REINFORCEMENT LEARNING IN SOCIAL NETWORKS

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ABSTRACT. We propose a model of network formation based on reinforcement learning, which can be seen as a generalization as the one proposed by Skyrms [1, 12] for signaling games. On a discrete graph, whose vertices represent individuals, at any time step each of them picks one of its neighbors with a probability proportional to their past number of communications; independently, Nature chooses, with an independent identical distribution in time, which ones are allowed to communicate. Communications occur when any two neighbors mutually pick each other and are both allowed by Nature to communicate.

Our results generalize the ones obtained by Hu, Skyrms and Tarrès in [12]. We prove that, up to an error term, the expected rate of communications increases in average, and thus a.s. converges. If we define the limit graph as the non-oriented subgraph on which edges are pairs of vertices communicating infinitely often, then, for stable configurations of the dynamics outside the boundary, the connected components of this limit graph are star-shaped. Conversely, any graph correspondence satisfying that property and a certain balance condition, and within which every vertex is connected to at least another one, is a limit configuration with positive probability.

1. INTRODUCTION

We introduce and analyse a model of network formation, based on one hand on a reciprocity condition - we can talk to somebody only if he conversely wants to talk to us - and on the other hand on a reinforcement learning procedure - we want to talk more to the ones we already talked to frequently.

Start with a weighted graph $G = (\mathbb{V}, E, \sim)$, where the vertices in $\mathbb{V}$ represent people, and nonoriented edges in E represent links between them. Assume that no vertex is isolated, i.e. that for all $i \in \mathbb{V}$, there exists $j \in \mathbb{V}$ such that $\{i, j\} \in E$ (also denoted by $i \sim j$).

Let $\mathcal{A} := (a_{ij})_{i,j \in \mathbb{V}}$ be a collection of nonnegative real numbers such that, for any $i, j \in \mathbb{V}$, $a_{ij} = a_{ji}$, and if $a_{ij} > 0$ then $\{i, j\} \in E$; a_{ij} represents the affinity between i and j. Let the network $G_{\mathcal{A}}$ be the graph G with weights $\mathcal{A}$.

Now consider the following game on the network $G_{\mathcal{A}}$, whose players are the vertices of G. The game is played in infinitely many rounds (one at each time step), and each of them consists in the following procedure:

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- each vertex $i \in \mathbb{V}$ chooses one, and only one, of its neighbours $j \sim i$;
- Nature independently picks a subset $V \subseteq \mathbb{V}$ of vertices, allowed to communicate;
- if $i, j \in V$ are neighbours, i.e. $i \sim j$, and if i and j mutually choose each other, then a communication occurs between them and they both receive a payoff equal to their affinity a_{ij}.

We model the choice by Nature of the set of vertices which are allowed to communicate at each time step by random independent identical Bernoulli distributions on the set of subsets of $\mathbb{V}$, denoted by $\mathcal{P}(\mathbb{V})$.

More precisely, we let $(p_V)_{V \in \mathcal{P}(\mathbb{V})}$ be a family of nonnegative real numbers such that

$$\sum_{V \in \mathcal{P}(\mathbb{V})} p_V = 1,$$

and assume that the sequence of subsets $V_n \in \mathcal{P}(\mathbb{V})$ chosen by Nature at time $n \in \mathbb{N}$ is an i.i.d. sequence of Bernoulli random variables with probability $(p_V)_{V \in \mathcal{P}(\mathbb{V})}$.

Note that, given any two adjacent vertices $i, j \in \mathbb{V}$, $i \sim j$,

$$p_{ij} := \sum_{V \in \mathcal{P}(\mathbb{V}) : i, j \in V} p_V$$

is the probability that i and j are both allowed to communicate by Nature at any time step. The numbers p_{ij}, $i \sim j$, will be important in the results. In general the nonoriented edges $\{i, j\}$ are not chosen independently of each other by Nature.

Let us describe three noticeable choices of $(p_V)_{V \in \mathcal{P}(\mathbb{V})}$.

The most obvious one is $p_{\mathbb{V}} = 1$, and $p_V = 0$ if $V \neq \mathbb{V}$, where any adjacent pair of vertices is allowed to communicate together at any time.

Another natural choice is that of a sequence $(p_V)_{V \in \mathcal{P}(\mathbb{V})}$, such that $p_V \neq 0$ if and only if (iff) $V = \{i\} \cup \{j : j \sim i\}$ for some $i \in \mathbb{V}$: at each time step only one particular (random) vertex i and its neighbours are allowed by Nature to communicate together.

The third choice is that of a signaling game, on which the same reinforcement learning procedure was studied in [12]. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two disjoint subsets spanning $\mathbb{V}$, i.e. $\mathbb{V} = \mathcal{S}_1 \cup \mathcal{S}_2$, $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$. Then assume $p_V \neq 0$ iff $V = V_i := \{i\} \cup \mathcal{S}_2$ for some $i \in \mathcal{S}_1$. Then $p_{ij} > 0$ only if $i \in \mathcal{S}_p$, $j \in \mathcal{S}_q$, p, $q \in \{1, 2\}$, $p \neq q$. In other words the graph on which communications occur is bipartite in this case, see Figure 1. In [12], the bipartite graph is complete, and it is assumed that $p_{V_i} = 1/M_1$, where M_1 is the number of vertices in $\mathcal{S}_1$, but the condition is not required in that paper.

The game being defined, let us now describe how the individuals pick each other. The model we consider here is reinforcement learning, a trial-and-error

State   1   2   3   4
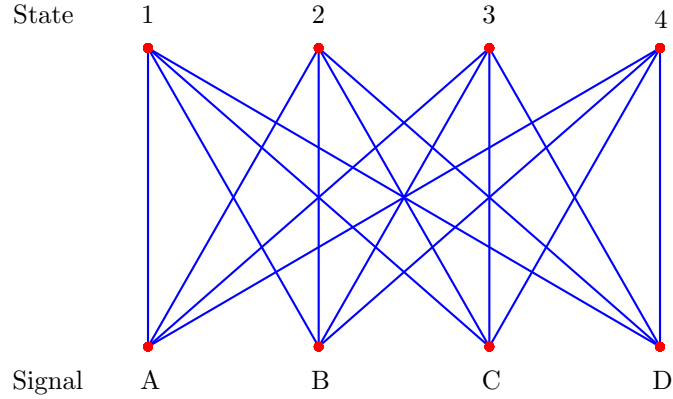
Signal   A   B   C   D

FIGURE 1. In the signaling game, $\mathcal{S}_1$ and $\mathcal{S}_2$ correspond respectively to the sets of states of Nature and signals: the agents learn to signal, i.e. they create a common language.

procedure in which agents are more likely to use the strategies that have given more payoff before, which in our case means that, *the more an agent has already talked to somebody, the more he will be likely to again talk to him.* At each round, each vertex will choose one of its neighbours with a probability that is proportional to the payoff they shared together so far.

More precisely, for all $i, j \in \mathbb{V}$, let us define the *cumulative payoff* V_{ij}^n (resp. V_i^n) on the edge $\{i, j\}$ (resp. vertex i) at time n by

$$V_{ij}^n = v_{ij}^0 + a_{ij} N_{ij}^n,$$

$$V_i^n = \sum_{j \in \mathbb{V}: j \sim i} V_{ij}^n,$$

where $v_{ij}^0 > 0$ iff $i \sim j$, v_{ij}^0 is a constant and N_{ij}^n is the number of communications that succeeded between i and j at time n. If $i \nsim j$, we set $V_{ij}^n = 0$ for all n.

Now we assume that, at each time step, any vertex $i \in \mathbb{V}$ chooses one of its neighbours j with probability

$$V_{ij}^n / V_i^n.$$

By symmetry, j also chooses i among its neighbours with probability

$$V_{ij}^n / V_j^n.$$

Let $\mathcal{F} := (\mathcal{F}_n)_n$ be the filtration generated by the process, i.e

$$\mathcal{F}_n := \sigma((V_{ij}^0, ..., V_{ij}^n), i, j \in \mathbb{V}), \quad n \in \mathbb{N}.$$

Then, the probability of a communication through an edge $ij \in E$ is

(1.1) $$\mathbb{P}\left(V_{ij}^{n+1} = V_{ij}^n + a_{ij} \,\middle|\, \mathcal{F}_n\right) = p_{ij} \frac{(V_{ij}^n)^2}{V_i^n V_j^n}.$$

We have now described our learning procedure of the network game. Observe that reinforcement learning is one of a number of models of strategic learning in games, in which players adapt their strategies, with the (possibly unconscious) aim to eventually maximize their payoffs, amongst for instance no-regret learning, fictitious play and its variants, and hypothesis testing, see [22].

In the spectrum of adaptive procedures, reinforcement learning certainly does not provide an optimal strategy, and rather assumes that players have bounded rationality. In particular they do not need to know which game the other agents are playing, or which strategies they are playing. It is therefore attractive as a simple behavioral model, since it does not require the agents to be entirely devoted to their task and, in our setting, even to be aware that they are involved in the formation of a network, but on the contrary just to observe their effective payoffs. Moreover it accumulates inertia, since the relative increase in payoff decreases in time, which would make it more stable with respect to randomness of the payoffs for instance, and in general avoid instability of the strategies.

Models of network formation using reinforcement learning were already proposed by Pemantle and Skyrms [18] as follows: each day, each individual chooses one of his neighbours and talks to him, the communication being always accepted by the neighbour; then both of them reinforce the probability to talk to each other. An important difference is that, in our model, a reciprocity assumption is made, in the sense that the communication is not necessarily accepted by the chosen neighbour.

Let us now summarize the results obtained in this paper, which are described in more detail in Section 2. As explained above, the network game model can be seen as a generalisation of the signaling game [17], on which the same reinforcement learning procedure was analyzed by Hu, Skyrms and Tarrès [12]. Our results are generalisations of the ones in [12], and the proofs adapt their techniques to a more general setting.

The main tools are stochastic approximation techniques. In particular, we will approach the behavior of some stochastic process by a deterministic ordinary differential equation (ODE), with the difficulty that the function governing the ODE is not continuous on the boundary of the simplex on which it is defined. One could instead show that the process, with a different system of coordinates, approximates a smooth dynamics, but it would then take place in an unbounded domain, which would lead to other technical difficulties.

First, we prove that there is a Lyapunov function for the deterministic ODE, which enables us to deduce in Theorem 2.2 the a.s. convergence of the expected payoff. Then we prove in Theorem 2.3 that the occupation measure of the communications converges a.s. to the set of equilibria of the ODE.

In Proposition 2.6, we show a property which characterizes the stable equilibria of the ODE: if we define the limit graph as the non-oriented subgraph on which edges are pairs of vertices communicating infinitely often, the connected components of this limit graph are star-shaped components, in other words *core and shell* (see Figure 2), satisfying furthermore a balance condition on the affinities a_{ij} and probabilities p_{ij}. In particular each connected component contains a nucleus vertex linked to (one or several) satellite vertices which are only linked to the nucleus.

In Theorem 2.8 we prove that any graph correspondence with the preceding property, such that no vertex is falling out of use, is a limit configuration with positive probability.

2. Main results

We let $\mathsf{Cst}(a_1, a_2, \ldots, a_p)$ denote a positive constant depending only on a_1, $a_2, \ldots a_p$, and let Cst denote a universal positive constant.

Given two real functions f and g defined on a set D, we write $f = O(g)$ iff there exists a constant $C > 0$ such that $|f(x)| \leqslant Cg(x)$ for all $x \in D$.

First, notice that if there is no $i, j \in \mathbb{V}$ such that $a_{ij}p_{ij} > 0$, then we are the trivial case where no successful communication can ever occur hence the system is just frozen in its initial configuration. Therefore, we assume that there exist $i, j \in \mathbb{V}$ such that $a_{ij}p_{ij} > 0$ for the rest of the paper.

For all $n \in \mathbb{N}$, define:

$$\text{(2.1)} \qquad T_n \;:=\; \sum_{i,j \in \mathbb{V}} V_{ij}^n,$$

$$x_{ij}^n \;:=\; \frac{V_{ij}^n}{T_n}, \text{ for all } i, j \in \mathbb{V},$$

$$x_i^n \;:=\; \frac{V_i^n}{T_n} = \sum_{j \in \mathbb{V}} x_{ij}^n, \text{ for all } i \in \mathbb{V}.$$

Note that $\sum_{i,j} x_{ij}^n = 1$. Let $x_n := (x_{ij}^n)_{i,j \in \mathbb{V}}$ be the *occupation measure* at time n, which belongs to the simplex:

$$\Delta := \left\{ (x_{ij})_{i,j \in \mathbb{V}} : \sum_{i,j \in \mathbb{V}} x_{ij} = 1, \text{ where } x_{ij} = x_{ji} \geqslant 0, \text{ and } x_{ij} = 0 \text{ if } i \nsim j \right\}.$$

In order to define the *expected payoff*, let us compute the conditional increment of T_n, for any $n \in \mathbb{N}$, recalling (1.1),

$$\text{(2.2)} \; \mathbb{E}\left(T_{n+1} - T_n \middle| \mathcal{F}_n\right) \;=\; \sum_{i,j \in \mathbb{V}} \mathbb{E}\left(V_{ij}^{n+1} - V_{ij}^n \middle| \mathcal{F}_n\right) = \sum_{i,j \in \mathbb{V}} a_{ij}p_{ij} \frac{(x_{ij}^n)^2}{x_i^n x_j^n}.$$

Definition 2.1. *Let $H : \mathbb{R}_+^{V \times V} \longrightarrow \mathbb{R}_+$ be the function defined, for all $x \in \Delta$, by*

$$H(x) := \sum_{i,j \in \mathbb{V} : x_{ij} > 0} a_{ij} p_{ij} \frac{x_{ij}^2}{x_i x_j}.$$

If $x \in \Delta$, we call $H(x)$ the expected payoff at x.

In this paper we use stochastic approximation techniques, namely we compare the evolution of the random process (x_n) to the behavior of the deterministic dynamics driven by the mean-field ODE

$$(2.3) \qquad\qquad\qquad \frac{dx}{dt} = F(x),$$

where F is a function from Δ to $T\Delta$, tangent space of Δ, which maps x to

$$F(x) = \left[x_{ij} \Big(a_{ij} p_{ij} \frac{x_{ij}}{x_i x_j} - H(x) \Big) \right]_{i,j \in \mathbb{V}},$$

with the convention that $F(x)_{ij} = 0$ if $x_{ij} = 0$. We will make this link explicit in Section 3.1.

As we will see, H is a Lyapunov function for the ODE (2.3), which should imply that, up to a small error term, the random process $(H(x_n))_n$ increases in average. We are indeed able to show the convergence of $(H(x_n))_n$, and hence the asymptotic linear growth of $(T_n)_n$ by conditional Borel-Cantelli Lemma, see Corollary 3.7. The proof is technical, since the function H is irregular on the boundary of the simplex.

Theorem 2.2. *The expected payoff process $(H(x_n))_{n \in \mathbb{N}}$ converges almost surely.*

The next Theorem 2.3 shows the convergence of $(x_n)_{n \in \mathbb{N}}$ towards the set of equilibria, i.e. the set of points x such that $F(x) = 0$. The equilibria are not isolated in general, and this result does not imply the a.s. convergence of $(x_n)_{n \in \mathbb{N}}$, which we could not prove in general.

Theorem 2.3. *The random processes $(F(x_n))_{n \in \mathbb{N}}$ and $(x_n)_{n \in \mathbb{N}}$ respectively converge to 0 and to the set of equilibria of the mean-field ODE (2.3) a.s.*

Our next result provides necessary and sufficient conditions for the stability of equilibria of the ODE.

Let us first define the *boundary* of the simplex as

$$\partial \Delta := \left\{ x \in \Delta : \exists i \in \mathbb{V} \text{ s.t. } x_i = \sum_{j \in \mathbb{V}} x_{ij} = 0 \right\}.$$

Note that $\partial \Delta$ is not the topological boundary of Δ. If $x_n \to \partial \Delta$, then one of the vertices *falls out of use*, in the sense that the frequency of its communications asymptotically goes to zero.

We call $x \in \Delta \setminus \partial\Delta$ a stable equilibrium of the ODE (2.3) iff the maximum real part of the eigenvalues of its Jacobian matrix at x is nonpositive, see Definition 5.6. This condition is close to the following one: if the solution of the ODE (2.3) starts near x, it will remain in its neighbourhood forever (see Section 5 for more detail).

We characterize in Proposition 2.6 the stable equilibria $x \in \Delta \setminus \partial\Delta$ in terms of a graph structure associated to x.

Let us first introduce some definitions. For any $x \in \Delta$, we define a subgraph G_x of G, with adjacency $\overset{x}{\sim}$, of possible communications between vertices associated to x. Then Definition 2.5 will introduce the property corresponding to the stability of equilibria in $\Delta \setminus \partial\Delta$.

Definition 2.4. *Given $x \in \Delta$, let G_x be the subgraph of G with vertices in $\mathbb{V}$ and adjacency $\overset{x}{\sim}$ such that $i \overset{x}{\sim} j$ if and only if $x_{ij} > 0$, for all $i, j \in \mathbb{V}$.*

Definition 2.5. *Consider a subgraph $\mathcal{G}$ of G, with adjacency $\overset{\mathcal{G}}{\sim}$, and let $\mathcal{C}_1, ..., \mathcal{C}_d$ be its connected components. Let $P_\mathcal{G}$ be the following property:*

  (1) *$\forall m \in \{1, ..., d\}$, $i, j, k, l \in \mathcal{C}_m$, s.t. $i \overset{\mathcal{G}}{\sim} j$ and $k \overset{\mathcal{G}}{\sim} l$, $a_{ij}p_{ij} = a_{kl}p_{kl}$;*
  (2) *$\forall m \in \{1, ..., d\}$, $\mathcal{C}_m$ contains at most one vertex with several neighbours;*
  (3) *each vertex has a corresponding edge within $\mathcal{G}$.*

Condition (1) in Definition 2.4 is a balance condition on the affinities within a connected component, whereas Condition (2) means that each connected component is star-shaped: if $P_\mathcal{G}$ holds thenn within each connected component, there exists a *nucleus vertex* i_0 such that if $j_1 \overset{\mathcal{G}}{\sim} i_0$ is a *satellite vertex*, then $j_1 \overset{\mathcal{G}}{\not\sim} i$ for any $i \neq i_0$, see Figure 2. Finally, Condition (3) applied to
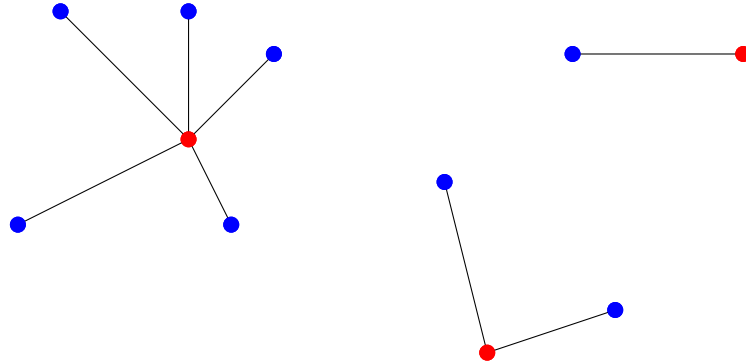

FIGURE 2. Stable configuration composed of three star-shaped connected components

$\mathcal{G} = G_x$ is equivalent to $x \in \Delta \setminus \partial\Delta$.

The following result is a consequence of Proposition 5.8.

Proposition 2.6. *An equilibrium $x \in \Delta \setminus \partial\Delta$ is stable if and only if its associated graph G_x satisfies the property P_{G_x}.*

Remark 2.7. *Note that it is straightforward that, if $P_{\mathcal{G}}$ holds, then there exists $x \in \Delta \setminus \partial\Delta$ such that $\mathcal{G} = G_x$.*

Theorem 2.8 implies that any subgraph $\mathcal{G}$ of G such that $P_{\mathcal{G}}$ holds, with positive probability x_n converges to some $x \in \Delta \setminus \partial\Delta$ with $G_x = \mathcal{G}$; and $\mathcal{G}$ is a limit graph in a strong sense, since after a (random) time communications will only occur within edges of this graph.

Theorem 2.8. *Let $q \in \Delta \setminus \partial\Delta$ be a stable equilibrium for the ODE (2.3) associated to the evolution of (x_n), and let $\mathcal{N}(q)$ be a neighbourhood of q in Δ. Then, with positive probability,*

 (1) $x_n \to x \in \mathcal{N}(q)$ *such that* $G_x = G_q$.
 (2) $\forall i,j \in \mathbb{V}, \, V_{ij}^\infty = \infty \iff \{i,j\}$ *is an edge of G_q.*

Theorems 2.2, 2.3 and 2.8, and Proposition 2.6, are generalisations of results of Hu, Skyrms and Tarrès [12], in the case of the so-called signaling game, see Section 1.

We believe but cannot prove that x_n converges a.s. to some stable equilibrium in Δ. The difficulty arises from the fact that convergence towards $\partial\Delta$ is also possible. In Section 4, we explain how several sites can a.s. asymptotically fall out of use (i.e. $x_i^n \to 0$) on a network, although of them performs infinitely many successful communications. The behavior in the neighborhood of $\partial\Delta$ is difficult to analyse, since the functions F and H are not continuous on $\partial\Delta$, and since the error in the stochastic approximation can be irregular.

The paper is organized as follows. In the following Section 3.1, we explicit the ODE associated to the dynamics of x_n, and prove that H is a Lyapunov function for the deterministic dynamics. Section 3.2 and Section 3.3 are respectively dedicated to the proofs of Theorem 2.2 and Theorem 2.3. Section 4 yields some explanation on the technical difficulties arising from the possible convergence to $\partial\Delta$. Section 5 is devoted to the classification of equilibria and analysis of stability, and Section 6 concerns the proof of Theorem 2.8.

3. Stochastic Approximation

Let us first compute the conditional increment of the process (x_n) and derive some mean-field ODE. Using stochastic approximation techniques, we will then study this ODE (in particular its equilibria in Section 5), following the idea that the random process approximates the solutions of the ODE.

In order to compute the increment of $(x_n)_{n \in \mathbb{N}}$ at time n we define, for all $i, j \in \mathbb{V}$, $i \sim j$, and for all $n \in \mathbb{N}$, the variables

$$(3.1) \qquad \Delta_{ij}^{n+1} \ := \ V_{ij}^{n+1} - V_{ij}^n,$$

$$(3.2) \qquad \Delta_T^{n+1} \ := \ T_{n+1} - T_n.$$

Now, for all $ij \in E$, we have

$$x_{ij}^{n+1} - x_{ij}^n = \frac{V_{ij}^n + \Delta_{ij}^{n+1}}{T_n + \Delta_T^{n+1}} - \frac{V_{ij}^n}{T^n} = \frac{T_n \cdot \Delta_{ij}^{n+1} - V_{ij}^n \cdot \Delta_T^{n+1}}{T_n \left(T_n + \Delta_T^{n+1} \right)}$$

$$(3.3) \qquad = \frac{\Delta_{ij}^{n+1} - x_{ij}^n \cdot \Delta_T^{n+1}}{T_{n+1}} = \frac{1}{T_n} \left(\Delta_{ij}^{n+1} - x_{ij}^n \cdot \Delta_T^{n+1} \right) + \widetilde{R}_{n+1}^{ij},$$

where we define $(\widetilde{R}_{n+1}) = (\widetilde{R}_{n+1}^{ij})_{i,j \in \mathbb{V}}$ by

$$(3.4) \qquad \widetilde{R}_{n+1}^{ij} := \left(\Delta_{ij}^{n+1} - x_{ij}^n \cdot \Delta_T^{n+1} \right) \left(\frac{1}{T_{n+1}} - \frac{1}{T_n} \right).$$

$\widetilde{R}_n^{ij}$ is the increment of a bounded and almost surely converging process: indeed, $0 \leqslant \Delta_{ij}^{n+1} \leqslant a_{ij}$, $0 \leqslant \Delta_T^{n+1} \leqslant \sum_{k,l \in \mathbb{V}} a_{kl}$, so that, for all $k \in \mathbb{N}$,

$$\sum_{n \geqslant k, \, i,j \in \mathbb{V}} \left| \widetilde{R}_{n+1}^{ij} \right| \leqslant 2 \sum_{n \geqslant k, \, i,j \in \mathbb{V}} a_{kl} \left(\frac{1}{T_n} - \frac{1}{T_{n+1}} \right) \leqslant \frac{2 \sum_{k,l \in \mathbb{V}} a_{kl}}{T_k}.$$

Taking the conditional expectation of (3.3) and using (1.1) and (2.2), we have

$$\mathbb{E}(x_{ij}^{n+1} - x_{ij}^n | \mathcal{F}_n) = \frac{x_{ij}^n}{T_n} \left(a_{ij} p_{ij} \frac{x_{ij}^n}{x_i^n x_j^n} - H(x_n) \right) + R_n^{ij} = \frac{F(x_n)_{i,j}}{T_n} + R_n^{ij},$$

where $R_n^{ij} := \mathbb{E} \left(\widetilde{R}_{n+1}^{ij} \Big| \mathcal{F}_n \right)$, and F is defined in (2). Note that $\sum_n |R_n^{ij}| < \infty$ a.s., using (3) and a generalised version of Conditional Borel-Cantelli Lemma, see [5, Lemma 2.7.33].

The following Lemma 3.1 can be deduced from the estimates above; we use the L^1 Euclidean norm $| \cdot |$ on $\mathbb{R}^{|\mathbb{V}| \times |\mathbb{V}|}$.

Lemma 3.1. *There exists an adapted martingale increment process* $(\eta_n)_{n \in \mathbb{N}}$ *such that, for all* $n \in N$,

$$(3.5) \qquad x_{n+1} - x_n = \frac{F(x_n)}{T_n} + \eta_{n+1} + \widetilde{R}_{n+1},$$

and, for all $k \in \mathbb{N}$,

$$(3.6) \qquad |\eta_{n+1}| \leqslant 2 \frac{\sum_{i,j \in \mathbb{V}} a_{ij}}{T_n}, \quad \sum_{n \geqslant k} |\widetilde{R}_{n+1}| \leqslant \frac{2 \sum_{k,l \in \mathbb{V}} a_{kl}}{T_k}.$$

Proof. Note that

$$(3.7) \qquad |x_{n+1} - x_n - \widetilde{R}_{n+1}| = \left| \frac{1}{T_n} \left(\Delta_{ij}^{n+1} - x_{ij}^n \cdot \Delta_T^{n+1} \right) \right| \leqslant \frac{\sum_{i,j\in\mathbb{V}} a_{ij}}{T_n},$$

which provides the upper bound on $|\eta_{n+1}|$; the upper bound on $(R_n)_{n\in\mathbb{N}}$ is given by (3). $\qquad\square$

Corollary 3.7, which proves asymptotic linear growth of T_n, will in particular imply that the martingale $(\sum_{k=1}^n \eta_k)_{n\in\mathbb{N}}$ converges a.s. by Doob's convergence theorem.

Equation (3.5) is a stochastic approximation of (x_n). We will show in Corollary 3.7 that $1/T_n$ is the step size and is of the order of $1/n$. Therefore it is reasonable to expect that (x_n) converges to the set of equilibria of (2.3), which we show in Theorem 2.3.

We first study the evolution of the deterministic process driven by (2.3). Let Γ be the *set of equilibria* of the ODE (2.3), i.e.

$$(3.8) \qquad \Gamma := \left\{ x \in \Delta \,:\, F(x) = 0 \right\}.$$

3.1. **Analysis of the mean-field ODE.** In this section we show that the function H is a Lyapunov function for the ODE (2.3), i.e. that H is nondecreasing along the paths of the ODE. It is natural to expect that result, in the sense that the overall expected payoff of the network should indeed increase in average; see [12] for more details.

We will then deduce that the random process $H(x_n)$ is a submartingale up to an error term (see Theorem 2.2). Recall that this stochastic result does not directly follow from the deterministic statements and classical results, since H is not continuous on the boundary $\partial\Delta$.

Let, for all $x \in \Delta$,

$$(3.9) \qquad p(x) := \sum_{i,j,k\in\mathbb{V}:x_{ij},x_{ik}>0} \frac{x_{ij}x_{ik}}{x_i} \left(y_{ij} - y_{ik} \right)^2,$$

where, given $x \in \Delta \setminus \partial\Delta$ we let, for all $i,j \in \mathbb{V}$ such that $i \sim j$,

$$(3.10) \qquad y_{ij} := a_{ij}p_{ij}\frac{x_{ij}}{x_i x_j}$$

be the weighted *efficiency* of the pair ij.

Proposition 3.2. *H is a Lyapunov function on $\Delta \setminus \partial\Delta$ for the mean-field ODE* (2.3)*, more precisely,*

$$(3.11) \qquad \nabla H \cdot F(x) = p(x) \geqslant 0,$$

where $\nabla H = (\partial H/\partial x_{ij})_{i,j\in\mathbb{V}}$.

Remark. H is not a strict Lyapunov function, i.e. $\nabla H \cdot F$ can vanish outside Γ.

Proof. Note that, in the definition of ∇H, x_{ij} and x_{ji} are independent variables. For any $x \in \Delta \setminus \partial\Delta$, we have

$$
\begin{aligned}
\nabla H \cdot F(x) \;=\; & \sum_{i,j \in \mathbb{V}} a_{ij} p_{ij} \left\{ 2 \frac{x_{ij}^2}{x_i x_j} \left(a_{ij} p_{ij} \frac{x_{ij}}{x_i x_j} - H(x) \right) \right. \\
& - \frac{x_{ij}^2}{x_i^2 x_j} \left(\sum_{k \sim i} a_{ik} p_{ik} \frac{x_{ik}^2}{x_i x_k} - x_{ik} H(x) \right) \\
& \left. - \frac{x_{ij}^2}{x_i x_j^2} \left(\sum_{l \sim j} a_{jl} p_{jl} \frac{x_{jl}^2}{x_j x_l} - x_{jl} H(x) \right) \right\} \\
\;=\; & 2 \sum_{i,j \in \mathbb{V}} a_{ij}^2 p_{ij}^2 \frac{x_{ij}^3}{(x_i x_j)^2} \\
& - \sum_{i,j,k \in \mathbb{V}} a_{ij} p_{ij} \frac{x_{ij}^2}{x_i^2 x_j} a_{ik} p_{ik} \frac{x_{ik}^2}{x_i x_k} - \sum_{i,j,l \in \mathbb{V}} a_{ij} p_{ij} \frac{x_{ij}^2}{x_i x_j^2} a_{jl} p_{jl} \frac{x_{jl}^2}{x_j x_l} \\
& - H(x) \times \sum_{i,j \in \mathbb{V}} a_{ij} p_{ij} \frac{x_{ij}^2}{x_i x_j} \left(2 - \frac{1}{x_i} \sum_{k \sim i} x_{ik} - \frac{1}{x_j} \sum_{l \sim j} x_{jl} \right) \\
\;=\; & 2 \sum_{i,j \in \mathbb{V}} a_{ij}^2 p_{ij}^2 \frac{x_{ij}^3}{(x_i x_j)^2} - 2 \sum_{i,j,k \in \mathbb{V}} a_{ij} p_{ij} a_{ik} p_{ik} \frac{x_{ij}^2 x_{ik}^2}{x_i^3 x_j x_k} \\
\;=\; & \sum_{i,j,k \in \mathbb{V}} a_{ij}^2 p_{ij}^2 \frac{x_{ij}^3 x_{ik}}{x_i^3 x_j^2} + \sum_{i,j,k \in \mathbb{V}} a_{ik}^2 p_{ik}^2 \frac{x_{ik}^3 x_{ij}}{x_i^3 x_k^2} \\
& - \sum_{i,j,k \in \mathbb{V}} 2 a_{ij} p_{ij} a_{ik} p_{ik} \frac{x_{ij}^2 x_{ik}^2}{x_i^3 x_j x_k} \\
\;=\; & \sum_{i,j,k \in \mathbb{V}} \frac{x_{ij} x_{ik}}{x_i} \left[a_{ij} p_{ij} \frac{x_{ij}}{x_i x_j} - a_{ik} p_{ik} \frac{x_{ik}}{x_i x_k} \right]^2 \\
\;=\; & p(x) \geqslant 0.
\end{aligned}
$$

We used, in the third equality, that $\sum_{k \sim i} x_{ik} = x_i$ (and the same equality with j instead of i), and we also used relevant substitutions and permutations of the indices. In the last equality, we use definition (3.10) of y_{ij}.

$\square$

Given $x \in \Delta \setminus \partial\Delta$, and $i, j \in \mathbb{V}$ such that $i \sim j$, let

$$(3.12) \qquad N_i(x) := \sum_{k \in \mathbb{V}} \frac{x_{ik}}{x_i} \cdot y_{ik}$$

be the *weighted efficiency* $N_i(x)$ of i.

Lemma 3.3. *For any $x \in \Delta \setminus \partial\Delta$:*

$$(3.13) \qquad p(x) = \nabla H \cdot F(x) = 2 \sum_{i,j \in \mathbb{V}} x_{ij} \Big(y_{ij} - N_i(x) \Big)^2.$$

Remark. In the context of communication systems, the above two formulas (3.11), (3.13) mean that the growth rate of the expected payoff is a function depending on the difference between efficiencies of different strategy pairs.

Proof of Lemma 3.3. Fix $i \in \mathbb{V}$, and define a measure $\mathbb{P}_i$ and a random variable Y such that, for all $k \in \mathbb{V}$, $\mathbb{P}_i(Y = y_{ik}) := \frac{x_{ik}}{x_i}$. We denote $\mathbb{E}_i$ the expectation associated with $\mathbb{P}_i$. Recalling (3.12), we have

$$\mathbb{E}_i(Y) = N_i(x).$$

This implies that, for all $j \sim i$,

$$\mathbb{E}_i\Big[\big(y_{ij} - Y\big)^2\Big] = \Big(y_{ij} - N_i(x)\Big)^2 + \mathbb{E}_i\Big[\big(Y - N_i(x)\big)^2\Big]$$
$$= \Big(y_{ij} - N_i(x)\Big)^2 + \sum_k \frac{x_{ik}}{x_i}\Big(N_i(x) - y_{ik}\Big)^2.$$

On the other hand,

$$\mathbb{E}_i\Big[\big(y_{ij} - Y\big)^2\Big] = \sum_k \frac{x_{ik}}{x_i}\big(y_{ij} - y_{ik}\big)^2.$$

Therefore, for any $x \in \Delta \setminus \partial\Delta$,

$$\nabla H \cdot F(x) = \sum_{i,j,k \in \mathbb{V}} \frac{x_{ij}x_{ik}}{x_i}\Big[y_{ij} - y_{ik}\Big]^2 = \sum_{i,j \in \mathbb{V}} x_{ij} \sum_k \frac{x_{ik}}{x_i}\Big[y_{ij} - y_{ik}\Big]^2$$
$$= \sum_{i,j \in \mathbb{V}} x_{ij}\Big(y_{ij} - N_i(x)\Big)^2 + \sum_{i,j,k \in \mathbb{V}} \frac{x_{ij}x_{ik}}{x_i}\Big(N_i(x) - y_{ik}\Big)^2$$
$$= 2 \sum_{i,j \in \mathbb{V}} x_{ij}\Big(y_{ij} - N_i(x)\Big)^2,$$

switching labels of k and j on the last sum of the penultimate line.   □

Let us define

$$(3.14) \qquad \Lambda := \{x \in \Delta : p(x) = 0\},$$

where p, defined in (3.9), is the derivative of H along a trajectory of the ODE (2.3). The proof of the following Lemma 3.4 is straightforward.

Lemma 3.4. $x \in \Lambda$ *if and only if*

$$y_{ij} = y_{ik}, \text{ for all } i, j, k \in \mathbb{V} \text{ s.t. } x_{ij} \neq 0, x_{ik} \neq 0.$$

Remark 3.5. $x \in \Lambda$ *iff the weighted efficiencies on edges y_e are constant over the connected components of $(G_x, \overset{x}{\sim})$. Now $x \in \Gamma$ iff y_e is equal to $H(x)$ for any edge e of G_x. Therefore $\Gamma \subseteq \Lambda$ but $\Lambda \neq \Gamma$ in general, and H is not a strict Lyapounov function.*

3.2. Proof of Theorem 2.2 and asymptotic linear growth of (T_n).

Proof of Theorem 2.2. In order to compute the conditional expectation of the increment of $(H(x_n))$, let us first estimate the increment

$$\xi_{ij}^{n+1} = \frac{(V_{ij}^{n+1})^2}{V_i^{n+1} V_j^{n+1}} - \frac{(V_{ij}^n)^2}{V_i^n V_j^n}.$$

By Taylor expansion,

$$\frac{(1+a)^2}{(1+b)(1+c)} = 1 + 2a - b - c + O(a^2 + b^2 + c^2)$$

for all a, b, $c > 0$.

We use notation Δ_{ij}^{n+1} from (3.1), and also define $\Delta_i^{n+1} := \sum_j \Delta_{ij}^{n+1}$. We have

$$
\begin{aligned}
\xi_{ij}^{n+1} &= \frac{(V_{ij}^n)^2}{V_i^n V_j^n} \left[\frac{(1 + \Delta_{ij}^{n+1}/V_{ij}^n)^2}{(1 + \Delta_i^{n+1}/V_i^n)(1 + \Delta_j^{n+1}/V_j^n)} - 1 \right] \\
&= \frac{(V_{ij}^n)^2}{V_i^n V_j^n} \left(2\frac{\Delta_{ij}^{n+1}}{V_{ij}^n} - \frac{\Delta_i^{n+1}}{V_i^n} - \frac{\Delta_j^{n+1}}{V_j^n} + \epsilon_{n+1}^{ij} \right) \\
&= \frac{2V_{ij}^n \Delta_{ij}^{n+1}}{V_i^n V_j^n} - \frac{(V_{ij}^n)^2 \Delta_i^{n+1}}{(V_i^n)^2 V_j^n} - \frac{(V_{ij}^n)^2 \Delta_j^{n+1}}{V_i^n (V_j^n)^2} + \zeta_{n+1}^{ij},
\end{aligned}
$$

where $|\epsilon_{n+1}^{ij}|$ and $|\zeta_{n+1}^{ij}|$ are upper bounded by

$$(3.15) \qquad C \left(\left(\frac{\Delta_{ij}^{n+1}}{V_{ij}^n} \right)^2 + \left(\frac{\Delta_i^{n+1}}{V_i^n} \right)^2 + \left(\frac{\Delta_j^{n+1}}{V_j^n} \right)^2 \right),$$

for some universal constant $C > 0$.

Note that, since the harmonic series converges we have, for all $i, j \in \mathbb{V}$,

$$\sum_{n \in \mathbb{N}} |\zeta_{n+1}^{ij}| < \infty.$$

Let $\zeta_{n+1} = \sum_{i,j\in\mathbb{V}} \zeta_{n+1}^{ij}$.

On the other hand, summing our estimate over i, j, we deduce that

$$
\begin{aligned}
H(x_{n+1}) \quad - \quad & H(x_n) - \zeta_{n+1} \\
= \quad & 2\sum_{i,j\in\mathbb{V}} a_{ij}p_{ij}\frac{V_{ij}^n \Delta_{ij}^{n+1}}{V_i^n V_j^n} - 2\sum_{i,j\in\mathbb{V}} a_{ij}p_{ij}\frac{(V_{ij}^n)^2 \Delta_i^{n+1}}{(V_i^n)^2 V_j^n} \\
= \quad & \sum_{i,j,k\in\mathbb{V}} a_{ij}p_{ij}\frac{V_{ij}^n V_{ik}^n}{(V_i^n)^2 V_j^n}\Delta_{ij}^{n+1} + \sum_{i,j,k\in\mathbb{V}} a_{ik}p_{ik}\frac{V_{ik}^n V_{ij}^n}{(V_i^n)^2 V_k^n}\Delta_{ik}^{n+1} \\
& -2\sum_{i,j,k\in\mathbb{V}} a_{ij}p_{ij}\frac{(V_{ij}^n)^2}{(V_i^n)^2 V_j^n}\Delta_{ik}^{n+1}.
\end{aligned}
$$

Recall that $\mathbb{E}\left(\Delta_{ik}^{n+1}|\mathcal{F}_n\right) = a_{ik}p_{ik}\frac{(x_{ik}^n)^2}{x_i^n x_k^n}$, for all $i,k\in\mathbb{V}$. Finally, taking the conditional expectation yields

$$
\mathbb{E}\left[H(x_{n+1}) - H(x_n) - \zeta_{n+1}\big|\mathcal{F}_n\right]
$$

$$
= \sum_{i,j,k\in\mathbb{V}} \frac{V_{ij}^n V_{ik}^n}{V_i^n}\left[\left(a_{ij}p_{ij}\frac{V_{ij}^n}{V_i^n V_j^n}\right)^2 + \left(a_{ik}p_{ik}\frac{V_{ik}^n}{V_i^n V_k^n}\right)^2 - 2a_{ij}p_{ij}a_{ik}p_{ik}\frac{V_{ij}^n V_{ik}^n}{\left(V_i^n V_j^n\right)^2}\right]
$$

$$
= \frac{1}{T_n}\sum_{i,j,k\in\mathbb{V}} \frac{x_{ij}^n x_{ik}^n}{x_i^n}\left[a_{ij}p_{ij}\frac{x_{ij}^n}{x_i^n x_j^n} - a_{ik}p_{ik}\frac{x_{ik}^n}{x_i^n x_k^n}\right]^2 = \frac{1}{T_n}p(x_n),
$$

where p is defined in (3.9). $\square$

We have proved that the expected payoff process converges almost surely. We can now prove that the process (x_n) converges to Λ, defined in (3.14).

Proposition 3.6. $(x_n)_{n\in\mathbb{N}}$ converges a.s. to Λ. More precisely, $(p(x_n))_{n\in\mathbb{N}}$ converges to 0 a.s.

Define for all $\varepsilon > 0$ the set

(3.16) $$\Delta_\varepsilon := \{x \in \Delta \setminus \partial\Delta : p(x) > \varepsilon\}.$$

Proof. This argument is similar to the proof of convergence to the set of equilibria in [1]. Let, for all $n \in \mathbb{N}$,

$$\xi_n = \sum_{k=0}^n \zeta_k.$$

We have proved that $(H(x_n) - \xi_n)$ is a bounded submartingale. Thanks to the Doob decomposition, we write it as the sum of a martingale (M_n) and an increasing predictable process (A_n). For all $n \in \mathbb{N}$, we have:

$$H(x_n) - \xi_n = M_n + A_n.$$

(M_n) is an upper-bounded martingale, hence a.s. converges. On the other hand,

$$A_{n+1} - A_n = \mathbb{E}\Big[H(x_{n+1}) - H(x_n) - \zeta_{n+1}\big|\mathcal{F}_n\Big] = \frac{1}{T_n}p(x_n).$$

Fix $\varepsilon > 0$ and define $\delta > 0$ the distance between the sets Δ_ε and $\Delta^c_{\varepsilon/2}$. First, notice that x_n cannot stay in Δ_ε for ever as $1/T_n$ is not summable. Then, assume that $x_n \in \Delta_\varepsilon$, $x_{n+1},...,x_{n+k-1} \in \Delta_{\varepsilon/2} \cap \Delta^c_\varepsilon, x_{n+k} \in \Delta^c_{\varepsilon/2}$, then:

$$A_{n+k} - A_n = \sum_{r=n}^{n+k-1} \frac{p(x_r)}{T_r} \geqslant \varepsilon \sum_{r=n}^{n+k-1} \frac{1}{2T_r}.$$

Therefore, using (3.7) and Lemma 3.1, we have

$$\delta \leqslant \sum_{r=n}^{n+k-1} |x_{r+1} - x_r| \leqslant \sum_{r=n}^{n+k-1} \frac{\mathsf{Cst}(a)}{T_r} \leqslant \frac{\mathsf{Cst}(a)}{\varepsilon}(A_{n+k} - A_n).$$

Then, if (x_n) were infinitely often away from Λ, A_n, and consequently $H(x_n)$, would explode. This implies the conclusion. $\qquad\square$

We end this section by stating a result that implies that the process (T_n) has a linear asymptotic growth.

Define the constant

(3.17) $$w_{\min} = \min\left\{a_{ij}p_{ij} : a_{ij}p_{ij} > 0\right\}.$$

Corollary 3.7. *There exists a constant $h_{\min} = \mathsf{Cst}((V^0_{ij})_{i,j\in\mathbb{V}}) > 0$ such that, a.s.*

$$\frac{T_n}{n} \to \lim_{n\to\infty} H(x_n) \in \left[h_{\min}w_{\min}, \sum_{i,j\in\mathbb{V}} a_{ij}p_{ij}\right] \quad as \ n \to \infty.$$

Moreover, for all $i \in \mathbb{V}$ such that $a_{ij}p_{ij} \neq 0$ for some $j \sim i$, $n \in \mathbb{N}$,

(3.18) $$N_i(x_n) = \sum_{j\in\mathbb{V},\, x_j>0} \frac{(x^n_{ij})^2}{(x^n_i)^2 x^n_j} \geqslant h_{\min}.$$

Proof. Given $x \in \Delta$, recall definition (3.12) of $N_i(x)$. Using that $\sum_{j\in\mathbb{V}} x_j = 1$, we have, by Cauchy-Schwarz inequality, if $x_i \neq 0$,

$$\left(\sum_{j\in\mathbb{V},\, x_j>0} \frac{x^2_{ij}}{x^2_i x_j}\right) \sum_{j\in\mathbb{V}} x_j \geqslant \left(\sum_{j:a_{ij}p_{ij}>0,\, x_j>0} \frac{x_{ij}}{x_i}\right)^2.$$

For all $j \sim i$ with $a_{ij}p_{ij} = 0$, we have $V_{ij}^n = v_{ij}^0$ a.s. for all $n \in \mathbb{N}$. Hence

(3.19)
$$\sum_{j:a_{ij}p_{ij}>0} \frac{x_{ij}^n}{x_i^n} = 1 - \sum_{j:a_{ij}p_{ij}=0} \frac{v_{ij}^0}{V_i^n} \geqslant 1 - \sum_{j:a_{ij}p_{ij}=0} \frac{v_{ij}^0}{V_i^0} = \sum_{j:a_{ij}p_{ij}>0} \frac{x_{ij}^0}{x_i^0} \geqslant h_{\min} \text{ a.s.}$$

for some constant $h_{\min} > 0$. Therefore, letting $B = \{i \in \mathbb{V} : a_{ij}p_{ij} > 0 \text{ for some } j \in \mathbb{V}\}$, for any $n \in \mathbb{N}$,

$$H(x_n) = \sum_{i \in \mathbb{V}} x_i^n N_i(x_n) \geqslant \sum_{i \in B} x_i^n N_i(x_n) \geqslant h_{\min} w_{\min} > 0 \text{ a.s. },$$

which gives us a lower bound on $\liminf_n H(x_n)$. The upper bound on H is trivial.

Finally, recall that

$$\mathbb{E}\left(T_{n+1} - T_n \,\middle|\, \mathcal{F}_n\right) = H(x_n)$$

and the result is now a direct consequence of Theorem 2.2 and a generalised version of Conditional Borel-Cantelli Lemma, see [5, Lemma 2.7.33].   □

3.3. **Proof of Theorem 2.3.** In the previous section, we proved that the occupation measure (x_n) converges a.s. to Λ, which is the set of points where the derivative of the Lyapunov function H vanishes. As we already mentioned in Remark 3.5, H is not a strict Lyapunov function, i.e. $\Gamma \subseteq \Lambda$ but $\Gamma \neq \Lambda$, where Γ is the set of equilibria of the ODE (2.3), defined in (3.8).

The aim of this section is to prove that the occupation measure converges a.s. to the set of equilibria of the ODE (2.3). To that end we show the following Proposition 3.8.

Proposition 3.8. *Let $i, j \in \mathbb{V}$, $i \sim j$. Then*

$$\limsup_{n \to \infty} x_{ij}^n \left(y_{ij}^n - H(x_n)\right)^- = 0.$$

Proposition 3.8 will enable us to conclude. Indeed, for all $n \in \mathbb{N}$, we have

$$H(x_n) = \sum_{i,j \in \mathbb{V}} y_{ij}^n x_{ij}^n = H(x_n) + \sum_{i,j \in \mathbb{V}} (y_{ij}^n - H(x_n))^+ x_{ij}^n - \sum_{i,j \in \mathbb{V}} (y_{ij}^n - H(x_n))^- x_{ij}^n$$

Now, for all $i, j \in \mathbb{V}$, Proposition 3.8 implies that $\lim_{n \to \infty} x_{ij}^n (y_{ij}^n - H(x_n))^- = 0$. Subsequently, for all $i, j \in \mathbb{V}$,

(3.20)
$$\lim_{n \to \infty} x_{ij}^n (y_{ij}^n - H(x_n)) = 0,$$

so that $F(x_n)$ converges to 0 as $n \to \infty$.

Let us show that this implies convergence of $(x_n)_{n\in\mathbb{N}}$ to the set of equilibria Γ. Fix $\epsilon > 0$ and let $(z_n)_{n\in\mathbb{N}}$ be defined as follows. For all $n \in \mathbb{N}$, i, $j \in \mathbb{V}$, let

$$h_n = \left(\sum_{i,j\in\mathbb{V}:\, x_n^{ij}>\epsilon} x_i^n x_j^n \right)^{-1},$$

and let $z_{ij}^n = x_i^n x_j^n h_n$ if $x_{ij}^n > \epsilon$, $z_{ij}^n = 0$ otherwise.

It is easy to check that $h_n = H(z_n)$ and $z_n \in \Gamma$ for all $n \in \mathbb{N}$. We easily deduce from (3.20) that $|h_n - H(x_n)| \leqslant \epsilon\mathsf{Cst}(a, |\mathbb{V}|)$ and $|x_n - z_n| \leqslant \epsilon\mathsf{Cst}(a, |\mathbb{V}|)$ for large $n \in \mathbb{N}$.

Let us now sketch the proof of Proposition 3.8, given in Section 3.3.3. Define

(3.21) $\qquad U_{ij}(\varepsilon) := \big\{ x \in \Delta : x_{ij} < \varepsilon \text{ or } y_{ij} - H(x) \geqslant -\varepsilon \big\},$

and recall the definitions of Δ_ε and $h_{\min}$ respectively introduced in (3.16) and in Corollary 3.7.

Proposition 3.8 will follow from Lemma 3.9, which implies that both x_{ij}^n (resp. y_{ij}^n) decreases (resp. increases) outside $U_{ij}(\varepsilon) \cup \Delta_{\varepsilon^4}$.

Lemma 3.9. *Let i, $j \in \mathbb{V}$, $i \sim j$, with $a_{ij}p_{ij} > 0$. Let*

$$Y_n \;\; := \;\; \sum_{m=m_0}^{n} \left(y_{ij}^m - y_{ij}^{m-1} - a_{ij}p_{ij}\frac{\varepsilon^2}{6T_{m-1}} \right) \mathbb{1}_{\{x_{m-1}\notin U_{ij}(\varepsilon)\cup\Delta_{\varepsilon^4}\}};$$

$$S_n \;\; := \;\; \sum_{m=m_0}^{n} \left(x_{ij}^m - x_{ij}^{m-1} + \frac{\varepsilon^2}{2T_{m-1}} \right) \mathbb{1}_{\{x_{m-1}\notin U_{ij}(\varepsilon)\}}.$$

Assume $T_{m_0} \geqslant \mathsf{Cst}(a_{ij}p_{ij}, a, \epsilon)$ and $\epsilon \in (0, 1/9)$. Then

(1) *$(Y_n)_{n\geqslant m_0}$ (resp. $(S_n)_{n\geqslant m_0}$) is a submartingale (resp. supermartingale);*

(2) *$\limsup_{n\geqslant m, m\to\infty}(Y_n - Y_m)^- = \limsup_{n\geqslant m, m\to\infty}(S_n - S_m)^+ = 0$.*

Lemma 3.9 is shown is Section 3.3.2. It follows from Lemma 3.1 for its part on x_{ij}^n, and from the following two Lemmas 3.10 and 3.11 for its part on y_{ij}^n.

Lemma 3.10 is proved in Section 3.3.1, and shows that the increment of y_{ij}^n is driven by a differential equation of the type $\dot{y} = G(x)$, where, for all $x \in \Delta$, we let

$$G(x) = \Big(\big(y_{ij}\big(y_{ij} - N_i(x) - N_j(x) + H(x) \big) \big) \Big)_{i,j\in\mathbb{V}:\, x_{ij}>0}.$$

Outside $U_{ij}(\varepsilon) \cup \Delta_{\varepsilon^4}$, Lemma 3.11 implies that

$$G(x)_{ij} \approx y_{ij}(H(x) - y_{ij}),$$

which indeed yields that y_{ij}^n increases on average.

Lemma 3.10. *For all i, $j \in \mathbb{V}$, the increment of $(y_{ij}^n)_n$ expands as follows:*

(3.22) $$y_{ij}^{n+1} - y_{ij}^n = \frac{1}{T_n}G(x_n)_{ij} + r_{ij}^{n+1} + \zeta_{ij}^{n+1},$$

where (r_{ij}^n) is predictable, $\mathbb{E}(\zeta_{ij}^{n+1}|\mathcal{F}_n) = 0$ and

$$|y_{ij}^{n+1} - y_{ij}^n|, \ |\zeta_{ij}^{n+1}| \leqslant \frac{c_0}{T_n x_i^n x_j^n}, \ \text{and} \ |r_{ij}^{n+1}| \leqslant \frac{c_0}{(T_n x_i^n x_j^n)^2},$$

where $c_0 = \mathsf{Cst}(a)$.

Lemma 3.11. *Assume that $\varepsilon \in (0, 1/9)$, and that $x \in \Delta_{\varepsilon^4}^c$. Then, for all vertices i and j, $i \sim j$, such that $x_{ij} > \varepsilon$, we have:*

$$|y_{ij} - N_i(x)| < \frac{\varepsilon}{3} \ \text{and} \ |y_{ij} - N_j(x)| < \frac{\varepsilon}{3}.$$

Proof. Follows directly from Lemma 3.3.   □

3.3.1. *Proof of Lemma 3.10.* Using $(1 + c)^{-1} = 1 - c + O(c^2)$ for all $c \geqslant 0$, we deduce that

(3.23) $$f(s, t, u, v) = \frac{(1 + s)(1 + t)}{(1 + u)(1 + v)} = 1 + s + t - u - v + O(\eta(\epsilon + \eta))$$

if $t \in [0, \epsilon]$ and s, u, $v \in [0, \eta]$.

Now recall $y_{ij} = a_{ij}p_{ij}T_n V_{ij}^n/(V_i^n V_j^n)$: therefore,

$$y_{ij}^{n+1} = y_{ij}^n f(s_{n+1}, t_{n+1}, u_{n+1}, v_{n+1}),$$

where

$$s_{n+1} = \frac{T_{n+1} - T_n}{T_n}, \ t_{n+1} = \frac{V_{ij}^{n+1} - V_{ij}^n}{V_{ij}^n},$$

$$u_{n+1} = \frac{V_i^{n+1} - V_i^n}{V_i^n}, \ v_{n+1} = \frac{V_j^{n+1} - V_j^n}{V_j^n}.$$

Let

$$\delta_{ij}^{n+1} = y_{ij}^n(s_{n+1} + t_{n+1} - u_{n+1} - v_{n+1}), \ \zeta_{ij}^{n+1} = \delta_{ij}^{n+1} - \mathbb{E}(\delta_{ij}^{n+1}|\mathcal{F}_n).$$

Note that ζ_{ij}^{n+1} uniquely determines r_{ij}^{n+1} from (3.22).

By definition of H, N_i and y_{ij}, we have

$$\mathbb{E}(s_{n+1}|\mathcal{F}_n) = \frac{H(x_n)}{T_n}, \ \mathbb{E}(t_{n+1}|\mathcal{F}_n) = \frac{y_{ij}^n}{T_n},$$

$$\mathbb{E}(u_{n+1}|\mathcal{F}_n) = \frac{N_i(x_n)}{T_n}, \ \mathbb{E}(v_{n+1}|\mathcal{F}_n) = \frac{N_j(x_n)}{T_n},$$

so that

$$\mathbb{E}(\delta_{ij}^{n+1}|\mathcal{F}_n) = \frac{1}{T_n}y_{ij}^n\Big(y_{ij}^n - N_i(x_n) - N_j(x_n) + H(x_n)\Big).$$

On the other hand, $s_{n+1}T_n$, $t_{n+1}V_{ij}^n$, $u_{n+1}V_i^n$ and $v_{n+1}V_j^n$ are all upper bounded by $\mathsf{Cst}(a)$, which implies that

$$|\delta_{ij}^{n+1}| \leqslant \mathsf{Cst}(a)\frac{y_{ij}^n}{V_{ij}^n} = \frac{\mathsf{Cst}(a)}{T_n x_i^n x_j^n}.$$

Also, assume w.l.o.g. that $V_i^n \leqslant V_j^n$; then it follows from (3.23) that

$$|r_{ij}^{n+1}| \leqslant \mathsf{Cst}(s_{n+1} \vee u_{n+1} \vee v_{n+1})t_{n+1}y_{ij}^n \leqslant \frac{\mathsf{Cst}(a)}{V_{ij}^n V_i^n}\frac{T_n V_{ij}^n}{V_i^n V_j^n} \leqslant \mathsf{Cst}(a)\left(\frac{T_n}{V_i^n V_j^n}\right)^2,$$

which enables us to conclude, noting that $|y_{ij}^{n+1} - y_{ij}^n| \leqslant |\delta_{ij}^{n+1}| + |r_{ij}^{n+1}|$.

3.3.2. *Proof of Lemma 3.9.* Assume $x_n \notin U_{ij}(\epsilon) \cup \Delta_{\epsilon^4}$.

Using that $|\tilde{R}_{n+1}^{ij}| \leqslant \mathsf{Cst}(a)/T_n^2$ from (3.4), Lemma 3.1 implies

$$\mathbb{E}(x_{ij}^{n+1} - x_{ij}^n|\mathcal{F}_n) = \frac{x_{ij}^n}{T_n}\big(y_{ij}^n - H(x_n)\big) + \mathbb{E}(\tilde{R}_{n+1}^{ij}|\mathcal{F}_n)$$

$$\leqslant -\frac{\varepsilon^2}{T_n} + \frac{\mathsf{Cst}(a)}{T_n^2} \leqslant -\frac{\varepsilon^2}{2T_n},$$

if $n \geqslant m_0$ and $T_{m_0} \geqslant \mathsf{Cst}(a, \epsilon)$, so that $(S_n)_{n \geqslant m_0}$ is a supermartingale.

Also, using Lemma 3.11,

$$y_{ij}^n - N_i(x_n) - N_j(x_n) + H(x_n) \geqslant \frac{\varepsilon}{3}.$$

Hence, using now Lemma 3.10,

$$\mathbb{E}\big(y_{ij}^{n+1} - y_{ij}^n|\mathcal{F}_n\big) \geqslant \frac{\epsilon}{3}\frac{y_{ij}^n}{T_n} - \frac{c_0}{(T_n x_i^n x_j^n)^2} = \frac{1}{T_n x_i^n x_j^n}\left(\frac{a_{ij}p_{ij}x_{ij}^n\varepsilon}{3} - \frac{c_0}{T_n x_i^n x_j^n}\right)$$

$$\geqslant \frac{1}{T_n x_i^n x_j^n}\left(\frac{a_{ij}p_{ij}\varepsilon^2}{3} - \frac{c_0}{T_n \varepsilon^2}\right) \geqslant \frac{\mathsf{Cst}(a_{ij}p_{ij}, a, \epsilon)}{T_n}$$

if $n \geqslant m_0$ and $T_{m_0} \geqslant \mathsf{Cst}(a_{ij}p_{ij}, a, \epsilon)$. This concludes the proof of (1).

In order to prove (2), define Π_n and Ξ_n as the martingale parts in the Doob decompositions of Y_n and S_n:

$$\Pi_n := Y_n - \sum_{m=m_0}^n \mathbb{E}\big[Y_m - Y_{m-1}|\mathcal{F}_n\big],$$

$$\Xi_n := S_n - \sum_{m=m_0}^n \mathbb{E}\big[S_m - S_{m-1}|\mathcal{F}_n\big].$$

By Lemma 3.10, we have for all $n \geqslant m_0$,

$$\mathbb{E}\big[\big(\Pi_{n+1} - \Pi_n\big)^2\big|\mathcal{F}_n\big] \leqslant \mathbb{E}\big[\big(\zeta_{ij}^{n+1}\big)^2\mathbb{1}_{\{x_n \notin U_{ij}(\varepsilon) \cup \Delta_{\varepsilon^4}\}}\big|\mathcal{F}_n\big]$$

$$\leqslant \frac{c_0^2}{(T_n x_i^n x_j^n)^2} \mathbb{1}_{\{x_n \notin U_{ij}(\varepsilon) \cup \Delta_{\varepsilon^4}\}} \leqslant \frac{c_0^2}{\varepsilon^4 T_n^2}.$$

Therefore, using Corollary 3.7, $(\Pi_n)_n$ is bounded in $\mathbb{L}^2$ and hence converges a.s. We conclude for $(\Xi_n)_n$ with similar computations, and (2) follows immediately.

3.3.3. *Proof of Proposition 3.8.* Let us first prove the following auxilliary Lemma 3.12.

Lemma 3.12. *Let $\varepsilon > 0$, i, $j \in \mathbb{V}$, $i \sim j$, and assume $T_n \geqslant \mathsf{Cst}(a, \epsilon)$. If $x_n \in U_{ij}(\varepsilon)$ and $|H(x_{n+1}) - H(x_n)| < \varepsilon/2$, then $x_{n+1} \in U_{ij}(2\varepsilon)$.*

Proof. Assume $x_n \in U_{ij}(\varepsilon)$. Then either $x_{ij}^n \leqslant \varepsilon$ or $y_{ij}^n - H(x_n) \geqslant -\varepsilon$.

If $x_{ij}^n \leqslant \varepsilon$ then, using (3.6)–(3.7), we deduce $|x_{ij}^{n+1} - x_{ij}^n| \leqslant \mathsf{Cst}(a)/T_n \leqslant \epsilon$ if $T_n \geqslant \mathsf{Cst}(a, \epsilon)$, so that $x_{ij}^{n+1} \leqslant 2\varepsilon$.

If $x_{ij}^n > \varepsilon$ and $y_{ij}^n - H(x_n) \geqslant -\varepsilon$, then $|y_{ij}^{n+1} - y_{ij}^n| \leqslant \mathsf{Cst}(a)/(\epsilon^2 T_n) \leqslant \epsilon/2$ if $T_n \geqslant \mathsf{Cst}(a, \epsilon)$ by Lemma 3.10. By assumption, $|H(x_{n+1}) - H(x_n)| < \varepsilon/2$, so that we conclude that $y_{ij}^{n+1} - H(x_{n+1}) \geqslant -2\varepsilon$. $\square$

Let us now show Proposition 3.8. We fix $\varepsilon > 0$ and $m_0 \in \mathbb{N}$, and assume $T_{m_0} \geqslant \mathsf{Cst}(a_{ij} p_{ij}, a, \epsilon)$, so that the assumptions of Lemmas 3.9 and 3.12 hold. Let τ_{m_0} be the stopping time

$$\tau_{m_0} = \inf \left\{ n \geqslant m_0 : x_n \in \Delta_{\varepsilon^4} \text{ or } |H(x_n) - H(x_{m_0})| > \frac{\varepsilon}{4} \right\}.$$

We want to prove that either $\tau_{m_0} < \infty$, or $x_n \in U_{ij}(3\varepsilon)$ for all large n. Recall that $U_{ij}(\cdot)$ is defined in (3.21). This will allow us to conclude, as Proposition 3.6 and Theorem 2.2 imply that there exists almost surely $m_0 \in \mathbb{N}$ s.t. $\tau_{m_0} = \infty$.

Let σ_{m_0} be the stopping time

$$\sigma_{m_0} := \inf\{n \geqslant m_0 : x_n \in U_{ij}(\varepsilon)\}.$$

Lemma 3.9 (2) implies that there exists a.s. a (random) $m_0 \in \mathbb{N}$ such that, for all $n \geqslant m \geqslant m_0$,

$$(3.24) \qquad (Y_n - Y_m)^- \leqslant \frac{\varepsilon}{2}, \qquad (S_n - S_m)^+ \leqslant \frac{\varepsilon}{2}.$$

Therefore $\sigma_{m_0} < \infty$: indeed, otherwise $y_{ij}^n \to \infty$ as $n \to \infty$ by Lemma 3.9 (recall $\tau_{m_0} = \infty$, since $\sum_{n \geqslant m_0} T_n^{-1} = \infty$, so that $x_n \in U_{ij}^{\epsilon}$ for large n, which leads to a contradiction.

For all $n \geqslant \sigma_{m_0}$, let ρ_n be the largest $k \leqslant n$ such that $x_k \in U_{ij}(\varepsilon)$. By (3.24),

$$(y_{ij}^n - y_{ij}^{\rho_n+1}) \geqslant -\frac{\varepsilon}{2}.$$

By Lemma 3.12, $x_{\rho_n+1} \in U_{ij}(2\varepsilon)$: let us assume for instance that $y_{ij}^{\rho_n+1} - H(x_{\rho_n+1}) \geqslant -2\varepsilon$. Together with $|H(x_n) - H(x_{\rho_n+1})| \leqslant \varepsilon/2$, as $\tau_{m_0} = \infty$, we deduce that

$$y_{ij}^n - H(x_n) \geqslant y_{ij}^{\rho_n+1} - H(x_{\rho_n+1}) - \varepsilon \geqslant -3\varepsilon.$$

With a similar argument, $x_{ij}^{\rho_n+1} \leqslant 2\varepsilon$ implies $x_{ij}^n \leqslant 3\varepsilon$. These two arguments together imply that $x_n \in U_{ij}(3\varepsilon)$ if $n \geqslant \sigma_{m_0}$, which enables us to conclude.

4. Asymptotic behavior near the boundary

The results presented in Sections 5 and 6 concern respectively the characterisation of stable equilibria and convergence towards them on $\Delta \setminus \partial\Delta$.

However, as we mentioned at the end of Section 2, a full analysis of the asymptotics of the process $(x_n)_{n\in\mathbb{N}}$ would require a better understanding of the behavior of the dynamics close to the boundary $\partial\Delta$. This can be understood through the following simple example.

Let G be a star-shaped connected graph consisting of $k + 1$ vertices $\mathbb{V} = \{v, u_1, ..., u_k\}$, where v is the core adjacent to all vertices u_l, $l = 1 \ldots k$, and any two vertices u_l and u_m, $l \neq m$, are not adjacent. Assume w.l.o.g that the sequence a_{vu_l}, $k = 1 \ldots k$, is nonincreasing in i, and consider the case where $p_\mathbb{V} = 1$.

Consider our process $((V_{ij}^n)_{i,j\in\mathbb{V}})_{n\in\mathbb{N}}$ on G. Then, for all $l = 1 \ldots k$, $V_{u_l,v} = V_{u_l}$, so that the probability to choose edge $\{u_l, v\}$ at time n is $V_{u_l,v}^n/V_v^n$ and, in that case, $V_{u_l,v}^{n+1} = V_{u_l,v}^n + a_{vu_l}$.

This corresponds to the so-called Friedman's urn with k colors: when a ball of color l is chosen, then it is put back into the urn, along with a_{vu_l} other balls of the same color. Thus $V_{u_l,v}^n/n^{a_{vu_l}/a_{vu_1}}$ converges a.s. towards a positive r.v. (see for instance [8]).

Assuming for instance that $a_{vu_l} < a_{vu_1}$ for all $l \neq 1$, we deduce in particular that $x_{u_2}^n, ..., x_{u_k}^n$ converge a.s. to 0, and that $x_{u_1}^n, x_v^n$ converge to $1/2$ a.s. Therefore x_n converges to the boundary $\partial\Delta$.

Note that, for any general graph $G = (\mathbb{V}, E, \sim)$, and for all $i \in \mathbb{V}$ such that $a_{ij}p_{ij} > 0$, the convergence to the boundary will occur at a rate at most $n^{\epsilon-1}$ for some $\epsilon > 0$. This is a consequence of (3.18) in Corollary 3.7: the probability that i receives a positive payoff at time $n + 1$ will be at least

$$x_i^n N_i(x_n) = x_i^n \sum_{j\in\mathbb{V}} \frac{x_{ij}^2}{(x_i^n)^2 x_j^n} \geqslant h_{\min} x_i^n = h_{\min} \frac{V_i^n}{T_n}.$$

If we let $L_i^n = h_{\min} V_i^n$, then we can couple the process $(L_i^n)_{n\in\mathbb{N}}$ with a Friedman's urn with 2 colors (i and i^c), so that when a ball of color i is chosen, then it is put back into the urn along with at least $h_{\min} \min_{j\in\mathbb{V}: a_{ij}>0} a_{ij}$ other

balls of color i, and such that in any case (either if i or i^c is chosen), at most $\mathsf{Cst}(a, h_{\min})$ balls of color i^c are added.

Now F is not continuous on the boundary $\partial\Delta$, so that Lemma 3.1 is not useful on the event that the trajectory of x_n has accumulation points in $\partial\Delta$. However it is possible to write the evolution of our network as a stochastic approximation of solutions of a smooth ODE: if we let $z = ((x_i)_{i\in\mathbb{V}}, (y_{ij})_{i,j\in\mathbb{V}})$, then the process $(z_n)_{n\in\mathbb{N}}$ can be seen as a Cauchy Euler approximation of the following ODE:

$$\dot{x}_i = x_i(N_i(x) - H(x)),$$
$$\dot{y}_{ij} = G(y)_{ij} = y_{ij}(y_{ij} - N_i(x) - N_j(x) + H(x)),$$
$$N_i(x) = \sum_{k\in\mathbb{V}} x_k y_{ik}^2.$$

However the perturbation in the evolution of y_{ij}^n in Lemma 3.10 depends on x_i and x_j, and the state space of the ODE on z is not compact anymore.

5. Classification of equilibria and stability

In this section, we analyze the deterministic dynamics associated to the ODE (2.3). In particular, we compute the Jacobian matrix of F and give a characterization of the stable equilibria in $\Delta \setminus \partial\Delta$.

5.1. Properties of Lyapunov function.
In this section, we show that H is constant on each connected component of Γ.

Proposition 5.1. *H is constant on each connected component of Γ.*

We first prove in Lemma 5.2 that H is constant on connected subsets of Γ with the same support (defined below) by a differentiability argument. Then we show in Lemma 5.3 that H is continuous on Γ (including $\Gamma \cap \partial\Delta$), which enables to conclude. Let Θ be the set of subsets of E. For any $x \in \Delta$, we define its support

(5.1) $$E_x := \{ij \in E : x_{ij} > 0\}.$$

Θ can be used as an index set to divide Δ or Γ into several subsets: for any $\theta \in \Theta$,

$$\Delta_\theta := \{x \in \Delta : E_x = \theta\},$$
$$\Gamma_\theta := \Delta_\theta \cap \Gamma.$$

Lemma 5.2. *For any $\theta \in \Theta$, H is constant on each connected component of Γ_θ.*

Proof. Given $q \in \Gamma_\theta$, let us differentiate H at q with respect to $x_{ij} = x_{ji}$, $ij \in \theta$ (i.e. $q_{ij} > 0$) without the constraint $x \in \Delta$:

$$
\begin{aligned}
\left[\frac{\partial H}{\partial x_{ij}}(x)\right]_{x=q} &= \left[\sum_{k,l:kl\in\theta} \frac{\partial}{\partial x_{ij}}\left(a_{kl}p_{kl}\frac{x_{kl}^2}{x_k x_l}\right)\right]_{x=q} \\
&= \left[2\frac{\partial}{\partial x_{ij}}\left(a_{ij}p_{ij}\frac{x_{ij}^2}{x_i x_j}\right) + 2\sum_{k\neq j, ik\in\theta} \frac{\partial}{\partial x_{ij}}\left(a_{ik}p_{ik}\frac{x_{ik}^2}{x_i x_k}\right)\right. \\
&\quad \left. +2\sum_{k\neq i, jk\in\theta} \frac{\partial}{\partial x_{ij}}\left(a_{jk}p_{jk}\frac{x_{jk}^2}{x_j x_k}\right)\right]_{x=q} \\
&= 2a_{ij}p_{ij}\frac{q_{ij}}{q_i q_j}\left(2 - \frac{q_{ij}}{q_i} - \frac{q_{ij}}{q_j}\right) \\
&\quad -2\sum_{k\neq j:ik\in\theta} a_{ik}p_{ik}\frac{q_{ik}}{q_i q_k}\cdot\frac{q_{ik}}{q_i} - 2\sum_{k\neq i:jk\in\theta} a_{jk}p_{jk}\frac{q_{jk}}{q_j q_k}\cdot\frac{q_{jk}}{q_j} \\
&= 2H(q)\left(2 - \frac{q_{ij}}{q_i} - \frac{q_{ij}}{q_j}\right) - 2H(q)\left(1 - \frac{q_{ij}}{q_i}\right) - 2H(q)\left(1 - \frac{q_{ij}}{q_j}\right) \\
&= 0.
\end{aligned}
$$

The penultimate equality comes from the fact that, for all r, $s \in \mathbb{V}$, $rs \in \theta$ (hence $q_{rs} > 0$, since $q \in \Gamma_\theta$), we have $(a_{rs}p_{rs}q_{rs})/(q_r q_s) = H(q)$. $\square$

Lemma 5.3. *H is continuous on Γ.*

Proof. Suppose that $q \in \Gamma$, and that $x \in \Gamma$ is in the neighbourhood of $q \in \Gamma$ within Δ, then $E_x \supseteq E_q$ and, using $x \in \Gamma$,

$$
H(x) = \sum_{i,j:ij\in E_q} a_{ij}p_{ij}\frac{x_{ij}^2}{x_i x_j} + \sum_{i,j:ij\in E_x\setminus E_q} x_{ij}H(x),
$$

so that

$$
H(x) = \frac{1}{1 - \sum_{i,j:ij\in E_x\setminus E_q} x_{ij}} \sum_{i,j:ij\in E_q} a_{ij}p_{ij}\frac{x_{ij}^2}{x_i x_j},
$$

and the conclusion follows. $\square$

5.2. **Jacobian matrix.** At any equilibrium $x \in (\Delta \setminus \partial\Delta) \cap \Gamma$ (F is not differentiable on $\partial\Delta$), we calculate the Jacobian matrix

$$
J(x) = \left(\frac{\partial F_{lk}}{\partial x_{ij}}\right)_{\{i,j\},\{l,k\}:ij,lk\in E},
$$

where, by a slight abuse of notation,

$$
F(x) = (F_{ij}(x))_{\{i,j\}:ij\in E}.
$$

For all ij, $kl \in E$, a simple extension of the calculation in the proof of Lemma 5.2 yields $\frac{\partial H}{\partial x_{ij}}(x) = 4H(x)(\mathbb{1}_{\{x_{ij}\neq 0\}} - 1)$, so that

$$
\begin{aligned}
\frac{\partial F_{lk}}{\partial x_{ij}}(x) &= -H(x)\mathbb{1}_{\{\{i,j\}=\{l,k\},x_{lk}=0\}} + x_{lk}\frac{\partial y_{lk}}{\partial x_{ij}}(x) - x_{lk}\frac{\partial H}{\partial x_{ij}}(x) \\
&= -H(x)\mathbb{1}_{\{\{i,j\}=\{l,k\},x_{lk}=0\}} + H(x)\mathbb{1}_{\{\{i,j\}=\{l,k\},x_{lk}\neq 0\}} - \frac{x_{lk}}{x_i}H(x)\mathbb{1}_{\{i\in\{l,k\}\}} \\
&\quad - \frac{x_{lk}}{x_j}H(x)\mathbb{1}_{\{j\in\{l,k\}\}} - 4x_{lk}H(x)\big(\mathbb{1}_{\{x_{ij}\neq 0\}} - 1\big).
\end{aligned}
$$

Therefore, for $x_{ij} \neq 0$, we deduce

$$
\begin{aligned}
\frac{\partial F_{ij}}{\partial x_{ij}} &= H(x)\left[1 - \frac{x_{ij}}{x_i} - \frac{x_{ij}}{x_j}\right]; \\
\frac{\partial F_{ik}}{\partial x_{ij}} &= -\frac{x_{ik}}{x_i}H(x), k \neq j; \\
\frac{\partial F_{jk}}{\partial x_{ij}} &= -\frac{x_{jk}}{x_j}H(x), k \neq i; \\
\frac{\partial F_{lk}}{\partial x_{ij}} &= 0, l \neq i, j; k \neq i, j.
\end{aligned}
$$

For any $ij \in E$ s.t. $x_{ij} = 0$, we have:

$$
\begin{aligned}
\frac{\partial F_{ij}}{\partial x_{ij}} &= -H(x) \\
\frac{\partial F_{lk}}{\partial x_{ij}} &= 0, l \neq i, j; k \neq i, j; x_{lk} = 0.
\end{aligned}
$$

Let $\mathcal{C}_1, \ldots, \mathcal{C}_d$ be the connected components of the subgraph $G_x := (\mathbb{V}, E_x)$, where E_x is defined in (5.1). Let

$$
J_x^m := \left(\frac{\partial F_{lk}}{\partial x_{ij}}\right)_{ij,kl\in\mathcal{C}_m}.
$$

Therefore, J_x can be written as follows, by putting first ij and kl coordinates such that $x_{ij} \neq 0$ and $x_{lk} \neq 0$ (in the same order, with increasing connected components $\mathcal{C}_1, \ldots, \mathcal{C}_d$)

$$
J(x) = \begin{pmatrix}
J_x^1 & & & & & \\
& \ddots & & & (*) & \\
& & J_x^d & & & \\
& & & -H(x) & & \\
& (0) & & & \ddots & \\
& & & & & -H(x)
\end{pmatrix}.
$$

5.3. **Classification of equilibria based on stability.** Let us introduce some definitions on stability for ordinary differential equations.

Definition 5.4. x is Lyapunov stable *if for any neighbourhood U_1 of x, there exists a neighbourhood $U_2 \subseteq U_1$ of x such that any solution $x(t)$ starting in U_2 is such that $x(t)$ remains in U_1 for all $t \geqslant 0$.*

Definition 5.5. x is asymptotically stable *if it is Lyapunov stable and there exists a neighbourhood U_1 such that any solution $x(t)$ starting in U_1 is such that $x(t)$ converges to x.*

An equilibrium that is Lyapunov stable but not asymptotically stable is sometimes called *neutrally stable*.

Definition 5.6. x is linearly stable *if all eigenvalues of the Jacobian matrix at x have nonpositive real part; otherwise, x is called* linearly unstable.

Remark that, with these definitions, linear stability allows for eigenvalues to have zero real part, and therefore does not necessarily imply Lyapunov stability. However the dynamics considered here makes these stable equilibria indeed Lyapunov stable: as in [12], as we will observe in Section 6 (in a stochastic version).

Definition 5.7. *Let*
$$\Gamma_0 := \Gamma \cap \Delta \setminus \partial\Delta,$$
$$\Gamma_b := \Gamma \cap \partial\Delta,$$
and let Γ_s (resp. Γ_u) be the set of linearly stable (resp. unstable) equilibria in Γ_0 for the mean-field ODE.

For any $x \in \Gamma_u$, let
$$\mathcal{E}_x := \{\theta \in \mathbb{R}^{|E|} : |\theta| = 1 \text{ and } \exists ij \in E_x \text{ s.t. } \theta \cdot \mathbf{e}_{ij} \neq 0\},$$
where E_x is defined in (5.1).

Recall Definition 2.4 of a graph G_x associated to some $x \in \Delta$.

Proposition 5.8. *We have*
  (a) $\Gamma_s = \{x \in \Gamma_0 : P_{G_x} \text{ holds}\}$.

  (b) *If $x \in \Gamma_u$, then there exists an eigenvector in $\mathcal{E}_x$ whose eigenvalue has positive real part.*

We are first going to prove following Lemma 5.9 giving a necessary condition on the elements of Γ_s. Its proof is widely inspired by that of a similar result in [4].

Lemma 5.9. *If x is in Γ_s, then for all i, j, k and l in the same connected component of G_x such that $i \overset{x}{\sim} j$, $k \overset{x}{\sim} l$, we have:*

$$a_{ij}p_{ij} = a_{kl}p_{kl}.$$

Let us denote $\mathcal{P}_a$ this last property.

Proof. Choose $x \in (\Delta \setminus \partial\Delta) \cap \Gamma$. It is sufficient to prove that, if $x_{i_0 j_0} > 0$ and $x_{i_0 k_0} > 0$, then $a_{i_0 j_0} p_{i_0 j_0} = a_{i_0 k_0} p_{i_0 k_0}$.

Assume the contrary, for some i_0, j_0, k_0. Then, in particular,

$$a_{i_0 j_0} p_{i_0 j_0} \frac{x_{i_0 j_0}}{x_{i_0} x_{j_0}} = H(x).$$

Recall that the entries of the Jacobian matrix $J(x)$, computed in Section 5.2, are such that, on the line corresponding to ij, we have:

$$\frac{\partial F_{ij}}{\partial x_{ij}} = H(x)\left[1 - \frac{x_{ij}}{x_i} - \frac{x_{ij}}{x_j}\right].$$

$$\frac{\partial F_{ij}}{\partial x_{ik}} = -\frac{x_{ij}}{x_i} H(x), k \neq j,$$

$$\frac{\partial F_{ij}}{\partial x_{jk}} = -\frac{x_{ij}}{x_j} H(x), k \neq i,$$

$$\frac{\partial F_{ij}}{\partial x_{kl}} = 0, \{k, l\} \cap \{i, j\} = \emptyset.$$

Recall also that $J(x)$ is block upper triangular with on its diagonal sub-matrices J_x^m, $1 \leqslant m \leqslant d$. Hence, the eigenvalues of J_x^m, $1 \leqslant m \leqslant d$, are eigenvalues of $J(x)$. Moreover, each J_x^m corresponds to the edges within the same connected component of the graph, where we called $\mathcal{C}_m$ this connected component. This means that for any coordinate kl of J_x^m, we have $kl \in \mathcal{C}_m$ and $x_{kl} > 0$.

Let us first prove that the matrices J_x^m can be written as the product of a diagonal matrix and a symmetric matrix. Define the diagonal matrix

$$D^m := \left(x_{ij} H(x) \mathbb{1}_{\{ij = kl\}}\right)_{ij, kl \in \mathcal{C}_m}.$$

Besides, define the symmetric matrix M^m, with entries of the coordinates (ij, kl), $ij, kl \in \mathcal{C}_m$, given by

$$M_{ij,ij}^m = \frac{1}{x_{ij}} - \frac{1}{x_i} - \frac{1}{x_j},$$

$$M_{ij,ik}^m = -\frac{1}{x_i}, k \neq j,$$

$$M_{ij,jk}^m = -\frac{1}{x_j}, k \neq i,$$

$$M_{ij,kl}^m = 0, \{i, j\} \cap \{k, l\} = \emptyset.$$

It is then straightforward to check that $J_x^m = D^m M^m$.

Now, the eigenvalues of J_x^m are all nonpositive if and only if the eigenvalues of M^m are all nonpositive. To have a proof of this fact, see the Claim in the proof of Lemma 1 in [4], p. 2199. Therefore, let us prove that M^m has a positive eigenvalue, which will enable us to conclude.

Let u be a vector in $\mathbb{R}^{|\mathcal{C}_m|}$, such that:

$$u^T = (0, ..., 0, u_{i_0 j_0}, 0..., 0, u_{i_0 k_0}, 0, ..., 0).$$

Dropping the 0 of i_0, j_0 and k_0 for simplicity, we have:

$$\left(M^m \cdot u\right)_{ij} = u_{ij}\Big[\frac{1}{x_{ij}} - \frac{1}{x_i} - \frac{1}{x_j}\Big] - u_{ik}\frac{1}{x_i}$$

$$\left(M^m \cdot u\right)_{ik} = u_{ik}\Big[\frac{1}{x_{ik}} - \frac{1}{x_i} - \frac{1}{x_k}\Big] - u_{ij}\frac{1}{x_i},$$

hence

$$u^T M^m u = u_{ij}^2\Big[\frac{1}{x_{ij}} - \frac{1}{x_i} - \frac{1}{x_j}\Big] + u_{ik}^2\Big[\frac{1}{x_{ik}} - \frac{1}{x_i} - \frac{1}{x_k}\Big]$$
$$- u_{ij}u_{ik}\Big(\frac{1}{x_i} + \frac{1}{x_i}\Big).$$

Then, we chose $u_{ij} = 1$ and $u_{ik} = -1$. So, we have:

$$u^T M^m u = \frac{1}{x_{ij}}\Big[1 - \frac{x_{ij}}{x_j}\Big] + \frac{1}{x_{ik}}\Big[1 - \frac{x_{ik}}{x_k}\Big].$$

Recall that $\frac{x_{ij}}{x_j} \leqslant 1$ and $\frac{x_{ik}}{x_k} \leqslant 1$, and notice that $\frac{x_{ij}}{x_j} \neq 1$ or $\frac{x_{ik}}{x_k} \neq 1$. Indeed, if $\frac{x_{ij}}{x_j} = \frac{x_{ik}}{x_k} = 1$, as $a_{ij}p_{ij}\frac{x_{ij}}{x_j} = a_{ik}p_{ik}\frac{x_{ik}}{x_k}$ (we are on the set of equilibria), then we would have $a_{ij}p_{ij} = a_{ik}p_{ik}$, which is not the case, by assumption.
Hence $u^T M^m u > 0$ and therefore M^m has a positive eigenvalue, as this matrix is symmetric. Finally, this implies that $J(x)$ has a positive eigenvalue and therefore x is not stable. $\square$

Lemma 5.9 implies that $\Gamma_s \subseteq \{x \in \Gamma_0 : \mathcal{P}_a \text{ holds}\} =: \Gamma_0^a$.
To prove Proposition 5.8, we need the following Lemma 5.10 on the structure of G_x when $x \in \Gamma_0^a$.

Lemma 5.10. *For all $x \in \Gamma_0^a$ such that P_{G_x} does not hold, then there is at least one connected component on which every vertex has at least two edges.*

Proof. Assume that x is in Γ_0^a and P_{G_x} does not hold. Assume by contradiction that, on any connected component, there exists a vertex j linked to only one

vertex, say i. So, we have $\frac{x_{ij}}{x_j} = 1$. Then, as $x \in \Gamma_0^a$, for all k such that $x_{ik} > 0$, we have:

$$y_{ik} = p_{ik} \frac{a_{ik} x_{ik}}{x_i x_k} = y_{ij} = p_{ij} \frac{a_{ij} x_{ij}}{x_i x_j} \Rightarrow \frac{x_{ik}}{x_k} = 1,$$

then k is only linked to i, and therefore each connected component of G_x is star-shaped, which contradicts the assumption that P_{G_x} does not hold, and allows us to conclude. $\square$

Proof of Proposition 5.8. First, assume that $x \in \Gamma_0^a$ and P_{G_x} does not hold. We want to prove that $x \in \Gamma_u$. Lemma 5.10 implies that G_x has a connected component on which each vertex is adjacent to at least two edges, which we assume w.l.o.g. to be $\mathcal{C}_1$. Let $V(\mathcal{C}_1)$ (resp. $E(\mathcal{C}_1)$) be its set of vertices (resp. edges). Let us show that J_x^1 has at least one eigenvalue with positive real part.

Compute the trace of J_x^1 :

$$
\begin{aligned}
Tr(J_x^1) &= H(x) \sum_{\{i,j\}:i \overset{x}{\sim} j} (1 - \frac{x_{ij}}{x_i} - \frac{x_{ij}}{x_j}) \\
&= H(x)(|E(\mathcal{C}_1)| - |V(\mathcal{C}_1)|) \geqslant 0.
\end{aligned}
$$

The last inequality comes from the fact that each vertex has at least two edges.

It is easy to check that $(1, ..., 1) \cdot J_x^1 = -H(x)(1, ..., 1) < 0$, hence $-H(x)$ is an eigenvalue of J_x^1, and subsequently there exists an eigenvalue with positive real part. This implies that **(a)**, using Lemma 5.9.

Now, assume that $x \in \Gamma_0^a$ and that P_{G_x} holds. Then, each connected component of G_x is star-shaped. Let us assume for instance that $\mathcal{C}_1$ is composed of a nucleus v and satellite vertices $1, ..., k$. Then, for all $i \in \{1, .., k\}$, $x_{iv} = x_i$, and we have:

$$
J_x^1 = -\frac{H(x)}{x_v}
\begin{pmatrix}
x_1 & \cdots & x_1 \\
x_2 & \cdots & x_2 \\
\vdots & & \vdots \\
x_k & \cdots & x_k
\end{pmatrix}.
$$

The rank of J_x^1 is 1, 0 and $-H(x) < 0$ are eigenvalues, as $(1, ..., 1) \cdot J_x^1 = -H(x)(1, ..., 1)$, so all its eigenvalues are nonpositive, which completes the proof.

$\square$

Remark 5.11. *Note that, in general, there may be no stable configuration on $\Delta \setminus \partial\Delta$. Now, if the number of vertices is even, every equilibrium x such that G_x yields a one-to-one correspondence between vertices (i.e. each connected component has only two vertices and one edge) is asymptotically stable.*

6. Proof of Theorem 2.8

Assume that $(G, (a_{ij}, p_{ij})_{ij \in E})$ is such that $\Gamma_s \neq \emptyset$.

Theorem 2.8 is a simple consequence of the following Proposition 6.1. Before stating it, we need some definitions.

Let $\mathcal{G} \subseteq G$ be a graph such that $P_{\mathcal{G}}$ holds (see Definition 2.5). In particular, each connected component is star-shaped and for all i, j, k and l in the same connected component, s.t. $ij \in \mathcal{G}$ and $kl \in \mathcal{G}$, $p_{ij}a_{ij} = p_{kl}a_{kl}$. Moreover, each connected component contains a nucleus vertex (i.e. any other vertex in the same component belongs to its neighbors), which is chosen arbitrarily if the connected component has only two vertices. We denote by $N_{\mathcal{G}}$ the set of nucleus vertices of $\mathcal{G}$.

Let $\pi := \pi_{\mathcal{G}} : \mathbb{V} \longrightarrow N_{\mathcal{G}}$ be a function mapping $i \in V$ to one nucleus vertex of the connected component that i belongs to.

For all $i \in \mathbb{V}$, $n \in \mathbb{N}$ and $\varepsilon > 0$, let us define

$$\alpha_i^n := x_i^n / x_{\pi(i)}^n,$$

$$H_n^1 := \bigcap_{i \in \mathbb{V}, i = \pi(i)} \{ V_i^n \geqslant 2\varepsilon n \},$$

$$H_n^2 := \bigcap_{i \in \mathbb{V}} \{ \alpha_i^n \geqslant \varepsilon \},$$

$$H_n^3 := \bigcap_{i,j \in \mathbb{V}, i \neq \pi(j), j \neq \pi(i)} \{ V_{ij}^n \leqslant \sqrt{n} \}.$$

Obviously, all these definitions depend on the graph $\mathcal{G}$, but for simplicity we do not write this dependency.

Proposition 6.1. *Let $\mathcal{G}$ be such that $\mathcal{P}_{\mathcal{G}}$ holds, and let $\pi := \pi_{\mathcal{G}}$. For all $\varepsilon \in (0, p_{\min})$, if H_n^1, H_n^2 and H_n^3 hold, and $n \geqslant \mathtt{Cst}(\varepsilon, (p_{ij}, a_{ij})_{i,j \in \mathbb{V}}, |\mathbb{V}|)$, then, with lower bounded probability (only depending on ε, $(p_{ij}, a_{ij})_{i,j \in \mathbb{V}}$, and $|\mathbb{V}|$), for all $i, j \in \mathbb{V}, k \geqslant n$,*

(6.1) $$V_{ij}^\infty = V_{ij}^n, \quad \text{when } i \neq \pi(j), j \neq \pi(i);$$

(6.2) $$\alpha_i^k / \alpha_i^n \in (1 - \varepsilon, 1 + \varepsilon);$$

(6.3) $$V_i^k \geqslant \varepsilon k, \quad \text{when } \pi(i) = i.$$

In the remainder of this section, we fix the graph $(\mathcal{G}, \overset{g}{\sim})$ (and thus $\pi = \pi_{\mathcal{G}}$ and the events H_n's) and $\varepsilon > 0$. The proof consists of the following Lemmas 6.2, 6.3 and 6.4.

Let, for all $i, j \in \mathbb{V}$, $n \in \mathbb{N}$,

$$\tau_n^{1,i,j} := \inf\{k \geqslant n : V_{ij}^k \neq V_{ij}^n\};$$

$$\tau_n^{2,i} := \inf\{k \geqslant n : \alpha_i^k / \alpha_i^n \notin (1 - \varepsilon, 1 + \varepsilon)\};$$

$$\tau_n^{3,i} := \inf\{k \geqslant n \,:\, V_i^k < \varepsilon k\},$$

and let

$$\tau_n^1 := \inf_{i,j\in\mathbb{V},i\neq\pi(j),j\neq\pi(i)} \tau_n^{1,i,j}, \quad \tau_n^2 := \inf_{i\in\mathbb{V}}\tau_n^{2,i}, \quad \tau_n^3 := \inf_{i\in\mathbb{V},\pi(i)=i}\tau_n^{3,i}, \quad \tau_n := \tau_n^1 \wedge \tau_n^2 \wedge \tau_n^3.$$

Lemma 6.2. *If* $n \geqslant \mathtt{Cst}(\varepsilon, (p_{ij}, a_{ij})_{i,j\in\mathbb{V}}, |\mathbb{V}|)$, *then*

$$\mathbb{P}\Big(\tau_n^1 > \tau_n^2 \wedge \tau_n^3 \,\big|\, \mathcal{F}_n, H_n^1, H_n^2, H_n^3\Big) \geqslant \exp\left(-4|\mathbb{V}|^2 \max_{i,j\in\mathbb{V}}\{a_{ij}\}\varepsilon^{-4}\right).$$

Proof. Assuming $n \geqslant \mathtt{Cst}(\varepsilon, (p_{ij}, a_{ij})_{i,j\in\mathbb{V}}, |\mathbb{V}|)$,

$$\mathbb{P}\Big(\tau_n^1 > \tau_n^2 \wedge \tau_n^3 \,\big|\, \mathcal{F}_n, H_n^1, H_n^2, H_n^3\Big) \geqslant \prod_{k\geqslant n}\left(1 - \sum_{i,j:i\neq\pi(j),j\neq\pi(i)} a_{ij}p_{ij}\frac{(V_{ij}^n)^2}{V_i^k V_j^k}\right)$$

$$\geqslant \exp\left(-\frac{3\max_{i,j\in\mathbb{V}}\{a_{ij}\}}{2}\sum_{i,j:i\neq\pi(j),j\neq\pi(i),k\geqslant n}\frac{n}{\varepsilon^4 k^2}\right)$$

$$\geqslant \exp\left(-4|\mathbb{V}|^2 \max_{i,j\in\mathbb{V}}\{a_{ij}\}\varepsilon^{-4}\right).$$

$\square$

Lemma 6.3. *If* $n \geqslant \mathtt{Cst}(\varepsilon, (p_{ij}, a_{ij})_{i,j\in\mathbb{V}}, |\mathbb{V}|)$ *then, for all* $i \in \mathbb{V}$,

$$\mathbb{P}\Big(\tau_n^{2,i} > \tau_n^1 \wedge \tau_n^3 \,\big|\, \mathcal{F}_n, H_n^1, H_n^2, H_n^3\Big) \geqslant 1 - 2\exp(-\mathtt{Cst}(\varepsilon)n).$$

Proof. Fix $i \in \mathbb{V}$, $n \in \mathbb{N}$, and assume w.l.o.g. that $\pi(i) \neq i$.
  Let, for all $j \in \mathbb{V}$ and $k \geqslant n$,

$$\hat{V}_j^k := V_j^n + \sum_{l\overset{g}{\sim}j}(V_{jl}^k - V_{jl}^n),$$

which is equal to V_j^k as long as $k < \tau_n^1$.
  Let, for all $k \geqslant n$,

$$W_k := \log\frac{\hat{V}_i^k}{\hat{V}_{\pi(i)}^k},$$

and let us consider the Doob decomposition of $(W_k)_{k\geqslant n}$:

$$W_k = W_n + \Delta_k + \Psi_k,$$

$$\Delta_k := \sum_{j=n+1}^k \mathbb{E}(W_j - W_{j-1}|\mathcal{F}_{j-1}).$$

In the following computation, we write $u = \square(v)$ if $|u| \leqslant v$, for all $u, v \in \mathbb{R}$. Assume that H_n^1, H_n^2 and H_n^3 hold, and that $k < \tau_n$: then, using that, for all $j \overset{g}{\sim} \pi(i)$, $p_{i\pi(i)}a_{i\pi(i)} = p_{j\pi(i)}a_{j\pi(i)}$,

$$
|\Delta_{k+1} - \Delta_k| = |\,\mathbb{E}\left[W_{k+1} - W_k \,|\, \mathcal{F}_k\right]\,|
$$

$$
= \left| a_{i\pi(i)}p_{i\pi(i)} \left(\frac{V_{i\pi(i)}^k}{V_i^k}\right)^2 \frac{1}{V_{\pi(i)}^k}(1 + \square((V_i^k)^{-1})) \right.
$$

$$
\left. - \sum_{j\overset{g}{\sim}\pi(i)} a_{j\pi(i)}p_{j\pi(i)} \frac{V_{\pi(i)j}^k}{V_j^k} \frac{V_{\pi(i)j}^k}{(V_{\pi(i)}^k)^2}(1 + \square((V_{\pi(i)}^k)^{-1})) \right|
$$

$$
= \frac{a_{i\pi(i)}p_{i\pi(i)}}{V_{\pi(i)}^k}\left| 1 + k^{-1/2}\square(\mathtt{Cst}(\varepsilon, (p_{ij}, a_{ij})_{i,j\in\mathbb{V}}, |\mathbb{V}|)) \right.
$$

$$
\left. - \sum_{j\overset{g}{\sim}\pi(i)} \left(1 + k^{-1/2}\square(\mathtt{Cst}(\varepsilon, (p_{ij}, a_{ij})_{i,j\in\mathbb{V}}, |\mathbb{V}|))\right) \frac{V_{\pi(i)j}^k}{V_{\pi(i)}^k} \right|
$$

$$
= k^{-3/2}\square\left(\mathtt{Cst}(\varepsilon, (p_{ij}, a_{ij})_{i,j\in\mathbb{V}}, |\mathbb{V}|)\right),
$$

where we use that, for all $j \overset{g}{\sim} \pi(i)$,

$$
(6.4) \quad \left| \frac{V_{\pi(i)j}^k}{V_j^k} - 1 \right|, \quad \left| \sum_{j\overset{g}{\sim}\pi(i)} \frac{V_{\pi(i)j}^k}{V_{\pi(i)}^k} - 1 \right| \leqslant k^{-1/2}\mathtt{Cst}(\varepsilon, (p_{ij}, a_{ij})_{i,j\in\mathbb{V}}, |\mathbb{V}|).
$$

Therefore, for all $k \geqslant n$,

$$
|\Delta_k| \leqslant n^{-1/2}\mathtt{Cst}(\varepsilon, (p_{ij}, a_{ij})_{i,j\in\mathbb{V}}, |\mathbb{V}|).
$$

Let us now estimate the martingale increment: $|\Psi_{k+1} - \Psi_k| \leqslant \mathtt{Cst}(\varepsilon)k^{-1}$ (since $|W_{k+1} - W_k| \leqslant \mathtt{Cst}(\varepsilon)k^{-1}$), so that [12, Lemma 7.4] (stated here as Lemma 6.5) implies

$$
\mathbb{P}\left(\sup_{k\geqslant n} |\Psi_k - \Psi_n|\mathbb{1}_{\{k\leqslant\tau_n\}} \leqslant \varepsilon/2 \right) \geqslant 1 - 2\exp(-\mathtt{Cst}(\varepsilon)n),
$$

which completes the proof. $\qquad\square$

Recall the definition (3.17) of $w_{\min}$.

Lemma 6.4. *If $\varepsilon \in (0, w_{\min})$ and $n \geqslant \mathtt{Cst}(\varepsilon, (p_{ij}, a_{ij})_{i,j\in\mathbb{V}}, |\mathbb{V}|)$ then, for all $i \in \mathbb{V}$ such that $\pi(i) = i$ (i.e. $i \in N_{\mathcal{G}}$),*

$$
\mathbb{P}\left(\tau_n^{3,i} > \tau_n^1 \wedge \tau_n^2 \,\big|\, \mathcal{F}_n, H_n^1, H_n^2, H_n^3 \right) > 1 - 2\exp(-\mathtt{Cst}(\varepsilon)n).
$$

Proof. Let $n \in \mathbb{N}$, assume that H_n^1, H_n^2 and H_n^3 hold, and fix $i \in \mathbb{V}$ such that $\pi(i) = i$. Let us consider the Doob decomposition of $(V_i^k)_{k \geqslant n}$:

$$V_i^k := V_i^n + \Phi_k + \Xi_k$$

$$\Phi_k := \sum_{j=n+1}^{k} \mathbb{E}\left(V_i^j - V_i^{j-1} \,\middle|\, \mathcal{F}_{j-1}\right).$$

Now, for all $\eta > 0$, if $n \geqslant \mathtt{Cst}(\eta, \varepsilon)$ and $k < \tau_n$, (6.4) implies

$$\Phi_{k+1} - \Phi_k = \mathbb{E}\left(V_i^{k+1} - V_i^k \,\middle|\, \mathcal{F}_k\right) \geqslant \sum_{j \overset{g}{\sim} i} a_{ij} p_{ij} \frac{(V_{ij}^k)^2}{V_i^k V_j^k} \geqslant w_{\min} - \eta,$$

if $n \geqslant \mathtt{Cst}(\epsilon, \eta, p, a, |\mathbb{V}|)$.

Let us now estimate the martingale increment: let, for all $p \geqslant n$,

$$\chi_p := \sum_{k=n}^{p-1} \frac{\Xi_{k+1} - \Xi_k}{k}.$$

Then, for all $p \geqslant n$,

$$\Xi_p = \sum_{n \leqslant k \leqslant p-1} (\chi_{k+1} - \chi_k) k = - \sum_{n \leqslant k \leqslant p-1} \chi_k + (p-1)\chi_p.$$

This implies, using [12, Lemma 7.4] (see Lemma 6.5) and $|\Xi_{k+1} - \Xi_k| \leqslant 1$ for all $k \geqslant n$, that for all $\varepsilon > 0$

$$\mathbb{P}\left(\forall k \geqslant n, \ V_i^k \geqslant (2\varepsilon - \eta)n + (k-n)(h_{\min} - \eta) \,\middle|\, \mathcal{F}_n\right)$$

$$\geqslant \mathbb{P}\left(\sup_{p \geqslant n} \left|\frac{\Xi_p}{p}\right| \leqslant \eta \,\middle|\, \mathcal{F}_n\right) \geqslant \mathbb{P}\left(\sup_{k \geqslant n} |\chi_k| \leqslant \frac{\eta}{2} \,\middle|\, \mathcal{F}_n\right) \geqslant 1 - 2\exp(-\mathtt{Cst}(\eta)n);$$

we choose $\eta = \min(\varepsilon, h_{\min} - \varepsilon)$, which completes the proof. $\qquad\square$

The following Lemma 6.5 states an exponential inequality for martingales (see for instance [12], Lemma 7.4 for a proof).

Lemma 6.5. *Let $(\gamma_k)_{k \in \mathbb{N}}$ be a deterministic sequence of positive reals, let $\mathbb{G} := (\mathcal{G}_n)_{n \in \mathbb{N}}$ be a filtration, and let $(M_n)_{n \in \mathbb{N}}$ be a $\mathbb{G}$-adapted martingale such that $|M_{n+1} - M_n| \leqslant \gamma_n$ for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$ and $\lambda > 0$,*

$$\mathbb{P}\left(\sup_{k \geqslant n}(M_k - M_n) \geqslant \lambda \,\middle|\, \mathcal{G}_n\right) \leqslant \exp\left(-\frac{\lambda^2}{2\sum_{k \geqslant n} \gamma_k^2}\right).$$

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