Approximation by generalized Szász operators involving Sheffer polynomials

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Abstract

The purpose of this article is to give a Chlodowsky type generalization of Szász operators defined by means of the Sheffer type polynomials. We obtain convergence properties of our operators with the help of Korovkin's theorem and the order of convergence by using a classical approach, the second order modulus of continuity and Peetre's K-functional. Moreover, we study the convergence of these operators in a weighted space of functions on a positive semi-axis and estimate the approximation by using a new type of weighted modulus of continuity introduced by *Gadjiev and Aral* in [12]. An algorithm is also given to plot graphical examples, and we have shown the convergence of these operators towards the function and these examples can be take as a comparison between the new operators with the previous one too. Finally, some numerical examples are also given.

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1. Introduction and preliminaries

In approximation theory, the positive approximation processes discovered by Korovkin play a central role and arise in a natural way in many problems connected with functional analysis, harmonic analysis, measure theory, partial differential equations and probability theory. The most useful examples of such operators are Szász [1] operators.

Szász [1] defined the positive linear operators:

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right) \tag{1.1}$$

where $x \geq 0$ and $f \in C[0,\infty)$ whenever the above sum converges. Motivated by this work, many authors have investigated several interesting properties of the operators (1.1).

Later, Jakimovski and Leviatan [9] obtained a generalization of Szász operators by means of Appell polynomials. Let $g(z) = \sum_{k=0}^{\infty} a_k z^k$ $(a_0 \neq 0)$ be an analytic function in the disk $|z| < R$, $(R > 1)$ and suppose that $g(1) \neq 0$. The Appell polynomials $p_k(x)$ have generating functions of the form

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k. \tag{1.2}$$

Under the assumption that $p_k(x) \geq 0$ for $x \in [0,\infty)$, Jakimovski and Leviatan introduced the positive linear operators $P_n(f;x)$ via

$$P_n(f;x) := \frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty} p_k(nx)f\left(\frac{k}{n}\right) \tag{1.3}$$

and gave the approximation properties of the operators.

Case 1. For $g(1)=1$, with the help of (1.2) we easily find $p_k(x)=\frac{x^k}{k!}$ and from (1.3) we meet again the Szász operators given by (1.1).

Then, Ismail [10] presented another generalization of Szász operators (1.1) and Jakimovski and Leviatan operators (1.3) by using Sheffer polynomials. Let $A(z)=\sum_{k=0}^{\infty} a_k z^k$ $(a_0 \neq 0)$ and $H(z)=\sum_{k=1}^{\infty} h_k z^k$ $(h_1 \neq 0)$ be analytic functions in the disk $|z|<R$ $(R>1)$ where a_k and h_k are real. The Sheffer polynomials $p_k(x)$ have generating functions of the type

$$A(t)e^{xH(t)}=\sum_{k=0}^{\infty} p_k(x)t^k, \quad |t|<R. \tag{1.4}$$

Using the following assumptions:

(i) for $x\in[0,\infty)$, $p_k(x)\geq 0$,

(ii) $A(1)\neq 0$ and $H'(1)=1$, (1.5)

Ismail investigated the approximation properties of the positive linear operators given by

$$T_n(f;x):=\frac{e^{-nxH(1)}}{A(1)}\sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \text{ for } n\in\mathbb{N}. \tag{1.6}$$

Case 1. For $H(t)=t$, it can be easily seen that the generating functions (1.4) return to (1.2) and, from this fact, the operators (1.6) reduce to the operators (1.3).

Case 2. For $H(t)=t$ and $A(t)=1$, one get the Szász operators from the operators (1.6).

In [9], Büyükyazıcı et al. introduced the Chlodowsky [2] variant of operators (1.3). Guided by their work we give the Chlodowsky type generalization of operators (1.6) as follows:

$$T_n^*(f;x):=\frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)}\sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) f\left(\frac{k}{n}b_n\right) \tag{1.7}$$

with b_n a positive increasing sequence with the properties

$$\lim_{n\to\infty} b_n=\infty, \qquad \lim_{n\to\infty}\frac{b_n}{n}=0 \tag{1.8}$$

and p_k are Sheffer polynomials defined by (1.4). For other generalization of operators (1.6) one can refer to [11].

The rest of the paper is organized as follows. In Section 2 we obtain some local approximation results by the generalized Szász operators given by (1.7). In particular, the convergence of operators is examined with the help of Korovkin's theorem. The order of approximation is established by means of a classical approach, the second-order modulus of continuity and Peetre's K-functional. An algorithm and some graphical examples are also given to in claim of convergence of operators towards the function. Section 3 is devoted to study the convergence of these operators in a weighted space of functions on a positive semi-axis and estimate the approximation by using a new type of weighted modulus of continuity introduced by *Gadjiev and Aral* in [12]. Finally, some numerical examples are also given in section 4.

Note that throughout the paper we will assume that the operators T_n^* are positive and we use the following test functions

$$e_i(x)=x^i, \quad i\in\{0,1,2\}.$$

2. Local approximation properties of $T_n^*(f;x)$

We denote by $C_E[o,\infty)$ the set of all continuous functions f on $[0,\infty)$ with the property that $|f(x)| \leq \beta e^{\alpha x}$ for all $x \geq 0$ and some positive finite α and β. For a fixed $r \in \mathbb{N}$ we denote by $C_E^r[0,\infty) = \{f \in C_E[0,\infty) : f', f'', \cdots, f^{(r)} \in C_E[0,\infty)\}$. Using equality (1.1) and the fundamental properties of the T_n^* operators, one can easily get the following lemmas:

Lemma 2.1. For all $x \in [0,\infty)$, we have

$$T_n^*(e_0;x) = 1; \tag{2.1}$$

$$T_n^*(e_1;x) = x + \frac{b_n}{n}\frac{A'(1)}{A(1)}; \tag{2.2}$$

$$T_n^*(e_2;x) = x^2 + \frac{b_n}{n}\frac{A(1)+2A'(1)+A(1)H''(1)}{A(1)}x + \frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)}. \tag{2.3}$$

It follows from Lemma 2.1 that,

$$T_n^*\big((e_1-x);x\big) = 0, \tag{2.4}$$

$$T_n^*\big((e_1-x)^2;x\big) = \frac{b_n}{n}\big(1+H''(1)\big)x + \frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)}. \tag{2.5}$$

Theorem 2.2. For $f \in C_E[0,\infty)$, the operators T_n^* converge uniformly to f on $[0,a]$ as $n \to \infty$.

Proof. According to (2.1)-(2.3), we have

$$\lim_{n\to\infty} T_n^*(e_i;x) = e_i(x), i \in \{0,1,2\}.$$

If we apply the Korovkin theorem [5], we obtain the desired result.

Algorithm

Graphically, to show the approximation of a given function $f(x)$ by positive linear operators $\mathbf{T}_n(f;x)$ and $\mathbf{T}_n^*(f;x)$ given by (1.7), the algorithm is summarized as below.

Step 1: Choose the functions $A(t)$ and $H(t)$ such that $A(t) \neq 0$ and $H'(1) = 1$.
Step 2: Find out the Sheffer polynomials $\mathbf{p}_k(x)$ with the help of relation (1.4).
Step 3: Check $\mathbf{p}_k(x) \geq 0$ for $x \geq 0$.
Step 4: Choose the sequence b_n under the condition given in (1.8).
Step 5: Plot the graph of function $f(x)$ and the operators $\mathbf{T}_n(f;x)$ and $\mathbf{T}_n^*(f;x)$ for the different values of n.

Example 2.3. For (i) $A(t) = e^t$ and $H(t) = t$, (ii) $A(t) = t$ and $H(t) = t$, the convergence of the two operators $T_n(f;x)$ and $T_n^*(f;x)$ to $f(x)$ are illustrated in Figs. 1, 2, 3 and 4 respectively, where $f(x) = -4xe^{-3x},\ n = 10,\ 50,\ 100,\ 200,\ 300,$ and $b_n = \sqrt{n}$.

Remark. From figure 1 and figure 3, we can see that when the value of n is increasing, the graph of operators $T_n(f;x)$ are going far away form the graph of the function $f(x)$ but with our proposed

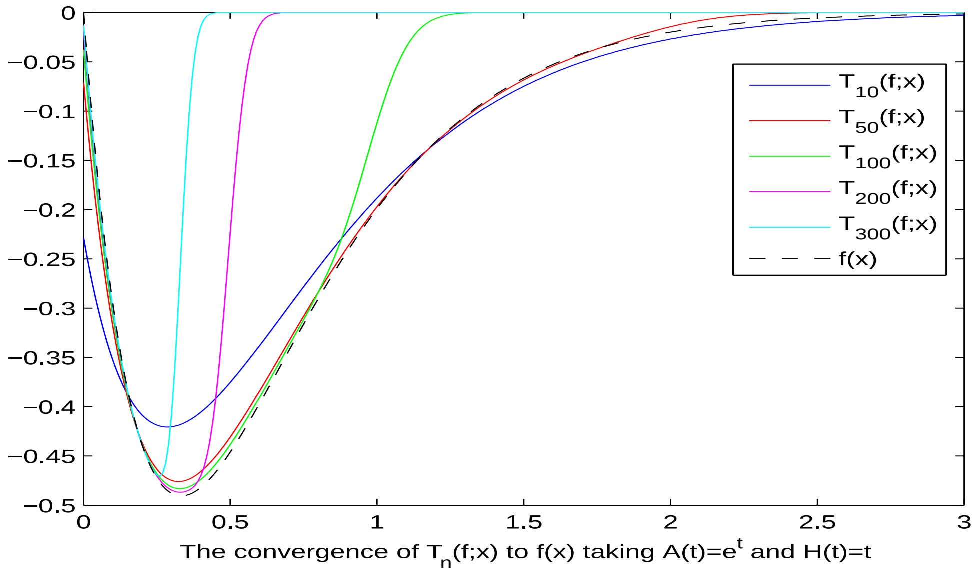

Figure 1:

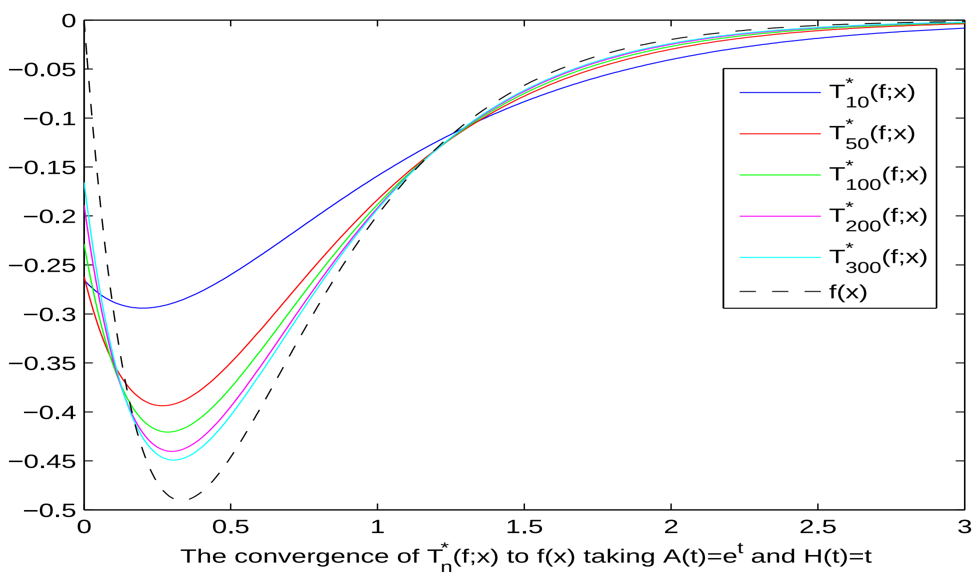

Figure 2:

operators $T_n^*(f;x)$, the convergence towards the function can be seen very clearly from figure 2 and figure 4. In a nut shell, we can claim that to approximate a function our operators $T_n^*(f;x)$ are better in comparison of $T_n(f;x)$.

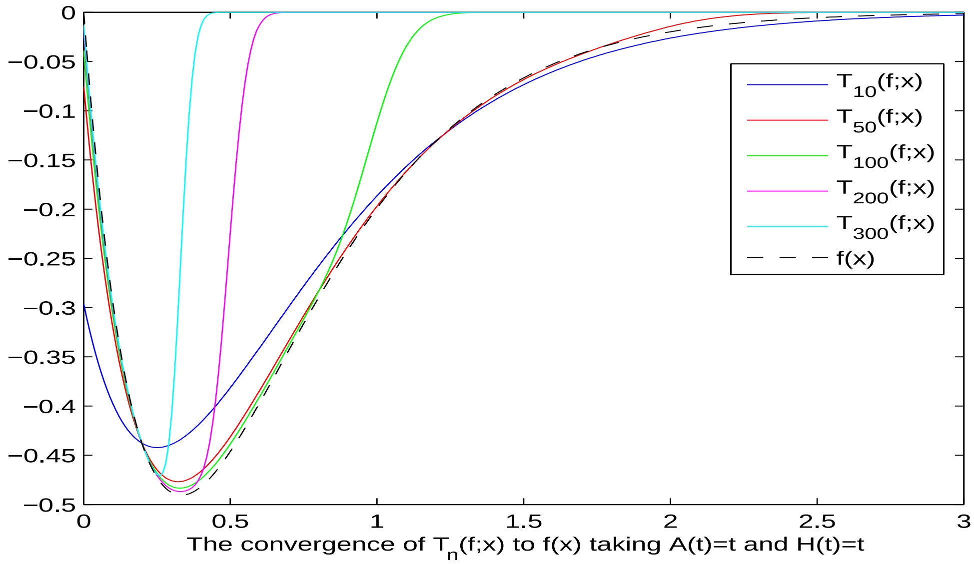

Figure 3:

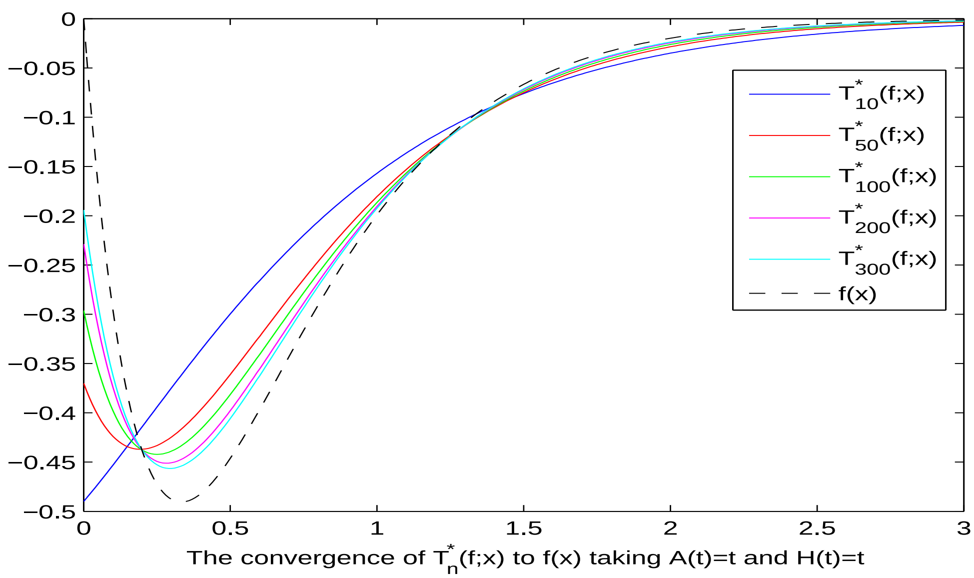

Figure 4:

Now, we concerned with the estimate of the order of approximation of a function f by means of the positive operators L_n^*, using the first and second order modulus of continuity [6].

If $\delta > 0$, the modulus of continuity $\omega(f,\delta)$ of $f \in C[a,b]$ is defined by

$$\omega(f,\delta) = \sup_{x,y\in[a,b],\ |x-y|\le\delta} |f(x)-f(y)|.$$

The second order modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$\omega_2(f,\delta) = \sup_{0<t\le\delta} \|f(.+2t) - 2f(.+t) + f(.)\|_{C_B}$$

where $C_B[0,\infty)$ is the class of real valued functions defined on $[0,\infty)$ which are bounded and uniformly continuous with the norm $\|f\|_{C_B} = \sup_{x\in[0,\infty)} |f(x)|$.

The Peetre's K-functional [6] of the function $f \in C_B[0,\infty)$ is defined by

$$K(f,\delta) := \inf_{g\in C_B^2[0,\infty)} \left\{ \|f-g\|_{C_B} + \delta\|g\|_{C_B^2} \right\}$$

where

$$C_B^2[0,\infty) := \left\{ g \in C_B[0,\infty) :\ g',\ g'' \in C_B[0,\infty) \right\}$$

and the norm $\|g\|_{C_B^2} := \|g\|_{C_B} + \|g'\|_{C_B} + \|g''\|_{C_B}$. It is clear that the following inequality:

$$K(f,\delta) \le M\left\{ \omega_2(f,\sqrt{\delta}) + \min(1,\delta)\|f\|_{C_B} \right\}$$

is valid, for all $\delta > 0$. The constant M is independent of f and δ.

Lemma 2.4 *([7]).* Let $g \in C^2[0,\infty)$ and $(P_n)_{n\ge0}$ be a sequence of positive linear operators with the property $P_n(1;x) = 1$. Then

$$|P_n(g;x) - g(x)| \le \|g'\|\sqrt{P_n\big((s-x)^2;x\big)} + \frac{1}{2}\|g''\|P_n\big((s-x)^2;x\big).$$

Lemma 2.5 *([8]).* Let $f \in C[a,b]$ and $h \in \left(0, \frac{b-a}{2}\right)$. Let f_h be the second-order Steklov function attached to the function f. Then the following inequalities are satisfied:

(i) $\|f_h - f\| \le \frac{3}{4}\omega_2(f,h)$,

(ii) $\|f_h''\| \le \frac{3}{2h^2}\omega_2(f,h)$.

Theorem 2.6. If $f \in C_E[0,\infty)$, then for any $x \in [0,a]$ we have

$$|T_n^*(f;x) - f(x)| \le \left\{ 1 + \sqrt{\big(1+H''(1)\big)a + \frac{b_n}{n}\frac{A'(1)+A''}{A(1)}} \right\} \omega\left(f, \sqrt{\frac{b_n}{n}}\right).$$

Proof. We will use the relation (2.5) and the well-known properties of the modulus of continuity. We have

$$\begin{aligned} |T_n^*(f;x) - f(x)| &\le \frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k\Big(\frac{n}{b_n}x\Big)\Big|f\Big(\frac{k}{n}b_n\Big) - f(x)\Big| \\ &\le \left\{ 1 + \frac{1}{\delta}\frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k\Big(\frac{n}{b_n}x\Big)\Big|\frac{k}{n}b_n - x\Big| \right\}\omega(f,\delta). \end{aligned}$$

Recalling the Cauchy-Schwartz inequality we obtain the formula below:

$$\begin{aligned} |T_n^*(f;x)-f(x)| &\leq \left\{1+\frac{1}{\delta}\left(\frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)}\sum_{k=0}^{\infty}p_k\Big(\frac{n}{b_n}x\Big)\Big(\frac{k}{n}b_n-x\Big)^2\right)^{\frac{1}{2}}\right\}\omega(f,\delta) \\ &= \left\{1+\frac{1}{\delta}\sqrt{T_n^*\big((e_1-x)^2;x\big)}\right\}\omega(f,\delta). \end{aligned} \tag{2.6}$$

By means of (2.5), for $0\leq x\leq a$, one gets

$$T_n^*\big((t-x)^2;x\big)\leq \frac{b_n}{n}\big(1+H''(1)\big)a+\frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)}. \tag{2.7}$$

Using (2.7) and taking $\delta=\sqrt{\frac{b_n}{n}}$ in (2.9), we obtain the desired result.

Theorem 2.7. For $f\in C[0,a]$, the following inequality:

$$|T_n^*(f;x)-f(x)|\leq \frac{2}{a}\|f\|h^2+\frac{3}{4}(a+2+h^2)\omega_2(f,h)$$

is satisfied where

$$h:=h_n(x)=\sqrt[4]{T_n^*\big((e_1-x)^2;x\big)}$$

and the second order modulus of continuity is given by $\omega_2(f,\delta)$ with the norm $\|f\|=\max\limits_{x\in[a,b]}|f(x)|$.

Proof. Let f_h be the second-order Steklov function attached to the function f. By virtue of the identity (2.1), we have

$$\begin{aligned} |T_n^*(f;x)-f(x)| &\leq |T_n^*(f-f_h;x)|+|T_n^*(f_h;x)-f_h(x)|+|f_h(x)-f(x)| \\ &\leq 2\|f_h-f\|+|T_n^*(f_h;x)-f_h(x)|. \end{aligned} \tag{2.8}$$

Taking into account the fact that $f_h\in C^2[0,a]$, it follows from Lemma 2.4 that

$$|T_n^*(f_h;x)-f_h(x)|\leq \|f_h'\|\sqrt{T_n^*\big((e_1-x)^2;x\big)}+\frac{1}{2}\|f_h''\|T_n^*\big((e_1-x)^2;x\big). \tag{2.9}$$

Combining the Landau inequality and Lemma 2.5, we can write

$$\begin{aligned} \|f_h'\| &\leq \frac{2}{a}\|f_h\|+\frac{a}{2}\|f_h''\| \\ &\leq \frac{2}{a}\|f\|+\frac{3a}{4}\frac{1}{h^2}\omega_2(f,h). \end{aligned}$$

From the last inequality, (2.9) becomes, on taking $h=\sqrt[4]{T_n^*\big((e_1-x)^2;x\big)}$,

$$|T_n^*(f_h;x)-f_h(x)|\leq \frac{2}{a}\|f\|h^2+\frac{3a}{4}\omega_2(f,h)+\frac{3}{4}h^2\omega_2(f,h). \tag{2.10}$$

Substituting (2.10) in (2.9), Lemma 2.5 hence gives the proof of the theorem.

Theorem 2.8. Let $f\in C_B^2[0,\infty)$. Then

$$|T_n^*(f;x)-f(x)|\leq \gamma_n(x)\|f\|_{C_B^2}$$

where

$$\gamma(x):=\gamma_n(x)=\frac{1}{2}\,T_n^*\big((t-x)^2;x\big).$$

Proof. Using the Taylor expansion of f, the linearity property of the operators T_n^* and (2.1), it follows that

$$T_n^*(f;x) - f(x) = f'(x)T_n^*(e_1 - x;x) + \frac{1}{2}f''(\eta)T_n^*\big((e_1-x)^2;x\big), \qquad \eta \in (x,t). \tag{2.11}$$

Taking into account the fact that

$$T_n^*\big((e_1 - x);x\big) \geq 0$$

for $x \leq t$, by combining Lemmas 2.1 and (2.5) in (2.11) we are led to

$$\begin{aligned} T_n^*(f;x) - f(x) &\leq \frac{1}{2}\left\{T_n^*\big((t-x)^2;x\big)\right\}\|f''\|_{C_B} \\ &\leq \frac{1}{2}\left\{T_n^*\big((t-x)^2;x\big)\right\}\|f\|_{C_B^2} \end{aligned}$$

which completes the proof.

Theorem 2.9. Let $f \in C_B[0,\infty)$. Then

$$|T_n^*(f;x) - f(x)| \leq 2M\big\{\omega_2(f,\sqrt{\delta}) + \min(1,\delta)\|f\|_{C_B}\big\}$$

where $\delta := \delta_n(x) = \frac{1}{4}\gamma_n(x)$ and $M > 0$ is a constant independent of the function f and δ. Note that $\gamma_n(x)$ is defined as in Theorem 2.8.

Proof. Let $g \in C_B^2[0,\infty)$. Theorem 2.8 allows us to write

$$\begin{aligned} |T_n^*(f;x) - f(x)| &\leq |T_n^*(f-g;x)| + |T_n^*(g;x) - g(x)| + |g(x) - f(x)| \\ &\leq 2\|f-g\|_{C_B} + \frac{1}{2}\left\{T_n^*\big((t-x)^2;x\big)\right\}\|g\|_{C_B^2} \\ &= 2\big\{\|f-g\|_{C_B} + \delta\|g\|_{C_B^2}\big\}. \end{aligned} \tag{2.12}$$

The left-hand side of inequality (2.12) does not depend on the function $g \in C_B^2[0,\infty)$, so

$$|T_n^*(f;x) - f(x)| \leq 2K(f,\delta). \tag{2.13}$$

By using the relation between Peetre's K-functional and second modulus of smoothness, (2.13) becomes

$$|T_n^*(f;x) - f(x)| \leq 2M\big\{\omega_2(f,\sqrt{\delta}) + \min(1,\delta)\|f\|_{C_B}\big\}.$$

3. Approximation properties in weighted spaces

Now we give approximation properties of the operators T_n^* of the weighted spaces of continuous functions with exponential growth on $\mathbb{R}_0^+ = [0,\infty)$ with the help of the weighted Korovkin type theorem proved by Gadjiev in [3, 4]. For this purpose, we consider the following weighted spaces of functions which are defined on the $\mathbb{R}_0^+ = [0,\infty)$.

Let $\rho(x)$ be the weighted function and M_f a positive constant, then we define

$$\begin{aligned} B_\rho(\mathbb{R}_0^+) &= \{f \in E(\mathbb{R}_0^+) : |f(x) \leq M_f\rho(x)|\}, \\ C_\rho(\mathbb{R}_0^+) &= \{f \in B_\rho(\mathbb{R}_0^+) : f \text{ is continuous}\}, \\ C_\rho^k(\mathbb{R}_0^+) &= \left\{f \in C_\rho(\mathbb{R}_0^+) : \lim_{n\to\infty}\frac{f(x)}{\rho(x)} = K_f < \infty\right\}. \end{aligned}$$

It is obvious that $C^k_\rho(\mathbb{R}^+_0) \subset C_\rho(\mathbb{R}^+_0) \subset B_\rho(\mathbb{R}^+_0)$. The space $B_\rho(\mathbb{R}^+_0)$ is a normed linear space with the following norm:

$$\|f\|_\rho = \sup_{x\in\mathbb{R}^+_0} \frac{|f(x)|}{\rho(x)}.$$

The following results on the sequence of positive linear operators in these spaces are given [3, 4].

Lemma 3.1 *([3, 4])*. The sequence of positive linear operators $(L_n)_{n\geq 1}$ which act from $C_\rho(\mathbb{R}^+_0)$ to $B_\rho(\mathbb{R}^+_0)$ if and only if there exists a positive constant k such that

$$L_n(\rho; x) \leq k\rho(x), \quad \text{i.e.}$$

$$\|L_n(\rho; x)\|_\rho \leq k.$$

Theorem 3.2 *([3, 4])*. Let $(L_n)_{n\geq 1}$ be the sequence of positive linear operators which act from $C_\rho(\mathbb{R}^+_0)$ to $B_\rho(\mathbb{R}^+_0)$ satisfying the conditions

$$\lim_{n\to\infty} \|L_n(t^i; x) - x^i\|_\rho = 0, \quad i \in \{0, 1, 2\},$$

then for any function $f \in C^k_\rho(\mathbb{R}^+_0)$

$$\lim_{n\to\infty} \|L_n f - f\|_\rho = 0.$$

Lemma 3.3. Let $\rho(x) = 1 + x^2$ be a weight function. If $f \in C_\rho(\mathbb{R}^+_0)$, then

$$\|T^*_n(\rho; x)\|_\rho \leq 1 + M.$$

Proof. Using (2.1) and (2.3), one has

$$\begin{aligned}
T^*_n(\rho; x) &= 1 + x^2 + \frac{b_n}{n}\frac{A(1) + 2A'(1) + A(1)H''(1)}{A(1)}x + \frac{b_n^2}{n^2}\frac{A'(1) + A''(1)}{A(1)} \\
\|T^*_n(\rho; x)\|_\rho &= \sup_{x\geq 0}\left\{\frac{1}{1+x^2}\left(1 + x^2 + \frac{b_n}{n}\frac{A(1) + 2A'(1) + A(1)H''(1)}{A(1)}x + \frac{b_n^2}{n^2}\frac{A'(1) + A''(1)}{A(1)}\right)\right\} \\
&\leq 1 + \frac{b_n}{n}\frac{A(1) + 2A'(1) + A(1)H''(1)}{A(1)} + \frac{b_n^2}{n^2}\frac{A'(1) + A''(1)}{A(1)}.
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{b_n}{n} = 0$, there exists a positive M such that

$$\|T^*_n(\rho; x)\|_\rho \leq 1 + M$$

so the proof is completed.

By using Lemma 3.3, we can easily see that the operators T^*_n defined by (1.7) act from $C_\rho(\mathbb{R}^+_0)$ to $B_\rho(\mathbb{R}^+_0)$.

Theorem 3.4. Let T^*_n be the sequence of positive linear operators defined by (1.7) and $\rho(x) = 1+x^2$, then for each $f \in C^k_\rho(\mathbb{R}^+_0)$

$$\lim_{n\to\infty} \|T^*_n(f; x) - f(x)\|_\rho = 0.$$

Proof. It is enough to prove that the conditions of the weighted Korovkin type theorem given by Theorem 3.2 are satisfied. From (2.1), it is immediately seen that

$$\lim_{n\to\infty} \|T^*_n(e_0; x) - e_0(x)\|_\rho = 0. \tag{3.1}$$

Using (2.2) we have

$$\|T_n^*(e_1;x)-e_1(x)\|_\rho=\frac{b_n}{n}\frac{A'(1)}{(1)} \tag{3.2}$$

this implies that

$$\lim_{n\to\infty}\|T_n^*(e_1;x)-e_1(x)\|_\rho=0. \tag{3.3}$$

By means of (2.3) we get

$$\begin{aligned}\|T_n^*(e_2;x)-e_2(x)\|_\rho &= \sup_{x\in R_0}\left|\frac{b_n}{n}\frac{A(1)+2A'(1)+A(1)H''(1)}{A(1)}\frac{x}{1+x^2}+\frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)}\frac{1}{1+x^2}\right|\\ &\le \frac{b_n}{n}\frac{A(1)+2A'(1)+A(1)H''(1)}{A(1)}+\frac{b_n^2}{n^2}\frac{A'(1)+A''(1)}{A(1)}. \end{aligned} \tag{3.4}$$

Using the conditions (1.8), it follows that

$$\lim_{n\to\infty}\|T_n^*(e_2;x)-e_2(x)\|_\rho=0. \tag{3.5}$$

From (3.1), (3.2) and (3.5), for $i\in\{0,1,2\}$ we have

$$\lim_{n\to\infty}\|T_n^*(e_i;x)-e_i(x)\|_\rho=0.$$

If we apply Theorem 3.2, we obtain the desired result.

Now, for any weighted function $\rho(x)$, we want to find the approximation and rate of approximation of the functions $f\in C_\rho^k(\mathbb{R}_0^+)$ by using the operators T_n^* on $\mathbb{R}_0^+=[0,\infty)$. For this we define new positive linear operators which are a generalization of the T_n^* operators. It is well-known that the usual first modulus of continuity does not tend to zero as $\delta\to 0$ on $\mathbb{R}_0^+$, so we use the following new type of weighted modulus of continuity introduced by Gadjiev and Aral in [12]:

$$\Omega_\rho(f,\delta)=\Omega(f,\delta)_{\mathbb{R}_0^+}=\sup_{x,t\in\mathbb{R}_0^+,\ |\rho(t)-\rho(x)|\le\delta}\frac{|f(t)-f(x)|}{[|\rho(t)-\rho(x)|+1]\rho(x)} \tag{3.6}$$

where ρ is satisfying the following assumptions:

(i) ρ is a continuously differentiable function on $\mathbb{R}_0^+$ and $\rho(0)=1$,

(ii) $\inf_{x\ge 0}\rho'(x)\ge 1$.

The weighted modulus of continuity $\Omega_\rho(f,\delta)$ given by (3.6) has some properties given in the following lemma (see [12]).

Lemma 3.5 *[12]* **.** For any $f\in C_\rho^k(\mathbb{R}_0^+)$ then

$$\lim_{\delta\to 0}\Omega_\rho(f,\delta)=0,$$

and for each $x,t\in\mathbb{R}_0^+$ the inequality

$$|f(t)-f(x)|\le 2\rho(x)(1+\delta^2)\left(1+\frac{\big(\rho(t)-\rho(x)\big)^2}{\delta^2}\right)\Omega_\rho(f,\delta)$$

holds, where δ is any fixed positive number.

The estimates of the approximation of functions by positive linear operators by means of the new type of modulus of continuity are given in the following theorem [12]:

Theorem 3.6 *[12]* **.** Let $\rho(x) \le \psi_k(x),\ k = 1, 2, 3$ and the sequences of the positive linear operators $(L_n)_{n\ge 1}$ satisfying the conditions

$$\|L_n(1; x) - 1\|_{\psi_1} = \alpha_n, \tag{3.7}$$

$$\|L_n(\rho; x) - \rho\|_{\psi_2} = \beta_n, \tag{3.8}$$

$$\|L_n(\rho^2; x) - \rho^2\|_{\psi_3} = \gamma_n, \tag{3.9}$$

where $\alpha_n,\ \beta_n$ and γ_n tend to zero as $n \to \infty$ and $\psi(x) = \max\big\{\psi_1(x), \psi_2(x), \psi_3(x)\big\}$. Then for all $f \in C^k_\rho(\mathbb{R}^+_0)$ the inequality

$$\|L_n(f; x) - f(x)\|_{\psi\rho^2} \le 16\Omega_\rho\big(f, \sqrt{\alpha_n + 2\beta_n + \gamma_n}\big) + \alpha_n\|f\|_\rho$$

holds for n large enough.

Now we define following a sequence of positive linear operators P^*_n with the help of T^*_n defined by (1.7)

$$P^*_n(f; x) := \frac{\rho^2(x)e^{-\frac{n}{b_n}xH(1)}}{A(1)} \sum_{k=0}^{\infty} \frac{f\big(\frac{k}{n}b_n\big)}{\rho^2\big(\frac{k}{n}b_n\big)} p_k\bigg(\frac{n}{b_n}x\bigg). \tag{3.10}$$

Theorem 3.7. Let P^*_n be the sequence of the positive linear operators defined by (3.10) and $\psi(x) = 1 + x^2$. If $f \in C^k_\rho(\mathbb{R}^+_0)$, then

$$\|P^*_n(f; x) - f(x)\|_{\rho^4\psi} \le 16\Omega_\rho\big(f, \sqrt{\alpha_n + 2\beta_n}\big) + \alpha_n\|f\|_\rho$$

Proof. By simple calculations we get

$$P^*_n(1; x) - 1 = \rho^2(x)\bigg[\frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)} \sum_{k=0}^{\infty} \frac{1}{\rho^2\big(\frac{k}{n}b_n\big)} p_k\bigg(\frac{n}{b_n}x\bigg) - \frac{1}{\rho^2(x)}\bigg], \tag{3.11}$$

$$P^*_n(\rho; x) - \rho(x) = \rho^2(x)\bigg[\frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)} \sum_{k=0}^{\infty} \frac{1}{\rho\big(\frac{k}{n}b_n\big)} p_k\bigg(\frac{n}{b_n}x\bigg) - \frac{1}{\rho(x)}\bigg], \tag{3.12}$$

$$P^*_n(\rho^2; x) - \rho^2(x) = 0. \tag{3.13}$$

From (3.3) and (3.5) we have

$$\lim_{n\to\infty}\bigg\|\frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)} \sum_{k=0}^{\infty} \frac{1}{\rho^2\big(\frac{k}{n}b_n\big)} p_k\bigg(\frac{n}{b_n}x\bigg) - \frac{1}{\rho^2(x)}\bigg\|_\psi = 0,$$

$$\lim_{n\to\infty}\bigg\|\frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)} \sum_{k=0}^{\infty} \frac{1}{\rho\big(\frac{k}{n}b_n\big)} p_k\bigg(\frac{n}{b_n}x\bigg) - \frac{1}{\rho(x)}\bigg\|_\psi = 0$$

using (3.4) and (3.11) we obtain

$$\begin{aligned}
\|P^*_n(1; x) - 1\|_{\rho^2\psi} &= \bigg\|\frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)} \sum_{k=0}^{\infty} \frac{1}{\rho^2\big(\frac{k}{n}b_n\big)} p_k\bigg(\frac{n}{b_n}x\bigg) - \frac{1}{\rho^2(x)}\bigg\|_\psi \\
&\le \frac{b_n}{n}\frac{A(1) + 2A'(1) + A(1)H''(1)}{A(1)} + \frac{b_n^2}{n^2}\frac{A'(1) + A''(1)}{A(1)} \\
&= \alpha_n.
\end{aligned}$$

By means of (3.2) and (3.12), one gets

$$\begin{aligned}\|P_n^*(\rho;x)-\rho\|_{\rho^2\psi} &= \left\| \frac{e^{-\frac{n}{b_n}xH(1)}}{A(1)}\sum_{k=0}^{\infty}\frac{1}{\rho(\frac{k}{n}b_n)}p_k\left(\frac{n}{b_n}x\right)-\frac{1}{\rho(x)}\right\|_{\psi} \\ &\le \frac{b_n}{n}\frac{A'(1)}{A(1)} \\ &= \beta_n.\end{aligned}$$

finally from (3.13), we obtain

$$\begin{aligned}\|P_n^*(\rho^2;x)-\rho^2\|_{\rho^2\psi} &= 0 \\ &= \gamma_n.\end{aligned}$$

Thus the (3.7)-(3.9) assumptions of Theorem 3.6 are satisfied for the operators (3.10). From Theorem 3.6, we have

$$\|P_n^*(f;x)-f(x)\|_{\rho^4\psi}\le 16\Omega_\rho\big(f,\sqrt{\alpha_n+2\beta_n+\gamma_n}\big)+\alpha_n\|f\|_\rho$$

for each $f\in C_\rho^k(\mathbb{R}_0^+)$. This completes the proof.

4. Numerical Examples

Example 4.1. The sequence $\{(1+x)^k\}_{k=1}^{\infty}$ which is the Sheffer sequence for $A(t)=e^t$ and $H(t)=t$ has the generating function of the following type

$$e^{(1+x)t}=\sum_{k=0}^{\infty}\frac{(1+x)^k}{k!}t^k.$$

Let us select $p_k(x)=\frac{(1+x)^k}{k!}$. Making use of above knowledge $p_k(x)\ge 0$ for $x\in[0,\infty)$, $A(1)\ne 0$ and $H'(1)=1$ are provided. Considering these polynomials in (1.7), we obtain operators as follows

$$T_n^*(f;x)=e^{-(\frac{n}{b_n}x+1)}\sum_{k=0}^{\infty}\frac{(\frac{n}{b_n}x+1)^k}{k!}f\left(\frac{k}{n}b_n\right).$$

The error bound for the function $f(x)=-4xe^{-3x}$ under the condition condition $A(t)=e^t$ and $H(t)=t$ is computed in the following Table 1:

Example 4.2. The sequence $\{x^{k-1}\}_{k=1}^{\infty}$ which is the Sheffer sequence for $A(t)=t$ and $H(t)=t$ has the generating function of the following type

$$te^{xt}=\sum_{k=1}^{\infty}\frac{x^{k-1}}{(k-1)!}t^k.$$

Let us select $p_k(x)=\frac{x^{k-1}}{(k-1)!}$. Making use of above knowledge $p_k(x)\ge 0$ for $x\in[0,\infty)$, $A(1)\ne 0$ and $H'(1)=1$ are provided. Considering these polynomials in (1.7), we obtain operators as follows

$$T_n^*(f;x)=e^{-\frac{n}{b_n}x}\sum_{k=1}^{\infty}\frac{(\frac{n}{b_n}x)^{k-1}}{(k-1)!}f\left(\frac{k}{n}b_n\right).$$

n	Error estimate by T_n^* operators including $\{(1+x)^k\}_{k=1}^{\infty}$ sequence
10	0.9481710727
10^3	0.8474426939
10^5	0.3806348279
10^7	0.1348930985
10^9	0.0442354247
10^{11}	0.0141505650
10^{13}	0.0044911482
10^{15}	0.0014218648
10^{17}	4.4979717260e-004
10^{19}	1.4225476356e-004

Table 1: The error bound of function $f(x) = -4xe^{-3x}$ by using modulus of continuity

n	Error estimate by T_n^* operators including $\{x^{k-1}\}_{k=1}^{\infty}$ sequence
10	0.8938844531
10^3	0.8409966996
10^5	0.3803350939
10^7	0.1348824385
10^9	0.0442350750
10^{11}	0.0141505538
10^{13}	0.0044911478
10^{15}	0.0014218647
10^{17}	4.4979717224e-004
10^{19}	1.4225476355e-004

Table 2: The error bound of function $f(x) = -4xe^{-3x}$ by using modulus of continuity

The error bound for the function $f(x) = -4xe^{-3x}$ under the condition condition $A(t) = t$ and $H(t) = t$ is computed in the following Table 2:

Conclusion

We introduced the Chlodowsky variant of generalized Szàsz operators by means of Sheffer polynomials and established different approximation results. We have also given an algorithm to plot the graphs of the positive linear operators and with the help of these graphical examples, we claimed that our operators are better than the old operators to approximate a given function. Some numerical examples are also provided and we found the error bound of a given function by using modulus of smoothness.

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