Homogenisation of thin periodic frameworks with high-contrast inclusions

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Abstract

We analyse a problem of two-dimensional linearised elasticity for a two-component periodic composite, where one of the components consists of disjoint soft inclusions embedded in a rigid framework. We consider the case when the contrast between the elastic properties of the framework and the inclusions, as well as the ratio between the period of the composite and the framework thickness increase as the period of the composite becomes smaller. We show that in this regime the elastic displacement converges to the solution of a special two-scale homogenised problem, where the microscopic displacement of the framework is coupled both to the slowly-varying "macroscopic" part of the solution and to the displacement of the inclusions. We prove the convergence of the spectra of the corresponding elasticity operators to the spectrum of the homogenised operator with a band-gap structure.

Introduction

The multi-scale extension of the notion of the weak L^2-limit was proposed in [10], [2], where a general theorem about two-scale compactness of L^2-bounded sequences was proved and a corrector-type result for the uniformly elliptic periodic homogenisation problem was established. Multi-scale convergence has proved to be an effective tool in the study of composite media with a complicated geometry of the periodic reference cell. Further, in problems where solutions do not converge in the strong L^2-sense, for example in the presence of degeneracies, see *e.g.* [12], the related techniques have the additional benefit of capturing the multi-scale structure of the limit, by providing a suitable generalised notion of strong convergence. As opposed to the uniformly elliptic case, where the limit function only depends on the macroscopic variable and is a solution to a single boundary-value problem, the multi-scale limit for degenerate homogenisation problems satisfies a coupled system of equations for the macroscopic and microscopic parts of the limit solution. This happens to be the case for periodic "thin structures", which are the subject of the present work.

We define a *thin structure* as an arrangement of rods of thickness $h > 0$ joined together at a number of junction points ("nodes"). Fig. 1 shows an example of a this structure, where the two panels show rods of thickness h (left) and the "singular" structure obtained by taking the mid-lines of the rods (right). In the literature, equations of elasticity on thin structures are studied either for a fixed rod thickness h or by treating it as a parameter linked to the typical rod length. In the context of homogenisation, the rods are often assumed to be arranged periodically with period ε, and the limit behaviour of the structure is studied as $\varepsilon \to 0$. The use of two-scale convergence for the study of periodic thin structures has been proposed in [13, 14, 4], where the two-scale approach of [10, 2] was extended to the setting of general Borel measures, and conditions on the measure sufficient for passing to the two-scale limit were determined. In addition, it was shown in [13] that the spectrum of the "double-porosity model", where the components of the composite have contrasting properties, is close to a band spectrum whose complement consists of an infinite set of disjoint intervals ("gaps"). This property is also possessed by the homogenised operator that we derive in the present work.

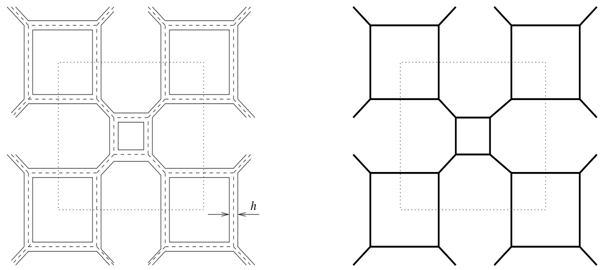

Figure 1: Example of a periodic network and unit cell.

If the thickness of the rods $h = h(\varepsilon)$ is a function of the period ε of the network, such that $\lim_{\varepsilon\to 0} h(\varepsilon) = 0$, then the limit behaviour depends on the asymptotics of the ratio h/ε^2 as $\varepsilon \to 0$. In particular, in the case when $\lim_{\varepsilon\to 0} h/\varepsilon^2 = \theta > 0$, sequences of symmetric gradients of the solutions are, in general, not compact with respect to strong two-scale convergence. As a consequence, the equation describing the limit energy balance is no longer obtained by setting the test function to be the solution of the homogenised equation for the corresponding "singular structure", obtained by considering the mid-lines of the rods with the measure induced by the thin structure (*cf.* Fig. 1). This problem was addressed [18], where the correct form of the energy equality was determined and the limit system of equations was derived. This study was followed by the analysis of Sobolev spaces for a variable measure [16, 11], Korn inequalities for periodic frames [17], and gaps in the spectrum of the elasticity operator on a high-contrast periodic structure [19] with non-vanishing volume fraction of the components as $\varepsilon \to 0$. In [19], the band-gap nature of the spectrum of the limit operator is analysed and conclude that the convergence of the spectra of the heterogeneous problems to the limit spectrum is proved.

In the present work we extend the techniques of [18, 19], for the study of an elasticity problem on a two-component periodic composite where the region occupied by the main material ("matrix") is a framework with $h/\varepsilon^2 \to \theta > 0$, and the complementary part of the space consisting of disjoint "inclusions" is filled by a less rigid material, so that ratio between the stiffness of inclusions and the matrix is of the order $O(\varepsilon^2)$. In other words, in addition to the assumption of high contrast, *cf.* [13, 17], we assume that the the stiff component is a thin structure so that its volume fraction is of the order $O(\varepsilon)$.

1 Problem formulation

We consider a periodic rod framework ("stiff" component of the composite) filled by a different material ("soft" component). We assume that the rod thickness $h > 0$ is a function of the period $\varepsilon > 0$, and consider the regime when $\lim_{\varepsilon\to 0} h/\varepsilon^2 = \theta > 0$. The ratio of the elastic moduli of the soft and stiff component is assumed to be of the order $O(\varepsilon^2)$. Denote by F_1^h the domain occupied by the rods and by F_1 the corresponding singular structure. Consider the "periodicity cell" $Q := [0,1]^2$ and denote $Q_1 := Q \cap F_1$ and $Q_0 := Q\backslash Q_1$. Consider also the "contraction" $F_1^{h,\varepsilon} := \varepsilon^{-1}F_1^h$ of the framework F_1^h. The soft component $\mathbb{R}^2 \setminus F_1^h$ and its contraction $\varepsilon^{-1}(\mathbb{R}^2 \setminus F_1^h)$ are denoted by F_0^h and $F_0^{h,\varepsilon}$, respectively. We denote by χ_1^h, $\chi_1^{h,\varepsilon}$ and χ_0^h, $\chi_0^{h,\varepsilon}$ the characteristic functions of the respective sets.

In what follows, we consider equations of two-dimensional elasticity in $\mathbb{R}^2$. These are obtained from the full system of linearised elasticity in three dimensions when there is a direction, say x_3, along which material properties are constant, assuming that the displacement does not depend on x_3. At each point $\mathbf{x} \in \mathbb{R}^2$, the fourth order tensor of the elastic moduli of the medium is given by

$$A^\varepsilon = \varepsilon^2 A_0(\cdot/\varepsilon)\chi_0(\cdot/\varepsilon) + A_1(\cdot/\varepsilon)\chi_1(\cdot/\varepsilon),$$

where A_0 and A_1 are periodic, bounded and positive definite[1]: $c_j\boldsymbol{\xi}^2 \le A_j\boldsymbol{\xi}\cdot\boldsymbol{\xi} \le c_j^{-1}\boldsymbol{\xi}^2$, $c_j > 0$, $j = 0, 1$. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, we denote by $\Omega_1^{\varepsilon,h} := \Omega \cap F_1^{h,\varepsilon}$ the stiff component and by $\Omega_0^{\varepsilon,h} := \Omega \cap F_0^{h,\varepsilon}$

[1] The scalar product of two symmetric matrices $\boldsymbol{\xi} = \{\xi_{ij}\}_{i,j=1}^2$ and $\boldsymbol{\eta} = \{\eta_{ij}\}_{i,j=1}^2$ is defined by $\boldsymbol{\xi}\cdot\boldsymbol{\eta} = \xi_{ij}\eta_{ij}$. The product of the fourth-order elasticity tensor A with a symmetric matrix $\boldsymbol{\xi}$ is defined as $A\boldsymbol{\xi} = a_{ijkl}\xi_{kl}$ and thus $A\boldsymbol{\xi}\cdot\boldsymbol{\xi} = a_{ijkl}\xi_{ij}\xi_{kl}$.

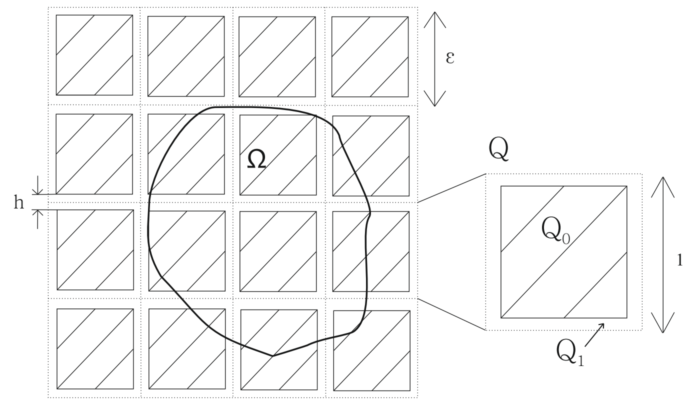

Figure 2: Periodic network with high-contrast.

the union of all soft inclusions in Ω. Consider the measures λ, λ^h defined on Q by

$$\lambda(B)=\frac{\mathcal{H}^1(B\cap F_1)}{\mathcal{H}^1(Q\cap F_1)},\qquad \lambda^h(B)=\frac{\mathcal{H}^2(B\cap F_1^h)}{\mathcal{H}^2(Q\cap F_1^h)},\qquad \forall\text{ Borel } B\subset Q,$$

where $\mathcal{H}^d$, $d=1,2$, is the d-dimensional Hausdorff measure (see *e.g.* [8]), and extended to $\mathbb{R}^2$ by Q-periodicity. Clearly, the weak convergence $\lambda^h\rightharpoonup\lambda$ holds as $h\to 0$, *i.e.* one has[2]

$$\lim_{h\to 0}\int_Q \boldsymbol{\varphi}\,\mathrm{d}\lambda^h=\int_Q\boldsymbol{\varphi}\,\mathrm{d}\lambda\quad \forall\boldsymbol{\varphi}\in\big[C^\infty_{\mathrm{per}}(Q)\big]^2.$$

Similarly, for the "composite" measures $\mu:=(1/2)\mathrm{d}\mathbf{x}+(1/2)\lambda$ and $\mu^h:=(1/2)\mathrm{d}\mathbf{x}+(1/2)\lambda^h$, where $\mathrm{d}\mathbf{x}$ is the plane Lebesgue measure, one has $\mu^h\rightharpoonup\mu$ as $h\to 0$. Further, we consider the "scaled" measure $\lambda^h_\varepsilon(B):=\varepsilon^2\lambda^h(\varepsilon^{-1}B)$ for all Borel $B\subset\mathbb{R}^2$, and $\mu^h_\varepsilon:=(1/2)\mathrm{d}\mathbf{x}+(1/2)\lambda^h_\varepsilon$, so that $\mu^h_\varepsilon\rightharpoonup\mathrm{d}\mathbf{x}$ as $\varepsilon\to 0$.

For $\varepsilon,h>0$ and $\mathbf{f}\in\big[C^\infty(\overline{\Omega})\big]^2$, we look for $\mathbf{u}\in\big[H^1_0(\Omega)\big]^2$ such that

$$\int_{\Omega_1^{\varepsilon,h}}A_1(\cdot/\varepsilon)\mathbf{e}(\mathbf{u}^h_\varepsilon)\cdot\mathbf{e}(\boldsymbol{\varphi})\,\mathrm{d}\mu^h_\varepsilon+\varepsilon^2\int_{\Omega_0^{\varepsilon,h}}A_0(\cdot/\varepsilon)\mathbf{e}(\mathbf{u}^h_\varepsilon)\cdot\mathbf{e}(\boldsymbol{\varphi})\,\mathrm{d}\mu^h_\varepsilon$$
$$+\int_\Omega\mathbf{u}^h_\varepsilon\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu^h_\varepsilon=\int_\Omega\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu^h_\varepsilon\quad\forall\boldsymbol{\varphi}\in\big[H^1_0(\Omega)\big]^2. \tag{1.1}$$

Define a bilinear form $\mathfrak{B}^h_\varepsilon(\cdot,\cdot)$ and a linear form $\mathfrak{L}^h_\varepsilon(\cdot)$ by

$$\mathfrak{B}^h_\varepsilon(\mathbf{u},\mathbf{v}):=\int_{\Omega_1^{\varepsilon,h}}A_1(\cdot/\varepsilon)\mathbf{e}(\mathbf{u})\cdot\mathbf{e}(\mathbf{v})\,\mathrm{d}\mu^h_\varepsilon+\varepsilon^2\int_{\Omega_0^{\varepsilon,h}}A_0(\cdot/\varepsilon)\mathbf{e}(\mathbf{u})\cdot\mathbf{e}(\mathbf{v})\,\mathrm{d}\mu^h_\varepsilon+\int_\Omega\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mu^h_\varepsilon,\qquad \mathfrak{L}^h_\varepsilon(\mathbf{v}):=\int_\Omega\mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mu^h_\varepsilon. \tag{1.2}$$

Notice that $\mathfrak{B}^h_\varepsilon$ is coercive and continuous, and $\mathfrak{L}^h_\varepsilon$ is continuous on $\big[H^1_0(\Omega)\big]^2$. It is a consequence of the Lax-Milgram lemma (see *e.g.* [7, Chapter 6]) that (1.1) has a unique solution $\mathbf{u}^h_\varepsilon$. In what follows we aim to describe the structure of the limit problem for the weak two-scale limit of the function $\mathbf{u}^h_\varepsilon$ as $\varepsilon\to 0$.

From the general theory of homogenisation on periodic rod structures, the following results hold regardless of the limit of the ratio h/ε^2:

1. There exists a vector function $\mathbf{u}(\mathbf{x},\mathbf{y})\in\big[C^\infty\big(\Omega,L^2_{\mathrm{per}}(Q,\mathrm{d}\mu)\big)\big]^2$ such that

$$\lim_{\varepsilon\to 0}\frac{1}{\big|\Omega_1^{h,\varepsilon}\big|}\int_{\Omega_1^{h,\varepsilon}}\big|\mathbf{u}^h_\varepsilon(\mathbf{x})-\mathbf{u}(\mathbf{x},\mathbf{x}/\varepsilon)\big|^2\mathrm{d}\mathbf{x}=0.$$

[2] We attach the superscript "per" to the notation for a function space when we refer to its subspace of Q-periodic functions.

2. The vector $\mathbf{u}(\mathbf{x},\cdot)$ is a "periodic rigid displacement" (see Definition 2.3): $\mathbf{u}(\mathbf{x},\mathbf{y}) = \mathbf{u}_0(\mathbf{x}) + \boldsymbol{\chi}(\mathbf{x},\mathbf{y})$, where $\mathbf{u}_0 \in \big[H_0^1(\Omega)\big]^2$ and $\boldsymbol{\chi}$ is the transverse displacement, see (2.6).

3. The equation $-\mathrm{div}\big(A^{\mathrm{hom}}\mathbf{e}(\mathbf{u}_0)\big) + \langle \mathbf{u}\rangle_{\mathbf{y}} = \mathbf{f}$ holds, where A^{hom} is the "homogenised tensor", see (2.12).

For each link I of the network F_1, let $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ be unit tangent and normal vectors that form a positively orientated system. Then all vectors $\mathbf{v} \in \mathbb{R}^2$ are written as $\mathbf{v} = v^{(\tau)}\boldsymbol{\tau} + v^{(\nu)}\boldsymbol{\nu}$, where $v^{(\tau)} = \mathbf{v}\cdot\boldsymbol{\tau}$ and $v^{(\nu)} = \mathbf{v}\cdot\boldsymbol{\nu}$. In the case when the tensor A_0 is isotropic[3], the vectors $\mathbf{U}$ and $\boldsymbol{\chi}$ are shown (see Section 3) to satisfy equations of the form

$$\mathcal{A}_0\mathbf{U} + \mathbf{u} = \mathbf{f}, \qquad \mathcal{L}_\tau \chi^{(\nu)} + \mathcal{T}_\nu U^{(\nu)} + u^{(\nu)} = f^{(\nu)}, \tag{1.3}$$

where $\mathcal{A}_0$ is a second-order differential operator in Q expressed in terns of the tensor A_0 only, $\mathcal{L}_\tau$ is a fourth-order differential operator in the "longitudinal" direction $\boldsymbol{\tau}$, and $\mathcal{T}_\nu$ is a first-order differential operator in the "transverse" direction $\boldsymbol{\nu}$ corresponding to each link I.

2 Two-scale structure of solution sequences

In this section we establish the structure of various two-scale limits on the soft and stiff components. This is achieved by taking the limits of integrals entering the identity (1.1), with suitably chosen test functions $\boldsymbol{\varphi}$.

2.1 Two-scale convergence: definition and properties

We first recall the notion of weak and strong two-scale convergence and their basic properties, see [14].

Definition 2.1 (Weak two-scale convergence)**.** Suppose that h is a function of ε and $\{\mathbf{u}_\varepsilon^h\} \subset \big[L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)\big]^2$ is a bounded sequence:

$$\limsup_{\varepsilon\to 0} \int_\Omega |\mathbf{u}_\varepsilon^h|^2 \, \mathrm{d}\mu_\varepsilon^h < \infty. \tag{2.1}$$

We refer to $\mathbf{u}(\mathbf{x},\mathbf{y}) \in \big[L^2(\Omega\times Q, \mathrm{d}\mathbf{x}\times\mathrm{d}\mu)\big]^2 =: \big[L^2(\Omega\times Q)\big]^2$ as the *weak two-scale limit* of $\mathbf{u}_\varepsilon^h$, denoted $\mathbf{u}_\varepsilon^h \stackrel{2}{\rightharpoonup} \mathbf{u}$, if

$$\lim_{\varepsilon\to 0}\int_\Omega \mathbf{u}_\varepsilon^h(\mathbf{x})\cdot\boldsymbol{\Phi}(\mathbf{x},\mathbf{x}/\varepsilon)\,\mathrm{d}\mu_\varepsilon^h = \int_\Omega\int_Q \mathbf{u}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\Phi}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mu(\mathbf{y})\mathrm{d}\mathbf{x} \qquad \forall \Phi \in \big[L^2(\Omega, C_{\mathrm{per}}(Q))\big]^2 \tag{2.2}$$

Proposition 2.1 (Two-scale compactness)**.** *If a sequence $\mathbf{u}_\varepsilon^h$ is bounded in $\big[L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)\big]^2$, then it is compact with respect to weak two-scale convergence.*

Proposition 2.2. *If $\mathbf{u}_\varepsilon^h \stackrel{2}{\rightharpoonup} \mathbf{u}$ then $\|\mathbf{u}\|_{[L^2(\Omega\times Q)]^2} \le \liminf_{\varepsilon\to0}\|\mathbf{u}\|_{[L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)]^2}$.*

Definition 2.2. Let $\mathbf{u}_\varepsilon^h$ be a bounded sequence in $\big[L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)\big]^2$. The function $\mathbf{u} = \mathbf{u}(\mathbf{x},\mathbf{y}) \in \big[L^2(\Omega\times Q)\big]^2$ is said to be the *strong two-scale limit* of $\mathbf{u}_\varepsilon^h$, denoted $\mathbf{u}_\varepsilon^h \stackrel{2}{\to} \mathbf{u}$, if for any weakly two-scale convergent sequence $\mathbf{v}_\varepsilon^h \stackrel{2}{\rightharpoonup} \mathbf{v}$ one has

$$\lim_{\varepsilon\to0}\int_\Omega \mathbf{u}_\varepsilon^h\cdot\mathbf{v}_\varepsilon^h\,\mathrm{d}\mu_\varepsilon^h = \int_\Omega\int_Q \mathbf{u}(\mathbf{x},\mathbf{y})\cdot\mathbf{v}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mu(\mathbf{y})\mathrm{d}\mathbf{x}. \tag{2.3}$$

Note that by setting $\mathbf{v}_\varepsilon^h = \mathbf{u}_\varepsilon^h$ one has

$$\lim_{\varepsilon\to0}\int_\Omega |\mathbf{u}_\varepsilon^h|^2\,\mathrm{d}\mu_\varepsilon = \int_\Omega\int_Q |\mathbf{u}|^2 \,\mathrm{d}\mu\mathrm{d}\mathbf{x}. \tag{2.4}$$

The next proposition shows that the converse also holds.

Proposition 2.3. *If $\mathbf{u}_\varepsilon^h \stackrel{2}{\rightharpoonup} \mathbf{u}$ and the convergence (2.4) holds, then $\mathbf{u}_\varepsilon^h \stackrel{2}{\to} \mathbf{u}$.*

Proposition 2.4. *For any arbitrary $a \in L^\infty(Q)$, the weak (resp. strong) two-scale convergence of $\mathbf{u}_\varepsilon^h$ to $\mathbf{u}(\mathbf{x},\mathbf{y})$ implies the weak (resp. strong) two-scale convergence of $a(\cdot/\varepsilon)\mathbf{u}_\varepsilon^h$ to $a(\mathbf{y})\mathbf{u}(\mathbf{x},\mathbf{y})$.*

[3] A tensor A_0 is said to be isotropic if for all symmetric matrices $\boldsymbol{\xi}$ one has $A_0\boldsymbol{\xi} = k_1\boldsymbol{\xi} + k_2(\mathrm{tr}\,\boldsymbol{\xi})I, \quad k_1, k_2 > 0$

2.2 Two-scale compactness of solutions to (1.1)

Consider the equation (1.1) with $\boldsymbol{\varphi} = \mathbf{u}_\varepsilon^h$:

$$\int_{\Omega_1^{\varepsilon,h}} A_1(\cdot/\varepsilon)\mathbf{e}(\mathbf{u}_\varepsilon^h)\cdot\mathbf{e}(\mathbf{u}_\varepsilon^h)\,\mathrm{d}\mu_\varepsilon^h + \varepsilon^2\int_{\Omega_0^{\varepsilon,h}} A_0(\cdot/\varepsilon)\mathbf{e}(\mathbf{u}_\varepsilon^h)\cdot\mathbf{e}(\mathbf{u}_\varepsilon^h)\,\mathrm{d}\mu_\varepsilon^h + \int_\Omega |\mathbf{u}_\varepsilon^h|^2\,\mathrm{d}\mu_\varepsilon^h = \int_\Omega \mathbf{f}\cdot\mathbf{u}_\varepsilon^h\,\mathrm{d}\mu_\varepsilon^h. \qquad (2.5)$$

Using ellipticity estimates on the left-hand side and the inequality $2ab \le a^2 + b^2$, $a, b \in \mathbb{R}$, on the right-hand side yields

$$c_0\varepsilon^2\int_{\Omega_0^{\varepsilon,h}}\left|\mathbf{e}(\mathbf{u}_\varepsilon^h)\right|^2\mathrm{d}\mu_\varepsilon^h + c_1\int_{\Omega_1^{\varepsilon,h}}\left|\mathbf{e}(\mathbf{u}_\varepsilon^h)\right|^2\mathrm{d}\mu_\varepsilon^h + \frac{1}{2}\int_\Omega|\mathbf{u}_\varepsilon^h|^2\,\mathrm{d}\mu_\varepsilon^h \le \frac{1}{2}\int_\Omega|\mathbf{f}|^2\,\mathrm{d}\mu_\varepsilon^h \qquad c_0, c_1 > 0.$$

Hence the following *a priori* bounds hold.

Proposition 2.5. *Let $\mathbf{u}_\varepsilon^h$ be a sequence in $\left[L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)\right]^2$ of solutions to (1.1). Then there exists $C > 0$ such that*

$$\|\mathbf{u}_\varepsilon^h\|_{[L^2(\Omega,\mathrm{d}\mu_\varepsilon^h)]^2} \le C, \qquad \left\|\mathbf{e}(\mathbf{u}_\varepsilon^h)\right\|_{[L^2(\Omega_1^{\varepsilon,h},\mathrm{d}\mu_\varepsilon^h)]^3} \le C, \qquad \varepsilon\left\|\mathbf{e}(\mathbf{u}_\varepsilon^h)\right\|_{[L^2(\Omega_0^{\varepsilon,h},\mathrm{d}\mu_\varepsilon^h)]^3} \le C.$$

Using two-scale compactness of L^2-bounded sets (see Proposition 2.1), we assume that the sequences

$$\mathbf{u}_\varepsilon^h, \quad \chi_1^{h,\varepsilon}\mathbf{u}_\varepsilon^h, \quad \chi_1^{h,\varepsilon}\mathbf{e}(\mathbf{u}_\varepsilon^h), \quad \varepsilon\chi_0^{h,\varepsilon}\mathbf{e}(\mathbf{u}_\varepsilon^h)$$

weakly two-sale converge to functions

$$\mathbf{u}(\mathbf{x},\mathbf{y}) \in \left[L^2(\Omega\times Q, \mathrm{d}\mathbf{x}\times\mathrm{d}\mu)\right]^2, \quad \widehat{\mathbf{u}}(\mathbf{x},\mathbf{y}),\ \mathbf{p}(\mathbf{x},\mathbf{y}) \in \left[L^2(\Omega\times Q, \mathrm{d}\mathbf{x}\times\mathrm{d}\lambda)\right]^2, \quad \widetilde{\mathbf{p}}(\mathbf{x},\mathbf{y}) \in \left[L^2(\Omega\times Q, \mathrm{d}\mathbf{x}\times\mathrm{d}\mathbf{y})\right]^2,$$

respectively, where $\left[L^2(\Omega\times Q, \mathrm{d}\mathbf{x}\times\mathrm{d}\mathbf{y})\right]^2$ is treated as a subspace of $\left[L^2(\Omega\times Q, \mathrm{d}\mathbf{x}\times\mathrm{d}\mu)\right]^2$.

2.3 Rigid displacements, potential and solenoidal matrices

Definition 2.3. A vector function $\mathbf{u} \in \left[L^2_{\mathrm{per}}(Q, \mathrm{d}\lambda)\right]^2$ is said to be a *periodic rigid displacement* (with respect the measure λ) if there exists a sequence $\{\mathbf{u}_n\} \subset \left[C^\infty_{\mathrm{per}}(Q)\right]^2$ such that $\left(\mathbf{u}_n, \mathbf{e}(\mathbf{u}_n)\right) \to (\mathbf{u}, 0)$ in $\left[L^2_{\mathrm{per}}(Q, \mathrm{d}\lambda)\right]^5$. We denote the set of periodic rigid displacements by $\mathcal{R}$, omitting the reference to the measure λ.

It is shown, see *e.g.* [14], that any $\mathbf{u} \in \mathcal{R}$ has a unique representation

$$\mathbf{u}(\mathbf{y}) = \mathbf{c} + \boldsymbol{\chi}(\mathbf{y}), \quad \mathbf{y} \in Q, \qquad (2.6)$$

where $\mathbf{c} \in \mathbb{R}^2$ and $\boldsymbol{\chi}$ is a periodic transverse displacement, *i.e.* on each link of the singular network F_1 it is orthogonal to the link. Denoting by $\widehat{\mathcal{R}}$ the set of transverse displacements, we thus have $\mathcal{R} = \mathbb{R}^2 \oplus \widehat{\mathcal{R}}$. The next definition characterises transverse displacements that occur in the study of rod networks with $h/\varepsilon^2 \to \theta > 0$ as $\varepsilon \to 0$.

Definition 2.4. Denote by $I_1, \dots, I_n$ the links of the network F_1 sharing an arbitrary node $\mathcal{O}$, and denote by $(\boldsymbol{\chi}\cdot\boldsymbol{\nu})'$ the derivative in the tangential direction: $(\boldsymbol{\chi}\cdot\boldsymbol{\nu})' = (\boldsymbol{\tau}\cdot\nabla)(\boldsymbol{\chi}\cdot\boldsymbol{\nu})$. The set $\widehat{\mathcal{R}}^0 \subset \widehat{\mathcal{R}}$ is defined to consist of periodic transverse displacements $\boldsymbol{\chi}$ satisfying the conditions:

(C1) The function $\boldsymbol{\chi}\cdot\boldsymbol{\nu}_j|_{I_j}$, $j = 1, 2, \dots, n$, has square integrable second derivatives on I_j, *i.e.* one has $\boldsymbol{\chi}\cdot\boldsymbol{\nu} \in H^2(I_j)$.

(C2) The first derivative along the link is continuous across each node: $(\boldsymbol{\chi}\cdot\boldsymbol{\nu}_1)'\big|_{\mathcal{O}} = (\boldsymbol{\chi}\cdot\boldsymbol{\nu}_2)'\big|_{\mathcal{O}} = \cdots = (\boldsymbol{\chi}\cdot\boldsymbol{\nu}_n)'\big|_{\mathcal{O}}$.

(C3) Each node is fastened: $\boldsymbol{\chi}|_{\mathcal{O}} = \mathbf{0}$.

The norm in $\widehat{\mathcal{R}}^0$ is defined to be the sum of the H^2-norms of $\boldsymbol{\chi}\cdot\boldsymbol{\nu}$ over all the links.

Definition 2.5. For a given Borel measure $\varkappa$ on Q, we define the space V^κ_{pot} of *$\varkappa$-potential matrices* as the closure of the set $\left\{\mathbf{e}(\mathbf{u}) \,|\, \mathbf{u} \in \left[C^\infty_{\mathrm{per}}(Q)\right]^2\right\}$ in the space $\left[L^2_{\mathrm{per}}(Q, \mathrm{d}\varkappa)\right]^3$. A symmetric matrix $\mathbf{v} \in \left[L^2_{\mathrm{per}}(Q, \mathrm{d}\varkappa)\right]^3$ is said to be *$\varkappa$-solenoidal* if

$$\int_Q \mathbf{v}\cdot\mathbf{e}(\mathbf{u})\,\mathrm{d}\varkappa = 0 \qquad \forall\,\mathbf{u} \in \left[C^\infty_{\mathrm{per}}(Q)\right]^2.$$

Denoting by $V_{\rm sol}^{\varkappa}$ the set of $\varkappa$-solenoidal matrices, we can write (see *e.g.* [14]) $\big[L^2_{\rm per}(Q,{\rm d}\varkappa)\big]^3=V_{\rm pot}^{\varkappa}\oplus V_{\rm sol}^{\varkappa}$. It follows that the orthogonal decomposition $\big[L^2(\Omega\times Q,{\rm d}\mathbf{x}\times\varkappa)\big]^2=L^2(\Omega,V_{\rm pot}^{\varkappa})\oplus L^2(\Omega,V_{\rm sol}^{\varkappa})$ holds, where the *two-scale* L^2-spaces of κ-potential and $\varkappa$-solenoidal vector fields are the closures of the linear spans of matrices $w\mathbf{e}(\mathbf{u})$, $w\in C_0^\infty(\Omega)$, $\mathbf{u}\in\big[C^\infty_{\rm per}(Q)\big]^2$ and $w\mathbf{v}$, $w\in C_0^\infty(\Omega)$, $\mathbf{v}\in V_{\rm sol}^{\varkappa}$, with respect to the norm of $\big[L^2(\Omega\times Q,{\rm d}\mathbf{x}\times{\rm d}\varkappa)\big]^3$. When $\varkappa$ is the Lebesgue measure on Q, we simply write $V_{\rm pot}$, $V_{\rm sol}$, $\big[L^2(\Omega\times Q)\big]^3$.

2.4 Convergence on the stiff component

We first study the relationship between $\mathbf{u}(\mathbf{x},\mathbf{y})$ and $\widehat{\mathbf{u}}(\mathbf{x},\mathbf{y})$.

Definition 2.6. Denote $\boldsymbol{\psi}^h_\varepsilon:=\boldsymbol{\psi}^h(\cdot/\varepsilon)$, where $\boldsymbol{\psi}^h\in\big[L^2_{\rm per}(Q,{\rm d}\mu^h)\big]^2$ extended to $\mathbb{R}^2$ by Q-periodicity.

1. We say that the sequence $\boldsymbol{\psi}^h_\varepsilon$ *weakly converges* to $\boldsymbol{\psi}\in\big[L^2_{\rm per}(Q,{\rm d}\mu)\big]^2$, and write $\boldsymbol{\psi}^h_\varepsilon\underset{\mu^h_\varepsilon}{\rightharpoonup}\boldsymbol{\psi}$, if

$$\int_Q\boldsymbol{\psi}^h_\varepsilon\cdot\boldsymbol{\xi}(\cdot/\varepsilon)\,{\rm d}\mu^h_\varepsilon\longrightarrow\int_Q\boldsymbol{\psi}\cdot\boldsymbol{\xi}\,{\rm d}\mu\qquad\forall\boldsymbol{\xi}\in\big[C^\infty_{\rm per}(Q)\big]^2,$$

where the test function ξ is extended to $\mathbb{R}^2$ by Q-periodicity.

2. We say that $\boldsymbol{\psi}^h_\varepsilon$ *strongly converge* to a function $\boldsymbol{\psi}\in\big[L^2_{\rm per}(Q,{\rm d}\mu)\big]^2$, and write $\boldsymbol{\psi}^h_\varepsilon\underset{\mu^h_\varepsilon}{\rightarrow}\boldsymbol{\psi}$, if

$$\int_Q\boldsymbol{\psi}^h_\varepsilon\cdot\boldsymbol{\xi}^h(\cdot/\varepsilon)\,{\rm d}\mu^h_\varepsilon\longrightarrow\int_Q\boldsymbol{\psi}\cdot\boldsymbol{\xi}\,{\rm d}\mu\qquad\text{if and only if}\quad\boldsymbol{\xi}^h_\varepsilon\underset{\mu^h_\varepsilon}{\rightharpoonup}\boldsymbol{\xi}.$$

Proposition 2.6. *If* $\mathbf{u}^h_\varepsilon(\mathbf{x})\overset{2}{\rightharpoonup}\mathbf{u}(\mathbf{x},\mathbf{y})$ *(see Appendix) and* $\boldsymbol{\psi}^h_\varepsilon\underset{\mu^h_\varepsilon}{\rightarrow}\boldsymbol{\psi}$, *then*

$$\int_\Omega\mathbf{u}^h_\varepsilon\cdot\boldsymbol{\psi}^h_\varepsilon\varphi\,{\rm d}\mu^h_\varepsilon\longrightarrow\int_\Omega\int_Q\mathbf{u}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\psi}(\mathbf{y})\varphi(\mathbf{x})\,{\rm d}\mu(\mathbf{y}){\rm d}\mathbf{x}\qquad\forall\varphi\in C_0^\infty(\Omega).$$

Proof. Since $\boldsymbol{\psi}^h_\varepsilon\underset{\mu^h_\varepsilon}{\rightarrow}\boldsymbol{\psi}$, it follows that for all $\boldsymbol{\zeta}\in\big[C_{\rm per}(Q)\big]^2$ the relation

$$\lim_{\varepsilon\to0}\int_\Omega\big|\boldsymbol{\psi}^h_\varepsilon-\boldsymbol{\zeta}(\cdot/\varepsilon)\big|^2\,{\rm d}\mu^h_\varepsilon=|\Omega|\int_Q|\boldsymbol{\psi}-\boldsymbol{\zeta}|^2\,{\rm d}\mu\tag{2.7}$$

holds. Notice further that, by the Hölder inequality, one has

$$\left|\int_\Omega\mathbf{u}^h_\varepsilon\cdot\big(\boldsymbol{\psi}^h_\varepsilon-\boldsymbol{\zeta}(\cdot/\varepsilon)\big)\varphi\,{\rm d}\mu^h_\varepsilon\right|\le\max_\Omega|\varphi|\|\mathbf{u}^h_\varepsilon\|_{[L^2(\Omega,{\rm d}\mu^h_\varepsilon)]^2}\left(\int_\Omega\big|\boldsymbol{\psi}^h-\boldsymbol{\zeta}(\cdot/\varepsilon)\big|^2{\rm d}\mu^h_\varepsilon\right)^{\frac12}.$$

The weak two-scale convergence of $\mathbf{u}^h_\varepsilon$ and the relation (2.7) imply that

$$\limsup_{\varepsilon\to0}\Big|\int_\Omega\mathbf{u}^h_\varepsilon\cdot\boldsymbol{\psi}^h_\varepsilon\varphi\,{\rm d}\mu^h_\varepsilon-\int_\Omega\int_Q\mathbf{u}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\zeta}(\mathbf{y})\varphi(\mathbf{x})\,{\rm d}\mu(\mathbf{y}){\rm d}\mathbf{x}\Big|$$

$$=\limsup_{\varepsilon\to0}\left|\int_\Omega\mathbf{u}^h_\varepsilon\cdot\boldsymbol{\psi}^h_\varepsilon\varphi\,{\rm d}\mu^h_\varepsilon-\int_\Omega\mathbf{u}^h_\varepsilon\cdot\boldsymbol{\zeta}(\cdot/\varepsilon)\varphi\,{\rm d}\mu^h_\varepsilon\right|\le C\left(\int_Q|\boldsymbol{\psi}-\boldsymbol{\zeta}|^2\,{\rm d}\mu\right)^{\frac12}\qquad\forall\boldsymbol{\zeta}\in\big[C_{\rm per}(Q)\big]^2.$$

The claim now follows by choosing an approximation sequence $\boldsymbol{\zeta}=\boldsymbol{\zeta}_k$ such that $\boldsymbol{\zeta}_k\to\boldsymbol{\psi}$ in $\big[L^2_{\rm per}(Q,{\rm d}\mu)\big]^2$. ☐

Theorem 2.1. *The function* $\widehat{\mathbf{u}}$ *is the trace of* $\mathbf{u}$ *on* F_1, *in the sense that* $\mathbf{u}(\mathbf{x},\mathbf{y})=\widehat{\mathbf{u}}(\mathbf{x},\mathbf{y})$ *a.e.* $\mathbf{x}\in\Omega$, λ*-a.e.* $\mathbf{y}\in F_1$.

Proof. For all functions $\widehat{\boldsymbol{\psi}}\in\big[L^2_{\rm per}(Q,{\rm d}\mu)\big]^2$ we define

$$\boldsymbol{\psi}(\mathbf{y}):=\begin{cases}\widehat{\boldsymbol{\psi}}(\mathbf{y}), & \mathbf{y}\in F_1\cap Q,\\ 0, & \mathbf{y}\in Q\backslash F_1,\end{cases}\qquad\qquad\boldsymbol{\psi}^h(\mathbf{y}):=\sum_j\boldsymbol{\psi}^h_j(\mathbf{y}),\quad\mathbf{y}\in F_1^h\cap Q.\tag{2.8}$$

where for each link I_j of $F_1 \cap Q$, we define $\boldsymbol{\psi}_j^h(y)$ to equal $\widehat{\boldsymbol{\psi}}(\mathbf{y}^*)$, whenever $\mathbf{y}$ is in the h-neighbouhood of I_j and $|\mathbf{y}-\mathbf{y}^*| = \mathrm{dist}(\mathbf{y}, I_j)$, $\mathbf{y}^* \in I_j$. Notice that for all $\varphi \in C_0^\infty(\Omega)$ one has

$$\int_\Omega \mathbf{u}_\varepsilon^h \cdot \boldsymbol{\psi}_\varepsilon^h \varphi \, \mathrm{d}\mu_\varepsilon^h = \int_\Omega \mathbf{u}_\varepsilon^h \chi_1^h(\cdot/\varepsilon) \cdot \boldsymbol{\psi}_\varepsilon^h \varphi \, \mathrm{d}\mu_\varepsilon^h + \int_\Omega \mathbf{u}_\varepsilon^h \chi_0^h(\cdot/\varepsilon) \cdot \boldsymbol{\psi}_\varepsilon^h \varphi \, \mathrm{d}\mu_\varepsilon^h. \tag{2.9}$$

Due to the fact that $\boldsymbol{\psi}_\varepsilon^h \underset{\mu_\varepsilon^h}{\to} \widehat{\boldsymbol{\psi}}$, one has

$$\int_\Omega \mathbf{u}_\varepsilon^h \cdot \boldsymbol{\psi}^h \varphi \, \mathrm{d}\mu_\varepsilon^h \longrightarrow \frac{1}{2} \int_\Omega \int_Q \mathbf{u}(\mathbf{x},\mathbf{y}) \cdot \widehat{\boldsymbol{\psi}}(\mathbf{y}) \varphi(\mathbf{x}) \, \mathrm{d}\lambda(\mathbf{y}) \mathrm{d}\mathbf{x}, \qquad \varepsilon \to 0. \tag{2.10}$$

Similarly, for the first integral on the right-hand side of (2.9), one has

$$\int_\Omega \mathbf{u}_\varepsilon^h \chi_1^h(\cdot/\varepsilon) \cdot \boldsymbol{\psi}_\varepsilon^h \varphi \, \mathrm{d}\mu_\varepsilon^h \longrightarrow \frac{1}{2} \int_\Omega \int_Q \widehat{\mathbf{u}}(\mathbf{x},\mathbf{y}) \cdot \widehat{\boldsymbol{\psi}}(\mathbf{y}) \varphi(\mathbf{x}) \, \mathrm{d}\lambda(\mathbf{y}) \mathrm{d}\mathbf{x}. \tag{2.11}$$

Finally, the second integral on the right-hand side of (2.9) goes to zero as $\varepsilon \to 0$, by virtue of the convergence $\mathbf{u}_\varepsilon^h \chi_0^h(\cdot/\varepsilon) \stackrel{2}{\rightharpoonup} \mathbf{u}(\mathbf{x},\mathbf{y})\chi_0(\mathbf{y})$. It follows that the limit integrals in (2.10) and (2.11) coincide, as required. □

The next theorem, proved in [14], describes the structure of the two-scale limit $\widehat{\mathbf{u}}$. Recall that on the stiff component $F_1^{h,\varepsilon}$ the symmetric gradient is bounded and hence $\varepsilon \chi_1^{\varepsilon,h} \mathbf{e}(\mathbf{u}_\varepsilon^h) \to 0$ in $\left[L^2(\Omega_1^{\varepsilon,h}, \mathrm{d}\lambda_\varepsilon^h)\right]^3$.

Theorem 2.2 (Theorem 12.2 in [14])**.** *1. It follows from* $\chi_1^{\varepsilon,h}\mathbf{u}_\varepsilon^h \stackrel{2}{\rightharpoonup} \widehat{\mathbf{u}}(\mathbf{x},\mathbf{y})$, $\varepsilon\chi_1^{\varepsilon,h}\mathbf{e}(\mathbf{u}_\varepsilon^h) \to 0$ *in* $\left[L^2(\Omega_1^{\varepsilon,h}, \mathrm{d}\lambda_\varepsilon^h)\right]^2$, *that* $\forall \mathbf{x} \in \Omega$, λ*-a.e.* $\mathbf{y} \in F_1$ *one has* $\widehat{\mathbf{u}}(\mathbf{x},\mathbf{y}) = \mathbf{u}_0(\mathbf{x}) + \boldsymbol{\chi}(\mathbf{x},\mathbf{y})$ *where* $\mathbf{u}_0 \in \left[H_0^1(\Omega)\right]^2$ *and* $\boldsymbol{\chi} \in L^2(\Omega, \widehat{\mathcal{R}})$.

2. Define the "λ-homogenised" tensor A_λ^{hom} *by the minimisation problem*

$$A_\lambda^{\mathrm{hom}} \boldsymbol{\xi} \cdot \boldsymbol{\xi} = \min_{v \in V_{\mathrm{pot}}^\lambda} \int_Q A_1(\boldsymbol{\xi} + v) \cdot (\boldsymbol{\xi} + v) \, \mathrm{d}\lambda \quad \forall \boldsymbol{\xi} \in \mathrm{Sym}_3. \tag{2.12}$$

Suppose that A_λ^{hom} *is positive-definite and that periodic rigid displacements take the form (2.6). If* $\chi_1^{\varepsilon,h}\mathbf{u}_\varepsilon^h \stackrel{2}{\rightharpoonup} \mathbf{u}_0(\mathbf{x}) + \boldsymbol{\chi}(\mathbf{x},\mathbf{y})$ *and the sequence* $\left\{\chi_1^{h,\varepsilon}\mathbf{e}(\mathbf{u}_\varepsilon^h(\mathbf{x}))\right\}$ *is bounded in* $\left[L^2(\Omega_1^{\varepsilon,h}, \mathrm{d}\lambda_\varepsilon^h)\right]^3$*, then, up passing to a subsequence, one has:*

i) $\mathbf{e}(\mathbf{u}_\varepsilon^h(\mathbf{x})) \stackrel{2}{\rightharpoonup} \mathbf{e}(\mathbf{u}_0(\mathbf{x})) + \mathbf{v}(\mathbf{x},\mathbf{y})$ *in* $\left[L^2(\Omega_1^{\varepsilon,h}, \mathrm{d}\lambda_\varepsilon^h)\right]^3$*, where* $\mathbf{v}(\mathbf{x},\mathbf{y}) \in L^2(\Omega, V_{\mathrm{pot}}^\lambda)$*;*

ii) $A_1(\cdot/\varepsilon)\mathbf{e}(\mathbf{u}_\varepsilon^h) \stackrel{2}{\rightharpoonup} A_\lambda^{\mathrm{hom}}\left\{\mathbf{e}\big(\mathbf{u}_0(\mathbf{x})\big) + \mathbf{v}(\mathbf{x},\mathbf{y})\right\} \in L^2(\Omega, V_{\mathrm{sol}})$.

The description of the structure of the two-scale limit of $\chi_1^{h,\varepsilon}\mathbf{u}_\varepsilon^h$ is a consequence of several statements proved in [18] Combining this with Theorem 2.1, we obtain the following result (*cf.* [18, Theorem 3.1]).

Theorem 2.3. *In the formula* $\widehat{\mathbf{u}}(\mathbf{x},\mathbf{y}) = \mathbf{u}_0(\mathbf{x}) + \boldsymbol{\chi}(\mathbf{x},\mathbf{y})$*, the transverse displacement* $\boldsymbol{\chi}$ *is an element of the space* $L^2(\Omega, \mathcal{R}_1^0)$.

2.5 Convergence on the soft component

Theorem 2.4. *For all sequences* $\{\mathbf{u}_\varepsilon^h\} \subset \left[H^1(\Omega)\right]^2$ *such that* $\mathbf{u}_\varepsilon^h \stackrel{2}{\rightharpoonup} \mathbf{u}(\mathbf{x},\mathbf{y})$ *in* $\left[L^2(\Omega_0^{\varepsilon,h}, \mathrm{d}\mu_\varepsilon^h)\right]^2$ *and* $\varepsilon\chi_0^{h,\varepsilon}\mathbf{e}(\mathbf{u}_\varepsilon^h) \stackrel{2}{\rightharpoonup} \widetilde{\mathbf{p}}(\mathbf{x},\mathbf{y})$ *in* $\left[L^2(\Omega_0^{\varepsilon,h}, \mathrm{d}\mu_\varepsilon^h)\right]^3$*, one has* $\mathbf{u} \in \left[L^2(\Omega, H^1(Q))\right]^2$ *and* $\widetilde{\mathbf{p}}(\mathbf{x},\mathbf{y}) = \mathbf{e}_{\mathbf{y}}\big(\mathbf{u}(\mathbf{x},\mathbf{y})\big)$ *a.e.* $\mathbf{x} \in \Omega$, $\mathbf{y} \in Q$.

Proof. For each $\delta > 0$, consider a C^∞-domain Q_δ such that $Q \cap F_0^{2\delta} \subset Q_\delta \subset Q \cap F_0^\delta$ and the set

$$\mathcal{X}_\delta := \left\{\mathbf{b} \in \left[C^\infty(Q_\delta)\right]^3 : \ \mathbf{b}\,\mathbf{n}|_{\partial Q_\delta} = 0\right\}$$

where $\mathbf{n}$ is the unit normal to ∂Q_δ. For $\mathbf{b} \in \mathcal{X}_\delta$, $\mathbf{a} = \mathrm{div}\,\mathbf{b}$ in Q_δ, consider the functions

$$\widetilde{\mathbf{a}}(\mathbf{y}) := \begin{cases} \mathbf{a}(\mathbf{y}), & \mathbf{y} \in Q_\delta, \\ 0, & \mathbf{0} \in Q \backslash Q_\delta, \end{cases} \qquad \widetilde{\mathbf{b}}(\mathbf{y}) := \begin{cases} \mathbf{b}(\mathbf{y}), & \mathbf{y} \in Q_\delta, \\ 0, & \mathbf{0} \in Q \backslash Q_\delta, \end{cases} \tag{2.13}$$

extended to $\mathbb{R}^2$ by Q-periodicity. Then for sufficiently small $\varepsilon > 0$ (recall that $h \to 0$ as $\varepsilon \to 0$) the following identity holds:

$$\varepsilon \int_{\Omega_0^{\varepsilon,h}} \widetilde{\mathbf{b}}(\cdot/\varepsilon) \cdot \mathbf{e}(\boldsymbol{\psi})\,\mathrm{d}\mu_\varepsilon^h = -\int_{\Omega_0^{\varepsilon,h}} \widetilde{\mathbf{a}}(\cdot/\varepsilon)\cdot\boldsymbol{\psi}\,\mathrm{d}\mu_\varepsilon^h \quad \forall \boldsymbol{\psi} \in \big[H_0^1(\Omega)\big]^2. \tag{2.14}$$

Setting $\boldsymbol{\psi} = \varphi \mathbf{u}_\varepsilon^h$, $\varphi \in C_0^\infty(\Omega)$, in (2.14) yields

$$\varepsilon \int_{\Omega_0^{\varepsilon,h}} \widetilde{\mathbf{b}}(\cdot/\varepsilon)\varphi \cdot \mathbf{e}(\mathbf{u}_\varepsilon^h)\,\mathrm{d}\mu_\varepsilon^h + \varepsilon \int_{\Omega_0^{\varepsilon,h}} \widetilde{\mathbf{b}}(\cdot/\varepsilon)\cdot \frac{1}{2}\big(\mathbf{u}_\varepsilon^h \otimes \nabla\varphi + \nabla\varphi \otimes \mathbf{u}_\varepsilon^h\big)\,\mathrm{d}\mu_\varepsilon^h = -\int_{\Omega_0^{\varepsilon,h}} \widetilde{\mathbf{a}}(\cdot/\varepsilon)\cdot\varphi\mathbf{u}_\varepsilon^h\,\mathrm{d}\mu_\varepsilon^h.$$

Passing to the limit in the last identity as $\varepsilon \to 0$ and using the fact that $\widetilde{\mathbf{a}}$, $\widetilde{\mathbf{b}}$ vanish in $Q \setminus Q_\delta$, we obtain

$$\int_\Omega \int_{Q_\delta} \widetilde{\mathbf{p}}(\mathbf{x},\mathbf{y})\varphi(\mathbf{x}) \cdot \mathbf{b}(\mathbf{y})\,\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} = -\int_\Omega \int_{Q_\delta} \mathbf{u}(\mathbf{x},\mathbf{y})\varphi(\mathbf{x})\cdot\mathbf{a}(\mathbf{y})\,\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}.$$

As $\varphi \in C_0^\infty(\Omega)$ is arbitrary, it follows that

$$\int_{Q_\delta} \widetilde{\mathbf{p}}(\mathbf{x},\mathbf{y}) \cdot \mathbf{b}(\mathbf{y})\,\mathrm{d}\mathbf{y} = -\int_{Q_\delta} \mathbf{u}(\mathbf{x},\mathbf{y})\cdot\mathbf{a}(\mathbf{y})\,\mathrm{d}\mathbf{y} \qquad \text{a.e. } \mathbf{x} \in \Omega. \tag{2.15}$$

Taking divergence-free fields $\mathbf{b} \in \mathcal{X}_\delta$ in (2.15) we infer (see *e.g.* [6]) the existence of $\mathbf{v} \in \big[L^2(\Omega, H^1(Q_\delta))\big]^2$ such that $\widetilde{\mathbf{p}}(\mathbf{x},\mathbf{y}) = \mathbf{e}_\mathbf{y}\big(\mathbf{v}(\mathbf{x},\mathbf{y})\big)$, $\mathbf{y} \in Q_\delta$, which implies

$$\int_{Q_\delta} \mathbf{v}(\mathbf{x},\mathbf{y})\cdot\mathbf{a}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{Q_\delta} \mathbf{u}(\mathbf{x},\mathbf{y})\cdot\mathbf{a}(\mathbf{y})\,\mathrm{d}\mathbf{y} \quad \text{a.e. } \mathbf{x} \in \Omega, \quad \forall \mathbf{a} \in \big\{\operatorname{div}\mathbf{b} \,\big|\, \mathbf{b} \in \mathcal{X}_\delta\big\} = \Big\{\mathbf{a} \in C^\infty(Q_\delta) : \int_{Q_\delta} \mathbf{a} = 0\Big\}.$$

Using the density in $\big[L^2(Q_\delta)\big]^2$ of vector functions $\mathbf{a}$ having the above representation implies that $\mathbf{v}(\mathbf{x},\mathbf{y})$ and $\mathbf{u}(\mathbf{x},\mathbf{y})$ differ by a constant for $\mathbf{y} \in Q_\delta$, hence $\widetilde{\mathbf{p}} = \mathbf{e}_\mathbf{y}(\mathbf{v}) = \mathbf{e}_\mathbf{y}(\mathbf{u})$, a.e. $\mathbf{y} \in Q_\delta$. By virtue of the arbitrary choice of the parameter δ, we conclude that $\widetilde{\mathbf{p}} = \mathbf{e}_\mathbf{y}(\mathbf{u})$ for a.e. $\mathbf{y} \in Q$. □

3 Homogenisation theorem

The proof of the homogenisation theorem is similar to the proof of the corresponding homogenisation theorem in [18]. However, modifications in the structure of the extension functions are required to prove the result in question. Accordingly, the limit equation includes two coupled microscopic equations which uniquely determine the function $\mathbf{U}$ on the soft inclusions and its trace $\boldsymbol{\chi}$ on the limit network.

3.1 Homogenised system of equations

Definition 3.1. We denote by V the *energy space* consisting of vectors

$$\mathbf{u}(\mathbf{x},\mathbf{y}) = \mathbf{u}_0(\mathbf{x}) + \mathbf{U}(\mathbf{x},\mathbf{y}), \quad \mathbf{u}_0 \in \big[H_0^1(\Omega)\big]^2, \quad \mathbf{U} \in \big[L^2\big(\Omega, H^1_{\mathrm{per}}(Q)\big)\big]^2,$$

$$\mathbf{U}(\mathbf{x},\mathbf{y}) = \boldsymbol{\chi}(\mathbf{x},\mathbf{y}), \quad \text{a.e. } \mathbf{x} \in \Omega,\ \lambda\text{-a.e. } \mathbf{y} \in Q, \qquad \boldsymbol{\chi} \in L^2(\Omega, \widehat{\mathcal{R}}^0).$$

We refer to $\mathbf{u} \in V$ as the *solution of the homogenised problem* if

$$\int_\Omega A_\lambda^{\mathrm{hom}}\mathbf{e}(\mathbf{u}_0)\cdot\mathbf{e}(\boldsymbol{\varphi}_0)\,\mathrm{d}\mathbf{x} + \frac{\theta^2}{6}\int_\Omega\int_Q \hat{k}\boldsymbol{\chi}''\cdot\boldsymbol{\Phi}''\,\mathrm{d}\lambda\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega\int_Q A_0\mathbf{e}_\mathbf{y}(\mathbf{U})\cdot\mathbf{e}_\mathbf{y}(\boldsymbol{\Phi})\,\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}$$
$$+\int_\Omega\int_Q \big(\mathbf{u}_0 + \mathbf{U}\big)\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu\mathrm{d}\mathbf{x} = \int_\Omega\int_Q \mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu\mathrm{d}\mathbf{x} \qquad \forall \boldsymbol{\varphi}(\mathbf{x},\mathbf{y}) = \boldsymbol{\varphi}_0(\mathbf{x}) + \boldsymbol{\Phi}(\mathbf{x},\mathbf{y}) \in V, \tag{3.1}$$

where A_λ^{hom} is given by (2.12), and $\hat{k} := \big\langle A_1^{-1}\eta\cdot\eta\big\rangle^{-1}$, $\eta = \boldsymbol{\tau}\otimes\boldsymbol{\tau}$.

The identity (3.1) is equivalent to a system of partial differential equations, which is obtained by considering various classes of test functions in (3.1). First, taking functions of the form $\boldsymbol{\varphi}(\mathbf{x},\mathbf{y}) = \boldsymbol{\varphi}_0(\mathbf{x})$ yields

$$-\operatorname{div} A^{\mathrm{hom}}\mathbf{e}(\mathbf{u}_0) + \mathbf{u}_0 + \langle\mathbf{U}\rangle = \mathbf{f}, \quad \mathbf{u}_0 \in \big[H_0^1(\Omega)\big]^2. \tag{3.2}$$

In order to obtain additional equations, test functions of the form $\boldsymbol{\varphi}(\mathbf{x},\mathbf{y}) = \boldsymbol{\Phi}(\mathbf{x},\mathbf{y})$ are considered where two restrictions of $\boldsymbol{\Phi}$ will be examined. Consider first those test functions which have support exclusively on the limiting network F_1 followed by those test functions that are supported by the soft component. If the tensor A_0 is isotropic, then the desired system of PDE's obtained take the form

$$\frac{\theta^2\hat{k}}{3}\partial_2^4\chi_1 + (k_1+k_2)\partial_1 U_2 + \big((u_0)_1 + \chi_1\big) = f_1, \tag{3.3}$$

$$-\frac{1}{2}\big\{k_1\boldsymbol{\Delta}\mathbf{U} + (k_1+2k_2)\nabla\mathrm{div}\,\mathbf{U}\big\} + \mathbf{u} = \mathbf{f}, \tag{3.4}$$

$$\mathbf{U}(\mathbf{x},\cdot) \in \big[H^1_{\mathrm{per}}(Q)\big]^2, \quad \mathbf{x}\in\Omega, \qquad \mathbf{U}(\mathbf{x},\mathbf{y}) = \boldsymbol{\chi}(\mathbf{x},\mathbf{y}) \quad \mathbf{x}\in\Omega, \quad \lambda\text{-a.e. } \mathbf{y}\in F_1, \qquad \boldsymbol{\chi}\in L^2(\Omega,\mathcal{R}_1^0). \tag{3.5}$$

For a general periodic framework F_1 where on each link there is a positively orientated pair of vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ with $\boldsymbol{\tau}$ pointing along the link and $\boldsymbol{\nu}$ orthogonal to the link. Then equation (3.3) in this case is given as

$$\frac{\theta^2\hat{k}}{3}\partial_\tau^4\chi^{(\nu)} + (k_1+k_2)\partial_\nu U^{(\nu)} + u^{(\nu)} = f^{(\nu)}, \tag{3.6}$$

where ∂_τ, ∂_ν denote differentiation along the links and in the direction normal to the links.

3.2 Extension theorem

Before proving the main result, we recall the description of a class of functions that extend periodic rigid displacements in $\mathcal{R}_1^0$ on the framework F_1 to the rod network F_1^h, introduced in [18].

Definition 3.2. Let D denote the space of functions $\mathbf{g}\in\mathcal{R}_1^0$ such that:

1. The function $\mathbf{g}$ is infinitely smooth outside a neighbourhood of the nodes of the network F_1;
2. In a neighbourhood of each node $\mathbf{y}_0$, the function $\mathbf{g}$ takes the form $\mathbf{g}(\mathbf{y}) = C\big(\boldsymbol{\omega}(\mathbf{y}) - \boldsymbol{\omega}(\mathbf{y}_0)\big)$, $\mathbf{y}\in F_1$, where C is a constant, $\boldsymbol{\omega}(\mathbf{y}) := (-y_2, y_1)$.

The following two statements are proved in [18].

Proposition 3.1. *The set D is dense in $\mathcal{R}_1^0$.*

Proposition 3.2. *For each $\mathbf{g}\in D$, there exists an extension $\mathbf{g}^h = \mathbf{g}^h(\mathbf{y})$ to the network F_1^h such that*

1. *The symmetric gradient $\mathbf{e}_{\mathbf{y}}(\mathbf{g}^h)$ is zero in a neighbourhood of each node,*
2. *The following asymptotic formula holds:*

$$A_1\mathbf{e}_{\mathbf{y}}(\mathbf{g}^h) = h\big[(\mathbf{g}\cdot\boldsymbol{\nu})''\rho\big]_h\boldsymbol{\sigma} + O(h^2), \quad h\to 0, \tag{3.7}$$

   where

$$\boldsymbol{\sigma} := -(\boldsymbol{\tau}\otimes\boldsymbol{\tau})\beta(\cdot/\varepsilon), \qquad \beta(\mathbf{y}) := h^{-1}\boldsymbol{\nu}\cdot(\mathbf{y}-\mathbf{y}_0) \text{ on the } h\text{-rod},$$

$$\rho := (A_1\boldsymbol{\eta}\cdot\boldsymbol{\eta})^{-1}, \qquad \boldsymbol{\eta} := \boldsymbol{\tau}\otimes\boldsymbol{\tau},$$

$$\big[(\mathbf{g}\cdot\boldsymbol{\nu})''\rho\big]_h \text{ is the natural extension of the function } (\mathbf{g}\cdot\boldsymbol{\nu})''\rho \text{ to } F_1^h.$$

3. *The convergence $\mathbf{g}^h\to\mathbf{g}$ holds in $\big[L^2(Q,\mathrm{d}\mu^h)\big]^2$.*

3.3 Convergence of solutions

Theorem 3.1. *Let $\mathbf{u}_\varepsilon^h$ solve the integral identity (1.1) with right-hand side $\mathbf{f} = \mathbf{f}^{h,\varepsilon}$, for all ε, h, and suppose that $h/\varepsilon \to \theta > 0$ as $\varepsilon \to 0$. If $\mathbf{f}^{h,\varepsilon} \overset{2}{\rightharpoonup} \mathbf{f}$ then $\mathbf{u}_\varepsilon^h \overset{2}{\rightharpoonup} \mathbf{u}$, and $\mathbf{u}$ satisfies (3.1). If $\mathbf{f}^{h,\varepsilon} \overset{2}{\to} \mathbf{f}$ then $\mathbf{u}_\varepsilon^h \overset{2}{\to} \mathbf{u}$ and, in addition, there is convergence of the elastic energies.*

Proof. Setting $\boldsymbol{\varphi} = \boldsymbol{\varphi}_0(\mathbf{x})$ in the identity (1.1) and using Theorems 2.2, 2.4, we obtain

$$\int_\Omega A_\lambda^{\text{hom}} \mathbf{e}(\mathbf{u}_0) \cdot \mathbf{e}(\boldsymbol{\varphi}_0)\, \mathrm{d}\mathbf{x} + \int_\Omega \int_Q \mathbf{u} \cdot \boldsymbol{\varphi}_0 \, \mathrm{d}\mu \mathrm{d}\mathbf{x} = \int_\Omega \int_Q \mathbf{f} \cdot \boldsymbol{\varphi}_0 \, \mathrm{d}\mu \mathrm{d}\mathbf{x}. \tag{3.8}$$

Suppose that $\mathbf{G} \in \left[H^1_{\text{per}}(Q)\right]^2$, $\mathbf{g} \in \mathcal{R}^0_1$ such that $\mathbf{G}(\mathbf{y}) = \mathbf{g}(\mathbf{y})$ for λ-a.e. $\mathbf{y} \in \partial Q$. Consider the function $\mathbf{G}_\varepsilon^h = \widetilde{\mathbf{G}}_\varepsilon^h + \widetilde{\mathbf{g}}^h$, where $\widetilde{\mathbf{g}}^h$ is the extension of $\mathbf{g}^h$ to Q such that $\widetilde{\mathbf{g}}^h\big|_{F_0^h}$ is A_0-harmonic, and $\widetilde{\mathbf{G}}_\varepsilon^h \in \left[H^1_0(Q)\right]^2$ solves the problem

$$\int_Q (A_0 \chi_0^h + \varepsilon^{-2} A_1 \chi_1^h) \mathbf{e}(\widetilde{\mathbf{G}}_\varepsilon^h) \cdot \mathbf{e}(\boldsymbol{\psi}) \mathrm{d}\mu^h = \int_Q A_0 \chi_0^h \mathbf{e}(\mathbf{G}) \cdot \mathbf{e}(\boldsymbol{\psi}) \mathrm{d}\mu^h \qquad \forall \boldsymbol{\psi} \in \left[H^1_0(Q)\right]^2. \tag{3.9}$$

Clearly, one has $\mathbf{G}_\varepsilon^h \in \left[H^1_{\text{per}}(Q)\right]^2$ for all ε, h. Taking in (1.1) test functions $\boldsymbol{\varphi} = \boldsymbol{\varphi}^{h,\varepsilon} = w \mathbf{G}_\varepsilon^h(\cdot/\varepsilon)$, where $w \in C_0^\infty(\Omega)$, yields

$$\begin{aligned} \varepsilon^{-1} \int_{\Omega_1^{\varepsilon,h}} A_1(\cdot/\varepsilon) \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}_{\mathbf{y}}(\mathbf{g}^h)(\cdot/\varepsilon) w \, \mathrm{d}\mu_\varepsilon^h + \int_{\Omega_1^{\varepsilon,h}} A_1(\cdot/\varepsilon) \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot (\mathbf{G}_\varepsilon^h(\cdot/\varepsilon) \otimes \nabla w) \, \mathrm{d}\mu_\varepsilon^h \\ + \varepsilon \int_{\Omega_0^{\varepsilon,h}} A_0(\cdot/\varepsilon) \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \mathbf{e}_{\mathbf{y}}(\mathbf{G})(\cdot/\varepsilon) w \, \mathrm{d}\mu_\varepsilon^h \\ + \varepsilon^2 \int_{\Omega_0^{\varepsilon,h}} A_0(\cdot/\varepsilon) \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot (\mathbf{G}_\varepsilon^h(\cdot/\varepsilon) \otimes \nabla w) \, \mathrm{d}\mu_\varepsilon^h = \int_\Omega (\mathbf{f}^{h,\varepsilon} - \mathbf{u}_\varepsilon^h) \cdot \mathbf{G}_\varepsilon^h(\cdot/\varepsilon) w \, \mathrm{d}\mu_\varepsilon^h, \end{aligned} \tag{3.10}$$

We denote the four integrals on the left-hand side of (3.10) by $I_j(\varepsilon)$, $j = 1, 2, 3, 4$. It follows from the L^2-boundedness of the sequence $\varepsilon \mathbf{e}(\mathbf{u}_\varepsilon^h)$ and the fact that $A_1\big(\mathbf{e}(\mathbf{u}_0) + \mathbf{v}(\mathbf{x}, \mathbf{y})\big)$ is pointwise orthogonal to the matrix $\mathbf{g} \otimes \nabla w$ (see [13, Lemma 5.3]) that the integrals $I_4(\varepsilon)$ and $I_2(\varepsilon)$ tend to zero as $\varepsilon \to 0$. By the convergence results on the soft component discussed in Section 2.5, it is seen that

$$\lim_{\varepsilon \to 0} I_3(\varepsilon) = \frac{1}{2} \int_\Omega \int_Q A_0 \mathbf{e}_{\mathbf{y}}(\mathbf{u}) \cdot \mathbf{e}_{\mathbf{y}}(\mathbf{g}) w \, \mathrm{d}\mathbf{y} \mathrm{d}\mathbf{x}.$$

The following lemma is proved in [18].

Lemma 3.1. *The two-scale convergence*

$$\frac{h}{\varepsilon} \mathbf{e}(\mathbf{u}_\varepsilon^h) \cdot \boldsymbol{\sigma}(\mathbf{y}) \overset{2}{\rightharpoonup} \frac{\theta^2}{3} (\boldsymbol{\chi} \cdot \boldsymbol{\nu})''_{\tau\tau} \tag{3.11}$$

holds, where $\boldsymbol{\sigma}$ is the function defined in Proposition 3.2.

It follows from the above lemma that

$$\lim_{\varepsilon \to 0} I_1(\varepsilon) = \frac{\theta^2}{6} \int_\Omega \int_Q \hat{k} \chi'' \cdot \mathbf{g}'' w \, \mathrm{d}\lambda \mathrm{d}\mathbf{x}. \tag{3.12}$$

Finally, we prove the following lemma.

Lemma 3.2. *The sequence $\mathbf{G}_\varepsilon^h$ converges to $\mathbf{G}$ in $L^2(Q)$ as $\varepsilon \to 0$.*

Proof. We start by using the Poincaré-type inequality

$$\|\mathbf{U}\|_{L^2(Q)} \leq C_\theta \int_Q (\varepsilon^2 \chi_0^h + h^{-1} \chi_1^h) \mathbf{e}(\mathbf{U}) \cdot \mathbf{e}(\mathbf{U}) \mathrm{d}\mathbf{x}, \qquad C_\theta > 0, \tag{3.13}$$

which holds whenever $h/\varepsilon^2 \to \theta$ as $\varepsilon \to 0$. Indeed, since the tensors A_0 and A_1 are positive definite, the inequality

$$\widetilde{C}^{-1}\|\mathbf{G}_\varepsilon^h - \mathbf{G}\|_{L^2(Q)} \le \int_{Q_1^h} A_1\big(\mathbf{e}(\mathbf{G}_\varepsilon^h) - \mathbf{e}(\mathbf{G})\big)\cdot\big(\mathbf{e}(\mathbf{G}_\varepsilon^h) - \mathbf{e}(\mathbf{G})\big)\mathrm{d}\mu_\varepsilon^h$$
$$+\varepsilon^2\int_{Q_0^h} A_0\big(\mathbf{e}(\mathbf{G}_\varepsilon^h) - \mathbf{e}(\mathbf{G})\big)\cdot\big(\mathbf{e}(\mathbf{G}_\varepsilon^h) - \mathbf{e}(\mathbf{G})\big)\mathrm{d}\mu_\varepsilon^h \tag{3.14}$$

holds, where the constant $\widetilde{C}$ depends of C_θ in (3.13) and the ellipticity constants c_j of A_j, $j = 0, 1$. Setting first $\boldsymbol{\psi} = \widetilde{\mathbf{G}}_\varepsilon^h$ and then $\boldsymbol{\psi} = \mathbf{G} - \tilde{\mathbf{g}}_\varepsilon^h$ in (3.9) we obtain

$$\int_Q (A_0\chi_0^h + \varepsilon^{-2}A_1\chi_1^h)\mathbf{e}(\widetilde{\mathbf{G}}_\varepsilon^h)\cdot\mathbf{e}(\widetilde{\mathbf{G}}_\varepsilon^h)\mathrm{d}\mu^h = \int_Q A_0\chi_0^h\mathbf{e}(\mathbf{G})\cdot\mathbf{e}(\widetilde{\mathbf{G}}_\varepsilon^h)\mathrm{d}\mu^h, \tag{3.15}$$

$$\int_Q (A_0\chi_0^h + \varepsilon^{-2}A_1\chi_1^h)\mathbf{e}(\widetilde{\mathbf{G}}_\varepsilon^h)\cdot\mathbf{e}(\mathbf{G} - \tilde{\mathbf{g}}_\varepsilon^h)\mathrm{d}\mu^h = \int_Q A_0\chi_0^h\mathbf{e}(\mathbf{G})\cdot\mathbf{e}(\mathbf{G} - \tilde{\mathbf{g}}_\varepsilon^h)\mathrm{d}\mu^h. \tag{3.16}$$

Using (3.15) and (3.16) to re-write the right-hand side of (3.14) yields the estimate

$$C^{-1}\|\mathbf{G}_\varepsilon^h - \mathbf{G}\|_{L^2(Q)} \le \varepsilon^2\int_{Q_0^h} A_0\big\{\mathbf{e}(\tilde{\mathbf{g}}^h)\cdot\mathbf{e}(\tilde{\mathbf{g}}^h - \mathbf{G}) + \mathbf{e}(\mathbf{G})\cdot\mathbf{e}(\tilde{\mathbf{G}}_\varepsilon^h + \tilde{\mathbf{g}}^h - \mathbf{G})\big\} + \int_{Q_1^h} A_1\mathbf{e}(\tilde{\mathbf{g}}^h - \mathbf{G})\cdot\mathbf{e}(\tilde{\mathbf{g}}^h - \mathbf{G}) \tag{3.17}$$

Further, in view of the uniform boundedness of the first integral in (3.17) and convergence

$$\int_Q A_1\mathbf{e}(\mathbf{G})\cdot\mathbf{e}(\mathbf{G})\mathrm{d}\mu^h \overset{h\to 0}{\longrightarrow} \int_Q A_1\mathbf{e}(\mathbf{g})\cdot\mathbf{e}(\mathbf{g})\mathrm{d}\mu = 0,$$

where the limit vanishes by virtue of $\mathbf{g} \in \mathcal{R}_1$, we infer the claim of the lemma. □

Passing to the limit in (3.10) as $\varepsilon \to 0$ we obtain

$$\frac{\theta^2}{6}\int_\Omega\int_Q \hat{k}\chi''\cdot\mathbf{g}''w\,\mathrm{d}\lambda\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega\int_Q A_0\mathbf{e}_\mathbf{y}(\mathbf{u})\cdot\mathbf{e}_\mathbf{y}(\mathbf{G})w\,\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} = \int_\Omega\int_Q (\mathbf{f} - \mathbf{u})\cdot\mathbf{G}w\,\mathrm{d}\mu\mathrm{d}\mathbf{x}, \tag{3.18}$$

Adding together the identities (3.8) and (3.18) and denoting $\boldsymbol{\varphi}(\mathbf{x}, \mathbf{y}) = \boldsymbol{\varphi}_0(\mathbf{x}) + \boldsymbol{\Phi}(\mathbf{x}, \mathbf{y})$, the homogenised formulation (3.1) follows.

In order to prove strong convergence of solutions when $\mathbf{f}^{h,\varepsilon} \xrightarrow{2} \mathbf{f}$, consider another version of problem (1.1) with right-hand sides $\mathbf{g}^{h,\varepsilon} \xrightarrow{2} \mathbf{g}$:

$$\mathbf{v}_\varepsilon^h \in \big[H_0^1(\Omega)\big]^2, \quad \int_{\Omega_1^{\varepsilon,h}} A_1(\cdot/\varepsilon)\mathbf{e}(\mathbf{v}_\varepsilon^h)\cdot\mathbf{e}(\boldsymbol{\varphi})\,\mathrm{d}\mu_\varepsilon^h + \varepsilon^2\int_{\Omega_0^{\varepsilon,h}} A_0(\cdot/\varepsilon)\mathbf{e}(\mathbf{v}_\varepsilon^h)\cdot\mathbf{e}(\boldsymbol{\varphi})\,\mathrm{d}\mu_\varepsilon^h$$
$$+\int_\Omega \mathbf{v}_\varepsilon^h\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu_\varepsilon^h = \int_\Omega \mathbf{g}^{h,\varepsilon}\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu_\varepsilon^h \quad \forall\boldsymbol{\varphi} \in \big[H_0^1(\Omega)\big]^2. \tag{3.19}$$

Setting $\boldsymbol{\varphi} = \mathbf{u}_\varepsilon^h$ in the above, $\boldsymbol{\varphi} = \mathbf{v}_\varepsilon^h$ in the original problem (1.1) and then subtracting one from the other yields

$$\lim_{\varepsilon\to 0}\int_\Omega \mathbf{u}_\varepsilon^h\cdot\mathbf{g}^{h,\varepsilon}\,\mathrm{d}\mu_\varepsilon^h = \int_\Omega \mathbf{v}_\varepsilon^h\cdot\mathbf{f}^{h,\varepsilon}\,\mathrm{d}\mu_\varepsilon^h = \int_\Omega\int_Q \mathbf{v}\cdot\mathbf{f}\,\mathrm{d}\mu\mathrm{d}\mathbf{x} = \int_\Omega\int_Q \mathbf{u}\cdot\mathbf{g}\,\mathrm{d}\mu\mathrm{d}\mathbf{x}, \tag{3.20}$$

where where $\mathbf{v}$ solves the homogenised equation with right-hand side $\mathbf{g}$.

Finally, using a standard two-scale convergence property (see *e.g.* [2]), we obtain

$$\lim_{\varepsilon\to 0}\left\{\int_{\Omega_1^{\varepsilon,h}} A_1(\cdot/\varepsilon)\mathbf{e}(\mathbf{u}_\varepsilon^h)\cdot\mathbf{e}(\mathbf{u}_\varepsilon^h)\,\mathrm{d}\mu_\varepsilon^h + \varepsilon^2\int_{\Omega_0^{\varepsilon,h}} A_0(\cdot/\varepsilon)\mathbf{e}(\mathbf{u}_\varepsilon^h)\cdot\mathbf{e}(\mathbf{u}_\varepsilon^h)\,\mathrm{d}\mu_\varepsilon^h\right\} = \int_\Omega\int_Q |\mathbf{f}|^2\,\mathrm{d}\mu\mathrm{d}\mathbf{x} - \int_\Omega\int_Q |\mathbf{u}|^2\,\mathrm{d}\mu\mathrm{d}\mathbf{x}$$
$$= \int_\Omega A^{\mathrm{hom}}\mathbf{e}(\mathbf{u}_0)\cdot\mathbf{e}(\mathbf{u}_0)\,\mathrm{d}\mathbf{x} + \frac{\theta^2}{6}\int_\Omega\int_Q \hat{k}\boldsymbol{\chi}''\cdot\boldsymbol{\chi}''\,\mathrm{d}\lambda\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega\int_Q A_0\mathbf{e}_\mathbf{y}(\mathbf{U})\cdot\mathbf{e}_\mathbf{y}(\mathbf{U})\,\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x},$$

as required. □

4 Convergence of spectra

Here we establish the convergence of the spectra of the operators associated with (1.1) to the spectrum given by the limit problem (3.1). We then calculate the spectrum on a model network and it to the spectrum of the analogous problem without high-contrast.

4.1 Spectrum of the limit operator

Consider the bilinear forms (*cf.* (3.1))

$$\mathfrak{b}_{\rm macro}(\mathbf{u}_0,\boldsymbol{\varphi}_0)=\int_\Omega A^{\rm hom}\mathbf{e}(\mathbf{u}_0)\cdot\mathbf{e}(\boldsymbol{\varphi}_0)\,{\rm d}\mathbf{x},\quad u_0,\boldsymbol{\varphi}_0\in\big[H_0^1(\Omega)\big]^2, \tag{4.1}$$

$$\mathfrak{b}_{\rm micro}(\mathbf{U},\boldsymbol{\Phi})=\frac{\theta^2}{6}\int_Q\hat{k}\boldsymbol{\chi}''\cdot\boldsymbol{\Phi}''\,{\rm d}\lambda+\frac{1}{2}\int_Q A_0\mathbf{e}_{\mathbf{y}}(\mathbf{U})\cdot\mathbf{e}_{\mathbf{y}}(\boldsymbol{\Phi})\,{\rm d}\mathbf{y},\quad \mathbf{U},\boldsymbol{\Phi}\in\widetilde{V}, \tag{4.2}$$

where the space $\tilde{V}$ consists of functions in $\big[H^1_{\rm per}(Q)\big]^2$ whose trace on $Q\cap F_1$ coincides with a rigid-body motion λ-a.e. The spectral problem associated with (3.1) can be written in the form

$$\begin{aligned}\mathfrak{b}_{\rm macro}(\mathbf{u}_0,\boldsymbol{\varphi}_0)&=s\big(\mathbf{u}_0+\langle\mathbf{U}\rangle,\boldsymbol{\varphi}_0\big)_{[L^2(\Omega)]^2}\quad\forall\,\boldsymbol{\varphi}_0\in\big[H_0^1(\Omega)\big]^2,\\ \mathfrak{b}_{\rm micro}(\mathbf{U},\boldsymbol{\Phi})&=s\big(\mathbf{u}_0+\mathbf{U},\boldsymbol{\Phi}\big)_{[L^2(Q,{\rm d}\mu)]^2}\quad\forall\,\boldsymbol{\Phi}\in\widetilde{V}.\end{aligned} \tag{4.3}$$

Let $\{\boldsymbol{\phi}_n\}_{n\in\mathbb{N}}$ be an orthonormal set of eigenvectors with non-zero average for the bilinear form $\mathfrak{b}_{\rm micro}$ with corresponding set of eigenvalues $\{\omega_n\}_{n\in\mathbb{N}}$:

$$\mathfrak{b}_{\rm micro}(\boldsymbol{\phi}_n,\boldsymbol{\Phi})=\omega_n\,(\boldsymbol{\phi}_n,\boldsymbol{\Phi})_{[L^2(Q,{\rm d}\mu)]^2}\quad\forall\boldsymbol{\Phi}\in\widetilde{V}. \tag{4.4}$$

Assuming that the values s is outside the spectrum $\sigma(\mathfrak{b}_{\rm micro})$ of the form $\mathfrak{b}_{\rm micro}$, the function $\mathbf{U}(\mathbf{x},\mathbf{y})$ is written as a series in terms of eigenfunctions $\{\boldsymbol{\phi}_n\}_{n\in\mathbb{N}}$:

$$\mathbf{U}(\mathbf{x},\mathbf{y})=s\sum_{n=1}^\infty\frac{\langle\boldsymbol{\phi}_n\rangle\cdot\mathbf{u}_0(\mathbf{x})}{\omega_n-s}\boldsymbol{\phi}_n(\mathbf{y}). \tag{4.5}$$

Substituting this expansion for $\mathbf{U}(\mathbf{x},\mathbf{y})$ into (4.3), we obtain

$$\mathfrak{b}_{\rm macro}(\mathbf{u}_0,\boldsymbol{\varphi}_0)=\big(\beta(s)\mathbf{u}_0,\boldsymbol{\varphi}_0\big)_{[L^2(\Omega)]^2}\ \ \forall\,\boldsymbol{\varphi}_0\in\big[H_0^1(\Omega)\big]^2,\qquad\beta(s):=s\left(I+s\sum_{n=1}^\infty\frac{\langle\boldsymbol{\phi}_n\rangle\otimes\langle\boldsymbol{\phi}_n\rangle}{\omega_n-s}\right). \tag{4.6}$$

Versions of the function β appear in the study of scalar [13] and vector ([12], [19], [20]) homogenisation problems. The following statement is a straightforward modification of a result in [20].

Proposition 4.1. *Denote by $\mathfrak{H}$ the closure in $\big[L^2(\Omega\times Q,{\rm d}\mathbf{x}\times{\rm d}\mu)\big]^2$ of the energy space V from Definition 3.1, and consider the operator $\mathfrak{A}$ whose domain consist of all solution pairs $(\mathbf{u}_0,\mathbf{U})$ to the identity*

$$\mathfrak{b}_{\rm macro}(\mathbf{u}_0,\boldsymbol{\varphi}_0)+\mathfrak{b}_{\rm micro}(\mathbf{U},\boldsymbol{\Phi})=(\mathbf{f},\boldsymbol{\varphi}_0+\boldsymbol{\Phi})_{[L^2(\Omega\times Q,{\rm d}\mathbf{x}\times{\rm d}\mu)]^2}\qquad\forall\boldsymbol{\varphi}_0+\boldsymbol{\Phi}\in V, \tag{4.7}$$

as the right-hand side $\mathbf{f}$ runs over all elements of $\mathfrak{H}$, and defined by $\mathbf{f}=\mathfrak{A}(\mathbf{u}_0+\mathbf{U})$ if and only if (4.7) holds.

Then $s\in\mathbb{C}$ belongs to the resolvent set $\rho(\mathfrak{A})$ of the operator $\mathfrak{A}$ if and only if $s\notin\sigma(\mathfrak{b}_{\rm micro})$ and the matrix $\beta(s)$ is negative definite:

$$\rho(\mathfrak{A})=\rho(\mathfrak{b}_{\rm micro})\cap\big\{s\,|\,\beta(s)<0\big\}, \tag{4.8}$$

where $\rho(\mathfrak{b}_{\rm micro})$ denotes the resolvent set of the operator generated by the form $\mathfrak{b}_{\rm micro}$ in the closure of[4] $\widetilde{V}$ in $\big[L^2(Q)\big]^2$.

[4]Note that the domain of this operator is dense in this closure.

Proof. Suppose that s belongs to the right-hand side of 4.8. We argue that the problem

$$\begin{cases} \mathfrak{b}_{\text{macro}}(\mathbf{u}_0, \boldsymbol{\varphi}_0) - s\big(\mathbf{u}_0 + \langle \mathbf{U} \rangle, \boldsymbol{\varphi}_0\big)_{[L^2(\Omega)]^2} = (\mathbf{f}, \boldsymbol{\varphi}_0)_{[L^2(\Omega)]^2}, \\ \mathfrak{b}_{\text{micro}}(\mathbf{U}, \boldsymbol{\Phi}) - s\big(\mathbf{u}_0 + \mathbf{U}, \boldsymbol{\Phi}\big)_{[L^2(Q,\mathrm{d}\mu)]^2} = (\mathbf{f}, \boldsymbol{\Phi})_{[L^2(Q,\mathrm{d}\mu)]^2}. \end{cases} \tag{4.9}$$

has a solution for every $\mathbf{f} \in \mathfrak{H}$ given that s satisfies the required assumptions of the lemma. Since $s \notin \mathrm{Sp}(\mathfrak{b}_{\text{micro}})$, it follows that $\mathbf{U}$ can be written in the form (4.5) with $\mathbf{u}_0$ replaced by $s\mathbf{u}_0 + \mathbf{f}$. Substituting this into the first equation given in system (4.9) yields

$$\mathfrak{b}_{\text{macro}}(\mathbf{u}_0, \boldsymbol{\varphi}_0) - \big(\beta(s)\mathbf{u}_0, \boldsymbol{\varphi}_0\big)_{[L^2(\Omega)]^2} = \big(s^{-1}\beta(s)\mathbf{f}, \boldsymbol{\varphi}_0\big)_{[L^2(\Omega)]^2}, \tag{4.10}$$

Since $\beta(s)$ is negative definite, the operator induced by the bilinear form on the left-hand side is invertible and thus (4.10) is solvable.

To prove the converse, assume that $s \in \rho(\mathfrak{A})$. It is clear that s cannot be in the spectrum of $\mathfrak{b}_{\text{micro}}$. Assume that $\beta(s)$ is positive definite for some $s \in \rho(\mathfrak{A})$. Hence problem (4.9) is uniquely solvable for any $\mathbf{f}$. Therefore (4.10) is solvable all right-hand sides, which is a contradiction. □

We note (see [19]) that all points of nontrivial spectrum for the periodic problem induced by the bilinear form $\mathfrak{b}_{\text{micro}}$ are at those points s where the matrix $\beta(s)$ is singular. The trivial spectral points are $\omega = 0$ which corresponds to constant eigenfunctions and those $\omega \in \mathrm{Sp}(\mathfrak{b}_{\text{micro}})$ such that the corresponding eigenfunctions have zero average. Moreover, as a consequence of the last result, when the matrix $\beta(s)$ is positive definite there is no solution and hence values of s for which $\beta(s)$ is negative definite correspond to a gap in the spectrum. The matrix $\beta(s) = \{\beta_{ij}(s)\}$ is negative definite if and only if the following conditions are satisfied:

$$\beta_{11}(s) = s\left(1 + s\sum_{n=1}^{\infty} \frac{\big(c_n^{(1)}\big)^2}{\omega_n - s}\right) < 0, \qquad \det \beta(s) > 0,$$

where $c_n^{(i)}$ is the ith component of the vector $\mathbf{c}_n := \langle \boldsymbol{\phi}_n \rangle$. Note that after simplification, it can be shown that

$$\det \beta(s) = s^3 \left\{ \sum_{n=1}^{\infty} \frac{\big(c_n^{(1)} - c_n^{(2)}\big)^2}{\omega_n - s} + s \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \frac{[\det(\mathbf{c}_n, \mathbf{c}_m)]^2}{(\omega_n - s)(\omega_m - s)} \right\}, \qquad \det(\mathbf{c}_n, \mathbf{c}_m) := \begin{vmatrix} c_n^{(1)} & c_m^{(1)} \\ c_n^{(2)} & c_m^{(2)} \end{vmatrix}.$$

If $\det \beta(s)$ has a zero on the interval (ω_n, ω_{n+1}) then, provided this zero is strictly less than the corresponding zero of $\beta_{11}(s)$ on the same interval, the interval between the these two zeros does not belong to the spectrum. Should $\det \beta(s)$ have a turning point in the interval (ω_n, ω_{n+1}) then, provided $\det \beta(s)$ is positive on the whole interval, there is no spectrum all the way up to the zero of $\beta_{11}(s)$ on the same interval.

Let $\{\nu_n\}_{n\in\mathbb{N}_0}$ denote the increasing sequence of values for which $\beta_{11}(s) = 0$. Let $\{\gamma_n\}_{n\in P}$ denote the increasing sequence of values for which $\det \beta(s) = 0$ where $P \subset \mathbb{N}_0$. Note that $\nu_0 = \gamma_0 = 0$. Hence, provided $\gamma_n < \nu_n$ for all $n \in P$, the spectrum of the limit operator $\mathcal{A}$ takes the form:

$$\mathrm{Sp}(\mathcal{A}) = \left(\bigcup_{n\in P} ([\omega_n, \gamma_n] \cup [\nu_n, \omega_{n+1}]) \right) \cup \left(\bigcup_{n \in \mathbb{N}_0 \setminus P} [\nu_n, \omega_{n+1}] \right) \cup \{\omega_1', \omega_2', \dots\}.$$

Therefore, the intervals (γ_n, ν_n), $n \in P$ and the intervals (ω_n, ν_n) $n \in \mathbb{N}_0 \setminus P$ are gaps in the spectrum of the operator $\mathcal{A}$ provided it does not contain a point from the set $\{\omega_1', \omega_2', \dots\}$ and $\gamma_n < \nu_n$.

4.2 Proof of convergence

Here we show that the spectra of the original problems converge to the spectrum of the limit problem (3.1).

Definition 4.1. We say that a sequence of sets $\mathcal{X}_\varepsilon \subset \mathbb{R}$, $\varepsilon > 0$, *converges in the sense of Hausdorff* to $\mathcal{X} \subset \mathbb{R}$ if the following two statements hold:

(H1) For each $\omega \in \mathcal{X}$, there exists a sequence $\omega_\varepsilon \in \mathcal{X}_\varepsilon$ such that $\omega_\varepsilon \to \omega$;

(H2) For all sequences $\omega_\varepsilon \in \mathcal{X}_\varepsilon$ such that $\omega_\varepsilon \to \omega \in \mathbb{R}$, it follows that $\omega \in \mathcal{X}$.

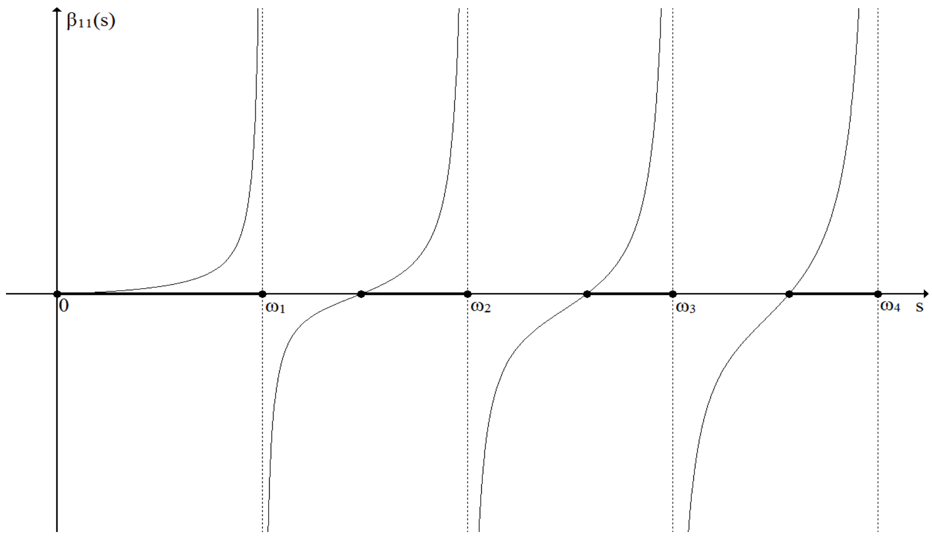

Figure 3: Sketch of the graph of β_{11}. The highlighted parts of the s-axis correspond to the regions where $\beta_{11}(s)$ is positive and hence there is the possibility for spectrum.

Definition 4.2. We say that a family of operators $\mathcal{A}_\varepsilon$ in $\left[L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)\right]^2$ *strongly two-scale resolvent convergent* as $\varepsilon \to 0$ to an operator $\mathcal{A}$ in $\left[L^2(\Omega \times Q, \mathrm{d}\mathbf{x} \times \mathrm{d}\mu)\right]^2$, and write $\mathcal{A}_\varepsilon \xrightarrow{2} \mathcal{A}$, if for all $\mathbf{f}$ in the range $R(\mathcal{A})$ of the operator $\mathcal{A}$ and for all sequences $\mathbf{f}_\varepsilon^h \in \left[L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)\right]^2$ such that $\mathbf{f}_\varepsilon^h \xrightarrow{2} \mathbf{f}$, the two-scale convergence $(\mathcal{A}_\varepsilon + I)^{-1}\,\mathbf{f}_\varepsilon^h \xrightarrow{2} (\mathcal{A} + I)^{-1}\,\mathbf{f}$ holds.

Proposition 4.2. *If $\mathcal{A}_\varepsilon \xrightarrow{2} \mathcal{A}$, then the property (H1) holds with $\mathcal{X}_\varepsilon = \mathrm{Sp}(\mathcal{A}_\varepsilon)$, $\mathcal{X} = \mathrm{Sp}(\mathcal{A})$.*

Proof. Let $T_\varepsilon := (\mathcal{A}_\varepsilon + I)^{-1}$ and $T := (\mathcal{A} + I)^{-1}$. If $s \in \mathrm{Sp}(\mathcal{A})$ then $t = (1+s)^{-1} \in \mathrm{Sp}(T)$. Therefore, for any $\delta > 0$, there exists a vector $\mathbf{f} \in R(\mathcal{A})$ such that

$$\|\mathbf{f}\|_{[L^2(\Omega \times Q, \mathrm{d}\mathbf{x} \times \mathrm{d}\mu)]^2} = 1, \qquad \big\|(T - t)\mathbf{f}\big\|_{[L^2(\Omega \times Q, \mathrm{d}\mathbf{x} \times \mathrm{d}\mu)]^2} \leq \delta/4.$$

Consider a sequence $\mathbf{f}_\varepsilon^h \in \left[L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)\right]^2$ such that $\mathbf{f}_\varepsilon^h \xrightarrow{2} \mathbf{f}$. Using the definition of strong two-scale resolvent convergence, one has

$$\lim_{\varepsilon \to 0} \big\|(T_\varepsilon - t)\mathbf{f}_\varepsilon^h\big\|_{[L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)]} = \big\|(T - t)\mathbf{f}\big\|_{[L^2(\Omega \times Q, \mathrm{d}\mathbf{x} \times \mathrm{d}\mu)]^2} \leq \delta/4.$$

Hence, $\|(T_\varepsilon - t)\mathbf{f}_\varepsilon^h\|_{L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)} \leq \delta/2$ and $\|\mathbf{f}_\varepsilon^h\|_{L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)} \geq 1/2$ for sufficiently small ε. Therefore, the interval $(-\delta + t, \delta + t)$ contains a point of the spectrum of the operator T_ε. Moreover, every interval centered at s contains a point of the spectrum of the operator $\mathcal{A}_\varepsilon$ for small enough ε, which completes the proof. □

Corollary 4.1. *For the operators $\mathfrak{A}_\varepsilon^h$ defined by the identity*

$$\mathfrak{B}_\varepsilon^h(\mathbf{u}, \mathbf{v}) = \mathfrak{L}_\varepsilon^h(\mathbf{v}),$$

where the forms $\mathfrak{B}_\varepsilon^h$, $\mathfrak{L}_\varepsilon^h$ are defined by (1.2), $\mathbf{f} = \mathfrak{A}_\varepsilon^h \mathbf{u}$, and the operator $\mathfrak{A}$ is defined in proposition 4.1, the property (H1) holds with $\mathcal{X}_\varepsilon = \mathrm{Sp}(\mathfrak{A}_\varepsilon^h)$, $\mathcal{X} = \mathrm{Sp}(\mathfrak{A})$, $h = h(\varepsilon)$.

The property (H2) of the Hausdorff convergence does not hold for spectra $\mathrm{Sp}(\mathfrak{A}_\varepsilon)$ in general, due to the fact that the soft component may have a non-empty intersection with the boundary of Ω. In addition, sequences of eigenfunctions of $\mathrm{Sp}(\mathfrak{A}_\varepsilon)$ may converge to the eigenfunctions of the "Bloch spectrum" associated with the expression (4.2) $\varkappa$-quasiperiodic functions, $\varkappa \in [0, 2\pi)^2$. However, a suitable version of (H2) does hold for a modified operator family, where the corresponding elements of the soft component are replaced by the stiff material. More precisely, for each ε, h, denote by $\widehat{\mathfrak{A}}^h_\varepsilon$ the operator defined similarly to $\mathfrak{A}^h_\varepsilon$, with $\Omega_0^{\varepsilon,h}$ and $\Omega_1^{\varepsilon,h}$ in (1.1) replaced by $\widehat{\Omega}_0^{\varepsilon,h}$ and $\Omega\setminus\widehat{\Omega}_0^{\varepsilon,h}$. Here, the set $\widehat{\Omega}_0^{\varepsilon,h}$ is the union of the sets $\varepsilon(Q\cap F_0^h+\mathbf{n})$ over all $\mathbf{n}\in\mathbb{Z}^2$ such that $\varepsilon(Q+\mathbf{n})\subset\Omega$.

Theorem 4.1. *For all $\varkappa\in[0,2\pi)$, denote by $\widetilde{V}^\varkappa$ the space of functions $\mathbf{U}(\mathbf{y})=\mathrm{e}^{\mathrm{i}\varkappa\cdot\mathbf{y}}\mathbf{U}_\#(\mathbf{y})$, $\mathbf{y}\in Q$, such that*

$$\mathbf{U}_\#\in\bigl[H^1_{\rm per}(Q)\bigr]^2,\qquad \mathbf{U}(\mathbf{y})=\boldsymbol{\chi}(\mathbf{y})\ \ \lambda\text{-a.e. }\mathbf{y}\in F_1,\qquad \boldsymbol{\chi}\in\widehat{\mathcal{R}}^0_\varkappa,$$

where $\widehat{\mathcal{R}}^0_\varkappa$ is the set of $\varkappa$-quasiperiodic rigid displacements, defined analogously to $\widehat{\mathcal{R}}_0$, see Definition 2.4. Consider the bilinear form

$$\mathfrak{b}^\varkappa_{\rm micro}(\mathbf{U},\boldsymbol{\Phi})=\frac{\theta^2}{6}\int_Q\hat{k}\boldsymbol{\chi}''\cdot\boldsymbol{\Phi}''\,\mathrm{d}\lambda+\frac12\int_Q A_0\mathbf{e}_\mathbf{y}(\mathbf{U})\cdot\mathbf{e}_\mathbf{y}(\boldsymbol{\Phi})\,\mathrm{d}\mathbf{y},\qquad \mathbf{U},\boldsymbol{\Phi}\in\widetilde{V}^\varkappa,$$

Suppose that of all ε, h, the function $\mathbf{u}^h_\varepsilon\in\bigl[H^1_0(\Omega)\bigr]^2$ is the L^2-normalised eigenfunction of $\widehat{\mathfrak{A}}_\varepsilon$:

$$\widehat{\mathfrak{A}}_\varepsilon\mathbf{u}^h_\varepsilon=\omega_\varepsilon\mathbf{u}^h_\varepsilon,\qquad \|\mathbf{u}^h_\varepsilon\|_{[L^2(\Omega,\mathrm{d}\mu^h_\varepsilon)]^2}=1.\tag{4.11}$$

If $\omega_\varepsilon\to\omega\notin\bigcup_\varkappa\mathrm{Sp}(\mathfrak{b}^\varkappa_{\rm micro})$, then the eigenfunction sequence $\mathbf{u}^h_\varepsilon$ is compact with respect to strong two-scale convergence on Ω.

Proof. The eigenvalue problem (4.11) is understood in the sense of the identity

$$\int_{\Omega\setminus\widehat{\Omega}_0^{\varepsilon,h}}A_1(\cdot/\varepsilon)\mathbf{e}(\mathbf{u}^h_\varepsilon)\cdot\mathbf{e}(\boldsymbol{\varphi})\,\mathrm{d}\mu^h_\varepsilon+\varepsilon^2\int_{\widehat{\Omega}_0^{\varepsilon,h}}A_0(\cdot/\varepsilon)\mathbf{e}(\mathbf{u}^h_\varepsilon)\cdot\mathbf{e}(\boldsymbol{\varphi})\,\mathrm{d}\mu^h_\varepsilon=\omega_\varepsilon\int_\Omega\mathbf{u}^h_\varepsilon\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu^h_\varepsilon\qquad\forall\boldsymbol{\varphi}\in\bigl[H^1_0(\Omega)\bigr]^2,$$

which implies, in particular, that

$$\int_{\Omega\setminus\widehat{\Omega}_0^{\varepsilon,h}}A_1(\cdot/\varepsilon)\mathbf{e}(\mathbf{u}^h_\varepsilon)\cdot\mathbf{e}(\mathbf{u}^h_\varepsilon)\,\mathrm{d}\mu^h_\varepsilon+\varepsilon^2\int_{\widehat{\Omega}_0^{\varepsilon,h}}A_0(\cdot/\varepsilon)\mathbf{e}(\mathbf{u}^h_\varepsilon)\cdot\mathbf{e}(\mathbf{u}^h_\varepsilon)\,\mathrm{d}\mu^h_\varepsilon=\omega_\varepsilon.$$

Denote by $\widehat{\Omega}_1^{\varepsilon,h}$ the union of $\varepsilon(Q\cap F_1^h+\mathbf{n})$ over all $\mathbf{n}\in\mathbb{Z}^2$ such that $\varepsilon(Q+\mathbf{n})\subset\Omega$. We claim that for all ε, h, there exists $\widetilde{\mathbf{u}}^h_\varepsilon$ such that

$$\mathbf{e}(\mathbf{u}^h_\varepsilon)=\mathbf{e}(\widetilde{\mathbf{u}}^h_\varepsilon)\text{ on }\widehat{\Omega}_1^{h,\varepsilon},\qquad \widetilde{\mathbf{u}}^h_\varepsilon\in\bigl[H^1_0(\Omega)\bigr]^2,\qquad \bigl\|\mathbf{e}(\widetilde{\mathbf{u}}^h_\varepsilon)\bigr\|_{[L^2(\widehat{\Omega}_0^{\varepsilon,h})]^2}\le C\bigl\|\mathbf{e}(\mathbf{u}^h_\varepsilon)\bigr\|_{[L^2(\widehat{\Omega}_1^{\varepsilon,h},\mathrm{d}\mu^h_\varepsilon)]^2},\tag{4.12}$$

$$\int_{\widehat{\Omega}_0^{h,\varepsilon}}A_0(\cdot/\varepsilon)\mathbf{e}(\widetilde{\mathbf{u}}^h_\varepsilon)\cdot\mathbf{e}(\boldsymbol{\varphi})\,\mathrm{d}\mu^h_\varepsilon=0\qquad\forall\,\boldsymbol{\varphi}\in\bigl[H^1_0(\Omega)\bigr]^2\text{ such that }\mathbf{e}(\boldsymbol{\varphi})=0\text{ in }\widehat{\Omega}_1^{\varepsilon,h},$$

where the constant $C>0$ is independent of ε, h. Indeed, we can consider $\widetilde{\mathbf{u}}^h_\varepsilon$ such that $\mathbf{z}^h_\varepsilon:=\mathbf{u}^h_\varepsilon-\widetilde{\mathbf{u}}^h_\varepsilon$ solves the minimisation problem

$$\frac12\int_{\widehat{\Omega}_0^{\varepsilon,h}}A_0(\cdot/\varepsilon)\mathbf{e}(\mathbf{v})\cdot\mathbf{e}(\mathbf{v})\,\mathrm{d}\mu^h_\varepsilon-\int_{\widehat{\Omega}_0^{\varepsilon,h}}A_0(\cdot/\varepsilon)\mathbf{e}(\mathbf{u}^h_\varepsilon)\cdot\mathbf{e}(\mathbf{v})\,\mathrm{d}\mu^h_\varepsilon\ \mapsto\ \min,\tag{4.13}$$

over all functions $\mathbf{v}\in\bigl[H^1_0(\Omega)\bigr]^2$ whose restriction to $\widehat{\Omega}_1^{h,\varepsilon}$ is a rigid-body motion with respect to the Lebesgue measure, *i.e.* one has $\mathbf{e}(\mathbf{v})=0$ in $\widehat{\Omega}_1^{h,\varepsilon}$. Clearly, one has $\mathbf{e}(\mathbf{z}^h_\varepsilon)=0$ in $\widehat{\Omega}_1^{\varepsilon,h}$ and

$$\varepsilon^2\int_{\widehat{\Omega}_0^{h,\varepsilon}}A_0(\cdot/\varepsilon)\mathbf{e}(\mathbf{z}^h_\varepsilon)\cdot\mathbf{e}(\boldsymbol{\varphi})\,\mathrm{d}\mu^h_\varepsilon-\omega_\varepsilon\int_\Omega\mathbf{z}^h_\varepsilon\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu^h_\varepsilon=\omega_\varepsilon\int_\Omega\widetilde{\mathbf{u}}^h_\varepsilon\cdot\boldsymbol{\varphi}\,\mathrm{d}\mu^h_\varepsilon\qquad\forall\boldsymbol{\varphi}\in\bigl[H^1_0(\Omega)\bigr]^2,\quad\mathbf{e}(\boldsymbol{\varphi})=0\text{ in }\widehat{\Omega}_1^{\varepsilon,h}.\tag{4.14}$$

It follows from the bound (4.12) that $\widetilde{\mathbf{u}}^h_\varepsilon$ is compact with respect to strong convergence in $\bigl[L^2(\Omega,\mathrm{d}\mu^h_\varepsilon)\bigr]^2$: there exists $\widetilde{\mathbf{u}}(\mathbf{x},\mathbf{y})$ such that, up to selecting a subsequence, one has

$$\widetilde{\mathbf{u}}^h_\varepsilon\to\widetilde{\mathbf{u}}\ \text{ in }\bigl[L^2(\Omega,\mathrm{d}\mu^h_\varepsilon)\bigr]^2,\qquad \chi_0^{\varepsilon,h}\widetilde{\mathbf{u}}^h_\varepsilon\overset{2}{\to}\chi_0(\mathbf{y})\widetilde{\mathbf{u}}(\mathbf{x}),\tag{4.15}$$

where we the second convergence follows from Proposition 2.4.

Lemma 4.1. *Suppose that* $\left[L^2(\Omega)\right]^2 \ni \mathbf{f}_\varepsilon^h \stackrel{2}{\rightharpoonup} \mathbf{f} \in \mathfrak{H}$, *where the space* $\mathfrak{H}$ *is defined in Proposition 4.1. For all* ε, h, *consider the function* $\mathbf{v}_\varepsilon^h \in \left[H_0^1(\Omega^{\varepsilon,h})\right]^2$ *such that* $\mathbf{e}(\mathbf{z}_\varepsilon^h) = 0$ *in* $\hat{\Omega}_1^{\varepsilon,h}$ *and the following resolvent identity holds (cf. (4.14)):*

$$\varepsilon^2 \int_{\Omega_0^{\varepsilon,h}} A_0 \mathbf{e}(\mathbf{v}_\varepsilon^h) \cdot \mathbf{e}(\boldsymbol{\varphi})\, \mathrm{d}\mu_\varepsilon^h - \omega_\varepsilon \int_\Omega \mathbf{v}_\varepsilon^h \cdot \boldsymbol{\varphi}\, \mathrm{d}\mu_\varepsilon^h = \int_\Omega \mathbf{f}_\varepsilon^h \cdot \boldsymbol{\varphi}\, \mathrm{d}\mu_\varepsilon^h \qquad \forall \boldsymbol{\varphi} \in \left[H_0^1(\Omega)\right]^2, \quad \mathbf{e}(\boldsymbol{\varphi}) = 0 \text{ in } \widehat{\Omega}_1^{\varepsilon,h}. \tag{4.16}$$

Then $\mathbf{v}_\varepsilon^h \stackrel{2}{\rightharpoonup} \mathbf{v} = \mathbf{v}(\mathbf{x}, \mathbf{y}) \in \left[L^2(\Omega, \widetilde{V})\right]^2$, *and*

$$\int_\Omega \int_Q A_0 \mathbf{e}_\mathbf{y}(\mathbf{v}) \cdot \mathbf{e}_\mathbf{y}(\boldsymbol{\varphi})\, \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} - \omega \int_\Omega \int_Q \mathbf{v} \cdot \boldsymbol{\varphi}\, \mathrm{d}\mu(\mathbf{y})\mathrm{d}\mathbf{x} = \int_\Omega \int_Q \mathbf{f} \cdot \boldsymbol{\varphi}\, \mathrm{d}\mu(\mathbf{y})\mathrm{d}\mathbf{x} \qquad \forall \boldsymbol{\varphi} \in \left[L^2(\Omega, \widetilde{V})\right]^2. \tag{4.17}$$

Proof. Note first that since ω_ε converges to a point outside the set $\bigcup_\varkappa \mathrm{Sp}(\mathfrak{b}_{\mathrm{micro}}^\varkappa)$, the identity (4.16) does not have non-zero solutions $\mathbf{v}_\varepsilon^h$ for $\mathbf{f}_\varepsilon^h = 0$ and ω_ϵ replaced by any value in some finite neighbourhood of the set $\{\omega_\varepsilon\}_{\varepsilon < \varepsilon_0}$ for some $\varepsilon_0 > 0$. Hence, for an L^2-bounded sequence of the right-hand sides $\mathbf{f}_\varepsilon^h$, the functions $\mathbf{v}_\varepsilon^h$ that satisfy (4.16) are uniformly bounded in $\left[L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)\right]^2$ for $\varepsilon < \varepsilon_0$.

Further, setting $\boldsymbol{\varphi} = \mathbf{v}_\varepsilon^h$ in (4.16) and using the fact that A_0 is uniformly positive definite yields the uniform estimate

$$\varepsilon \left\| \chi_0^{h,\varepsilon} \mathbf{e}(\mathbf{v}_\varepsilon^h) \right\|_{[L^2(\Omega_0^{\varepsilon,h}, \mathrm{d}\mu_\varepsilon^h)]^2} \leq C$$

for some positive constant C. Proceeding as in Section 2, and using the fact that $\widehat{\Omega}_0^{\varepsilon,h} \cup \widehat{\Omega}_1^{\varepsilon,h} \to \Omega$ as $\varepsilon \to 0$, we extract a subsequence of $\mathbf{v}_\varepsilon^h$ that weakly two-scale converges to a function $\mathbf{v} \in \left[L^2(\Omega, \tilde{V})\right]^2$ and such that $\chi_0^{h,\varepsilon} \mathbf{e}(\mathbf{v}_\varepsilon^h) \stackrel{2}{\rightharpoonup} \mathbf{e}_\mathbf{y}(\mathbf{v})$ in $\left[L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)\right]^2$.

Finally, passing to the limit as $\varepsilon \to 0$ in (4.16) yields the identity (4.17). By the uniqueness of solution to (4.16), the whole sequence $\mathbf{v}_\varepsilon^h$ weakly two-scale converges to $\mathbf{v}$. □

Lemma 4.1 implies that the sequence $\mathbf{z}_\varepsilon^h$ is compact with respect to weak two-scale convergence, its two-scale limit $\mathbf{z} = \mathbf{z}(\mathbf{x}, \mathbf{y})$ is a rigid-body motion on F_1 and satisfies the identity

$$\int_\Omega \int_Q A_0 \mathbf{e}_\mathbf{y}(\mathbf{z}) \cdot \mathbf{e}_\mathbf{y}(\boldsymbol{\varphi})\, \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} - \omega \int_\Omega \int_Q \mathbf{z} \cdot \boldsymbol{\varphi}\, \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} = \omega \int_\Omega \int_Q \widetilde{\mathbf{u}} \cdot \boldsymbol{\varphi}\, \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} \qquad \forall \boldsymbol{\varphi} \in \left[L^2(\Omega, \widetilde{V})\right]^2. \tag{4.18}$$

Setting $\boldsymbol{\varphi} = \mathbf{v}_\varepsilon^h$ in the identiy (4.14) and $\boldsymbol{\varphi} = \mathbf{z}_\varepsilon^h$ in (4.16) yields

$$\int_\Omega \mathbf{z}_\varepsilon^h \cdot \mathbf{f}_\varepsilon^h\, \mathrm{d}\mu_\varepsilon^h = \omega_\varepsilon \int_\Omega \mathbf{v}_\varepsilon^h \cdot \left(\chi_0^{\varepsilon,h} \widetilde{\mathbf{u}}_\varepsilon^h\right) \mathrm{d}\mu_\varepsilon^h \qquad \forall \varepsilon, h. \tag{4.19}$$

Taking the limit on both sides of (4.19) as $\varepsilon \to 0$, $h = h(\varepsilon)$, and using the above convergence properties, we obtain

$$\int_\Omega \int_Q \mathbf{z}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{f}(\mathbf{x}, \mathbf{y})\, \mathrm{d}\mu(\mathbf{y})\mathrm{d}\mathbf{x} = \omega \int_\Omega \int_Q \mathbf{v}(\mathbf{x}, \mathbf{y}) \cdot \widetilde{\mathbf{u}}(\mathbf{x})\, \mathrm{d}\mu(\mathbf{y})\mathrm{d}\mathbf{x}.$$

In particular, setting $\mathbf{f}_\varepsilon^h = \mathbf{z}_\varepsilon^h$ and using (4.17) with $\mathbf{f} = \mathbf{z}$, $\boldsymbol{\varphi} = \tilde{\mathbf{u}}$, and (4.18) with $\boldsymbol{\varphi} = \mathbf{z}$, we infer the convergence $\|\mathbf{z}_\varepsilon^h\|_{[L^2(\Omega, \mathrm{d}\mu_\varepsilon^h)]^2} \to \|\mathbf{z}\|_{[L^2(\Omega \times Q, \mathrm{d}\mathbf{x} \times \mathrm{d}\mu)]^2}$. Therefore, the sequence $\mathbf{z}_\varepsilon^h$ strongly two-scale converges to $\mathbf{z}$, see Proposition 2.3. □

4.3 Limit spectrum for model framework

In the following, an explicit calculation of the spectrum for the periodic bilinear form $\mathfrak{b}_{\mathrm{micro}}$ defined by equation (4.2) will be given for the case shown in Fig. 2, where the unit cell Q has links labelled I_1, I_2, I_3, I_4. Consider the spectral problem

$$\mathfrak{b}_{\mathrm{micro}}(\mathbf{U}, \phi) = s(\mathbf{U}, \phi)_{[L^2(Q, \mathrm{d}\mu)]^2} \quad \forall \phi, \quad \mathbf{U} \in \left[H_{\mathrm{per}}^1(Q)\right]^2, \quad \mathbf{U}|_{\mathcal{O}_j} = \mathbf{0}, \quad (\mathbf{U} \cdot \boldsymbol{\tau}_j)|_{I_j} = 0, \quad j = 1, 2, 3, 4. \tag{4.20}$$

Its spectrum can be calculated explicitly by using the Fourier method (see *e.g.* [3]). Let $\mathbf{U}$ be written as a Fourier series:

$$\mathbf{U}(\mathbf{y}) = \sum_{\mathbf{n} \in \mathbb{Z}^2} \mathbf{b}(\mathbf{n}) \mathrm{e}^{2\pi \mathrm{i} \mathbf{n} \cdot \mathbf{y}}, \quad \mathbf{n} = (n_1, n_2), \tag{4.21}$$

where $\mathbf{b}(\mathbf{n}) = \big(b_1(\mathbf{n}), b_2(\mathbf{n})\big)^\top$ are the Fourier coefficients. For completeness, the conditions on the Fourier coefficients required to be satisfied in order the the eigenfunction $\mathbf{U}$ to be in the right space are

$$\sum_{n_1\in\mathbb{Z}} b_2(n_1,n_2) = 0 \quad \forall n_2 \in \mathbb{Z}, \qquad \sum_{n_2\in\mathbb{Z}} b_1(n_1,n_2) = 0 \quad \forall n_1 \in \mathbb{Z}.$$

Hence, by substituting the Fourier series (4.21) into (4.20) and using the test functions $\phi_j(\mathbf{y}) = \mathbf{e}_j \mathrm{e}^{2\pi i \mathbf{m}\cdot\mathbf{y}}$, $\mathbf{m} \in \mathbb{Z}^2$, $j = 1, 2$ in (4.20). two algebraic equations are obtained:

$$\sum_{\mathbf{n}\in\mathbb{Z}^2} B_{\mathrm{micro}}\big(\mathbf{b}(\mathbf{n})\mathrm{e}^{2\pi i\mathbf{n}\cdot\mathbf{y}}, \mathbf{e}_j\mathrm{e}^{2\pi i\mathbf{m}\cdot\mathbf{y}}\big) = s\sum_{\mathbf{n}\in\mathbb{Z}^2}\big\langle \mathbf{b}(\mathbf{n})\mathrm{e}^{2\pi i\mathbf{n}\cdot\mathbf{y}}, \mathbf{e}_j\mathrm{e}^{2\pi i\mathbf{m}\cdot\mathbf{y}}\big\rangle, \qquad j = 1, 2, \tag{4.22}$$

Using the definition of the bilinear form where the tensor A_0 is chosen to be isotropic and applying Fourier theory, the above equations can be expressed as $A(\mathbf{m}, s)\mathbf{b}(\mathbf{m}) = \mathbf{0}$, where

$$A(\mathbf{m}, s) := \begin{pmatrix} \dfrac{16\pi^4\theta^2\hat{k}}{3}m_2^4 + 2\pi^2\big[(k_1+k_2)m_1^2 + \dfrac{k_1}{2}m_2^2\big] - s, & \pi^2(k_1+2k_2)m_1m_2 \\ \pi^2(k_1+2k_2)m_1m_2, & \dfrac{16\pi^4\theta^2\hat{k}}{3}m_1^4 + 2\pi^2\big[(k_1+k_2)m_2^2 + \dfrac{k_1}{2}m_1^2\big] - s \end{pmatrix}. \tag{4.23}$$

Recall that k_1, k_2 are the Lamé constants and that $\hat{k} = k_1(k_1+2k_2)(k_1+k_2)^{-1}$. Clearly the above system has a non-trivial solution if and only if $\det A(\mathbf{m}, s) = 0$. It should also be obvious that the eigenvalues of the matrix $A(\mathbf{m}, s)$ are also the points of the spectrum of the bilinear form B_{micro}.

The characteristic polynomial associated with the matrix $A(\mathbf{m}, s)$ is given by the equation

$$\begin{aligned} s^2 &- \left\{\frac{16\pi^4\theta^2\hat{k}}{3}(m_1^4+m_2^4) + \pi^2(3k_1+2k_2)(m_1^2+m_2^2)\right\}s - \pi^4(k_1+2k_2)^2m_1^2m_2^2 \\ &+ \left[\frac{16\pi^4\theta^2\hat{k}}{3}m_2^4 + 2\pi^2\big[(k_1+k_2)m_1^2 + \frac{k_1}{2}m_2^2\big]\right]\left[\frac{16\pi^4\theta^2\hat{k}}{3}m_1^4 + 2\pi^2\big[(k_1+k_2)m_2^2 + \frac{k_1}{2}m_1^2\big]\right] = 0. \end{aligned}$$

In the general case (θ, k_1 and k_2 arbitrary), it follows that the spectral values for all $m_1, m_2 \in \mathbb{Z}^2$ are given by

$$s(m_1, m_2) = \frac{1}{2}\left\{\frac{16\pi^4\theta^2\hat{k}}{3}(m_1^4+m_2^4) + \pi^2(3k_1+2k_2)(m_1^2+m_2^2) \pm \sqrt{D(m_1,m_2)}\right\}, \tag{4.24}$$

where

$$\begin{aligned} D(m_1, m_2) := (m_1^2+m_2^2)\Bigg\{&\left(\frac{16\pi^4\theta^2\hat{k}}{3}\right)^2(m_1^2-m_2^2)^2(m_1^2+m_2^2) - \\ &- \frac{32\pi^6\theta^2\hat{k}}{3}(k_1+2k_2)(m_1^2-m_2^2)^2 + \pi^4(k_1+2k_2)^2(m_1^2+m_2^2)\Bigg\}. \end{aligned}$$

The spectrum calculated for the high-contrast, critically scaled model will now be compared with the spectrum for the model which is critically scaled only. The bilinear form of consideration in this case is simply the microscopic bilinear form without the second integral, *i.e.*

$$\widetilde{B}_{\mathrm{micro}}(\mathbf{U}, \mathbf{\Phi}) = \frac{\theta^2}{3}\int_Q \hat{k}\mathbf{U}''\cdot\mathbf{\Phi}''\,\mathrm{d}\lambda.$$

Consider the same geometric setting, i.e. the model network and let the Lamé constants be set $k_1 = 1$ and $k_2 = 0$. Hence when the critical scale parameter $\theta = 1$, it can easily be seen that the spectral points are at

$$\widetilde{s}_1(\mathbf{m}) = \frac{16\pi^4}{3}m_1^4, \quad \widetilde{s}_2(\mathbf{m}) = \frac{16\pi^4}{3}m_2^4, \quad m_1, m_2 \in \mathbb{Z}.$$

In this case, $s = 0$ has multiplicity two and every other point of the spectrum has multiplicity four.

The spectrum in the critically scaled case is distributed (up to a factor) at every fourth power of an integer with multiplicity four, whereas, in the critically scaled high-contrast case [19], the spectrum is less regularly distributed with varying degrees of multiplicity.

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