

# Bounding $f(R)$ gravity by particle production rate

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(Dated: January 11, 2016)

Several models of  $f(R)$  gravity have been proposed in order to address the dark side problem in cosmology. However, these models should be constrained also at ultraviolet scales in order to achieve some correct fundamental interpretation. Here we analyze this possibility comparing quantum vacuum states in given  $f(R)$  cosmological backgrounds. Specifically, we compare the Bogolubov transformations associated to different vacuum states for some  $f(R)$  models. The procedure consists in fixing the  $f(R)$  free parameters by requiring that the Bogolubov coefficients can be correspondingly minimized to be in agreement with both high redshift observations and quantum field theory predictions. In such a way, the particle production is related to the value of the Hubble parameter and then to the given  $f(R)$  model. The approach is developed in both metric and Palatini formalism.

PACS numbers: 04.50.Kd; 98.80.Qc; 95.36.+x; 98.80.-k

Keywords: Alternative Theories of Gravity; Cosmology; Quantum Field Theory; Dark Energy.

## I. INTRODUCTION

Extensions of General Relativity can contribute to achieve a comprehensive cosmological picture by giving a geometrical interpretation of the dark side [1]. Although General Relativity is accurately bounded at the Solar System scales [2, 3], several cosmological indications seem to point out the need of extending the Hilbert-Einstein action in order to achieve a comprehensive and self-consistent description of the universe expansion history, i.e. at far infrared scales [4–6]. Rephrasing it differently, we wonder whether General Relativity is effectively the final paradigm to address consistently the *universe dynamical problem* [7–9]. In particular, the issue related to the existence of a dark energy, which drives the late-time dynamics, needs the introduction of additional fluids capable of dominating over the standard pressureless matter [10]. Moreover, the issue of dark matter is needed to address the clustering structures and asks for some additional particles resulting hard to find out at fundamental level by direct and indirect detection [11–15].

Among the possible proposals,  $f(R)$  gravity seems quite promising as a straightforward extension of General Relativity since the strict request of linearity in the Ricci scalar  $R$  of the Hilbert-Einstein action is relaxed. The paradigm consists in the fact that observations and phenomenology, in principle, could fix the action of gravitational interaction whose action is assumed as a generic function of the curvature invariants. In this perspective, the simplest generalization is [16]

$$\mathcal{A} = \frac{1}{2} \int d^4x \sqrt{-g} [f(R) + \mathcal{L}_m], \quad (1)$$

where  $g$  is the determinant of the metric  $g_{\mu\nu}$  and  $\mathcal{L}_m$  is the standard perfect fluid matter Lagrangian. We adopt the conventions  $8\pi G = c = 1$ . The variation of (1) with respect to  $g_{\mu\nu}$  gives the field equations [7, 17–21]:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu \nabla_\nu - g_{\mu\nu} \square] f'(R) = T_{\mu\nu}, \quad (2)$$

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where  $T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$  is the energy momentum tensor of matter and the prime indicates derivative with respect to  $R$ . The adopted signature is  $(+, -, -, -)$ . The dynamical system is completed by considering also the contracted Bianchi identities

$$\nabla^\mu T_{\mu\nu} = 0. \quad (3)$$

Enlarging the geometric sector can be useful in view of the dark side since the further gravitational degrees of freedom have, in principle, a role in addressing dark energy and dark matter issues [7, 22].

Varying the action with respect to the metric is not the only choice. It is also possible to vary with respect to the affine connection, considering it independent from the metric itself [23, 25]. This is the so called *Palatini approach* that produces different field equations. The advantage of the latter approach is that field equations remain of second order [24–26]. According to the Palatini formalism, an important remark is in order. The Ricci scalar is  $R \equiv R(g, \Gamma) = g^{\alpha\beta} R_{\alpha\beta}(\Gamma)$  being a *generalized Ricci scalar* and  $R_{\mu\nu}(\Gamma)$  is the Ricci tensor of a torsion-less connection  $\Gamma$ , which, *a priori*, has no relations with the metric  $g$  of spacetime. Field equations, derived from the Palatini variational principle are:

$$f'(R)R_{(\mu\nu)}(\Gamma) - \frac{1}{2}f(R)g_{\mu\nu} = T_{\mu\nu} \quad (4)$$

$$\nabla_\alpha^\Gamma(\sqrt{-g}f'(R)g^{\mu\nu}) = 0 \quad (5)$$

where  $\nabla^\Gamma$  is the covariant derivative with respect to  $\Gamma$ . We denote  $R_{(\mu\nu)}$  as the symmetric part of  $R_{\mu\nu}$ , i.e.  $R_{(\mu\nu)} \equiv \frac{1}{2}(R_{\mu\nu} + R_{\nu\mu})$ .

In order to get (5), one has to additionally assume that the above  $\mathcal{L}_m$  is functionally independent of  $\Gamma$ ; however it may contain metric covariant derivatives  $\overset{g}{\nabla}$  of the fields. This means that the matter stress-energy tensor  $T_{\mu\nu} = T_{\mu\nu}(g, \Psi)$  depends on the metric  $g$  and matter fields  $\Psi$ , and their derivatives with respect to the Levi-Civita connection of  $g$ . It is natural to define a new metric  $h_{\mu\nu}$ , such that

$$\sqrt{-g}f'(R)g^{\mu\nu} = \sqrt{-h}h^{\mu\nu}. \quad (6)$$

This choice impose  $\Gamma$  to be the Levi-Civita connection of  $h$  and the only restriction is that  $\sqrt{-g}f'(R)g^{\mu\nu}$  is non-degenerate. In the case of Hilbert-Einstein Lagrangian, it is  $f'(R) = 1$  and the statement is trivial.

In both metric and Palatini approaches, the function  $f(R)$  is not fixed *a priori*. Thus, its determination, according to data and phenomenology, represents the main challenge for the  $f(R)$  theory [27].

From a different point of view, reliable classes of  $f(R)$  models should be constrained at fundamental level [28]. Specifically, bounds on  $f(R)$  models could be derived by taking into account different vacuum states via Bogolubov transformations [29–31]. In fact, in quantum field theories, the Bogolubov coefficients drive the different choices of vacuum states. So, requiring that different classes of  $f(R)$  functions change vacuum states according to Bogolubov transformations is a basic requirement to guarantee the  $f(R)$  viability at fundamental level. This procedure somehow fixes the  $f(R)$  free parameters and so it is of some help in reconstructing the  $f(R)$  form by means of basic requirements of quantum field theory [32, 33]. To this end, one has to confront with the problem of quantizing the space-time in a curved background and then to provide relations between quantization and  $f(R)$  gravity at least at semiclassical level. Hence, the Bogolubov coefficients allow to pass from a vacuum state to another through a semiclassical procedure where the rate of particle production is minimized. If the rate is minimized, one can fix, indeed, the free parameters of a given  $f(R)$  model. We assume that the rate is minimized to be consistent with cosmological high redshift observations, from one side, and with quantum field theory predictions, from the other side.

Furthermore, one can relate the rate of particle production with the Hubble parameter and thus with the redshift  $z$ . In so doing, it is possible to frame the Bogolubov coefficients in terms of observable cosmological quantities as  $H_0$ , the today observed Hubble constant, or  $R_0 \sim \rho_0$ , the value of the today curvature or density.

The paper is organized as follows. In Sec. II, we sketch the derivation of the Bogolubov coefficients as semiclassical quantities in the context of quantum field theory. In Sec. III the rate of particle production is investigated assuming a constant (de Sitter) curvature  $R_0$  in the framework of non-minimally coupled theories of gravity. Such theories are the prototype of  $f(R)$  models and then a generalization is straightforward. In Sec. IV, we minimize the Bogolubov coefficients to get constraints for  $f(R)$  free parameters in both metric and Palatini formalism. Applications to some cosmological models are discussed. Outlooks and conclusions are reported in Sec. V.

## II. A SEMICLASSICAL APPROACH FOR PARTICLE PRODUCTION RATE

A strategy to derive the particle production rate in curved space is to fix a background with a constant curvature i.e.  $R = R_0$ . This situation is usually named as the *de-Sitter phase* [34, 35]. From the above field Eqs. (2), it is easy to derive an effective cosmological constant term  $\Lambda_{eff} = \frac{f(R_0)}{2f'(R_0)} = \frac{R_0}{4}$ , which, in principle, depending on the value of  $R_0$ , can give rise to an accelerating expansion phase [36]. The choice of  $R_0$  allows to simplify the calculations thanks to the symmetries of de Sitter spacetime. In order to constrain the form of  $f(R)$  function, a method is to fix the range of free parameters by the transition to different vacuum states. Such a procedure relies on the definition of the Bogolubov coefficients. To define them, let us consider the quantization on a curved background.

Since we are considering modified theories of gravity, a model where a scalar field  $\phi$  is non-minimally coupled to geometry, i.e.  $\propto R\phi$  can be assumed. The related Klein-Gordon equation is

$$[\square - m^2 + \xi R(x)] \phi = 0, \quad (7)$$

where  $m$  is the effective mass of the field,  $\xi$  is the coupling<sup>1</sup>.

The general solution can be expressed as a complete set of mode-solutions for the field  $\phi$  [37]:

$$\phi(x) = \sum_i [a_i u_i(x) + a_i^\dagger u_i^*(x)], \quad (8)$$

where it is possible to adopt a specific set of mode solutions  $u_i(x)$ , albeit it is always possible to rewrite  $\phi(x)$  for a different set  $\bar{u}_j$  as

$$\phi(x) = \sum_j [\bar{a}_j \bar{u}_j(x) + \bar{a}_j^\dagger \bar{u}_j^*(x)]. \quad (9)$$

In other words, one can pass through different decompositions of  $\phi$ , defining a corresponding form of the vacuum solution that is, in general,  $\bar{a}_j|0\rangle \neq 0$ , in a curved space background. In fact, expressing the new modes,  $\bar{u}_j$  in terms of the old ones  $u_i$ , one can write

$$\begin{aligned} \bar{u}_j &= \sum_i (\alpha_{ji} u_i + \beta_{ji} u_i^*), \\ u_i &= \sum_j (\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*), \end{aligned} \quad (10)$$

where  $\alpha_{ji}$  and  $\beta_{ji}$  are defined as  $\alpha_{ij} = (\bar{u}_i, u_j)$ ,  $\beta_{ij} = -(\bar{u}_i, u_j^*)$  and satisfy the relations

$$\begin{aligned} \sum_k (\alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^*) &= \delta_{ij}, \\ \sum_k (\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk}) &= 0. \end{aligned} \quad (11)$$

In particular, if we consider the vacuum  $|0\rangle$  then  $a_i|0\rangle = 0, \forall i$ , but, in general, it is  $a_i|\bar{0}\rangle \neq 0$ . Thence, it results from the definition of the particle number given by  $N_i \equiv a_i^\dagger a_i$ , that is

$$\langle \bar{0} | N_i | \bar{0} \rangle = \sum_j |\beta_{ji}|^2. \quad (12)$$

It follows that the physical meaning of such coefficients is associated to the rate of particle production. In fact, generic coefficients  $\beta_{ji}$  are associated to the particle number count for given set of modes  $i$ . Specifically,  $\alpha_{ji}$  and  $\beta_{ji}$  are referred to as the *Bogolubov coefficients* which identify the *Bogolubov transformations* and allow to pass from a vacuum state to another one.

Since the form of the  $f(R)$  function is not known *a priori*, by adopting the above semiclassical procedure and evaluating the different vacuum states for some classes of  $f(R)$ , we can minimize the rate of particle production. In so doing, we can constrain the free parameters of a given  $f(R)$  model. In particular, we will see that Bogolubov coefficients strictly depend on the form of  $f(R)$ .

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<sup>1</sup> It is worth noticing that any  $f(R)$  theory of gravity can be recast as a non-minimally coupled theory through the identification  $\phi \rightarrow f'(R)$  and the coupling  $f'(R)^{-1}$ .

### III. PARTICLE PRODUCTION IN NON-MINIMALLY COUPLED THEORIES OF GRAVITY

The generic prototype of alternative theories of gravity are the non-minimally coupled scalar-tensor theories. As we said before,  $f(R)$  theories and any extended theory can be recast as General Relativity with some non-minimal coupling and a further contribution in the stress-energy momentum tensor (see [39] for the general procedure). In this section, we discuss the Bogolubov transformations in the context of homogeneous and isotropic cosmologies, in order to define the rate of particle production and then to constrain the functional form of  $f(R)$  gravity.

The particle production rate is a mostly universal feature, in the sense that it has not to depend on the particular gravitational background. Indeed, assuming a different gravitational theory<sup>2</sup> we expect that it is the same and can be consistently used to fix the parameters of the theory itself. In other words, this property is extremely relevant since it allows to consider the Bogolubov transformations for different gravitational backgrounds. For the purposes of this work, we limit to the case of  $f(R)$  gravity.

As we said, the simplest choice to construct the Bogolubov transformations is assuming a de Sitter phase with a constant curvature  $R_0$ . A spatially flat Friedman-Robertson-Walker (FRW) conformal metric is [38]

$$ds^2 = \frac{1}{H^2\eta^2} (d\eta^2 - dx_1^2 - dx_2^2 - dx_3^2), \quad (13)$$

where we adopted the conformal time  $\eta = -\frac{1}{Ha(t)}$ , which varies in the interval  $-\infty < \eta < 0$ . Introducing a scalar field  $\phi(\eta, \mathbf{x})$  depending on  $\eta$  and  $\mathbf{x} \equiv (x_1, x_2, x_3)$ , the corresponding Klein-Gordon equation Eq. (7), in terms of space-time modes, can be rewritten as

$$(\square - m^2 + \xi R)\phi(\eta, \mathbf{x}) = 0. \quad (14)$$

This equation is formally equivalent to Eq. (7), albeit the functional dependence on the variables  $\eta$  and  $\mathbf{x}$  is explicit. Without choosing  $\xi$  *a priori*, the corresponding class of solutions is

$$\phi(\eta, \mathbf{x}) = \phi_k(\eta)e^{i(\mathbf{k}\cdot\mathbf{x})}, \quad (15)$$

where the wave vector is decomposed as  $\mathbf{k} \equiv (k_1, k_2, k_2)$ . By scaling  $\phi(\eta, \mathbf{x}) = \frac{\tilde{\phi}(\eta, \mathbf{x})}{a}$ , the Klein-Gordon differential equation for the FRW metric (13) is

$$\tilde{\phi}_k''(\eta) + \omega^2(\eta, k, \xi)\tilde{\phi}_k(\eta) = 0, \quad (16)$$

where the prime stands for the derivative with respect to the conformal time  $\eta$ .

The above equation is analogue to the harmonic oscillator with  $\omega$  depending on the conformal time  $\eta$ . The  $\omega$  parameter takes the form:

$$\omega(\eta) = \sqrt{k^2 + a^2 [m^2 + 2f(\xi)H^2]}, \quad (17)$$

where  $f(\xi) \equiv 6\xi - 1$ . For our purposes, the function  $f(\xi)$  can be conventionally positive-definite assuming  $\xi > \frac{1}{6}$ . Moreover, it is convenient to define an effective mass  $M_{eff}$  as

$$\frac{M_{eff}^2}{H^2} \equiv \frac{m^2}{H^2} + 2f(\xi), \quad (18)$$

where  $m$  is the state of mass of the scalar field and  $H \equiv \frac{\dot{a}}{a}$  is the Hubble parameter. Since  $\xi > \frac{1}{6}$ ,  $M_{eff}$  is always positive. The frequency dependence is

$$\omega(\eta) = \sqrt{k^2 + \frac{M_{eff}^2}{H^2\eta^2}}. \quad (19)$$

which is always positive for  $\xi \geq \frac{1}{6}$ . The functions  $k(\eta)$  and  $\omega(\eta)$  are sketched in Fig. (1) for some cases of interest.

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<sup>2</sup> For example,  $f(R, G)$ , General Relativity,  $f(T)$ , and so forth.

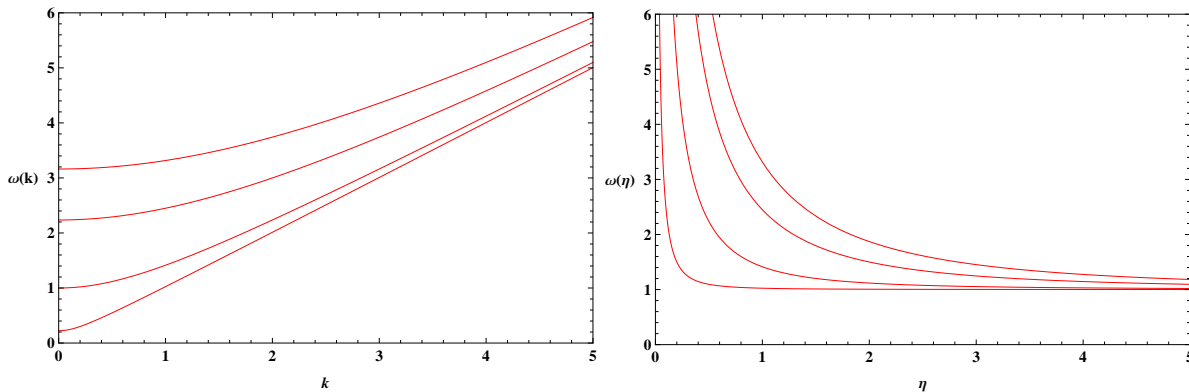


FIG. 1: Plots of the  $\omega$  profiles in function of  $k$  and  $\eta$  respectively on the left and right. The left plots show the  $\omega(k)$  behaviors for four indicative ratios  $\frac{M_{eff}^2}{H^2\eta^2} = 0.05, 1, 5, 10$ . The figures on the right, on contrary, show the  $\omega(\eta)$  behaviors in function of  $\eta$ , with indicative ratios  $\frac{M_{eff}^2}{H^2} = 0.05, 1, 5, 10$  and conventionally with  $k = 1$ .

The general solution  $\tilde{\phi}_k(\eta)$  reads

$$\tilde{\phi}_k(\eta) = \sqrt{\eta} \left( A_k H_{k,\nu}^{(1)}(\eta) + B_k H_{k,\nu}^{(2)}(\eta) \right), \quad (20)$$

where  $H_{k,\nu}^{(1)}$  and  $H_{k,\nu}^{(2)}$  are Hankel's functions of first and second type respectively, with the position [40, 41]

$$\nu \equiv \sqrt{\frac{1}{4} - \frac{M_{eff}^2}{H^2}}. \quad (21)$$

The corresponding asymptotic behavior is relevant to infer the particle production rate. In the case  $\eta \rightarrow 0^-$ , we obtain

$$\tilde{\phi}_k(\eta) \rightarrow \frac{\sqrt{|\eta|}}{\pi\nu} \left\{ \sin(\pi\nu) \Gamma(1-\nu) \left( \frac{k\eta}{2} \right)^\nu - \Gamma(1+\nu) \left( \frac{k\eta}{2} \right)^{-\nu} \right\}, \quad (22)$$

and the square modulus of  $\beta_k$  is [42, 43]

$$|\beta_k(\eta)|^2 = \frac{\omega_k}{2} \left| \phi_k(\eta) - \frac{i}{\omega_k(\eta)} \dot{\phi}_k(\eta) \right|^2. \quad (23)$$

We are interested in the case  $\frac{M_{eff}}{H} \gg 1$ , which corresponds either to the situation where the effective mass dominates over the Hubble rate or  $H$  is small at late times of the universe evolution. Thus, we find out

$$|\beta_k(\eta)|^2 \sim \frac{H^3}{32\pi m^3} \left| \Gamma \left( 1 - i \frac{m}{H} \right) \right|^2 \exp(\pi m), \quad (24)$$

where  $\Gamma$  is the Euler function.

In the case  $\frac{M_{eff}}{H} \ll 1$ , we obtain that the Bogolubov coefficients are negligibly small [44], that is

$$|\beta_k(\eta)|^2 \ll 1. \quad (25)$$

Hence, by assuming that  $\frac{M_{eff}}{H} \ll 1$ , we can consider two different cases. The first is  $H \gg m$ , with  $m \rightarrow 0$ . The second is un-physical, since it provides a diverging Bogolubov coefficient  $\beta_k$  [45]. Thus, by assuming the validity of the above results, we are able to relate the  $f(R)$  gravity to Bogolubov coefficients constraining the free parameters of the models. To this goal, we assume to pass through different vacuum states. Clearly, different  $f(R)$  gravity models means different couplings  $\xi$ .

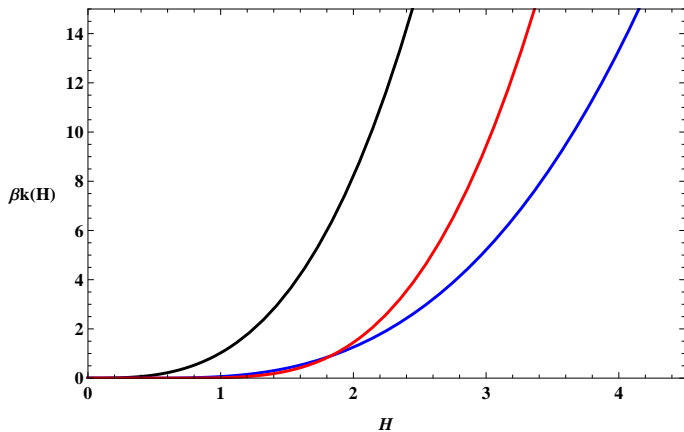


FIG. 2: Plot of the Bogolubov coefficient  $|\beta_k|^2$  varying in the range  $H = 0 \dots 5$ , with  $m = 0.01; 1; 2$  respectively for the black, blue and red lines.

#### IV. MINIMIZING THE RATE OF PARTICLE PRODUCTION IN $f(R)$ GRAVITY

Let us focus on the physical case  $\frac{m}{H} \ll 1$  in the context of  $f(R)$  gravity. We investigate such a case in the de Sitter phase with  $R = R_0$  and minimize the Bogolubov coefficients obtaining, correspondingly, the minimum of particle production rate.

Such a quantity has to be minimized essentially for two reasons. The first concerns cosmological observations at high energy regimes. For example, taking into account the cosmic microwave background, cosmological measurements could not be compatible with particle production rate, so the condition  $\frac{m}{H} \ll 1$  is required to guarantee that any theory of gravity works at high  $z$ . In addition, assuming to pass from different vacuum states, it is important to test if cosmological models describe the vacuum according to observations. In this perspective, minimizing Bogolubov coefficients is a powerful tool to discriminate among different models (see also [52]).

In particular, our purpose is to infer physical bounds on the free parameters of some classes of  $f(R)$  models by Bogolubov transformations. The first step is to write  $H$  from the cosmological equations derived in the  $f(R)$  framework. In a FRW universe we obtain in metric and Palatini formalism respectively:

$$H^2 = \frac{1}{3} \left[ \rho_{curv} + \frac{\rho_m}{f'(R)} \right], \quad (26a)$$

$$H^2 = \frac{1}{6f'(R)} \left[ \frac{2\rho + Rf'(R) - f(R)}{G(R)} \right]. \quad (26b)$$

where  $\rho_m$  is the standard matter density. The effective curvature density term is [49]

$$\rho_{curv} = \frac{1}{2} \left[ \frac{f(R)}{f'(R)} - R \right] - 3H\dot{R} \left[ \frac{f''(R)}{f'(R)} \right], \quad (27)$$

and the function  $G(R)$  is given by

$$G(R) = \left[ 1 - \frac{3f''(R)(Rf'(R) - 2f(R))}{2f'(R)(Rf''(R) - f'(R))} \right]^2. \quad (28)$$

As said before, clearly  $f(R)$  gravity can be recast in term of a scalar-tensor theory as soon as the identifications  $\phi \rightarrow f'(R)$ , for the field, and  $G_{eff} \rightarrow f'(R)^{-1}$ , for the coupling, are adopted (see [20] for an extended discussion on this point).

Besides, the particle production rate can be achieved at first order by a Taylor expansion of the Bogolubov coefficient  $\beta_k$ . It is

$$|\beta_k|^2 = e^{\pi m} \left[ \frac{H^3}{32\pi m^3} + \gamma^2 \frac{H}{32\pi m} \right], \quad (29)$$

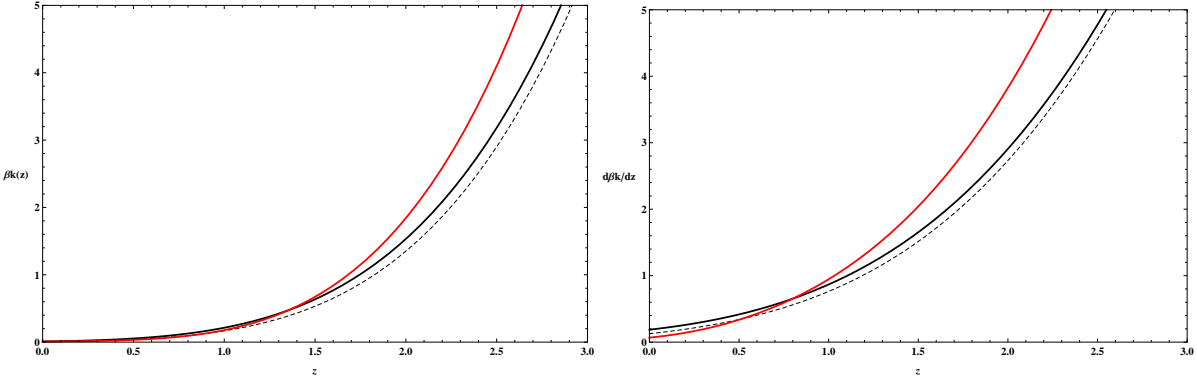


FIG. 3: Bogolubov coefficients (left figure) evaluated for the  $\Lambda$ CDM model (black line), the cosmographic expansion (dashed line) and the Chaplygin gas (red line), with matter density  $\rho = 0.27$ , and the Chaplygin gas coefficients:  $A = 0.9$ ,  $\beta = 0.8$ , (see [46–48]) with the normalized Hubble rate  $H_0 = 0.68$ . Derivatives of  $\beta_k(z)$  (right figure) have been reported for the same cases, i.e.  $\Lambda$ CDM, cosmographic and Chaplygin gas.

where we adopted the  $\Gamma(1 - ix)$  function and its Taylor series in case  $x \ll 1$ , obtaining  $\Gamma \sim 1 + i\gamma x$ . The constant  $\gamma$  is the Euler constant and reads  $\gamma \sim 0.577$ .

As a first step, one can compare such Bogolubov coefficients with the Hubble rate expressed as function of the redshift  $z$ . In such a way, the form of  $\beta_k$  becomes a function of the redshift as well. This has been reported in the left plot of Fig. III, whereas in the right plot we draw the variation of  $\beta_k$  as the redshift increases, i.e. its first derivative with respect to the redshift  $z$ . The reported three models are: (i) the  $\Lambda$ CDM model [50]; (ii) a cosmographic expansion where the deceleration parameter variation, namely the jerk parameter, is  $j(z) \geq 1$  [51]; (iii) the Chaplygin gas where dark energy and dark matter are considered under the standard of a single fluid [46–48]. These models can be considered as the most relevant paradigms for describing dark energy [53].

In the case of constant curvature  $R = R_0$  related to a de-Sitter phase, we obtain

$$|\beta_k|^2 = e^{\pi m} \left[ \frac{1}{3^{\frac{3}{2}} 32\pi m^3} \left( \frac{\rho_0}{f_0'} - \Lambda_{eff} \right)^{3/2} + \frac{\gamma^2}{32\sqrt{3}\pi m} \left( \frac{\rho_0}{f_0'} - \Lambda_{eff} \right)^{1/2} \right], \quad (30)$$

and

$$|\beta_k|^2 = e^{\pi m} \left[ \frac{1}{32\pi m^3} \left( \frac{(2\rho_0 + R_0 - 2\Lambda_{eff})(R_0 - \frac{f_0'}{f_0})}{R_0 - \frac{f_0'}{f_0} - \frac{3}{2}R_0 + 6\Lambda_{eff}} \right)^{3/2} + \frac{\gamma^2}{32\pi m} \left( \frac{(2\rho_0 + R_0 - 2\Lambda_{eff})(R_0 - \frac{f_0'}{f_0})}{R_0 - \frac{f_0'}{f_0} - \frac{3}{2}R_0 + 6\Lambda_{eff}} \right)^{1/2} \right] \quad (31)$$

respectively for metric and Palatini formalism. Hereafter  $f_0 \equiv f(R = R_0)$ ,  $f_0' \equiv f'(R = R_0)$ , and  $f_0'' \equiv f''(R = R_0)$  and  $\rho_0$  is the value of standard matter-energy density for  $R_0$ . Since the form of  $f(R)$  is not known *a priori*, we need to consider cases of particular interest [54, 55] as

$$f_1(R) = R^{1+\delta} + \Lambda, \quad (32a)$$

$$f_2(R) = R + \epsilon R^2 \dots, \quad (32b)$$

$$f_3(R) = R + R^n + \sigma R^{-m}, \quad (32c)$$

$$f_4(R) = R - \frac{\alpha(R)^n}{1 + \beta(R)^n}. \quad (32d)$$

We therefore need to fix via Eq. The coefficients  $\delta, \epsilon, \sigma, n, m, \alpha$  and  $\beta$  can be fixed by Eq.(29). To do so, we require that the rate of particle production is negligible or essentially as small as possible [56]. Hence, the strategy to follow is to assume that the free parameters of Eqs. (32) minimize the Bogolubov coefficients. Following this procedure, we obtain the results of Tabs. I and II.

One can calibrate the constraints over the free parameters in Tabs. I and II using also late-time and CMBR cosmological constraints [51, 52]. In general, any consistent choice of  $f(R)$  gravity leads to

$$f'(R) \leq \frac{4\rho_0}{R_0}, \quad (33)$$

$f_n(R)$	$num.param.$	$minimiz.$
$f_{1M}(R)$	1	$1 + \delta \leq \frac{4\rho_0}{R_0^{1+\delta}}$
$f_{2M}(R)$	1	$\epsilon \leq \frac{2\rho_0}{R_0^2} - \frac{1}{2R_0}$
$f_{3M}(R)$	3	$nR_0^n \left  1 - \frac{m}{n} \sigma R_0^{-(m+n)} \right  \leq 4\rho_0 - R_0$
$f_{4M}(R)$	3	$(R_0 - nR_0^n \alpha + 2R_0^{1+n} \beta + R_0^{1+2n} \beta^2) (1 + R_0^n \beta^2)^{-1} \leq 4\rho_0 - R_0$

TABLE I: Table of minimizing conditions for the free parameters of  $f(R)$  models from Eq. (32) in the metric formalism. Here the subscript  $M$  stands for *metric*. The above equalities correspond to the case of vanishing Bogolubov coefficients, whereas the inequalities to more general cases where the Bogolubov coefficients are not zero.

$f_n(R)$	$num.param.$	$minimiz.$
$f_{1P}(R)$	1	$\forall \delta < 0, \forall \delta \geq 1, \delta \neq -1$
$f_{2P}(R)$	1	$\epsilon < 0$
$f_{3P}(R)$	3	$\left(1 + nR_0^{n-1} - m\sigma R_0^{-(m+1)}\right) \left(n(n-1)R_0^{n-2} + m(m+1)\sigma R_0^{-(m+2)}\right)^{-1} \leq R_0$
$f_{4P}(R)$	3	$R_0^{1-n} (1 + R_0^n \beta) (R_0 + R_0^{1+2n} \beta^2 + R_0^n (2\beta R_0 - \alpha n)) (n\alpha(1-n + (1+n)R_0^n \beta))^{-1} \leq R_0$

TABLE II: Table of minimizing conditions for the free parameters of  $f(R)$  models from Eq. (32) in the Palatini formalism. Here the subscript  $P$  stands for *Palatini*. Inequalities and equalities follow the same considerations of Tab. I. Here, we assumed that  $\rho_0 + \frac{R_0}{4} > 0$ .

in the metric formalism (where the equality requires vanishing Bogolubov coefficients) and

$$R_0 \geq -4\rho_0, \quad (34a)$$

$$f_0' \neq R_0 f_0'', \quad (34b)$$

in the Palatini formalism. Again, the equalities lead to vanishing Bogolubov coefficients. In addition, Eq. (34a) represents a natural constraint on  $R_0$ . These conditions have to be satisfied, if one wants to pass through different vacuum states, without a significant particle production rate. In principle, once evaluated the above constraints, it would be also possible to numerically constrain the derivatives of  $f(R)$  models. For example, to guarantee that the gravitational constant does not significantly depart from the Solar System limits, one needs that  $4\rho_0 \sim R_0$ . Thus, observations on  $\rho_0$  open the possibility to constrain  $R_0$  and may be compared to cosmological constraints over  $R_0$  itself [57, 58]. Analogously, in the Palatini case,  $R_0$  is somehow comparable to  $-4\rho_0$ . Hence, a correct determination of the limits over  $R_0$  could also discriminate between the metric and Palatini approaches.

## V. OUTLOOKS AND PERSPECTIVES

We investigated the role played by the particle production rate in the context of  $f(R)$  gravity. To this aim, we considered both the metric and Palatini approaches reproducing particle production in both cases. Specifically, we derived the Bogolubov coefficients, which permit to pass from a vacuum state to another. These coefficients can be related to the Hubble parameter which strictly depends on the functional form of a given  $f(R)$  model. In this sense, the particle production rate depends on the specific form of  $f(R)$  gravity. Hence, this is a method to constrain the free parameters of a given model, invoking a semiclassical scheme. Indeed, since the function  $f(R)$  is not defined *a priori*, it is necessary to determine some theoretical conditions on  $f(R)$  parameters at some fundamental level. Thus, we assumed to minimize the Bogolubov coefficients, i.e. the particle production rate, allowing us to pass through different vacuum states, once postulated the background. In particular, we considered a the de-Sitter phase  $R = R_0$  and derived the Bogolubov transformations for some class of  $f(R)$  models taking advantage from the fact that such theories can be easily recast as scalar-tensor models. The Bogolubov coefficients have been evaluated for a homogeneous and isotropic universe, postulating that the particle production rate is negligibly small. This provided conditions on the form of  $f(R)$  which have been reported in Tabs. I and II. As result, constraints can be derived on free parameters of different classes of  $f(R)$  functions. Such constraints can be combined with Solar System constraints under suitable conditions. In particular, we demonstrated that cosmological measurements of  $R_0$  would discriminate between metric or Palatini approaches. A straightforward generalization of the method would be to consider the Bogolubov transformations for



space-times with variable curvature. Also in those cases one may check how to minimize the rate of particle production in order to pass from a vacuum to another one. Furthermore, it would be possible to evaluate Bogolubov coefficients in other modified gravity theories to seek for constraints on free parameters [59]. These topics will be the argument of future developments.

### Acknowledgements

SC and MP acknowledge financial support of INFN *iniziative specifiche* QGSKY and TEONGRAV. SC thanks the TSPU for being awarded as Honorary Professor.

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