AN ERGODIC THEOREM FOR THE QUASI-REGULAR REPRESENTATION OF THE FREE GROUP

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Abstract. In [BM11], an ergodic theorem à la Birkhoff-von Neumann for the action of the fundamental group of a compact negatively curved manifold on the boundary of its universal cover is proved. A quick corollary is the irreducibility of the associated unitary representation. These results are generalized [Boy15] to the context of convex cocompact groups of isometries of a CAT(-1) space, using Theorem 4.1.1 of [Rob03], with the hypothesis of non arithmeticity of the spectrum. We prove all the analog results in the case of the free group $\mathbb{F}_r$ of rank r even if $\mathbb{F}_r$ is not the fundamental group of a closed manifold, and may have an arithmetic spectrum.

1. Introduction

In this paper, we consider the action of the free group $\mathbb{F}_r$ on its boundary $\mathbf{B}$, a probability space associated to the Cayley graph of $\mathbb{F}_r$ relative to its canonical generating set. This action is known to be *ergodic* (see for example [FTP82] and [FTP83]), but since the measure is not preserved, no theorem on the convergence of means of the corresponding unitary operators had been proved. Note that a close result is proved in [FTP83, Lemma 4, Item (i)].
We formulate such a convergence theorem in Theorem 1.2. We prove it following the ideas of [BM11] and [Boy15] replacing [Rob03, Theorem 4.1.1] by Theorem 1.1.

1.1. Geometric setting and notation.

We will denote $\mathbb{F}_r = \langle a_1, ..., a_r\rangle$ the free group on r generators, for $r \geq 2$. For an element $\gamma \in \mathbb{F}_r$, there is a unique reduced word in $\{a_1^{\pm 1}, ..., a_r^{\pm 1}\}$ which represents it. This word is denoted $\gamma_1 \cdots \gamma_k$ for some integer k which is called the *length* of γ and is denoted by $|\gamma|$. The set of all elements of length k is denoted S_n and is called the *sphere of radius* k. If $u \in \mathbb{F}_r$ and $k \geq |u|$, let us denote $Pr_u(k) := \{\gamma \in \mathbb{F}_r \mid |\gamma| = k,\ u \text{ is a prefix of } \gamma\}$.
Let X be the Cayley graph of $\mathbb{F}_r$ with respect to the set of generators $\{a_1^{\pm 1}, ..., a_r^{\pm 1}\}$, which is a $2r$-regular tree. We endow it with the (natural) distance, denoted by d, which gives length 1 to every edge ; for this distance, the natural action of $\mathbb{F}_r$ on X is isometric and freely transitive on the vertices ; the space X is uniquely geodesic, the geodesics between vertices being finite sequences of successive edges. We denote by $[x, y]$ the unique geodesic joining x to y.
We fix, once and for all, a vertex x_0 in X. For $x \in X$, the vertex of X which is the closest to x in $[x_0, x]$, is denoted by $\lfloor x \rfloor$; because the action is free, we can identify $\lfloor x \rfloor$ with the element γ that brings x_0 on it, and this identification is an isometry.

The Cayley tree and its boundary. As for any other CAT(-1) space, we can construct a boundary of X and endow it with a distance and a measure. For a general construction, see [Bou95]. The construction we provide here is elementary.

Let us denote by $\mathbf{B}$ the set of all right-infinite reduced words on the alphabet $\{a_1^{\pm 1}, ..., a_r^{\pm 1}\}$. This set is called the **boundary** of X.

We will consider the set $\overline{X} := X \cup \mathbf{B}$.

For $u = u_1 \cdots u_l \in \mathbb{F}_r \setminus \{e\}$, we define the sets

$$X_u := \{x \in X \mid u \text{ is a prefix of } \lfloor x \rfloor\}$$
$$\mathbf{B}_u := \{\xi \in \mathbf{B} \mid u \text{ is a prefix of } \xi\}$$

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$$C_u := X_u \cup \mathbf{B}_u$$

We can now define a natural topology on $\overline{X}$ by choosing as a basis of neighborhoods

(1) for $x \in X$, the set of all neighborhoods of x in X
(2) for $\xi \in \mathbf{B}$, the set $\{C_u \mid u \text{ is a prefix of } \xi\}$

For this topology, $\overline{X}$ is a compact space in which the subset X is open and dense. The induced topology on X is the one given by the distance. Every isometry of X continuously extend to a homeomorphism of $\overline{X}$.

Distance and measure on the boundary. For ξ_1 and ξ_2 in $\mathbf{B}$, we define the **Gromov product** of ξ_1 and ξ_2 with respect to x_0 by

$$(\xi_1|\xi_2)_{x_0} := \sup\{k \in \mathbb{N} \mid \xi_1 \text{ and } \xi_2 \text{ have a common prefix of length } k\}$$

and

$$d_{x_0}(\xi_1, \xi_2) := e^{-(\xi_1|\xi_2)_{x_0}}.$$

Then d defines an ultrametric distance on $\mathbf{B}$ which induces the same topology ; precisely, if $\xi = u_1u_2u_3\cdots$, then the ball centered in ξ of radius e^{-k} is just $\mathbf{B}_{u_1\ldots u_k}$.

On $\mathbf{B}$, there is at most one Borel regular probability measure which is invariant under the isometries of X which fix x_0; indeed, such a measure μ_{x_0} must satisfy

$$\mu_{x_0}(\mathbf{B}_u) = \frac{1}{2r(2r-1)^{|u|-1}}$$

and it is straightforward to check that the $\ln(2r-1)$-dimensional Hausdorff measure verifies this property.

If $\xi = u_1\cdots u_n\cdots \in \mathbf{B}$, and $x, y \in X$, then $(d(x, u_1\cdots u_n) - d(y, u_1\cdots u_n))_{n\in\mathbb{N}}$ is stationary. We denote this limit $\beta_\xi(x, y)$. The function β_ξ is called the **Busemann function** at ξ.

Let us denote, for $\xi \in \mathbf{B}$ and $\gamma \in \mathbb{F}_r$ the function

$$P(\gamma, \xi) := (2r-1)^{\beta_\xi(x_0, \gamma x_0)}$$

The measure μ_{x_0} is, in addition, quasi-invariant under the action of $\mathbb{F}_r$. Precisely, the Radon-Nikodym derivative is given for $\gamma \in \Gamma$ and for a.e. $\xi \in \mathbf{B}$ by

$$\frac{d\gamma_*\mu_{x_0}}{d\mu_{x_0}}(\xi) = P(\gamma, \xi),$$

where $\gamma_*\mu_{x_0}(A) = \mu_{x_0}(\gamma^{-1}A)$ for any Borel subset $A \subset \mathbf{B}$.

The quasi-regular representation. Denote the unitary representation, called the quasi-regular representation of $\mathbb{F}_r$ on the boundary of X by

$$\begin{array}{rcl} \pi : \mathbb{F}_r & \to & \mathcal{U}(L^2(\mathbf{B})) \\ \gamma & \mapsto & \pi(\gamma) \end{array}$$

defined as

$$\big(\pi(\gamma)g\big)(\xi) := P(\gamma, \xi)^{\frac{1}{2}} g(\gamma^{-1}\xi)$$

for $\gamma \in \mathbb{F}_r$ and for $g \in L^2(\mathbf{B})$. We define the *Harish-Chandra* function

$$\Xi(\gamma) := \langle \pi(\gamma)\mathbf{1_B}, \mathbf{1_B}\rangle = \int_{\mathbf{B}} P(\gamma, \xi)^{\frac{1}{2}} d\mu_{x_0}(\xi), \tag{1.1}$$

where $\mathbf{1_B}$ denotes the characteristic function on the boundary.

For $f \in C(\overline{X})$, we define the operators

$$M_n(f) : g \in L^2(\mathbf{B}) \mapsto \frac{1}{|S_n|} \sum_{\gamma \in S_n} f(\gamma x_0) \frac{\pi(\gamma)g}{\Xi(\gamma)} \in L^2(\mathbf{B}). \tag{1.2}$$

We also define the operator

$$M(f) := m(f_{|\mathbf{B}}) P_{\mathbf{1_B}} \tag{1.3}$$

where $m(f_{|\mathbf{B}})$ is the multiplication operator by $f_{|\mathbf{B}}$ on $L^2(\mathbf{B})$, and $P_{\mathbf{1_B}}$ is the orthogonal projection on the subspace of constant functions.

Results. The analog of Roblin's equidistribution theorem for the free group is the following.

Theorem 1.1. *We have, in $C(\overline{X}\times\overline{X})^*$, the weak-$*$ convergence*

$$\frac{1}{|S_n|}\sum_{\gamma\in S_n} D_{\gamma x_0}\otimes D_{\gamma^{-1}x_0} \rightharpoonup \mu_{x_0}\otimes\mu_{x_0}$$

where D_x denotes the Dirac measure on a point x.

Remark 1. *It is then straightforward to deduce the weak-$*$ convergence*

$$\|m_\Gamma\| e^{-\delta n}\sum_{|\gamma|\leq n} D_{\gamma x_0}\otimes D_{\gamma^{-1}x_0} \rightharpoonup \mu_{x_0}\otimes\mu_{x_0}$$

m_Γ denoting the Bowen-Margulis-Sullivan measure on the geodesic flow of SX/Γ (where SX is the "unit tangent bundle") and δ denoting $\ln(2r-1)$, *the Hausdorff measure of* $\mathbf{B}$.

(1) *Notice that in our case, the spectrum is $\mathbb{Z}$ so the geodesic flow is not topologically mixing, according to* [Dal99] *or directly by* [CT01, Ex 1.3].
(2) *Notice also that our multiplicative term is different of that of* [Rob03, Theorem 4.1.1], *which shows that the hypothesis of non-arithmeticity of the spectrum cannot be removed.*

We use the above theorem to prove the following convergence of operators.

Theorem 1.2. *We have, for all f in $C(\overline{X})$, the weak operator convergence*

$$M_n(f)\underset{n\to+\infty}{\longrightarrow} M(f).$$

In other words, we have, for all f in $C(\overline{X})$ and for all g, h in $L^2(\mathbf{B})$, the convergence

$$\frac{1}{|S_n|}\sum_{\gamma\in S_n} f(\gamma x_0)\frac{\langle\pi(\gamma)g,h\rangle}{\Xi(\gamma)}\underset{n\to+\infty}{\longrightarrow}\langle M(f)g,h\rangle.$$

We deduce the irreducibility of π, and give an alternative proof of this well known result (see [FTP82, Theorem 5]).

Corollary 1.3. *The representation π is irreducible.*

Proof. Applying Theorem 1.2 to $f=\mathbf{1}_{\overline{X}}$ shows that the orthogonal projection onto the space of constant functions is in the von Neumann algebra associated with π. Then applying Theorem 1.2 to $g=\mathbf{1}_{\mathbf{B}}$ shows that the vector $1_{\mathbf{B}}$ is cyclic. Then, the classical argument of [Gar14, Lemma 6.1] concludes the proof. □

Remark 2. *For $\alpha\in\mathbb{R}_+^*$, let us denote by W_α the wedge of two circles, one of length 1 and the other of length α. Let $p:T_\alpha\twoheadrightarrow W_\alpha$ the universal cover, with T_α endowed with the distance making p a local isometry. Then $\mathbb{F}_2\simeq\pi_1(W_\alpha)$ acts freely properly discontinously and cocompactly on the 4-regular tree T_α (which is a CAT(-1) space) by isometries. For $\alpha\in\mathbb{R}\setminus\mathbb{Q}$, the analog of Theorem 1.2 for the quasi-regular representation π_α of $\mathbb{F}_2$ on $L^2(\partial T_\alpha,\mu_\alpha)$ for a Patterson-Sullivan measure associated to a Bourdon distance is known to hold (*[Boy15]*) because* [Rob03, Theorem 4.1.1] *is true in this setting. Now if α_1 and α_2 are such that $\alpha_1\neq\alpha_2^{\pm1}$, then the representations π_α are not unitarily equivalent (*[Gar14, Theorem 7.5]*). For $\alpha\in\mathbb{Q}_+^*\setminus\{1\}$, it would be interesting to formulate and prove an equidistribution result like Theorem 1.1 in order to prove Theorem 1.2 for π_α.*

2. Proofs

2.1. Proof of the equidistribution theorem.

For the proof of Theorem 1.1, let us denote

$$E:=\left\{f:C(\overline{X}\times\overline{X})\mid\frac{1}{|S_n|}\sum_{\gamma\in S_n} f(\gamma x_0,\gamma^{-1}x_0)\to\int_{\overline{X}\times\overline{X}} f d(\mu_{x_0}\otimes\mu_{x_0})\right\}$$

The subspace E is clearly closed in $C(\overline{X}\times\overline{X})$; it remains only to show that it contains a dense subspace of it.

Let us define a modified version of certain characteristic functions : for $u \in \mathbb{F}_r$ we define

$$\chi_u(x) := \begin{cases} \max\{1 - d_X(x, C_u), 0\} & \text{if} \quad x \in X \\ 0 & \text{if} \quad x \in \mathbf{B} \setminus \mathbf{B}_u \\ 1 & \text{if} \quad x \in \mathbf{B}_u \end{cases}$$

It is easy to check that he function χ_u is a continuous function which coincides with χ_{C_u} on $\mathbb{F}_r x_0$ and $\mathbf{B}$.

The proof of the following lemma is straightforward.

Lemma 2.1. *Let $u \in \mathbb{F}_r$ and $k \geq |u|$, then $\chi_u - \sum_{\gamma \in Pr_u(k)} \chi_\gamma$ has compact support included in X.*

Proposition 2.2. *The set $\chi := \{\chi_u \mid u \in \mathbb{F}_r \setminus \{e\}\}$ separates points of $\mathbf{B}$, and the product of two such functions of χ is either in χ, the sum of a function in χ and of a function with compact support contained in X, or zero.*

Proof. It is clear that χ separates points. It follows from Lemma 2.1 that $\chi_u \chi_v = \chi_v$ if u is a proper prefix of v, that $\chi_u^2 - \chi_u$ has compact support in X, and that $\chi_u \chi_v = 0$ if none of u and v is a proper prefix of the other. □

Proposition 2.3. *The subspace E contains all functions of the form $\chi_u \otimes \chi_v$.*

Proof. We make the useful observation that

$$\frac{1}{|S_n|} \sum_{\gamma \in S_n} (\chi_u \otimes \chi_v)(\gamma x_0, \gamma^{-1} x_0) = \frac{|S_n^{u,v}|}{|S_n|}$$

where $S_n^{u,v}$ is the set of reduced words of length n with u as a prefix and v^{-1} as a suffix. We easily see that this set is in bijection with the set of all reduced words of length $n - (|u| + |v|)$ that do not begin by the inverse of the last letter of u, and that do not end by the inverse of the first letter of v^{-1}. So we have to compute, for $s, t \in \{a_1^{\pm 1}, ..., a_r^{\pm 1}\}$ and $m \in \mathbb{N}$, the cardinal of the set $S_m(s,t)$ of reduced words of length m that do not start by s and do not finish by t.

Now we have

$$S_m = S_m(s,t) \cup \{x \mid |x| = m \text{ and starts by } s\} \cup \{x \mid |x| = m \text{ and ends by } t\}.$$

Note that the intersection of the two last sets is the set of words both starting by s and ending by t, which is in bijection with $S_{m-2}(s^{-1}, t^{-1})$.

We have then the recurrence relation :

$$\begin{array}{rcl} |S_m(s,t)| & = & 2r(2r-1)^{m-1} - 2(2r-1)^{m-1} + |S_{m-2}(s^{-1}, t^{-1})| \\ & = & 2(r-1)(2r-1)^{m-1} + 2(r-1)(2r-1)^{m-3} + |S_{m-4}(s,t)| \\ & = & (2r-1)^m \dfrac{2(r-1)\left((2r-1)^2+1\right)}{(2r-1)^3} + |S_{m-4}(s,t)| \end{array}.$$

We set $C := \frac{2(r-1)\left((2r-1)^2+1\right)}{(2r-1)^3}$, $n = 4k + j$ with $0 \leq j \leq 3$ and we obtain

$$\begin{array}{rcl} |S_{4k+j}^{s,t}| & = & C(2r-1)^{4k+j} + |S_{4(k-1)+j}^{s,t}| \\ & = & C(2r-1)^{4k+j} + C(2r-1)^{4(k-1)+j} + |S_{4(k-2)+j}^{s,t}| \\ & & \\ & = & C \displaystyle\sum_{i=1}^{k} (2r-1)^{4i+j} + |S_j^{s,t}| \\ & = & C(2r-1)^{4+j} \dfrac{(2r-1)^{4k} - 1}{(2r-1)^4 - 1} + |S_j(s,t)| \\ & & \\ & = & (2r-1)^{1+j} \dfrac{(2r-1)^{4k} - 1}{2r} + |S_j(s,t)| \end{array}$$

Now we can compute

$$\begin{aligned}
\frac{|S^{u,v}_{4k+j}|}{|S_{4k+j}|} &= \frac{\left|S_{4k+j-(|u|+|v|)}(u_{|u|}, v^{-1}_{|v|})\right|}{|S_{4k+j}|} \\
&= \frac{(2r-1)^{1+j}\,\dfrac{(2r-1)^{4k-(|u|+|v|)}-1}{2r} + \left|S_j(u_{|u|}, v^{-1}_{|v|})\right|}{2r(2r-1)^{4k+j-1}} \\
&= \frac{1}{2r(2r-1)^{|u|-1}}\frac{1}{2r(2r-1)^{|v|-1}} + o(1) \\
&= \mu_{x_0}(\mathbf{B}_u)\mu_{x_0}(\mathbf{B}_v) + o(1)
\end{aligned}$$

when $k \to \infty$, and this proves the claim. □

Corollary 2.4. *The subspace E is dense in $C(\overline{X} \times \overline{X})$.*

Proof. Let us consider E', the subspace generated by the constant functions, the functions which can be written as $f \otimes g$ where f, g are continuous functions on $\overline{X}$ and such that one of them has compact support included in X, and the functions of the form $\chi_u \otimes \chi_v$. By Proposition 2.2, it is a subalgebra of $C(\overline{X} \times \overline{X})$ containing the constants and separating points, so by the Stone-Weierstraß theorem, E' is dense in $C(\overline{X} \times \overline{X})$. Now, by Proposition 2.3, we have that $E' \subseteq E$, so E is dense as well. □

2.2. **Proof of the ergodic theorem.** The proof of Theorem 1.2 consists in two steps:

Step 1: Prove that the sequence M_n is bounded in $\mathcal{L}(C(\overline{X}), \mathcal{B}(L^2(\mathbf{B})))$.

Step 2: Prove that the sequence converges on a dense subset.

2.2.1. *Boundedness.* In the following $\mathbf{1}_{\overline{X}}$ denotes the characteristic function of $\overline{X}$. Define

$$F_n := \left[M_n(\mathbf{1}_{\overline{X}})\right] \mathbf{1}_{\mathbf{B}}.$$

We denote by $\Xi(n)$ the common value of Ξ on elements of length n.

Corollary 2.5. *The function $\xi \mapsto \sum_{\gamma \in S_n} (P(\gamma, \xi))^{\frac{1}{2}}$ is constant equal to $|S_n| \times \Xi(n)$.*

Proof. This function is constant on orbits of the action of the group of automorphisms of X fixing x_0. Since it is transitive on $\mathbf{B}$, the function is constant. By integrating, we find

$$\begin{aligned}
\sum_{\gamma \in S_n} (P(\gamma, \xi))^{\frac{1}{2}} &= \int_{\mathbf{B}} \sum_{\gamma \in S_n} (P(\gamma, \xi))^{\frac{1}{2}}\, d\mu_{x_0}(\xi) \\
&= \sum_{\gamma \in S_n} \int_{\mathbf{B}} (P(\gamma, \xi))^{\frac{1}{2}}\, d\mu_{x_0}(\xi) \\
&= \sum_{\gamma \in S_n} \Xi(n) \\
&= |S_n|\Xi(n),
\end{aligned}$$

□

Lemma 2.6. *The function F_n is constant, equal to $\mathbf{1}_{\mathbf{B}}$.*

Proof. Because Ξ depends only on the length, we have that

$$\begin{aligned}
F_n(\xi) &:= \frac{1}{|S_n|} \sum_{\gamma \in S_n} \frac{(P(\gamma, \xi))^{\frac{1}{2}}}{\Xi(\gamma)} \\
&= \frac{1}{|S_n|\Xi(n)} \sum_{\gamma \in S_n} (P(\gamma, \xi))^{\frac{1}{2}} \\
&= 1,
\end{aligned}$$

and the proof is done. □

It is easy to see that $M_n(f)$ induces continuous linear transformations of L^1 and L^∞, which we also denote by $M_n(f)$.

Proposition 2.7. *The operator $M_n(\mathbf{1}_{\overline{X}})$, as an element of $\mathcal{L}(L^\infty, L^\infty)$, has norm 1; as an element of $\mathcal{B}(L^2(\mathbf{B}))$, it is self-adjoint.*

Proof. Let $h \in L^\infty(\mathbf{B})$. Since $M_n(\mathbf{1}_{\overline{X}})$ is positive, we have that

$$\begin{aligned}\left\| \left[M_n(\mathbf{1}_{\overline{X}})\right] h \right\|_\infty &\leq \left\| \left[M_n(\mathbf{1}_{\overline{X}})\right] \mathbf{1}_{\mathbf{B}} \right\|_\infty \|h\|_\infty \\ &= \|F_n\|_\infty \|h\|_\infty \\ &= \|h\|_\infty\end{aligned}$$

so that $\|M_n(\mathbf{1}_{\overline{X}})\|_{\mathcal{L}(L^\infty, L^\infty)} \leq 1$.

The self-adjointness follows from the fact that $\pi(\gamma)^* = \pi(\gamma^{-1})$ and that the set of summation is symmetric. ☐

Let us briefly recall one useful corollary of Riesz-Thorin's theorem :

Let (Z, μ) be a probability space.

Proposition 2.8. *Let T be a continuous operator of $L^1(Z)$ to itself such that the restriction T_2 to $L^2(Z)$ (resp. T_∞ to $L^\infty(Z)$) induces a continuous operator of $L^2(Z)$ to itself (resp. $L^\infty(Z)$ to itself).*

Suppose also that T_2 is self-adjoint, and assume that $\|T_\infty\|_{\mathcal{L}(L^\infty(Z), L^\infty(Z))} \leq 1$.

Then $\|T_2\|_{\mathcal{L}(L^2(Z), L^2(Z))} \leq 1$.

Proof. Consider the adjoint operator T^* of $(L^1)^* = L^\infty$ to itself. We have that

$$\|T^*\|_{\mathcal{L}(L^\infty, L^\infty)} = \|T\|_{\mathcal{L}(L^1(Z), L^1(Z))}.$$

Now because T_2 is self-adjoint, it is easy to see that $T^* = T_\infty$. This implies

$$1 \geq \|T^*\|_{\mathcal{L}(L^\infty, L^\infty)} = \|T\|_{\mathcal{L}(L^1(Z), L^1(Z))}.$$

Hence the Riesz-Thorin's theorem gives us the claim.

☐

Proposition 2.9. *The sequence $(M_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{L}(C(\overline{X}), \mathcal{B}(L^2(\mathbf{B})))$.*

Proof. Because $M_n(f)$ is positive in f, we have, for every positive $g \in L^2(\mathbf{B})$, the inequality

$$-\|f\|_\infty [M_n(\mathbf{1}_{\overline{X}})] g \leq [M_n(f)] g \leq \|f\|_\infty [M_n(\mathbf{1}_{\overline{X}})] g$$

from which we deduce, for every $g \in L^2(\mathbf{B})$

$$\begin{aligned}\|[M_n(f)] g\|_{L^2} &\leq \|f\|_\infty \|[M_n(\mathbf{1}_{\overline{X}})] g\|_{L^2} \\ &\leq \|f\|_\infty \ \|M_n(\mathbf{1}_{\overline{X}})\|_{\mathcal{B}(L^2)} \ \|g\|_{L^2}\end{aligned}$$

which allows us to conclude that

$$\|M_n(f)\|_{\mathcal{B}(L^2)} \leq \|M_n(\mathbf{1}_{\overline{X}})\|_{\mathcal{B}(L^2)} \|f\|_\infty.$$

This proves that $\|M_n\|_{\mathcal{L}(C(\overline{X}), \mathcal{B}(L^2))} \leq \|M_n(\mathbf{1}_{\overline{X}})\|_{\mathcal{B}(L^2)}$.

Now, it follows from Proposition 2.7 and Proposition 2.8 that the sequence $(M_n(\mathbf{1}_{\overline{X}}))_{n\in\mathbb{N}}$ is bounded by 1 in $\mathcal{B}(L^2)$, so we are done. ☐

2.2.2. *Estimates for the Harish-Chandra function.* The values of the Harish-Chandra are known (see for example [FTP82, Theorem 2, Item (iii)]). We provide here the simple computations we need.

We will calculate the value of

$$\langle \pi(\gamma) \mathbf{1}_{\mathbf{B}}, \mathbf{1}_{\mathbf{B}_u} \rangle = \int_{\mathbf{B}_u} P(\gamma, \xi)^{\frac{1}{2}} d\mu_{x_0}(\xi).$$

Lemma 2.10. *Let* $\gamma = s_1 \cdots s_n \in \mathbb{F}_r$. *Let* $l \in \{1, ..., |\gamma|\}$, *and* $u = s_1 \cdots s_{l-1} t_l t_{l+1} \cdots t_{l+k}$[1], *with* $t_l \neq s_l$ *and* $k \geq 0$, *be a reduced word. Then*

$$\langle \pi(\gamma) \mathbf{1}_{\mathbf{B}}, \mathbf{1}_{\mathbf{B}_u} \rangle = \frac{1}{2r(2r-1)^{\frac{|\gamma|}{2}+k}}$$

and

$$\langle \pi(\gamma) \mathbf{1}_{\mathbf{B}}, \mathbf{1}_{\mathbf{B}_\gamma} \rangle = \frac{2r-1}{2r(2r-1)^{\frac{|\gamma|}{2}}}$$

Proof. The function $\xi \mapsto \beta_\xi(x_0, \gamma x_0)$ is constant on $\mathbf{B}_u$ equal to $2(l-1) - |\gamma|$.

So $\langle \pi(\gamma)\mathbf{1}_{\mathbf{B}}, \mathbf{1}_{\mathbf{B}_u} \rangle$ is the integral of a constant function:

$$\begin{aligned} \int_{\mathbf{B}_u} P(\gamma, \xi)^{\frac{1}{2}} d\mu_{x_0}(\xi) &= \mu_{x_0}(\mathbf{B}_u)\, e^{\log(2r-1)\left((l-1)-\frac{|\gamma|}{2}\right)} \\ &= \frac{1}{2r(2r-1)^{\frac{|\gamma|}{2}+k}}. \end{aligned}$$

The value of $\langle \pi(\gamma)\mathbf{1}_{\mathbf{B}}, \mathbf{1}_{\mathbf{B}_\gamma} \rangle$ is computed in the same way. ☐

Lemma 2.11. *(The Harish-Chandra function)*

Let $\gamma = s_1 \cdots s_n$ *in* S_n *written as a reduced word. We have that*

$$\Xi(\gamma) = \left(1 + \frac{r-1}{r}|\gamma|\right)(2r-1)^{-\frac{|\gamma|}{2}}.$$

Proof. We decompose $\mathbf{B}$ into the following partition:

$$\mathbf{B} = \bigsqcup_{u_1 \neq s_1} \mathbf{B}_{u_1} \sqcup \left(\bigsqcup_{l=2}^{|\gamma|} \bigsqcup_{\substack{u = s_1 \cdots s_{l-1} t_l \\ t_l \notin \{s_l, (s_{l-1})^{-1}\}}} \mathbf{B}_u \right) \sqcup \mathbf{B}_\gamma$$

and Lemma 2.10 provides us the value of the integral on the subsets forming this partition. A simple calculation yields the announced formula. ☐

The proof of the following lemma is then obvious :

Lemma 2.12. *If* $\gamma, w \in \mathbb{F}_r$ *are such that* w *is not a prefix of* γ, *then there is a constant* C_w *not depending on* γ *such that*

$$\frac{\langle \pi(\gamma) \mathbf{1}_{\mathbf{B}}, \mathbf{1}_{\mathbf{B}_w} \rangle}{\Xi(\gamma)} \leq \frac{C_w}{|\gamma|}.$$

2.2.3. *Analysis of matrix coefficients.* The goal of this section is to compute the limit of the *matrix coefficients* $\langle M_n(\chi_u)\mathbf{1}_{\mathbf{B}_v}, \mathbf{1}_{\mathbf{B}_w} \rangle$.

Lemma 2.13. *Let* $u, w \in \mathbb{F}_r$ *such that none of them is a prefix of the other (i.e.* $\mathbf{B}_u \cap \mathbf{B}_w = \emptyset$*). Then*

$$\lim_{n\to\infty} \langle M_n(\chi_u) \mathbf{1}_{\mathbf{B}}, \mathbf{1}_{\mathbf{B}_w} \rangle = 0$$

Proof. Using Lemma 2.12, we get

$$\begin{aligned} \langle M_n(\chi_u)\mathbf{1}_{\mathbf{B}}, \mathbf{1}_{\mathbf{B}_w} \rangle &= \frac{1}{|S_n|} \sum_{\gamma \in S_n} \chi_u(\gamma x_0) \frac{\langle \pi(\gamma)\mathbf{1}_{\mathbf{B}}, \mathbf{1}_{\mathbf{B}_w} \rangle}{\Xi(\gamma)} \\ &= \frac{1}{|S_n|} \sum_{\gamma \in C_u \cap S_n} \frac{\langle \pi(\gamma)\mathbf{1}_{\mathbf{B}}, \mathbf{1}_{\mathbf{B}_w} \rangle}{\Xi(\gamma)} \\ &\leq \frac{1}{|S_n|} \sum_{\gamma \in C_u \cap S_n} \frac{C_w}{|\gamma|} \\ &= O\left(\frac{1}{n}\right) \end{aligned}$$

[1]For $l = 1$, $s_1 \cdots s_{l-1}$ is e by convention.

□

Lemma 2.14. *Let $u, v \in \mathbb{F}_r$. Then*

$$\limsup_{n\to\infty} \langle M_n(\chi_u) \mathbf{1}_{\mathbf{B}_v}, \mathbf{1}_{\mathbf{B}} \rangle \leq \mu_{x_0}(\mathbf{B}_u)\mu_{x_0}(\mathbf{B}_v)$$

Proof.

$$\begin{aligned}
\langle M_n(\chi_u)\mathbf{1}_{\mathbf{B}_v}, \mathbf{1}_{\mathbf{B}} \rangle &= \langle M_n(\chi_u)^* \mathbf{1}_{\mathbf{B}}, \mathbf{1}_{\mathbf{B}_v} \rangle \\
&= \frac{1}{|S_n|} \sum_{\gamma \in S_n} \chi_u(\gamma^{-1}x_0) \frac{\langle \pi(\gamma)\mathbf{1}_{\mathbf{B}}, \mathbf{1}_{\mathbf{B}_v} \rangle}{\Xi(\gamma)} \\
&\leq \frac{1}{|S_n|} \sum_{\gamma \in S_n} \chi_u(\gamma^{-1}x_0)\chi_v(\gamma x_0) \\
&\quad + \frac{1}{|S_n|} \sum_{\substack{\gamma \in S_n \\ \gamma \notin C_v}} \chi_u(\gamma^{-1}x_0) \frac{\langle \pi(\gamma)\mathbf{1}_{\mathbf{B}}, \mathbf{1}_{\mathbf{B}_v} \rangle}{\Xi(\gamma)} \\
&= \frac{1}{|S_n|} \sum_{\gamma \in S_n} \chi_u(\gamma^{-1}x_0)\chi_v(\gamma x_0) \\
&\quad + O\left(\frac{1}{n}\right)
\end{aligned}$$

Hence, by taking the lim sup and using Theorem **I**, we obtain the desired inequality. □

Proposition 2.15. *For all $u, v, w \in \mathbb{F}_r$, we have*

$$\lim_{n\to\infty} \langle M_n(\chi_u) \mathbf{1}_{\mathbf{B}_v}, \mathbf{1}_{\mathbf{B}_w} \rangle = \mu_{x_0}(\mathbf{B}_u \cap \mathbf{B}_w)\mu_{x_0}(\mathbf{B}_v)$$

Proof. We first show the inequality

$$\limsup_{n\to\infty} \langle M_n(\chi_u)\mathbf{1}_{\mathbf{B}_v}, \mathbf{1}_{\mathbf{B}_w} \rangle \leq \mu_{x_0}(\mathbf{B}_u \cap \mathbf{B}_w)\mu_{x_0}(\mathbf{B}_v).$$

If none of u and w is a prefix of the other, we have nothing to do according to Lemma 2.13. Let us assume that u is a prefix of w (the other case can be treated analogously). We have, by Lemma 2.14, that

$$\begin{aligned}
\mu_{x_0}(\mathbf{B}_w)\mu_{x_0}(\mathbf{B}_v) &\geq \limsup_{n\to\infty} \langle M_n(\chi_w)\mathbf{1}_{\mathbf{B}_v}, \mathbf{1}_{\mathbf{B}} \rangle \\
&\geq \limsup_{n\to\infty} \langle M_n(\chi_w)\mathbf{1}_{\mathbf{B}_v}, \mathbf{1}_{\mathbf{B}_w} \rangle \\
&\geq \limsup_{n\to\infty} \langle M_n(\chi_w)\mathbf{1}_{\mathbf{B}_v}, \mathbf{1}_{\mathbf{B}_w} \rangle + \sum_{\gamma \in Pr_u(|w|)\setminus\{w\}} \limsup_{n\to\infty} \langle M_n(\chi_\gamma)\mathbf{1}_{\mathbf{B}_v}, \mathbf{1}_{\mathbf{B}_w} \rangle \\
&= \limsup_{n\to\infty} \langle M_n(\chi_u)\mathbf{1}_{\mathbf{B}_v}, \mathbf{1}_{\mathbf{B}_w} \rangle
\end{aligned}$$

We now compute the expected limit. Let us define

$$S_{u,v,w} := \{(u', v', w') \in \mathbb{F}_r \mid |u| = |u'|, |v| = |v'|, |w| = |w'|\}.$$

Then

$$\begin{aligned}
1 &= \liminf_{n\to\infty} \langle M_n(\mathbf{1}_{\overline{X}})\mathbf{1}_{\mathbf{B}}, \mathbf{1}_{\mathbf{B}} \rangle \\
&\leq \liminf_{n\to\infty} \langle M_n(\chi_u)\mathbf{1}_{\mathbf{B}_v}, \mathbf{1}_{\mathbf{B}_w} \rangle + \sum_{(u',v',w')\in S_{u,v,w}\setminus\{u,v,w\}} \limsup_{n\to\infty} \langle M_n(\chi_{u'})\mathbf{1}_{\mathbf{B}_{v'}}, \mathbf{1}_{\mathbf{B}_{w'}} \rangle \\
&\leq \limsup_{n\to\infty} \langle M_n(\chi_u)\mathbf{1}_{\mathbf{B}_v}, \mathbf{1}_{\mathbf{B}_w} \rangle + \sum_{(u',v',w')\in S_{u,v,w}\setminus\{u,v,w\}} \limsup_{n\to\infty} \langle M_n(\chi_{u'})\mathbf{1}_{\mathbf{B}_{v'}}, \mathbf{1}_{\mathbf{B}_{w'}} \rangle \\
&\leq \mu_{x_0}(\mathbf{B}_u \cap \mathbf{B}_w)\mu_{x_0}(\mathbf{B}_v) + \sum_{(u',v',w')\in S_{u,v,w}\setminus\{u,v,w\}} \mu_{x_0}(\mathbf{B}_{u'} \cap \mathbf{B}_{w'})\mu_{x_0}(\mathbf{B}_{v'}) \\
&= 1
\end{aligned}$$

This proves that all the inequalities above are in fact equalities, and moreover proves that the inequalities

$$\liminf_{n\to\infty} \langle M_n(\chi_u)\mathbf{1}_{\mathbf{B}_v}, \mathbf{1}_{\mathbf{B}_w} \rangle \leq \limsup_{n\to\infty} \langle M_n(\chi_u)\mathbf{1}_{\mathbf{B}_v}, \mathbf{1}_{\mathbf{B}_w} \rangle \leq \mu_{x_0}(\mathbf{B}_u \cap \mathbf{B}_w)\mu_{x_0}(\mathbf{B}_v)$$

are in fact equalities. □

Proof of Theorem 1.2. Because of the boundedness of the sequence $(M_n)_{n\in\mathbb{N}}$ proved in Proposition 2.9, it is enough to prove the convergence for all (f, h_1, h_2) in a dense subset of $C(\overline{X}) \times L^2 \times L^2$, which is what Proposition 2.15 asserts. □

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