# UNITARY DILATION OF FREELY INDEPENDENT CONTRACTIONS

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Abstract. Inspired by the Sz.-Nagy-Foias dilation theorem we show that $n$ freely independent contractions dilate to $n$ freely independent unitaries. A new free product of unital completely positive maps is defined to obtain this result.

## 1. Introduction

The Sz.-Nagy-Foias dilation theorem is a celebrated result in classical dilation theory. It says that $n$ doubly commuting contractions can be simultaneously dilated to $n$ doubly commuting unitaries. This was the original multivariable dilation theory context suggested by Sz.-Nagy [17] until Andô [1] proved that one can do this for just commuting and not doubly commuting contractions when $n = 2$. However, it was subsequently shown in [13] and [18] that there are three commuting contractions which do not dilate to three commuting unitaries. This obstruction spurred on dilation theories in other contexts [2, 6, 8, 10, 15] and many other generalizations.

Doubly commuting is one of two ingredients in the notion of tensor independence (or classical independence). It is natural then to ask whether $n$ tensor independent contractions can be dilated to $n$ tensor independent unitaries. Then answer is yes (Theorem 2.2) and begs the question whether this can be done with other notions of non-commutative probability, namely free probability.

Stemming from the notion of reduced free product [3, 19] Voiculescu developed the theory of free probability in the 1980's with the goal of solving the free group factor problem. While this still remains unsolved, free probability has become a very important field of mathematical research. For further reading see [11, 12].

The paper culminates in Theorem 4.3, that $n$ freely independent contractions do indeed dilate to $n$ freely independent unitaries. This result relies upon the construction of the universal and reduced free products and the construction of a new free product of unital completely positive maps which differs from [5].

**Acknowledgements.** We would like to thank David Sherman and Stuart White for some very helpful discussions during White's visit to the University of Virginia sponsored by the Institute of Mathematical Science.

2010 *Mathematics Subject Classification.* 47A20, 46L54, 46L09.

*Key words and phrases:* Dilation, non-commutative probability, tensor independence, free independence, free product.

## 2. Dilation theory of tensor independence

We first turn to the classical setting for inspiration. There are many great proofs of the Sz.-Nagy-Foias dilation theorem and the following constructive method is undoubtedly known but the authors have never seen it written down. Recall that two operators $S, T \in B(\mathcal{H})$ doubly commute if $ST = TS$ and $S^*T = TS^*$. This is equivalent to requiring that $C^*(1, S)$ and $C^*(1, T)$ commute. Note that this does not require that $S$ and $T$ are normal.

**Theorem 2.1** (Sz.-Nagy-Foias)**.** *Let $T_1, \cdots, T_n$ be doubly commuting contractions in $B(\mathcal{H})$. Then there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and doubly commuting unitaries $U_1, \ldots, U_n \in B(\mathcal{K})$ such that*

$$T_1(k_1)\cdots T_n(k_n) = P_{\mathcal{H}} U_1^{k_1} \cdots U_n^{k_n}|_{\mathcal{H}}, \quad \textit{where} \;\; T(k) = \begin{cases} T^k, & k \geq 0 \\ T^{*-k}, & k < 0 \end{cases}.$$

*Furthermore, this dilation is unique up to unitary equivalence when $\mathcal{K}$ is minimal, meaning that it is the smallest reducing subspace of $U_1, \cdots, U_n$ containing $\mathcal{H}$.*

*Proof.* Let $\mathcal{H}_1 = \ell_2(\mathbb{Z}) \otimes \mathcal{H}$ and set

$$T_1^{(1)} = \begin{bmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & & \\ & I & 0 & & & & \\ & & D_{T_1^*} & T_1 & & & \\ & & -T_1^* & D_{T_1} & 0 & & \\ & & & & I & 0 & \\ & & & & & \ddots & \ddots \end{bmatrix} \quad \text{and} \;\; T_j^{(1)} = I_{\ell_2(\mathbb{Z})} \otimes T_j, \;\; 2 \leq j \leq n.$$

Note that $T_1^{(1)}$ is the classic Schäffer form of the Sz.-Nagy dilation of $T_1$ and so is a unitary [16]. Since $T_1, \cdots, T_n$ doubly commute then so do $T_1^{(1)}, \cdots, T_n^{(1)}$, this is immediate after noticing that the defect operators $D_{T_1} = (I - T_1^*T_1)^{1/2}$ and $D_{T_1^*}$ are in $C^*(1, T_1)$.

In the second step, let $\mathcal{H}_2 = \ell_2(\mathbb{Z}) \otimes \mathcal{H}_1$, $T_2^{(2)}$ be the Schäffer-Sz.-Nagy dilation of $T_2^{(1)}$ and $T_j^{(2)} = I_{\ell_2(\mathbb{Z})} \otimes T_j^{(1)}$ for $j \neq 2$. Then $T_1^{(2)}$ and $T_2^{(2)}$ are unitaries and $T_1^{(2)}, \cdots, T_n^{(2)}$ are doubly commuting.

Continuing in this way one arrives at the nth step with doubly commuting unitaries $T_1^{(n)}, \cdots, T_n^{(n)} \in B(\mathcal{H}_n)$ that are easily seen to satisfy the joint power dilation condition.

Uniqueness when the dilation is minimal follows in the same way as in the one variable setting. It is proven by way of the uniqueness of the minimal Stinespring representation. □

This can be rephrased into a non-commutative probability context. Recall that a non-commutative C*-probability space $(\mathcal{A}, \varphi)$ is a C*-algebra $\mathcal{A}$ along with a state $\varphi \in S(\mathcal{A})$. We say that the operators $T_1, \cdots, T_n \in \mathcal{A}$ are tensor (or classically) independent in $(\mathcal{A}, \varphi)$ (or with respect to $\varphi$) if $C^*(1, T_1), \cdots, C^*(1, T_n)$ pairwise commute and given $a_i \in C^*(1, T_i)$ we have the following factorization

$$\varphi\left(\prod_{i=1}^n a_i\right) = \prod_{i=1}^n \varphi(a_i).$$

**Theorem 2.2.** *Let $T_1, \cdots, T_n \in B(\mathcal{H})$ be tensor independent contractions in the non-commutative probability space $(B(\mathcal{H}), \varphi)$. Then there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and unitaries $U_1, \cdots, U_n \in B(\mathcal{K})$ that are tensor independent with respect to the state $\psi = \varphi \circ \mathrm{ad}_{P_\mathcal{H}}$ such that*

$$T_1(k_1)\cdots T_n(k_n) = P_\mathcal{H} U_1^{k_1} \cdots U_n^{k_n}|_\mathcal{H}, \quad \text{where} \quad T(k) = \begin{cases} T^k, & k \geq 0 \\ T^{*-k}, & k < 0 \end{cases}.$$

*Furthermore, this dilation is unique up to unitary equivalence when $\mathcal{K}$ is minimal, meaning that it is the smallest reducing subspace of $U_1, \cdots, U_n$.*

*Proof.* All that needs to be shown is that the unitaries arising from Theorem 2.1 are tensor independent with respect to $\psi$. To this end first assume that $\mathcal{K} = \mathcal{H}_n$ and $U_j = T_j^{(n)}$ as in the proof of Theorem 2.1.

Let $a_i \in C^*(1, T_i^{(1)}), 1 \leq i \leq n$. This implies that $P_\mathcal{H} a_i|_\mathcal{H} \in C^*(1, T_i)$ for $1 \leq i \leq n$ and $P_\mathcal{H}$ and $a_i$ commute for $2 \leq i \leq n$ since $\mathcal{H}$ is a reducing subspace for both $C^*(1, T_i^{(1)})$. Hence,

$$\begin{aligned} \varphi\left(P_\mathcal{H} \prod_{i=1}^n a_i P_\mathcal{H}\right) &= \varphi\left(\prod_{i=1}^n (P_\mathcal{H} a_i P_\mathcal{H})\right) \\ &= \prod_{i=1}^n \varphi(P_\mathcal{H} a_i P_\mathcal{H}). \end{aligned}$$

Thus, $T_1^{(1)}, \cdots, T_n^{(1)}$ are tensor independent with respect to $\varphi \circ \mathrm{ad}_{P_\mathcal{H}}$. Continuing in this fashion one gets that $T_1^{(n)}, \cdots, T_n^{(n)}$ are tensor independent with respect to $\varphi \circ \mathrm{ad}_{P_\mathcal{H}} \circ \mathrm{ad}_{P_{\mathcal{H}_1}} \circ \cdots \circ \mathrm{ad}_{P_{\mathcal{H}_{n-1}}} = \varphi \circ \mathrm{ad}_{P_\mathcal{H}} = \psi$ where the last copy of $\mathcal{H}$ is in $\mathcal{H}_n$.

Uniqueness of this dilation is given by Theorem 2.1. ∎

## 3. A free product of ucp maps

Recall that the universal free product of unital C*-algebras $\mathcal{A}_1, \cdots, \mathcal{A}_n$ is the universal C*-algebra amalgamated over $\mathbb{C}$ generated by $\mathcal{A}_1, \cdots, \mathcal{A}_n$ and is denoted $\check{*}_{i=1}^n \mathcal{A}_i$. In particular, whenever one has $*$-homomorphisms $\pi_i : \mathcal{A}_i \to \mathcal{B}$ then there exists a $*$-homomorphism $\pi : \check{*}_{i=1}^n \mathcal{A}_i \to \mathcal{B}$ such that $\pi|_{\mathcal{A}_i} = \pi_i$.

Suppose there are unital completely positive maps $\theta_i : \mathcal{A}_i \to \mathcal{B}$. In [5] Boca proves that there exists a ucp map $\theta = \theta_1 * \cdots * \theta_n : \check{*}_{i=1}^n \mathcal{A}_i \to \mathcal{B}$ such that $\theta|_{\mathcal{A}_i} = \theta_i$. This is defined on *reduced words* with respect to the expectation onto $\mathbb{C}1$, called $E$. Namely, when $a_j \in \mathcal{A}_{i_j}$ with $E(a_j) = 0$ and $i_j \neq i_{j-1}$ then

$$\theta(a_1 \cdots a_m) = \theta_{i_1}(a_1) \cdots \theta_{i_m}(a_m).$$

This completely determines $\theta$ as (reduced words + $\mathbb{C}1$) is dense in $\check{*}_{i=1}^n \mathcal{A}_i$.

In the following theorem we will show that one can define a free product of ucp maps depending on any state and not just $E$. This new ucp map is really the universal free product version of the reduced free product of ucp maps given by Choda [7]. The proof of the main theorem of this paper hinges upon this point.

**Theorem 3.1.** *For $1 \leq i \leq n$, let $\theta_i : \mathcal{A}_i \to B(\mathcal{H})$ be a unital completely positive map, and let $\psi$ be a state on $\check{*}_{i=1}^n \mathcal{A}_i$. There exists a ucp map $\theta_\psi : \check{*}_{i=1}^n \mathcal{A}_i \to B(\mathcal{H})$ such that whenever $a_j \in \mathcal{A}_{i_j}, \psi(a_j) = 0, 1 \leq j \leq m$ with $i_j \neq i_{j-1}$ then*

$$\theta_\psi(a_1 \cdots a_m) = \theta_{i_1}(a_1) \cdots \theta_{i_m}(a_m).$$

*This gives that $\theta_\psi|_{\mathcal{A}_i} = \theta_i$.*

*Proof.* One need only make a few minor modifications and notes to Boca's proof [5, Proposition 3.2] to obtain this result. The main difference mentioned previously is that $\theta_1 * \cdots * \theta_n = \theta_E$ where $E$ is the expectation onto the identity of the free product.

Let $W_\psi = \{a_1 \cdots a_m : a_j \in \mathcal{A}_{i_j}, \psi(a_j) = 0, i_j \neq i_{j-1}\}$ be the set of reduced words with respect to $\psi$ and $W_E$ be the reduced words with respect to $E$. An important property of these reduced words given by the definition of the free product is that if $a_1 \cdots a_m, b_1 \cdots b_k \in W_E$ with $a_1 \cdots a_m = b_1 \cdots b_k$ then $k = m$ and $a_i = \lambda_i b_i$ for $\lambda_i \in \mathbb{C} \setminus \{0\}, 1 \leq i \leq m$ such that $\lambda_1 \cdots \lambda_m = 1$. Moreover, if $w_1, \cdots, w_m, v_1, \cdots, v_k \in W_E$ with $\sum_{i=1}^m w_i = \sum_{i=1}^k v_i$ then one has that for each $N \geq 1$ one has

$$\sum_{1 \leq i \leq m, |w_i| = N} w_i = \sum_{1 \leq j \leq k, |v_j| = N} v_j$$

where $|w|$ is the length of the reduced word, meaning $|w| = N$ if and only if $w = a_1 \cdots a_N \in W_E$.

Suppose now one has reduced words with respect to $\psi$, $a_1 \cdots a_m = b_1 \cdots b_k \in W_\psi$. Hence, one can write

$$\begin{aligned} a_1 \cdots a_m = &(a_1 - E(a_1)) \cdots (a_m - E(a_m)) \\ &+ \sum_{j=1}^m (a_1 - E(a_1)) \cdots (a_{j-1} - E(a_{j-1})) \\ &\qquad E(a_j)(a_{j+1} - E(a_{j+1})) \cdots (a_m - E(a_m)) \\ &+ \cdots \end{aligned}$$

as the sum of reduced words in $W_E$ and $\{1\}$ by the usual centering method. But $a_1 \cdots a_m$ has only one reduced word summand of length $m$ and so $a_1 \cdots a_m = b_1 \cdots b_k$ implies that

$$(a_1 - E(a_1)) \cdots (a_m - (E(a_m)) = (b_1 - E(b_1)) \cdots (b_k - E(b_k)).$$

So $m = k$ and $a_i - E(a_i) = \lambda_i(b_i - E(b_i))$ which implies $a_i = \lambda b_i + E(a_i - \lambda b_i)$ and applying $\psi$ to both sides gives that $E(a_i - \lambda b_i) = 0$, since $\psi(a_i) = \psi(b_i) = 0$. Thus, every reduced word in $W_\psi$ has a unique form.

Consider the operator system

$$\mathcal{S} = \mathrm{Alg}\{\mathcal{A}_1, \cdots, \mathcal{A}_n\} = \mathbb{C}1 \cup \mathrm{span}\{w : w \in W_\psi\} = \mathbb{C}1 \cup \mathrm{span}\{w : w \in W_E\}$$

which is dense in $\check{*}_{i=1}^n \mathcal{A}_i$. It is straightforward to see that $\theta_\psi$ is well defined, unital and self-adjoint after assuming that $\theta_\psi(1) = 1$ and $\theta_\psi(a_1 \cdots a_m) = \theta_{i_1}(a_1) \cdots \theta_{i_m}(a_m)$ for all $a_1 \cdots a_m \in W_\psi$. One needs to show that $\theta_\psi$ is completely positive on $\mathcal{S}$ and then the result follows by Arveson's extension theorem.

By careful checking one can use the exact same proof of complete positivity that Boca gives. □

This shows that the free product situation is different than that of the tensor product, that is, there will not be a unique ucp map from the free product extending ucp maps in each component.

To end this section we show that this new free product of ucp maps works well with dilation theory.

**Lemma 3.2.** *Suppose $T_1, \cdots, T_n \in B(\mathcal{H})$ are contractions, $U_1, \cdots, U_n \in B(\mathcal{K})$ are unitaries and $\theta_i : C(U_i) \to C^*(1, T_i)$ are ucp maps such that $p(U_i) \mapsto p(T_i)$. If $\psi$ is a state on $\check{*}_{i=1}^n C^*(U_i)$ then the ucp map $\theta_\psi$ is a homomorphism on the subalgebra $\overline{\mathrm{Alg}}\{1, U_1, \cdots, U_n\}$ of $\check{*}_{i=1}^n C^*(U_i)$.*

*Proof.* The result can be established by induction. By definition each $\theta_i$ is already a homomorphism on $\overline{\mathrm{Alg}}\{1, U_i\} \subseteq C^*(U_i)$. Now for $m \geq 1$ assume that for all $1 \leq k \leq m$ and for any $b_j \in \overline{\mathrm{Alg}}\{1, U_{i_j}\}, 1 \leq j \leq k$ we have that $\theta_\psi\left(\prod_{j=1}^k b_j\right) = \prod_{j=1}^k \theta_{i_j}(b_j)$.

Suppose now we have $a_j \in \overline{\mathrm{Alg}}\{1, U_{i_j}\}, 1 \leq j \leq m+1$. If a pair of neighbouring terms belongs to the same algebra, say $i_j = i_{j-1}$, by the inductive hypothesis and since $\theta_{i_j}$ is a homomorphism we have

$$\begin{aligned}\theta_\psi(a_1 \cdots a_{j-1} a_j \cdots a_{m+1}) &= \theta_{i_1}(a_1) \cdots \theta_{i_j}(a_{j-1}a_j) \cdots \theta_{i_{m+1}}(a_{m+1}) \\ &= \theta_{i_1}(a_1) \cdots \theta_{i_{j-1}}(a_{j-1}) \theta_{i_j}(a_j) \cdots \theta_{i_{m+1}}(a_{m+1}).\end{aligned}$$

Otherwise assume that $i_{j-1} \neq i_j$ for $1 < j \leq m+1$ and then we have

$$\begin{aligned}\theta_\psi\left(\prod_{j=1}^{m+1} a_j\right) = \theta_\psi&\left(\prod_{j=1}^{m+1}(a_j - \psi(a_j))\right) + \\ &\theta_\psi\left(\psi(a_1)\prod_{j=2}^{m+1}(a_j - \psi(a_j))\right) + \\ &\theta_\psi\left(a_1\psi(a_2)\prod_{j=3}^{m+1}(a_j - \psi(a_j))\right) + \cdots + \\ &\theta_\psi(a_1 \cdots a_m \psi(a_{m+1})) \\ = &\prod_{j=1}^{m+1}\theta_{i_j}(a_j - \psi(a_j)) + \\ &\psi(a_1)\prod_{j=2}^{m+1}\theta_{i_j}(a_j - \psi(a_j)) + \\ &\theta_{i_1}(a_1)\psi(a_2)\prod_{j=3}^{m+1}\theta_{i_j}(a_j - \psi(a_j)) + \cdots + \\ &\theta_{i_1}(a_1) \cdots \theta_{i_m}(a_m)\psi(a_{m+1}) \\ = &\prod_{j=1}^{m+1}\theta_{i_j}(a_j)\end{aligned}$$

by the definition of $\theta_\psi$ and the inductive hypothesis.

The result follows as it is a simple matter now to show that $\theta_\psi(ab) = \theta_\psi(a)\theta_\psi(b)$ for all $a, b \in \overline{\mathrm{Alg}}\{1, U_1, \cdots, U_n\}$. □

## 4. Dilation theory of free independence

In this section we will prove a theorem very similar to Theorem 2.2 in another non-commutative probability context. Recall that the operators $T_1, \cdots, T_n \in \mathcal{A}$ are freely independent (or $*$-free) in $(\mathcal{A}, \varphi)$ if their C*-algebras $C^*(1, T_1), \cdots, C^*(1, T_n)$ are freely independent, that is whenever $a_j \in C^*(1, T_{i_j})$ such that $\varphi(a_j) = 0$ for $1 \le i_j \le n$ and $i_j \ne i_{j-1}$ for $1 < j \le m$ then

$$\varphi(a_1 a_2 \cdots a_m) = 0.$$

Another proof of the Sz.-Nagy-Foias Theorem (Theorem 2.1 above) can be found in Paulsen [14, Theorem 12.10]. Here one gets ucp maps $\theta_i : C(\mathbb{T}) \to C^*(1, T_i)$ given by dilation theory. Now one can extend this to the ucp map $\theta_1 \otimes \cdots \otimes \theta_n$ on $C(\mathbb{T}) \otimes_{\max} \cdots \otimes_{\max} C(\mathbb{T}) \simeq C(\mathbb{T}^n)$. By taking the Stinespring representation of $\theta$ one gets the desired doubly commuting unitaries that jointly dilate $T_1, \cdots, T_n$.

This provides a roadmap for an attempt to prove the free analogue of Theorem 2.2. Namely, by using the free product of ucp maps given in Theorem 3.1 and then taking the Stinespring representation of this map it will be shown that one gets unitaries that jointly dilate $T_1, \cdots, T_n$. One then hopes that these unitaries will be $*$-free with respect to a natural state.

The main theorem will follow after some technical lemmas.

**Lemma 4.1.** *Suppose $T \in B(\mathcal{H})$ and $\tilde{T} \in B(\tilde{\mathcal{H}})$ such that there is a $*$-homomorphism $\pi : C^*(1, T) \to C^*(1, \tilde{T})$ with $\pi(T) = \tilde{T}$. If $U \in B(\mathcal{K})$ and $\tilde{U} \in B(\tilde{\mathcal{K}})$ are the minimal unitary dilations of $T$ and $\tilde{T}$ respectively then there exists a $*$-homomorphism $\tilde{\pi} : C^*(U) \to C^*(\tilde{U})$ with $\tilde{\pi}(U) = \tilde{U}$ such that the following diagram commutes*

$$\begin{array}{ccc} C^*(U) & \xrightarrow{\tilde{\pi}} & C^*(\tilde{U}) \\ {\scriptstyle \mathrm{ad}\, P_{\mathcal{H}}} \big\downarrow & & {\scriptstyle \mathrm{ad}\, P_{\tilde{\mathcal{H}}}} \big\downarrow \\ C^*(1, T) & \xrightarrow{\pi} & C^*(1, \tilde{T}) \end{array}$$

*Proof.* Let $(\gamma, \mathcal{K}')$ be the minimal Stinespring representation of the unital completely positive map $\pi \circ \mathrm{ad}\, P_{\mathcal{H}} : C^*(U) \to C^*(1, \tilde{T})$. Because the $*$-homomorphism is unital we can assume that $\tilde{\mathcal{H}} \subset \mathcal{K}'$ and so $\mathrm{ad}\, P_{\tilde{\mathcal{H}}} \circ \gamma(a) = \pi \circ \mathrm{ad}\, P_{\mathcal{H}}(a)$, for all $a \in C^*(U)$. Now, for the unitary $V := \gamma(U)$

$$P_{\tilde{\mathcal{H}}} V^n|_{\tilde{\mathcal{H}}} = \pi(P_{\mathcal{H}} U^n|_{\mathcal{H}}) = \pi(T(n)) = \tilde{T}(n), \quad n \in \mathbb{Z},$$

where $T(k) = \begin{cases} T^k, & k \ge 0 \\ T^{*-k}, & k < 0 \end{cases}$, and by the minimality of the Stinespring representation

$$\overline{\mathrm{span}}\{V^n P_{\tilde{\mathcal{H}}} : n \in \mathbb{Z}\} = \mathcal{K}'$$

Thus, $V$ is a minimal unitary dilation of $\tilde{T}$ and so there exists a unitary $W : \mathcal{K}' \to \tilde{\mathcal{K}}$ such that $Wh = h, \forall h \in \tilde{\mathcal{H}}$ and $WVW^* = \tilde{U}$.

Therefore, define $\tilde{\pi} : C^*(U) \to C^*(\tilde{U})$ by $\tilde{\pi} = \mathrm{ad}\, W \circ \gamma$, which is a $*$-homomorphism such that

$$\tilde{\pi}(U) = \mathrm{ad}\, W \circ \gamma(U) = WVW^* = \tilde{U}$$

and

$$\operatorname{ad} P_{\tilde{\mathcal{H}}} \circ \tilde{\pi}(U^n) = \tilde{T}(n) = \pi(T(n)) = \pi \circ \operatorname{ad} P_{\mathcal{H}}(U^n).$$

This establishes the result. □

**Lemma 4.2.** *Let $T \in B(\mathcal{H})$ be a contraction and $\varphi$ be a faithful state on $C^*(1,T)$. If $U \in B(\mathcal{K})$ with $\mathcal{H} \subset \mathcal{K}$ is the minimal unitary dilation of $T$ then $\varphi \circ \operatorname{ad} P_{\mathcal{H}}$ is a faithful state on $C^*(U)$.*

*Proof.* Let $(\pi, \mathcal{K}', \xi)$ be the GNS representation of $(C^*(U), \varphi \circ \operatorname{ad} P_{\mathcal{H}})$. Then $\pi(U)$ is still a unitary, $\varphi \circ \operatorname{ad} P_{\mathcal{H}}(a) = \langle \pi(a)\xi, \xi\rangle$ and $\langle \cdot \xi, \xi \rangle$ is a faithful state on $\pi(C^*(U)) = C^*(\pi(U))$.

Suppose $a$ is a positive element in $\ker \pi$. Then $0 = \langle \pi(a)\xi, \xi\rangle = \varphi \circ \operatorname{ad} P_{\mathcal{H}}(a)$ which implies that the positive element $P_{\mathcal{H}}(a) = 0$ since compression to $\mathcal{H}$ is a completely positive map and $\varphi$ is faithful. The same argument holds in reverse, so $\ker \pi = \ker \operatorname{ad} P_{\mathcal{H}}$. This induces a ucp map $\theta : C^*(\pi(U)) \to C^*(1,T)$ by sending $\pi(a) \mapsto P_{\mathcal{H}} a|_{\mathcal{H}}$. Notably this gives $\theta(\pi(U)^n) = T^n, n \geq 0$.

Let $(\tilde{\pi}, \mathcal{K}'')$ with $\mathcal{H} \subset \mathcal{K}''$ be the minimal Stinespring representation of $\theta$. That is, $\tilde{\pi} : C^*(\pi(U)) \to B(\mathcal{K}'')$ is a $*$-homomorphism (in fact a $*$-isomorphism) such that $\theta(a) = P_{\mathcal{H}}\tilde{\pi}(a)|_{\mathcal{H}}$ and $\mathcal{K}''$ is the closed linear span of $\tilde{\pi}(C^*(\pi(U)))\mathcal{H}$ by minimality. Define $V := \tilde{\pi}(\pi(U))$ a unitary and note that $P_{\mathcal{H}} V^n|_{\mathcal{H}} = \theta(\pi(U^n)) = T^n$. Thus, because of this and the minimality of the Stinespring representation we have that $V$ is a minimal unitary dilation of $T$.

Consider now the state $\psi := \varphi \circ \operatorname{ad} P_{\mathcal{H}}$ on $C^*(V)$. Now

$$\begin{aligned}
\psi \circ \tilde{\pi}\left(\sum_{i=-n}^{n} \alpha_i \pi(U)^n\right) &= \psi\left(\sum_{i=-n}^{n} \alpha_i V^n\right) \\
&= \varphi\left(\sum_{i=-n}^{n} \alpha_i T(n)\right) \\
&= \varphi \circ \operatorname{ad} P_{\mathcal{H}}\left(\sum_{i=-n}^{n} \alpha_i U^n\right) \\
&= \left\langle \left(\sum_{i=-n}^{n} \alpha_i \pi(U)^n\right)\xi, \xi\right\rangle.
\end{aligned}$$

Hence, $\psi \circ \tilde{\pi}(\cdot) = \langle \cdot \xi, \xi\rangle$ is a faithful state on $C^*(\pi(U))$ and so $\psi$ is a faithful state on $C^*(V)$.

By minimality of the dilations there exists a unitary $W : \mathcal{K} \to \mathcal{K}''$ such that $Wh = h$ for all $h \in \mathcal{H}$ and $WUW^* = V$. This implies that $\varphi \circ \operatorname{ad} P_{\mathcal{H}}$ on $C^*(U)$ is equal to $\psi \circ \operatorname{ad} W$ which is faithful. Therefore, $\varphi \circ \operatorname{ad} P_{\mathcal{H}}$ was a faithful state on $C^*(U)$ all along. □

One last ingredient before the main theorem is the reduced free product of C*-probability spaces $(\mathcal{A}_i, \varphi_i)$, denoted $(\mathcal{A}, \varphi)$ with $\mathcal{A} = *_{i=1}^n \mathcal{A}_i$. As mentioned in the introduction this was introduced by both Avitzour [3] and Voiculescu [19] in the 1980's.

**Theorem 4.3.** *Let $T_1, \cdots, T_n \in B(\mathcal{H})$ be freely independent contractions in the non-commutative probability space $(B(\mathcal{H}), \varphi)$. Then there exists a Hilbert space $\mathcal{K}$*

*containing $\mathcal{H}$ and unitaries $U_1, \cdots, U_n \in B(\mathcal{K})$ that are freely independent with respect to $\varphi \circ \operatorname{ad} P_{\mathcal{H}}$ such that*

$$T_{i_1}^{k_1} \cdots T_{i_m}^{k_n} = P_{\mathcal{H}} U_{i_1}^{k_1} \cdots U_{i_m}^{k_n}|_{\mathcal{H}}, \quad 1 \le i_j \le n \text{ and } k_j \in \mathbb{N} \cup \{0\}.$$

*Proof.* For each $1 \le i \le n$, let $V_i \in B(\mathcal{K}_i)$ with $\mathcal{H} \subset \mathcal{K}_i$ be the minimal unitary dilation of $T_i$ and $\theta_i : C^*(V_i) \to C^*(1, T_i)$ given by $\theta_i = \operatorname{ad} P_{\mathcal{H}}$ is a unital completely positive map.

Let $(\pi, \tilde{\mathcal{H}}, \xi)$ be the GNS representation of $\varphi$ on $C^*(1, T_1, \cdots, T_n)$. Define $\tilde{T}_i := \pi(T_i), 1 \le i \le n$ and $\tilde{\varphi} := \langle \cdot \xi, \xi \rangle$. Hence, $\tilde{T}_1, \cdots, \tilde{T}_n$ are freely independent contractions with respect to the faithful state $\tilde{\varphi}$ on $C^*(1, \tilde{T}_1, \cdots, \tilde{T}_n)$.

Consider the reduced free product of non-commutative probability spaces

$$(*_{i=1}^n C^*(1, \tilde{T}_i), \tilde{\varphi}) = *_{i=1}^n (C^*(1, \tilde{T}_i), \tilde{\varphi})$$

which contains $*$-isomorphic copies of each $C^*(1, \tilde{T}_i)$ by the faithfulness of $\tilde{\varphi}$. The faithfulness of these states is a very strong condition and thus by [9, Lemma 1.3] there exists a $*$-isomorphism $*_{i=1}^n C^*(1, \tilde{T}_i) \to C^*(1, \tilde{T}_1, \cdots, \tilde{T}_n)$ such that $\tilde{T}_i \mapsto \tilde{T}_i, 1 \le i \le n$ and the states $\tilde{\varphi}$ are equal. By abuse of notation, assume that

$$\pi : C^*(1, T_1, \cdots, T_n) \to *_{i=1}^n C^*(1, \tilde{T}_i)$$

such that $\varphi = \tilde{\varphi} \circ \pi$.

Next, for $1 \le i \le n$, let $(\tilde{V}_i, \tilde{\mathcal{K}}_i)$ be the minimal unitary dilation of $\tilde{T}_i$. This implies that there are ucp maps $\tilde{\theta}_i : C^*(\tilde{V}_i) \to C^*(1, \tilde{T}_i)$ defined by $\tilde{\theta}_i(\tilde{V}_i^n) = \tilde{T}_i(n), n \in \mathbb{Z}$, namely $\tilde{\theta}_i = \operatorname{ad} P_{\tilde{\mathcal{H}}}$. Define the states $\tilde{\psi}_i = \tilde{\varphi} \circ \tilde{\theta}_i$ on $C^*(\tilde{V}_i)$ which are faithful by Lemma 4.2. Now consider the reduced free product

$$(*_{i=1}^n C^*(\tilde{V}_i), \tilde{\psi}) = *_{i=1}^n (C^*(\tilde{V}_i), \tilde{\psi}_i)$$

and note that $\tilde{\psi}$ is faithful since each of the $\tilde{\psi}_i$ are faithful, this also implies that there are $*$-isomorphic copies of each $C^*(\tilde{V}_i)$ in the reduced free product. Hence, by the construction of the reduced free product $\tilde{V}_1, \cdots, \tilde{V}_n$ are $*$-free with respect to $\tilde{\psi}$. As in the universal free product case, Choda [7] showed that there is a reduced free product of ucp maps (and Blanchard-Dykema [4] that there is an amalgamated reduced free product of ucp maps). Thus, there exists a ucp map

$$\tilde{\theta} : *_{i=1}^n C^*(\tilde{V}_i) \to *_{i=1}^n C^*(1, \tilde{T}_i)$$

extending the $\tilde{\theta}_i$ in each component and we get $\tilde{\psi} = \tilde{\varphi} \circ \tilde{\theta}$. Moreover, by [7] this map $\tilde{\theta}$ splits over what she calls *reduced words*. That is, if $b_j \in C^*(\tilde{V}_{i_j})$ with $\psi_j(b_j) = 0, 1 \le j \le m$ and $i_j \ne i_{j-1}, 1 < j \le m$ then

$$\tilde{\theta}\left(\prod_{j=1}^m b_j\right) = \prod_{j=1}^m \tilde{\theta}_{i_j}(b_j). \tag{1}$$

By exactly the same proof as Lemma 3.2 $\tilde{\theta}$ is a homomorphism on $\overline{\operatorname{Alg}}\{1, \tilde{V}_1, \cdots, \tilde{V}_n\}$.

In each component of the universal and reduced free products we have by Lemma 4.1 that there exist $*$-homomorphisms $\tilde{\pi}_i : C^*(V_i) \to C^*(\tilde{V}_i)$ with $V_i \mapsto \tilde{V}_i$ and such that the following diagram commutes

$$\begin{array}{ccc} C^*(V_i) & \xrightarrow{\tilde{\pi}_i} & C^*(\tilde{V}_i) \\ {\scriptstyle \theta_i}\downarrow & & {\scriptstyle \tilde{\theta}_i}\downarrow \\ C^*(1, T_i) & \xrightarrow{\pi} & C^*(1, \tilde{T}_i) \end{array} \tag{2}$$

By universality there exists a $*$-homomorphism

$$\tilde{\pi} := *_{i=1}^n \tilde{\pi}_i : \check{*}_{i=1}^n C^*(V_i) \to *_{i=1}^n C^*(\tilde{V}_i).$$

Let $\psi = \tilde{\psi} \circ \tilde{\pi}$ be a state on $\check{*}_{i=1}^n C^*(V_i)$ and note that $V_1, \cdots, V_n$ are $*$-free with respect to $\psi$. By Theorem 3.1 the map

$$\theta_\psi : \check{*}_{i=1}^n C^*(V_i) \to C^*(1, T_1, \cdots, T_n)$$

exists and is ucp, and by Lemma 3.2 is a homomorphism on $\overline{\mathrm{Alg}}\{1, V_1, \cdots, V_n\}$. For $1 \leq j \leq m$ with $a_j \in C^*(V_{i_j}), \psi(a_j) = 0$ and $i_j \neq i_{j-1}$ we have

$$\begin{aligned} \pi \circ \theta_\psi(a_1 \cdots a_m) &= \pi(\theta_{i_1}(a_1) \cdots \theta_{i_m}(a_m)) \\ &= \pi \circ \theta_{i_1}(a_1) \cdots \pi \circ \theta_{i_m}(a_m) \\ &= \tilde{\theta}_{i_1} \circ \tilde{\pi}(a_1) \cdots \tilde{\theta}_{i_m} \circ \tilde{\pi}(a_m) && \text{by (2)} \\ &= \tilde{\theta}(\tilde{\pi}(a_1) \cdots \tilde{\pi}(a_m)) && \text{by (1) } \tilde{\psi}(\pi(a_i)) = \psi(a_i) = 0 \\ &= \tilde{\theta} \circ \tilde{\pi}(a_1 \cdots a_m). \end{aligned}$$

Thus, the following diagram commutes

$$\begin{array}{ccc} \check{*}_{i=1}^n C^*(V_i) & \xrightarrow{\tilde{\pi}} & *_{i=1}^n C^*(\tilde{V}_i) \\ {\scriptstyle \theta_\psi}\downarrow & & {\scriptstyle \tilde{\theta}}\downarrow \\ C^*(1, T_1, \cdots, T_n) & \xrightarrow{\pi} & C^*(1, \tilde{T}_1, \cdots, \tilde{T}_n) \end{array}$$

Lastly, let $(\gamma, \mathcal{K})$ be the Stinespring representation of $\theta_\psi$ with $\mathcal{H} \subset \mathcal{K}$ which gives $\theta_\psi(a) = P_\mathcal{H} \gamma(a)|_\mathcal{H}$. Define unitaries $U_i := \gamma(V_i)$ and note that for $m \geq 1$ and $k_i \in \geq 0$ then

$$\begin{aligned} P_\mathcal{H} U_{i_1}^{k_1} \cdots U_{i_m}^{k_m}|_\mathcal{H} &= P_\mathcal{H} \gamma(V_{i_1}^{k_1} \cdots V_{i_m}^{k_m})|_\mathcal{H} \\ &= \theta_\psi(V_{i_1}^{k_1} \cdots V_{i_m}^{k_m}) \\ &= T_{i_1}^{k_1} \cdots T_{i_m}^{k_m} && \text{by Lemma 3.2.} \end{aligned}$$

Furthermore,

$$\begin{aligned} \varphi \circ \mathrm{ad}\, P_\mathcal{H} \circ \gamma &= \varphi \circ \theta_\psi \\ &= \tilde{\varphi} \circ \pi \circ \theta_\psi \\ &= \tilde{\varphi} \circ \tilde{\theta} \circ \tilde{\pi} \\ &= \tilde{\psi} \circ \tilde{\pi} \\ &= \psi. \end{aligned}$$

Therefore, because $V_1, \cdots, V_n$ are $*$-free with respect to $\psi$ then $U_1, \cdots, U_n$ are $*$-free with respect to $\varphi \circ \mathrm{ad}\, P_\mathcal{H}$. $\square$

The first thing to note in the setting of the previous theorem is that a state $\psi$ is completely determined by the values $\psi(U_i^k) = \varphi(T_i^k)$ since $U_1, \cdots, U_n$ are $*$-free unitaries and by [12, Lemma 5.13]. Hence, $\psi = \varphi \circ \mathrm{ad}\, P_{\mathcal{H}}$ on $C^*(U_1, \cdots, U_n)$.

This leads one to ask whether this free unitary dilation is unique when it is minimal, meaning that $\mathcal{K}$ is the smallest reducing subspace of $U_1, \cdots, U_n$ containing $\mathcal{H}$. One would need that $\theta_\psi$ from Theorem 3.1 is the unique ucp map extending the $\theta_i$ such that $\psi = \varphi \circ \theta_\psi$. The answer is unknown to the authors even in the case of the reduced free product of ucp maps.

Finally, by [20, Proposition 2.5.3] $\varphi \circ \mathrm{ad}\, P_{\mathcal{H}}$ is a tracial state on $C^*(U_1, \cdots, U_n)$ since $U_1, \cdots, U_n$ are $*$-free and $\varphi \circ \mathrm{ad}\, P_{\mathcal{H}}$ is a trace on each $C^*(U_i)$ (the algebra is commutative).

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