\langle k,\omega\rangle+D_{[a]})\hat S'^{[b]}_{[a]}(k)-S'^{[b]}_{[a]}(k)D_{[b]}=i\hat F'^{[b]}_{[a]}(k)-i\hat R'^{[b]}_{[a]}(k). \tag{5.27}$$

This equation can be solved term by term:

$$\hat R'_{j\ell}(k)=\hat F'_{j\ell}(k),\quad j\in[a],\ \ell\in[b],\ |k|>N \tag{5.28}$$

11. Actually (5.20) decomposes into four scalar equations but the fourth one is the transpose of the third one.

and

$$\hat{S}'_{j\ell}(k) = \frac{i}{\langle k, \omega(\rho)\rangle \ - \alpha_j(\rho) - \beta_\ell(\rho)} \hat{F}'_{j\ell}(k), \quad j \in [a],\ \ell \in [b],\ |k| \le N \tag{5.29}$$

where $\alpha_j(\rho)$ and $\beta_\ell(\rho)$ denote eigenvalues of $Q_{[a]}(\rho)$ and $Q_{[b]}(\rho)$, respectively. First notice that by (5.28) one has

$$|R(\theta)|_{s,\beta} = |R'(\theta)|_{s,\beta} \le \frac{Ce^{-\frac{1}{2}(\sigma-\sigma')N}}{(\sigma-\sigma')^n} \sup_{|\Im\theta|<\sigma} |F(\theta)|_{s,\beta}.$$

To estimate S we want to use Lemma A.3 below. As $Q_{[a]} = \operatorname{diag}\{\lambda_a \ : a \in [a]\} + B_{[a]}$ with B Hermitian, using hypothesis (5.19) we get that

$$|(\alpha_j(\rho) + \beta_\ell(\rho)) - (\lambda_a + \lambda_b)| \le \left(\frac{\delta_0}{4} + \frac{\delta_0}{4}\right) \frac{1}{(w_a w_b)^\beta} \le \frac{\delta_0}{2(w_a w_b)^\beta}. \tag{5.30}$$

Moreover, in order to apply Lemma A.3 we have to estimate $|\alpha_j(\rho) - \lambda_a|$ and $|\beta_l(r) - \lambda_b|$, this is done thanks to assumption (5.19) :

$$|\alpha_j - \lambda_a| \le \|Q_{[a]}(\rho) - \lambda_{[a]}\mathrm{Id}\| \le \frac{1}{w_{[a]}^{2\beta}} |A(\rho) - A_0|_{s,\beta} \le \frac{\delta_0}{4w_{[a]}^{2\beta}}$$

and the corresponding estimate holds for $|\beta_l(r) - \lambda_b|$.

It follows as in the proof of Proposition 5.3, using Lemma 5.2, relation (2.5), Assumption A2 and (5.19), that there exists a subset $\mathcal{D}_2 = \mathcal{D}_2(h, \kappa, N) \subset \mathcal{D}$, satisfying

$$\operatorname{meas}(\mathcal{D} \setminus \mathcal{D}_2) \le C\Big(\frac{N}{c_0}\Big)^{\exp} \frac{\kappa}{\delta_0},$$

such that

$$|\langle k, \omega(\rho)\rangle \ - \alpha_j(\rho) - \beta_\ell(\rho)| \ge \kappa(1 + |w_a + w_b|),$$

holds for all $\rho \in \mathcal{D}_2$, all $|k| \le N$, all $j \in [a],\ \ell \in [b]$ and all $[a], [b] \in \hat{\mathcal{L}}$. Thus for $\rho \in \mathcal{D}_2$ we obtain by Lemma A.3 that $\hat{S}'(k) \in \mathcal{M}^+_{s,\beta}$ for all $|k| \le N$ and

$$|\hat{S}'(k)|_{s,\beta+} \le C\kappa^{-1-d^*/(4\beta)} N^{d^*/(2\gamma)} |\hat{F}'(k)|_{s,\beta}.$$

Therefore we obtain a solution S satisfying for any $|\Im\theta| < \sigma'$

$$|S(\theta)|_{s,\beta+} \le \frac{CN^{d^*/(2\gamma)}}{\kappa^{1+d^*/(4\beta)}(\sigma-\sigma')^n} \sup_{|\Im\theta|<\sigma} |F(\theta)|_{s,\beta}.$$

The estimates for the derivatives with respect to ρ are obtained by differentiating (5.26) which leads to (here we drop all the indices to simply the formula)

$$L(\partial_\rho \hat{S}_{[a]}^{[b]}(k,\rho)) = -(\partial_\rho L)\hat{S}_{[a]}^{[b]}(k,\rho) + i\partial_\rho \hat{F}_{[a]}^{[b]}(k,\rho) - i\partial_\rho \hat{R}_{[a]}^{[b]}(k,\rho)\,,$$

which is an equation of the same type as (5.26) for $\partial_\rho \hat{S}_{[a]}^{[b]}(k,\rho)$ and $\partial_\rho \hat{R}_{[a]}^{[b]}(k,\rho)$ where $i\hat{F}_{[a]}^{[b]}(k,\rho)$ is replaced by $B_{[a]}^{[b]}(k,\rho) = -(\partial_\rho L)\hat{S}_{[a]}^{[b]}(k,\rho) + i\partial_\rho \hat{F}_{[a]}^{[b]}(k,\rho)$. This equation is solved by defining

$$\begin{aligned}\partial_\rho \hat{S}_{[a]}^{[b]}(k,\rho) =& \chi_{|k|\le N}(k) L(k,[a],[b],\rho)^{-1} B_{[a]}^{[b]}(k,\rho),\\ \partial_\rho \hat{R}_{[a]}^{[b]}(k,\rho) =& -i\chi_{|k|>N}(k) B_{[a]}^{[b]}(k,\rho) = \chi_{|k|>N}(k)\partial_\rho \hat{F}_{[a]}^{[b]}(k,\rho)\,.\end{aligned}$$

Since

$$|(\partial_\rho L)\hat{S}(k,\rho)|_{s,\beta} \le C(N(|\partial_\rho \omega_0| + \delta_0) + 2\delta_0)|\hat{S}(k,\rho)|_{s,\beta} \le CN|\hat{S}(k,\rho)|_{s,\beta}\,,$$

we obtain

$$|B(k,\rho)|_{s,\beta} \le CN\kappa^{-1-d^*/(4\beta} N^{d^*/2\gamma} |\hat{F}(k)|_{s,\beta}$$

and thus following the same strategy as in the resolution of (5.26) we get

$$\mu^2|\partial_\rho S(\theta)|_{s,\beta+} \leq \frac{CN^{1+d^*/\gamma}}{\kappa^{2+d^*/(2\beta)}(\sigma-\sigma')^n}[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}},$$
$$\mu^2|\partial_\rho R(\theta)|_{s,\beta} \leq \frac{Ce^{-\frac{1}{2}(\sigma-\sigma')N}}{(\sigma-\sigma')^n}[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}}.$$

Equation (5.25). It remains to consider (5.25) which decomposes into the "components" over the product blocks $[a]\times[b]$ (we have suppressed the upper index $\xi\eta$):

$$\begin{aligned}\langle k,\omega(\rho)\rangle\ \hat{S}^{[b]}_{[a]}(k) + Q_{[a]}(\rho)\hat{S}^{[b]}_{[a]}(k) - \hat{S}^{[b]}_{[a]}(k)Q_{[b]}(\rho)\\ = -i\delta_{k,0}B^{[b]}_{[a]} + i\hat{F}^{[b]}_{[a]}(k,\rho) - i\hat{R}^{[b]}_{[a]}(k).\end{aligned} \tag{5.31}$$

First we solve this case when $k=0$ and $w_a=w_b$ by defining

$$\hat{S}^{[a]}_{[a]}(0)=0,\quad \hat{R}^{[a]}_{[a]}(0)=0 \text{ and } B^{[a]}_{[a]}=\hat{F}^{[a]}_{[a]}(0).$$

Then we impose $B^{[b]}_{[a]}=0$ for $w_a\neq w_b$ in such a way $B\in\mathcal{M}_{s,\beta}\cap\mathcal{NF}$ and satisfies

$$|B|_{s,\beta}\leq|\hat{F}(0)|_{s,\beta}.$$

The estimates of the derivatives with respect to ρ are obtained by differentiating the expressions for B.

Then, when $k\neq 0$ or $w_a\neq w_b$, with the same definition of S', F' as in (5.27) we obtain

$$(\langle k,\omega\rangle + D_{[a]})\hat{S'}^{[b]}_{[a]}(k) - S'^{[b]}_{[a]}(k)D_{[b]} = i\hat{F'}^{[b]}_{[a]}(k) - i\hat{R'}^{[b]}_{[a]}(k). \tag{5.32}$$

This equation can be solved term by term:

$$\hat{R'}_{j\ell}(k)=\hat{F'}_{j\ell}(k),\quad j\in[a],\ \ell\in[b],\ |k|>N \tag{5.33}$$

and

$$\hat{S'}_{j\ell}(k)=\frac{i}{\langle k,\omega(\rho)\rangle\ -\alpha_j(\rho)-\beta_\ell(\rho)}\hat{F'}_{j\ell}(k),\quad j\in[a],\ \ell\in[b],\ |k|\leq N \tag{5.34}$$

where $\alpha_j(\rho)$ and $\beta_\ell(\rho)$ denote eigenvalues of $Q_{[a]}(\rho)$ and $Q_{[b]}(\rho)$, respectively. First notice that by (5.33) one has

$$|R(\theta)|_{s,\beta}=|R'(\theta)|_{s,\beta}\leq\frac{Ce^{-\frac{1}{2}(\sigma-\sigma')N}}{(\sigma-\sigma')^n}\sup_{|\Im\theta|<\sigma}|F(\theta)|_{s,\beta}.$$

To solve (5.34) we face the small divisors

$$\langle k,\omega(\rho)\rangle+\alpha_j(\rho)-\beta_\ell(\rho),\quad j\in[a],\ \ell\in[b]. \tag{5.35}$$

To estimate them, we have to distinguish between the case $k=0$ and $k\neq 0$.

The case $k=0$. In that case we know that $w_a\neq w_b$ and we use (5.19) and (2.6) to get

$$|\alpha_j(\rho)-\beta_\ell(\rho)|\geq c_0|w_a-w_b|-\frac{\delta_0}{4w_a^{2\beta}}-\frac{\delta_0}{4w_b^{2\beta}}\geq\kappa(1+|w_a-w_b|).$$

This last estimate allows us to use Lemma A.3 to conclude that

$$|\hat{S}(0)|_{\beta+}\leq\frac{C}{\kappa^{d^*+1}}|\hat{F}(0)|_\beta.$$

The case $k\neq 0$. If $k\neq 0$ we face the small divisors (5.35) with non-trivial $\langle k,\omega\rangle$. Using Hypothesis A3, there is a set $\mathcal{D}_2'=\mathcal{D}(\omega,2\eta,N)$,

$$\text{meas}(\mathcal{D}\setminus\mathcal{D}_2')\leq CN^{\alpha_1}(\frac{\eta}{\delta_0})^{\alpha_2},$$

such that for all $\rho \in \mathcal{D}_2'$ and $0 < |k| \le N$

$$|\langle k, \omega(\rho)\rangle \ - \lambda_a + \lambda_b| \ge 2\eta(1 + |w_a - w_b|).$$

By (5.19) this implies

$$\begin{aligned}|\langle k, \omega(\rho)\rangle \ - \alpha_j(\rho) + \beta_\ell(\rho)| &\ge 2\eta(1 + |w_a - w_b|) - \frac{\delta_0}{4w_a^{2\beta}} - \frac{\delta_0}{4w_b^{2\beta}} \\ &\ge \eta(1 + |w_a - w_b|)\end{aligned}$$

if

$$w_b \ge w_a \ge \Big(\frac{\delta_0}{2\eta}\Big)^{\frac{1}{2\beta}}.$$

Let now $w_a \le (\frac{\delta_0}{2\eta})^{\frac{1}{2\beta}}$. We note that $|\langle k, \omega(\rho)\rangle \ - \lambda_a + \lambda_b| \le 1$ implies that

$$w_b^\delta \le (\frac{\delta_0}{2\eta})^{\frac{\delta}{2\beta}} + C|k| \le (\frac{\delta_0}{2\eta})^{\frac{\delta}{2\beta}} + N\,.$$

As in Section 5.3, we obtain that

$$|\langle k, \omega(\rho)\rangle + \alpha_j(\rho) - \beta_\ell(\rho)| \ge \kappa(1 + |w_a - w_b|) \quad \forall j \in [a],\ \forall \ell \in [b] \tag{5.36}$$

holds outside a set $F_{[a],[b],k}$ of measure $w_a^{d^*} w_b^{d^*}(1+|w_a - w_b|)\kappa\delta_0^{-1}$. This can be done considering equation (5.34) as the multiplication of a vector of size $d_{[a]}d_{[b]}$ called $\hat{F}'_{jl}(k)$ by a real diagonal (hence hermitian) square $d_{[a]}d_{[b]} \times d_{[a]}d_{[b]}$ matrix, and using Hypothesis A2, Condition (5.19) and Lemma 5.2.

If F is the union of $F_{[a],[b],k}$ for $|k| \le N$, $[a],[b] \in \hat{\mathcal{L}}$ such that $w_a \le (\frac{\delta_0}{2\eta})^{\frac{1}{2\beta}}$ and $w_b^\delta \le (\frac{\delta_0}{2\eta})^{\frac{\delta}{2\beta}} + N$ respectively, we have

$$\begin{aligned}\mathrm{meas}(F) &\le C(\frac{\delta_0}{2\eta})^{\frac{d^*+1}{2\beta}}\big((\frac{\delta_0}{2\eta})^{\frac{\delta}{2\beta}} + N\big)^{(d^*+2)/\delta}\frac{\kappa}{\delta_0}N^n \\ &\le CN^{n+(d^*+2)/\delta}(\frac{\delta_0}{\eta})^{\frac{2d^*+3}{2\beta}}\frac{\kappa}{\delta_0}\,.\end{aligned}$$

Now we choose η so that

$$(\frac{\eta}{\delta_0})^{\alpha_2} = (\frac{\delta_0}{\eta})^{\frac{2d^*+3}{2\beta}}\frac{\kappa}{\delta_0} \quad \text{i.e. } \frac{\eta}{\delta_0} = \big(\frac{\kappa}{\delta_0}\big)^{\frac{2\beta}{2d^*+3+2\beta\alpha_2}}.$$

Then, as $\beta \le 1$, $\eta \le \kappa$ and $\delta \ge 1$, we have

$$\mathrm{meas}(F) \le CN^{n+d^*+2}\big(\frac{\kappa}{\delta_0}\big)^{\frac{2\beta\alpha_2}{2d^*+3+2\beta\alpha_2}}\,.$$

Let $\mathcal{D}_3 = \mathcal{D}_2 \cap \mathcal{D}_2' \setminus F$, we have

$$\mathrm{meas}(\mathcal{D} \setminus \mathcal{D}_3) \le CN^{exp}\big(\frac{\kappa}{\delta_0}\big)^{\frac{2\beta\alpha_2}{2d^*+3+2\beta\alpha_2}}$$

and by construction for all $\rho \in \mathcal{D}_3$, $0 < |k| \le N$, $a, b \in \mathcal{L}$ and $j \in [a],\ \ell \in [b]$ we have

$$|\langle k, \omega(\rho)\rangle \ - \alpha_j(\rho) + \beta_\ell(\rho)| \ge \kappa(1 + |w_a - w_b|).$$

Hence using Lemma A.3 once again we obtain from (5.32) that $\hat{S}'(k) \in \mathcal{M}^+_{s,\beta}$ and

$$|\hat{S}'(k)|_{s,\beta+} \le C\kappa^{-1-d^*/2\delta}N^{d^*/2\gamma}|\hat{F}'(k)|_{s,\beta}.$$

Therefore we obtain a solution S satisfying for any $|\Im\theta| < \sigma'$

$$|S(\theta)|_{s,\beta+} \le \frac{CN^{d^*/2\gamma}}{\kappa^{1+d^*/2\delta}(\sigma - \sigma')^n}\sup_{|\Im\theta|<\sigma}|F(\theta)|_{s,\beta},$$

The estimates of the derivatives with respect to ρ are obtained by differentiating (5.31) and proceeding as at the end of the resolution of equation (5.24).

In this way we have constructed a solution $S_{\zeta\zeta}, R_{\zeta\zeta}, B$ of the fourth component of the homological equation which satisfies all required estimates. To guarantee that it is real, as at the end of Section 5.3 we replace $S_{\zeta\zeta}, R_{\zeta\zeta}, B$ by their real parts and extend it analytically to $\mathbb{T}^n_{\sigma'}$ (e.g, replace $S_{\zeta\zeta}(\theta,\rho)$ by $\frac{1}{2}(S_{\zeta\zeta}(\theta,\rho)+\bar{S}_{\zeta\zeta}(\bar{\theta},\rho)))$. □

5.5. Summing up. — Let

$$h=\omega(\rho)\cdot r+\frac{1}{2}\langle\zeta, A(\rho)\zeta\rangle$$

where $\rho\to\omega(\rho)$ and $\rho\to A(\rho)$ are C^1 on $\mathcal{D}$ and A is on normal form.

Proposition 5.5. — *Assume*

$$|\partial^j_\rho(A(\rho)-A_0)|_{s,\beta}\le\frac{\delta_0}{4},\quad |\partial^j_\rho(\omega-\omega_0)|\le\delta_0 \tag{5.37}$$

for $j=0,1$ and $\rho\in\mathcal{D}$. Let $f\in\mathcal{T}^{s,\beta}(\sigma,\mu,\mathcal{D})$, $0<\kappa\le\frac{\delta_0}{2}$ and $N\ge 1$. Then there exists a subset $\mathcal{D}'=\mathcal{D}'(h,\kappa,N)\subset\mathcal{D}$, satisfying

$$\text{meas}(\mathcal{D}\setminus\mathcal{D}')\le CN^{\exp}\Big(\frac{\kappa}{\delta_0}\Big)^{\exp'},$$

and there exist real jet-functions $S\in\mathcal{T}^{s,\beta+}(\sigma',\mu,\mathcal{D}')$, $R\in\mathcal{T}^{s,\beta}(\sigma',\mu,\mathcal{D}')$ and a normal form

$$\hat{h}=[\![f(\cdot,0;\rho)]\!]+[\![\nabla_r f(\cdot,0;\rho)]\!]\cdot r+\frac{1}{2}\langle\zeta,B(\rho)\zeta\rangle,$$

such that

$$\{h,S\}+f^T=\hat{h}+R.$$

Furthermore, for all $0\le\sigma'<\sigma$

$$|\partial^j_\rho B(\rho)|_{s,\beta}\le C\ [f]^{s,\beta}_{\sigma,\mu,\mathcal{D}'},\quad j=0,1 \text{ and } \rho\in\mathcal{D}' \tag{5.38}$$

$$[S]^{s,\beta+}_{\sigma',\mu,\mathcal{D}'}\le C\frac{N^{1+d^*/\gamma}}{\kappa^{2+d^*/2\beta}(\sigma-\sigma')^n}[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}'} \tag{5.39}$$

$$[R]^{s,\beta}_{\sigma',\mu,\mathcal{D}'}\le C\frac{e^{-\frac{1}{2}(\sigma-\sigma')N}}{(\sigma-\sigma')^n}[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}'}. \tag{5.40}$$

The two exponents exp *and* exp′ *are positive numbers depending on c_0, n, d^*, α_1, a_2, γ, β. The constant C also depends on $|\omega_0|_{C^1(\mathcal{D})}$.*

Proof. — We define S by

$$S(\theta,r,\zeta)=S_\theta(\theta)+\langle S_r(\theta),r\rangle+\langle S_\zeta(\theta),\zeta\rangle+\frac{1}{2}\langle S_{\zeta\zeta}(\theta)\zeta,\zeta\rangle.$$

where S_θ, S_r, S_ζ and $S_{\zeta\zeta}$ are constructed in Propositions 5.1, 5.3 and 5.4. Hamiltonians R and B are also constructed in these 3 propositions. Then all the statements in Proposition 5.5 are satisfied and in particular we notice that

$$\nabla_\zeta S=S_\zeta+S_{\zeta\zeta}\zeta$$

belongs to $Y_{s+\beta}$ as a consequence of Propositions 5.3, 5.4 and Lemma 4.1 (iii). □

6. Proof of the KAM Theorem.

The Theorem 2.2 is proved by an iterative KAM procedure. We first describe the general step of this KAM procedure.

6.1. The KAM step. — Let h be a normal form Hamiltonian

$$h = \omega \cdot r + \frac{1}{2}\langle \zeta, A(\omega)\zeta\rangle$$

with A on normal form, $A - A_0 \in \mathcal{M}_\beta$ and satisfying (5.37). Let $f \in \mathcal{T}^{s,\beta}(\sigma,\mu,\mathcal{D})$ be a (small) Hamiltonian perturbation. Let $S = S^T \in \mathcal{T}^{s,\beta+}(\sigma',\mu,\mathcal{D}')$ be the solution of the homological equation

$$\{h, S\} + f^T = \hat{h} + R. \tag{6.1}$$

defined in Proposition 5.5. Then defining

$$h^+ := h + \hat{h},$$

we get

$$h \circ \Phi_S^1 = h^+ + f^+$$

with

$$f^+ = R + (f - f^T) \circ \Phi_S^1 + \int_0^1 \{(1-t)(\hat{h} + R) + tf^T, S\} \circ \Phi_S^t \, \mathrm{d}t. \tag{6.2}$$

The following Lemma gives an estimation of the new perturbation:

Lemma 6.1. — *Let $\kappa > 0$, $N \geq 1$, $0 < \sigma' < \sigma \leq 1$ and $0 < 2\mu' < \mu \leq 1$. Assume that $\mathcal{D}' \subset \mathcal{D}$, that $f \in \mathcal{T}^{s,\beta}(\sigma,\mu,\mathcal{D})$, that R satisfies (5.40) and that $S = S^T$ belongs to $\mathcal{T}^{s,\beta+}(\sigma'',\mu,\mathcal{D}')$ with $\sigma'' = \frac{\sigma+\sigma'}{2}$ and satisfies*

$$[S]^{s,\beta+}_{\sigma'',\mu,\mathcal{D}'} \leq \frac{1}{16}\mu^2(\sigma - \sigma'). \tag{6.3}$$

Then the function f^+ given by formula (6.2) belongs to $\mathcal{T}^{s,\beta}(\sigma',\mu',\mathcal{D}')$ and

$$[f^+]^{s,\beta}_{\sigma',\mu',\mathcal{D}'} \leq M\left(\frac{e^{-\frac{1}{2}(\sigma-\sigma')N}}{(\sigma-\sigma')^n} + \left(\frac{\mu'}{\mu}\right)^3 + \frac{N^{1+d^*/\gamma}}{\kappa^{2+d^*/2\beta}\mu^2(\sigma-\sigma')^{n+1}}[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}}\right)[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}} \tag{6.4}$$

where M is a constant depending on n, d^*, α_1, α_2, c_0, γ and β.

Proof. — Let us denote the three terms in the r.h.s. of (6.2) by f_1^+, f_2^+ and f_3^+. In view of (5.40), we have that $[f_1^+]^{s,\beta}_{\sigma',\mu',\mathcal{D}'}$ is controlled by the first term in r.h.s. of (6.4).
By Proposition 4.2, we get

$$[f - f^T]^{s,\beta}_{\sigma,2\mu',\mathcal{D}'} \leq C\left(\frac{\mu'}{\mu}\right)^3 [f]^{s,\beta}_{\sigma,\mu,\mathcal{D}}.$$

By hypothesis $S = S^T$ belongs to $\mathcal{T}^{s,\beta+}(\sigma',\mu,\mathcal{D}')$ and satisfies (6.3) which implies $[S]^{s,\beta+}_{\sigma'',\mu,\mathcal{D}'} \leq \frac{1}{2}(\mu-\mu')^2(\sigma''-\sigma')$ since $2\mu' < \mu$. Therefore by Lemma 4.6 and since $2\mu' \leq 2(\mu-\mu')$, $[f_2^+]^{s,\beta}_{\sigma',\mu',\mathcal{D}'}$ is controlled by the second term in r.h.s. of (6.4).
It remains to control $[f_3^+]^{s,\beta}_{\sigma',\mu',\mathcal{D}'}$. To begin with, $g_t := (1-t)(\hat{h}+R) + tf^T$ is a jet function in $\mathcal{T}^{s,\beta}(\sigma',\mu,\mathcal{D})$. Furthermore, defining for $j = 1, 2$,

$$\sigma_j = \sigma' + j\frac{\sigma - \sigma'}{3}$$

and using (5.40) we get (for N large enough)

$$[g_t]^{s,\beta}_{\sigma_2,\mu,\mathcal{D}'} \leq C\left(1 + 3^n\frac{e^{-(\sigma-\sigma')N/6}}{(\sigma-\sigma')^n}\right)[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}} \leq C[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}}.$$

On the other hand $S \in \mathcal{T}^{s,\beta+}(\sigma_2,\mu,\mathcal{D}')$ is also a jet function and satisfies

$$[S]^{s,\beta+}_{\sigma_2,\mu,\mathcal{D}'} \leq \frac{CN^{1+d^*/\gamma}}{\kappa^{2+d^*/2\beta}(\sigma-\sigma')^n}[f]^{s,\beta}_{\sigma,\mu,\mathcal{D}}.$$

Then using Lemma 4.3 we have

$$[\{g_t, S\}]^{s,\beta}_{\sigma_1,\mu,\mathcal{D}'} \leq C \frac{N^{1+d^*/\gamma}}{\kappa^{2+d^*/2\beta}\mu^2(\sigma-\sigma')^{n+1}}([f]^{s,\beta}_{\sigma,\mu,\mathcal{D}})^2.$$

We conclude the proof by Proposition 4.6. □

6.2. Choice of parameters. — To prove the main theorem we construct the transformation Φ as the composition of infinitely many transformations S as in Theorem 5.5, i.e. for all $k \geq 1$ we construct iteratively S_k, h_k, f_k following the general scheme (6.1)–(6.2) as follows :

$$(h+f)\circ \Phi^1_{S_1}\circ\cdots\circ\Phi^1_{S_k} = h_k + f_k.$$

At each step $f_k \in \mathcal{T}^{s,\beta}(\sigma_k,\mu_k,\mathcal{D}_k)$ with $[f_k]^{s,\beta}_{\sigma_k,\mu_k,\mathcal{D}_k} \leq \varepsilon_k$, $h_k = \langle\omega_k, r\rangle + \frac{1}{2}\langle\zeta, A_k\zeta\rangle$ is on normal form, the Fourier series are truncated at order N_k and the small divisors are controlled by κ_k. In this section we specify the choice of all the parameters for $k \geq 1$.
First we fix

$$\kappa_0 = \varepsilon^{\frac{1}{24(2+d^*/2\beta)}}.$$

We define $\varepsilon_0 = \varepsilon$, $\sigma_0 = \sigma$, $\mu_0 = \mu$ and for $j \geq 1$ we choose

$$\begin{aligned}\sigma_{j-1} - \sigma_j &= C_*\sigma_0 j^{-2},\\ N_j &= 2(\sigma_j - \sigma_{j+1})^{-1}\ln\varepsilon_j^{-1},\\ \kappa_j &= \varepsilon_j^{\frac{1}{24(2+d^*/2\beta)}}\\ \mu_j &= \left(\frac{\varepsilon_j}{(2M)^j\varepsilon^{6/5}}\right)^{\frac{1}{3}},\end{aligned}$$

where M is the absolute constant defined in (6.4) and $(C_*)^{-1} = 2\sum_{j\geq 1}\frac{1}{j^2}$, and

$$\varepsilon_j = (\varepsilon_{j-1})^{\frac{5}{4}}. \tag{6.5}$$

Observe that with this choice, (μ_j) satisfies $2\mu_{j+1} \leq \mu_j$. Then the only unfixed parameter is $\varepsilon = \varepsilon_0$, that will be fixed next section. Nevertheless, ε will be small enough to ensure the property $\kappa_j \leq \frac{\delta_0}{2}$ that is necessary to apply Proposition 5.5. This is guaranteed if

$$\varepsilon^{\frac{1}{24(2+d^*/2\beta)}} \leq \frac{\delta_0}{2}. \tag{6.6}$$

6.3. Iterative lemma. — Let set $\mathcal{D}_0 = \mathcal{D}$, $h_0 = \langle\omega_0(\rho), r\rangle + \frac{1}{2}\langle\zeta, A_0\zeta\rangle$ and $f_0 = f$ in such a way $[f_0]^{s,\beta}_{\sigma_0,\mu_0,\mathcal{D}_0} \leq \varepsilon_0$. For $k \geq 0$ let us denote

$$\mathcal{O}_k = \mathcal{O}^s(\sigma_k,\mu_k).$$

Lemma 6.2. — *For ε sufficiently small depending on μ_0, σ_0, n,s, β and $|\omega_0|_{C^1(\mathcal{D})}$ we have the following:*
For all $k \geq 1$ there exist $\mathcal{D}_k \subset \mathcal{D}_{k-1}$, $S_k \in \mathcal{T}^{s,\beta+}(\sigma_k,\mu_k,\mathcal{D}_k)$, $h_k = \langle\omega_k, r\rangle + \frac{1}{2}\langle\zeta, A_k\zeta\rangle$ on normal form and $f_k \in \mathcal{T}^{s,\beta}(\sigma_k,\mu_k,\mathcal{D}_k)$ such that

(i) The mapping

$$\Phi_k(\cdot,\rho) = \Phi^1_{S_k} \ : \ \mathcal{O}_k \to \mathcal{O}_{k-1}, \quad \rho\in\mathcal{D}_k,\ k = 1,2,\cdots \tag{6.7}$$

is an analytic symplectomorphism linking the hamiltonian at step $k-1$ and the hamiltonian at the step k, i.e.

$$(h_{k-1} + f_{k-1})\circ\Phi_k = h_k + f_k.$$

(ii) we have the estimates

$$\begin{aligned}
\text{meas}(\mathcal{D}_{k-1}\setminus\mathcal{D}_k) &\le \varepsilon_{k-1}^{\alpha},\\
[h_k-h_{k-1}]^{s,\beta}_{\sigma_k,\mu_k,\mathcal{D}_k} &\le C\varepsilon_{k-1},\\
[f_k]^{s,\beta}_{\sigma_k,\mu_k,\mathcal{D}_k} &\le \varepsilon_k,\\
\|\Phi_k(x,\rho)-x\|_s &\le \varepsilon^{4/5}.\varepsilon_{k-1}^{1/4}, \quad \text{for } x\in\mathcal{O}_k,\ \rho\in\mathcal{D}_k.
\end{aligned}$$

The exponents α is a positive number depending on n, d^*, α_1, a_2, γ, β. The constant C also depends on $|\omega_0|_{C^1(\mathcal{D})}$.

Proof. — At step 1, $h_0=\langle\omega_0(\rho),r\rangle+\frac{1}{2}\langle\zeta,A_0\zeta\rangle$ and thus hypothesis (5.37) is trivially satisfied and we can apply Proposition 5.5 to construct S_1, R_0, B_0 and $\mathcal{D}_1$ such that for $\rho\in\mathcal{D}_1$

$$\{h_0,S_0\}+f_0^T=\hat{h}_0+R_0.$$

Then we see that, using (5.39) and defining $\sigma_{1/2}=\frac{\sigma_0+\sigma_1}{2}$, we have

$$[S_1]^{s,\beta+}_{\sigma_{1/2},\mu_0,\mathcal{D}_1}\le C\frac{\varepsilon_0 N_0^{1+d^*/\gamma}}{\kappa_0^{2+d^*/2\beta}(\sigma_0-\sigma_{1/2})^n}\le\frac{1}{16}\mu_0^2(\sigma_0-\sigma_1)$$

for $\varepsilon=\varepsilon_0$ small enough in view of our choice of parameters. Therefore both Proposition 4.7 and Lemma 6.2 apply and thus for any $\rho\in\mathcal{D}_1$, $\Phi_1(\cdot,\rho)=\Phi^1_{S_1}:\mathcal{O}_1\to\mathcal{O}_0$ is an analytic symplectomorphism such that

$$(h_0+f_0)\circ\Phi_1=h_1+f_1$$

with h_1, f_1, $\mathcal{D}_1$ and Φ_1 satisfying the estimates $(ii)_{k=1}$. In particular we have

$$\|\Phi_1(x)-x\|_s\le\frac{C}{\sigma_0\mu_0^2}[S_1]^{s,\beta+}_{\sigma_{1/2},\mu_0,\mathcal{D}_1}\le\frac{CN_0^{1+d^*/\gamma}}{\sigma_0^{n+1}\mu_0^2\kappa_0^{2+d^*/2\beta}}\varepsilon_0\le\frac{C(\ln\varepsilon_0)^{1+d^*/\gamma}}{\sigma_0^{n+2+d^*/\gamma}\mu_0^2}\varepsilon_0^{23/24}\le\frac{1}{2}\varepsilon_0^{11/12}$$

for ε_0 small enough.

Now assume that we have completed the iteration up to step j. We want to perform the step $j+1$. We first note that by construction (see Proposition 5.5)

$$A_j=A_0+B_0+\cdots+B_{j-1}$$

and by (5.38)

$$|A_j|_\beta\le\varepsilon_0+\cdots+\varepsilon_{j-1}\le 2\varepsilon_0\le\frac{1}{4}\delta_0$$

for ε_0 small enough. Similarly

$$\omega_j=\omega_0+[\![\nabla_r f_0(\cdot,0;\rho)]\!]+\cdots+[\![\nabla_r f_{j-1}(\cdot,0;\rho)]\!]$$

and thus $|\partial_r^j(\omega_j-\omega_0)|\le\delta_0$ for ε_0 small enough.
Therefore (5.37) is satisfied at rank j and we can apply Proposition 5.5 in order to construct S_{j+1}, B_j, R_j and $\mathcal{D}_j$.

Then we construct f_{j+1} as in (6.2), i.e.

$$f_{j+1}=R_j+(f_j-f_j^T)\circ\Phi^1_{S_{j+1}}+\int_0^1\{(1-t)(\hat{h}_j+R_j)+tf_j^T,S_{j+1}\}\circ\Phi^t_{S_{j+1}}\,dt.$$

To control f_{j+1} we may apply Lemma 6.1 since, defining $\sigma_{j+1/2}=\frac{\sigma_j+\sigma_{j+1}}{2}$,

$$[S_{j+1}]^{s,\beta+}_{\sigma_{j+1/2},\mu_j,\mathcal{D}_{j+1}}\le C\frac{\varepsilon_j N_j^{1+d^*/\gamma}}{\kappa_j^{2+d^*/2\beta}(\sigma_j-\sigma_{j+1})^n}\le\frac{1}{8}\mu_j^2(\sigma_j-\sigma_{j+1}).$$

Therefore we can apply Lemma 6.1 and, using the preceding choice of parameters, we may bound all the terms of the r.h.s. of (6.4). Let us start with the second term:

$$M\left(\frac{\mu_{j+1}}{\mu_j}\right)^3\varepsilon_j=\frac{1}{2}\varepsilon_{j+1}\,. \tag{6.8}$$

The third term may be computed as
(6.9)

$$M\left(\frac{2(j+1)^2\ln(\varepsilon_j^{-1})}{C_*\sigma_0}\right)^{1+d^*/\gamma}\left(\frac{(j+1)^2}{C_*\sigma_0}\right)^{n+1}\frac{\varepsilon_j^{2-1/24}}{\mu_j^2}=C(j+1)^{2n+3+2d^*/\gamma}(2M)^{2j/3}\varepsilon^{4/5}(\varepsilon_j)^{1/24}\varepsilon_{j+1}$$

and there exists $\bar\varepsilon_1>0$ such that for $0<\varepsilon\le\bar\varepsilon_1$ we have for any $j\ge1$

$$C(j+1)^{2n+3+2d^*/\gamma}(2M)^{2j/3}(\varepsilon)^{\frac45+\frac1{24}\cdot(\frac54)^j}\le\frac14\,.$$

The first term gives

$$M\frac{\varepsilon_j^2}{C_*\sigma_0}(j+1)^{2n}=M\frac{(j+1)^{2n}}{C_*\sigma_0}(\varepsilon)^{\frac34\cdot(\frac54)^j}\varepsilon_{j+1}\,, \tag{6.10}$$

and there exists $\bar\varepsilon_2>0$ such that for $0<\varepsilon\le\bar\varepsilon_2$ we have for any $j\ge1$

$$M\frac{(j+1)^{2n}}{C_*\sigma_0}(\varepsilon)^{\frac34\cdot(\frac54)^j}\le\frac14\,.$$

Take $\varepsilon_0\ \le\bar\varepsilon=\min(\bar\varepsilon_1,\bar\varepsilon_2)>0$ and we conclude that

$$[f_{j+1}]^{s,\beta}_{\sigma_{j+1},\mu_{j+1},\mathcal{D}_{j+1}}\le\varepsilon_{j+1}. \tag{6.11}$$

On the other hand by Proposition 5.5 the domain $\mathcal{D}_{j+1}$ satisfies

$$\text{meas}(\mathcal{D}_j\setminus\mathcal{D}_{j+1})\le CN_j^{\exp}\left(\frac{\kappa_j}{\delta_0}\right)^{\exp'}\le\varepsilon_j^\alpha$$

for some $\alpha>0$ and for $\varepsilon_0=\varepsilon$ small enough. The estimate concerning $h_{k+1}-h_k$ follows from (5.38) and (6.11) for the infinite dimensional part, from (6.11) for the control of $[\![f_{j+1}(\cdot,0;\rho)]\!]$ and a straightforward Cauchy estimate for the control of the mean value $[\![\nabla_r f_{j+1}(\cdot,0;\rho)]\!]$. Concerning the flow, we have for $j\ge1$,

$$\begin{aligned}\|\Phi_{j+1}(x)-x\|_s\ &\le\ \frac{C}{\sigma_j\mu_j^2}[S_{j+1}]^{s,\beta+}_{\sigma_{j+1/2},\mu_j,\mathcal{D}_{j+1}}\le\frac{CN_j^{1+d^*/\gamma}}{\sigma_j^{n+1}\mu_j^2\kappa_j^{2+d^*/2\beta}}\varepsilon_j\\ &\le\ C'(\ln\varepsilon_j)^{1+d^*/\gamma}(2M)^{2j/3}j^{n+2+d^*/\gamma}\varepsilon^{4/5}\varepsilon_j^{7/24}\le\varepsilon^{4/5}\frac12\varepsilon_j^{1/4}\,,\end{aligned}$$

for ε small enough.

□

6.4. Transition to the limit and proof of Theorem 2.2. — Let

$$\mathcal{D}'=\cap_{k\ge0}\mathcal{D}_k.$$

In view of the iterative lemma, this is a Borel set satisfying

$$\text{meas}(\mathcal{D}\setminus\mathcal{D}')\le2\varepsilon^\alpha.$$

Let us set

$$Q_l=\mathcal{O}^s(\sigma/\ell,\mu/\ell),\ \ \mathcal{Z}_s=\mathbb{T}^n_\sigma\times\mathbb{C}^n\times Y_s$$

where $\ell\ge2$, and recall that $\|\cdot\|_s$ denotes the natural norm on $\mathbb{C}^n\times\mathbb{C}^n\times Y_s$. It defines the distance on $\mathcal{Z}_s$. We used the notations introduced in Lemma 6.2. By Proposition 4.5 assertion 2 and since $\sigma_k>\sigma/2$, for each $\rho\in\mathcal{D}'$ and $k\ge2$, the map Φ_k extends to Q_2 and satisfies on Q_2 the same estimate as on $\mathcal{O}_k$:

$$\Phi_k:\ Q_2\to\mathcal{Z}_s,\quad\|\Phi_k-\text{Id}\|_s\le C\mu_k^{-2}(\sigma_{k-1}-\sigma_k)^{-1}\varepsilon_k\le\frac{k^2}{C_*\sigma_0}(2M)^{2k/3}\varepsilon_k^{1/3}\varepsilon^{4/5}. \tag{6.12}$$

Now for $0 \le j \le N$ let us denote $\Phi_N^j = \Phi_{j+1} \circ \cdots \circ \Phi_N$. Due to (6.7), it maps $\mathcal{O}_N$ to $\mathcal{O}_j$. Again using Proposition 4.5, this map extends analytically to a map $\Phi_N^j : Q_2 \to \mathcal{Z}_s$, and by (6.12), for $M > N$, $\|\Phi_N^j - \Phi_M^j\|_s \le C\varepsilon_N^{1/4}\varepsilon^{4/5}$, i.e. $(\Phi_N^j)_N$ is a Cauchy sequence. Thus when $N \to \infty$ the maps Φ_N^j converge to a limiting mapping $\Phi_\infty^j : Q_2 \to \mathcal{Z}_s$. Furthermore we have

$$\|\Phi_\infty^j - \mathrm{Id}\|_s \le C\varepsilon^{4/5}\sum_{k\ge j}\varepsilon_k^{1/4} \le C\varepsilon^{4/5}\varepsilon_j^{1/4}, \ \forall j \ge 1. \tag{6.13}$$

By the Cauchy estimate the linearized map satisfies

$$\|D\Phi_\infty^j(x) - \mathrm{Id}\|_{\mathcal{L}(Y_s,Y_s)} \le C\varepsilon^{4/5}\varepsilon_j^{1/4}, \quad \forall x \in Q_3, \ \forall j \ge 1. \tag{6.14}$$

By construction, the map Φ_N^0 transforms the original hamiltonian

$$H_0 = \langle \omega, r\rangle + \frac{1}{2}\langle \zeta, A_0\zeta\rangle + f$$

into

$$H_N = \langle \omega_N, r\rangle + \frac{1}{2}\langle \zeta, A_N(\omega)\zeta\rangle + f_N.$$

Here

$$\omega_N = \omega + [\![\nabla_r f_0(\cdot,0;\rho)]\!] + \cdots + [\![\nabla_r f_{N-1}(\cdot,0;\rho)]\!]$$

and

$$A_N = A_0 + B_0 + \cdots + B_{N-1}$$

where B_k is built from $\langle \nabla^2_{\zeta\zeta} f_k(\cdot,0)\rangle$ as in the proof of Proposition 5.4.
Clearly, $\omega_N \to \omega'$ and $A_N \to A$ where the vector $\omega' \equiv \omega'(\rho)$ and the operator $A \equiv A(\rho)$ satisfy the assertions of Theorem 2.2.
Let us denote $\Phi = \Phi_\infty^0$, consider the limiting hamiltonian $H' = H_0 \circ \Phi$ and write it as

$$H' = \langle \omega', r\rangle + \frac{1}{2}\langle \zeta, A(\rho)\zeta\rangle + f'.$$

The function f' is analytic in the domain Q_2. Since $H' = H_k \circ \Phi_\infty^k$, we have

$$\nabla H'(x) = D\Phi_\infty^k(x) \cdot \nabla H_k(\Phi_\infty^k(x)).$$

As $[f_k]^{s,\beta}_{\sigma_k,\mu_k,\mathcal{D}_k} \le \varepsilon_k$, we deduce

$$\nabla_r H_k(\Phi_\infty^k(\theta,0,0)) = \omega_k + O(\varepsilon_k^{1/4}) \quad \theta \in \mathbb{T}^n_{\frac{\sigma}{3}}.$$

Since the map Φ_∞^k satisfies (6.14), then

$$\nabla_r H'(\theta,0,0) = \omega' + O(\varepsilon_k^{1/4}) \quad \text{for all } k \ge 1 \text{ and } \theta \in \mathbb{T}^n_{\frac{\sigma}{3}}.$$

Hence, $\nabla_r H'(\theta,0,0) = \omega'$ and thus

$$\nabla_r f'(\theta,0,0) \equiv 0 \quad \text{for } \theta \in \mathbb{T}^n_{\frac{\sigma}{3}}.$$

Similar arguments lead to

$$\nabla_{\zeta_a} f'(\theta,0,0) \equiv 0 \text{ and } \nabla_{\zeta_a}\nabla_r f'(\theta,0,0) \equiv 0 \quad \text{for } \theta \in \mathbb{T}^n_{\frac{\sigma}{3}}.$$

Now consider $\nabla_{\zeta_a}\nabla_{\zeta_b} H'(x)$. To study this matrix let us write it in the form (4.29), with $h = H_k$ and $x(1) = \Phi_\infty^k(x)$. Repeating the arguments used in the proof of Proposition 4.6 we get that

$$\nabla_{\zeta_a}\nabla_{\zeta_b} H'(\theta,0,0) = (A_k)_{ab} + O(\varepsilon_k^{1/4}) \quad \text{for all } k \ge 1 \text{ and } \theta \in \mathbb{T}^n_{\frac{\sigma}{3}}.$$

Therefore $\nabla_{\zeta_a}\nabla_{\zeta_b} H'(\theta,0,0) = A_{ab}$ i.e.

$$\nabla_{\zeta_a}\nabla_{\zeta_b} f'(\theta,0,0) = 0 \quad \text{for } \theta \in \mathbb{T}^n_{\frac{\sigma}{3}}.$$

This concludes the proof of Theorem 2.2.

Appendix A
Some calculus

Lemma A.1. — *Let $j, k, \ell \in \mathbb{N} \setminus \{0\}$ then*

$$\frac{\min(j,k)}{\min(j,k)+|j^2-k^2|}\frac{\min(k,\ell)}{\min(k,\ell)+|k^2-\ell^2|} \le \frac{\min(j,\ell)}{\min(j,\ell)+|j^2-\ell^2|}. \tag{A.1}$$

Proof. — Without lost of generality we can assume $j \le \ell$.
If $k \le j$ then $|k^2-\ell^2| \ge |j^2-\ell^2|$ and thus

$$\frac{\min(j,\ell)}{\min(j,\ell)+|j^2-\ell^2|} = \frac{j}{j+|j^2-\ell^2|} \ge \frac{j}{j+|k^2-\ell^2|} \ge \frac{k}{k+|k^2-\ell^2|} = \frac{\min(k,\ell)}{\min(k,\ell)+|k^2-\ell^2|}$$

which leads to (A.1). The case $\ell \le k$ is similar.
In the case $j \le k \le \ell$ we have

$$\frac{\min(j,k)}{\min(j,k)+|j^2-k^2|}\frac{\min(k,\ell)}{\min(k,\ell)+|k^2-\ell^2|} \le \frac{j}{j+|j^2-k^2|+|k^2-\ell^2|} \le \frac{j}{j+|j^2-\ell^2|} = \frac{\min(j,\ell)}{\min(j,\ell)+|j^2-\ell^2|}.$$

□

Lemma A.2. — *Let $j \in \mathbb{N}$ then*

$$\sum_{k\in\mathbb{N}} \frac{1}{k^\beta(1+|k-j|)} \le C$$

for a constant C depending only on $\beta > 0$.

Proof. — We note that

$$\sum_{k\in\mathbb{N}} \frac{1}{k^\beta(1+|k-j|)} = a \star b(j)$$

where $a_k = \frac{1}{k}$ for $k \ge 1$, $a_k = 0$ for $k \le 0$ and $b_k = \frac{1}{1+|k|}$, $k \in \mathbb{Z}$. We have that $b \in \ell^p$ for any $1 < p \le +\infty$ and that $a \in \ell^q$ for any $\frac{1}{\beta} < q \le +\infty$. Thus by Young inequality $a \star b \in \ell_r$ for r such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. In particular choosing $q = \frac{2}{\beta}$ and $p = \frac{2}{2-\beta}$ we conclude that $a \star b \in \ell_\infty$. □

The following Lemma is a variant of Proposition 2.2.4 in [**11**].

Lemma A.3. — *Let $A \in \mathcal{M}_{s,\beta}$ and let $B(k)$ defined by*

$$B(k)_j^l = \frac{i}{\langle k,\omega\rangle \ + \varepsilon\mu_j - \mu_l} A_j^l, \quad j \in [a],\ \ell \in [b] \tag{A.2}$$

where $\varepsilon = \pm 1$, $(\mu_a)_{a\in\mathcal{L}}$ is a sequence of real numbers satisfying

$$|\mu_a - \lambda_a| \le \min\left(\frac{C_\mu}{w_a^\delta}, \frac{c_0}{4}\right), \qquad \text{for all } a \in \mathcal{L} \tag{A.3}$$

for a given $C_\mu > 0$ and $\delta > 0$, and such that for all $a, b \in \mathcal{L}$ and all $|k| \le N$

$$|\langle k, \omega(\rho)\rangle \ + \varepsilon\mu_a - \mu_b| \ge \kappa(1 + |w_a - w_b|). \tag{A.4}$$

Then $B \in \mathcal{M}^+_{s,\beta}$ and there exists a constant $C > 0$ depending only on C_μ, $\mathcal{A}$ and δ such that

$$|B(k)|_{s,\beta+} \le C\frac{|A|_{s,\beta} N^{\frac{d^*}{2\gamma}}}{\kappa^{1+\frac{d^*}{2\delta}}} \quad \text{for all } |k| \le N.$$

Proof. — We first remark that the claimed property only concerns the operator norms of the blocks $B_{[a]}^{[b]}$, which can be computed separately. Let k_1 and k_2 be positive integers that will be fixed later. We define the following decomposition in $\mathcal{M}_{s,\beta}$, according to the weights w_a and w_b :

$$\mathcal{M}_{s,\beta} = \Upsilon^1_{s,\beta}(k_1,k_2) \oplus \Upsilon^2_{s,\beta}(k_1,k_2) \oplus \Upsilon^3_{s,\beta}(k_1,k_2)\,,$$

where

$$\Upsilon^1_{s,\beta}(k_1,k_2) = \left\{M \in \mathcal{M}_{s,\beta}, M_{[a]}^{[b]} = 0 \text{ if } \max(w_a,w_b) \leq k_1 \min(w_a,w_b)\right\},$$

$$\Upsilon^2_{s,\beta}(k_1,k_2) = \left\{M \in \mathcal{M}_{s,\beta}, M_{[a]}^{[b]} = 0 \text{ if } \max(w_a,w_b) > k_1 \min(w_a,w_b) \text{ or } \max(w_a,w_b) \leq k_2\right\},$$

$$\Upsilon^3_{s,\beta}(k_1,k_2) = \left\{M \in \mathcal{M}_{s,\beta}, M_{[a]}^{[b]} = 0 \text{ if } \max(w_a,w_b) > k_1 \min(w_a,w_b) \text{ or } \max(w_a,w_b) > k_2\right\},$$

and we prove the desired estimates according to this decomposition. Since we estimate the operator norm of $B_{[a]}^{[b]}$, we need to rewrite the definition (A.2) in a operator way : denoting by $D_{[a]}$ the diagonal (square) matrix with entries μ_j, for $j \in [a]$ and $D'_{[a]}$ the diagonal (square) matrix with entries $\langle k, \omega(\rho)\rangle + \varepsilon\mu_j$, for $j \in [a]$, equation (A.2) reads

$$D'_{[a]}B_{[a]}^{[b]} - B_{[a]}^{[b]}D_{[b]} = iA_{[a]}^{[b]}\,. \tag{A.5}$$

Step 1 : suppose $A \in \mathcal{M}_{s,\beta} \cap \Upsilon^1_{s,\beta}(k_1,k_2)$. The only nonzero blocks $A_{[a]}^{[b]}$ correspond to weights w_a and w_b such that

$$\max(w_a,w_b) > k_1 \min(w_a,w_b)$$

take for instance $w_a > k_1 w_b$. Then $|w_a - w_b| \geq w_a(1-\frac{1}{k_1})$, $w_a \geq k_1$ and

$$|\langle k,\omega(\rho)\rangle \ + \varepsilon\mu_a| \geq c_0\left(w_a^\gamma - \frac{1}{4}\right) - nN\max(\omega_k(\rho)) \geq \frac{c_0}{2}w_a^\gamma\,, \tag{A.6}$$

for

$$k_1 \geq \left(4nNc_0^{-1}\max(\omega_k(\rho))\right)^{1/\gamma} := C_1, \tag{A.7}$$

that proves that $D'_{[a]}$ is invertible and gives an upper bound for the operator norm of its inverse. Then (A.5) is equivalent to

$$B_{[a]}^{[b]} - {D'_{[a]}}^{-1}B_{[a]}^{[b]}D_{[b]} = i{D'_{[a]}}^{-1}A_{[a]}^{[b]}\,. \tag{A.8}$$

Next consider the operator $\mathscr{L}^1_{[a]\times[b]}$ acting on matrices of size $[a]\times[b]$ such that

$$\mathscr{L}^1_{[a]\times[b]}\left(B_{[a]}^{[b]}\right) := {D'_{[a]}}^{-1}B_{[a]}^{[b]}D_{[b]}\,. \tag{A.9}$$

We have

$$\|\mathscr{L}^1_{[a]\times[b]}\left(B_{[a]}^{[b]}\right)\| \leq \frac{4w_b}{w_a}\|B_{[a]}^{[b]}\| \leq \frac{4}{k_1}\|B_{[a]}^{[b]}\|\,, \tag{A.10}$$

hence, in operator norm, $\|\mathscr{L}^1_{[a]\times[b]}\| \leq \frac{1}{2}$ if $k_1 \geq 8$. Then the operator $\mathrm{Id} - \mathscr{L}^1_{[a]\times[b]}$ is invertible and

$$\begin{aligned}\|B_{[a]}^{[b]}\| &\leq \|\left(\mathrm{Id} - \mathscr{L}_{[a]\times[b]}\right)^{-1}\|\|i{D'_{[a]}}^{-1}A_{[a]}^{[b]}\|\\ &\leq \frac{4}{w_a}\|A_{[a]}^{[b]}\|\\ &\leq \frac{4k_1}{k_1-1}\frac{1}{1+|w_a-w_b|}\|A_{[a]}^{[b]}\|\end{aligned}$$

We have obtained that, for $k_1 \geq \max(C_1, 8)$, $B \in \mathcal{M}^+_{s,\beta}$ and

$$|B|_{s,\beta+} \leq 8|A|_{s,\beta} \tag{A.11}$$

Step 2 : suppose $A \in \mathcal{M}_{s,\beta} \cap \Upsilon^2_{s,\beta}(k_1,k_2)$. The only nonzero blocks $A_{[a]}^{[b]}$ correspond to weights w_a and w_b such that

$$\max(w_a,w_b) \le k_1 \min(w_a,w_b) \text{ and } \max(w_a,w_b) > k_2\,.$$

Notice that these two conditions imply that

$$\min(w_a,w_b) \ge \frac{k_2}{k_1}.$$

We define the square matrix $\tilde{D}_{[a]} = \lambda_a \mathbf{1}_{[a]}$, where $\mathbf{1}_{[a]}$ is the identity matrix. Then

$$\|D_{[a]} - \tilde{D}_{[a]}\| \le \frac{C_\mu}{w_a^\delta}\,, \tag{A.12}$$

and equation (A.2) may be rewritten as

$$\mathscr{L}^2_{[a]\times[b]}\left(B_{[a]}^{[b]}\right) - \varepsilon(\tilde{D}_{[a]} - D_{[a]})B_{[a]}^{[b]} + B_{[a]}^{[b]}(\tilde{D}_{[b]} - D_{[b]}) = A_{[a]}^{[b]}\,, \tag{A.13}$$

where we denote by $\mathscr{L}^2_{[a]\times[b]}$ the operator acting on matrices of size $[a]\times[b]$ such that

$$\mathscr{L}^2_{[a]\times[b]}\left(B_{[a]}^{[b]}\right) := \left(\langle k,\omega(\rho)\rangle + \varepsilon\lambda_a - \lambda_b\right) B_{[a]}^{[b]}\,. \tag{A.14}$$

This dilation is invertible and (A.4) then gives, in operator norm,

$$\|\left(\mathscr{L}^2_{[a]\times[b]}\right)^{-1}\| \le \frac{1}{\kappa(1+|w_a-w_b|)}\,. \tag{A.15}$$

This allows to write (A.13) as

$$B_{[a]}^{[b]} - \left(\mathscr{L}^2_{[a]\times[b]}\right)^{-1}\mathscr{K}_{[a]\times[b]}\left(B_{[a]}^{[b]}\right) = \left(\mathscr{L}^2_{[a]\times[b]}\right)^{-1}\left(A_{[a]}^{[b]}\right)\,, \tag{A.16}$$

where $\mathscr{K}_{[a]\times[b]}\left(B_{[a]}^{[b]}\right) = \varepsilon(\tilde{D}_{[a]} - D_{[a]})B_{[a]}^{[b]} - B_{[a]}^{[b]}(\tilde{D}_{[b]} - D_{[b]})$. We have, thanks to (A.3), in operator norm,

$$\|\mathscr{K}_{[a]\times[b]}\| \le C_\mu\left(\frac{1}{w_a^\delta} + \frac{1}{w_b^\delta}\right) \le C_\mu\left(\frac{k_1}{k_2}\right)^\delta\,. \tag{A.17}$$

Then for

$$k_2 \ge k_1\left(\frac{2C_\mu}{\kappa}\right)^{1/\delta}, \tag{A.18}$$

the operator $\mathrm{Id} - (\mathscr{L}^2_{[a]\times[b]})^{-1}\mathscr{K}_{[a]\times[b]}$ is invertible and from (A.16) we get

$$\begin{aligned}\|B_{[a]}^{[b]}\| &= \|\left(\mathrm{Id} - (\mathscr{L}^2_{[a]\times[b]})^{-1}\mathscr{K}_{[a]\times[b]}\right)^{-1}\|\|\left(\mathscr{L}^2_{[a]\times[b]}\right)^{-1}\left(A_{[a]}^{[b]}\right)\| \\ &\le 2\|\left(\mathscr{L}^2_{[a]\times[b]}\right)^{-1}\left(A_{[a]}^{[b]}\right)\|\,,\end{aligned}$$

Hence in this case

$$|B|_{s,\beta+} \le \frac{2}{\kappa}|A|_{s,\beta} \tag{A.19}$$

Step 3 : suppose $A \in \mathcal{M}_{s,\beta} \cap \Upsilon^3_{s,\beta}(k_1,k_2)$. The only nonzero blocks $A_{[a]}^{[b]}$ correspond to weights w_a and w_b such that

$$\max(w_a,w_b) \le k_1 \min(w_a,w_b) \text{ and } \max(w_a,w_b) \le k_2\,,$$

hence there are only finitely many such blocks. In this case, for any $j \in [a]$ and $l \in [b]$ we have

$$|B_j^l| = \left|\frac{i}{\langle k,\omega(\rho)\rangle \;+ \varepsilon\mu_j - \mu_l}\right| |A_j^l| \le \frac{1}{\kappa(1+|w_a-w_b|)}|A_j^l| \tag{A.20}$$

A majoration of the coefficients gives a poor majoration of the operator norm of a matrix, but it is sufficient here since the number of nonzero blocks (and their size, see (2.1)) is finite :

$$\|B_{[a]}^{[b]}\| \le (\frac{(C_b \max(w_a, w_b))^{d^*/2}}{\kappa(1+|w_a - w_b|)}\|A_{[a]}^{[b]}\|\,, \tag{A.21}$$

hence $B \in \mathcal{M}^+_{s,\beta}$ and

$$|B|_{s,\beta+} \le \frac{(C_b k_2)^{d^*/2}}{\kappa}|A|_{s,\beta}\,. \tag{A.22}$$

Collecting (A.11), (A.19) and (A.22) and taking into account (A.7), (A.18) leads to the result. □

References

[1] V.I. Arnold, *Mathematical methods in classical mechanics; 3d edition.* Springer-Verlag, Berlin, 2006.

[2] D. Bambusi, *A Birkhoff normal form theorem for some semilinear PDEs*, Hamiltonian Dynamical Systems and Applications, Springer, 2007, pp. 213–247.

[3] D. Bambusi, J.-M. Delort, B. Grébert, and J. Szeftel, *Almost global existence for Hamiltonian semilinear Klein-Gordon equations with small Cauchy data on Zoll manifolds*, Comm. Pure Appl. Math. **60** (2007), no. 11, 1665–1690.

[4] M. Berti, P. Bolle, *Sobolev quasi periodic solutions of multidimensional wave equations with a multiplicative potential*, Nonlinearity **25** (2012), 2579-2613.

[5] M. Berti, P. Bolle, *Quasi-periodic solutions with Sobolev regularity of NLS on T^d with a multiplicative potential*, J. Eur. Math. Soc. **15** (2013), 229-286.

[6] M. Berti, L. Corsi, M. Procesi *An Abstract Nash-Moser Theorem and Quasi-Periodic Solutions for NLW and NLS on Compact Lie Groups and Homogeneous Manifolds. Comm. Math. Phys* (2014).

[7] M. Berti, M. Procesi *Nonlinear wave and Schrödinger equations on compact Lie groups and Homogeneous spaces. Duke Math. J.* **159**, 479-538 (2011).

[8] J. Bourgain *Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equation*, *Ann. Math.* **148** (1998), 363-439.

[9] J. Bourgain *Green's function estimates for lattice Schrödinger operators and applications*, *Annals of Mathematical Studies*, Princeton, 2004.

[10] M. Bouthelja *KAM pour l'équation des ondes nonlinéaire sur le cercle*, *preprint*, hal-01249664.

[11] J. M. Delort and J. Szeftel, *Long–time existence for small data nonlinear Klein–Gordon equations on tori and spheres*, Internat. Math. Res. Notices **37** (2004), 1897–1966.

[12] L.H. Eliasson, B. Grébert and S.B. Kuksin. KAM for the nonlinear beam equation 1: small amplitude solutions.*preprint*, arXiv:1412.2803.

[13] L.H. Eliasson, B. Grébert and S.B. Kuksin. KAM for the nonlinear beam equation 2 : a normal form theorem. *preprint*, arXiv:1502.02262.

[14] L.H. Eliasson and S.B. Kuksin. Infinite Töplitz-Lipschitz matrices and operators. *Z. Angew. Math. Phys.* **59** (2008), 24-50.

[15] L.H. Eliasson and S.B. Kuksin. KAM for the nonlinear Schrödinger equation. *Ann. Math* **172** (2010), 371-435.

[16] J. Geng and J. You. A KAM theorem for Hamiltonian partial differential equations in higher dimensional spaces. *Comm. Math. Phys.*, **262** (2006), 343–372.

[17] J. Geng and J. You. KAM tori for higher dimensional beam equations with constant potentials, *Nonlinearity*, **19** (2006), 2405–2423.

[18] B. Grébert, R. Imekraz and É. Paturel. Normal forms for semilinear quantum harmonic oscillators. *Commun. Math. Phys.* 291, 763–798 (2009).

[19] B. Grébert and É. Paturel. On reducibility of the quantum harmonic oscillator with quasiperiodic in time potential on $\mathbb{R}^d$.

[20] B. Grébert and L. Thomann, KAM for the Quantum Harmonic Oscillator, *Comm. Math. Phys.*, **307** (2011), 383–427.

[21] T. Kappeler and J. Pöschel. KAM & KdV. Springer-Verlag, Berlin, 2003.

[22] S. B. Kuksin. Hamiltonian perturbations of infinite-dimensional linear systems with an imaginary spectrum. *Funct. Anal. Appl.*, **21** (1987), 192–205.

[23] S. B. Kuksin. Nearly integrable infinite-dimensional Hamiltonian systems. *Lecture Notes in Mathematics, 1556.* Springer-Verlag, Berlin, 1993.

[24] S. B. Kuksin and J. Pöschel. Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation. *Ann. Math.* **143** (1996), 149–179.

[25] J. Pöschel. A KAM-theorem for some nonlinear partial differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 23 (1996), no. 1, 119–148.

Benoît Grébert, Laboratoire de Mathématiques Jean Leray, Université de Nantes, UMR CNRS 6629, 2, rue de la Houssinière, 44322 Nantes Cedex 03, France • *E-mail :* benoit.grebert@univ-nantes.fr

Eric Paturel, Laboratoire de Mathématiques Jean Leray, Université de Nantes, UMR CNRS 6629, 2, rue de la Houssinière, 44322 Nantes Cedex 03, France • *E-mail :* eric.paturel@univ-nantes.fr