Estimation for Entanglement Negativity of Free Fermions

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Abstract

In this letter we study the negativity of one dimensional free fermions. We derive the general form of the $\mathbb{Z}_N$ symmetric term in moments of the partial transposed (reduced) density matrix, which is an algebraic function of the end points of the system. Such a path integral turns out to be a convenient tool for making estimations for the negativity.

1 Introduction

Measures of quantum entanglement have become a focus of intense research activity at the boundaries between quantum information, quantum field theory, condensed matter physics, general relativity and string theory (see refs. [1, 2, 3] for reviews). One key quantity, the entanglement entropy, measures the quantum entanglement between two complementary pieces of a system in a pure state. However the entanglement entropy is no longer a good measure of quantum entanglement if the initial state of the system is mixed. Negative eigenvalues in the partial transpose of the density matrix ρ^{T_2} implies quantum entanglement even in a (bipartite) mixed state scenario [4, 5]. This observation led to the proposal of the negativity [6], which was later demonstrated to be a good entanglement measure [7].

Like the entanglement entropy, the negativity in a quantum field theory can be computed by employing the replica trick [8, 9]. In this setting, the negativity is the $N \to 1$ limit of the partition function of an N-sheeted spacetime. In practice, these partition functions can only be computed in special cases [8, 9]. For conformal field theories in $1 + 1$ dimensions, the negativity of the single interval and the two adjacent interval cases is determined by conformal symmetry.[1] Another special case where the negativity can be determined, at least for $N > 1$, is a massless free scalar field in 1+1 dimensions. In this case, the N-sheeted partition functions are known in terms of Riemann-Siegel theta functions although it is not known in general how to continue the result away from

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[1]See ref. [10] for an extension to the massive case.

integer $N > 1$ and in particular to $N = 1$. Since the partial transposed reduced density matrix is Gaussian, the negativity for a free scalar can be checked through a lattice computation by using Wick's Theorem [9, 11].

The case of free fermions in 1+1 dimensions appears to be more difficult than the case of free scalars however. The partial transpose of the reduced density matrix is no longer Gaussian but a sum of two, generically non-commuting, Gaussian matrices [12]:

$$\rho^{T_2} = \frac{1}{\sqrt{2}} \left(e^{i\pi/4} O_+ + e^{-i\pi/4} O_- \right) . \tag{1}$$

(We will define $O_\pm$ in section 2.) This fact brings additional complication to both the lattice and field theoretical calculations. On the lattice side, eigenvalues of $(\rho^{T_2})^N$ cannot be simply derived from eigenvalues of a covariance matrix as in the Gaussian case. In a field theory setting, one has to sum over partition functions with different spin structure, corresponding to different terms in the expansion of $(e^{i\pi/4} O_+ + e^{-i\pi/4} O_-)^N$. Various efforts have been made to tame the difficulties in deriving the negativity of free fermions: On the lattice side, algebraic simplification and numerical diagonalization of products of these two Gaussian matrices yields the $N > 1$ moments of negativity for the two disjoint interval case [12, 13].[2] (Monte-Carlo and tensor network methods have also been used to calculate negativity for the Ising model [14, 15, 16] which, although not identical to the Dirac fermion, is closely related.) The analytical form of such moments are derived by evaluation of the corresponding path integrals [17, 18]. However in the existing results the sheet number N does not appear as a continuous variable; it remains an open problem how to take the $N \to 1$ limit to get the negativity.

In this letter we shall introduce a $\mathbb{Z}_N$-symmetric free fermion with specific choice of spin structure. This fermion has several nice features that we believe will help us explore and understand the features of free fermion negativity. 1) The partition function explicitly reproduces the correct adjacent interval limit. 2) The $N \to 1$ limit of the N sheeted path integral can be easily derived. 3) There exists a natural generalization to multiple interval cases, nonzero temperature, and nonzero chemical potential. 4) While such a partition function is not an Nth moment of ρ^{T_2} (except in the special case $N = 2$), it appears to be a useful quantity for bounding these N^{th} moments including the negativity itself.

The rest of this letter is arranged as follows: In section 2 we review previous results. Section 3 contains a derivation of the partition function for the $\mathbb{Z}_N$-symmetric free fermion system and in particular $\text{tr}(O_+^N)$ and $\text{tr}[(O_+O_-)^{N/2}]$. In section 4, we discuss bounds on the negativity and its Nth moments. We conclude in section 5 with remarks on possible generalizations of our results and future directions. An appendix contains a discussion of a two-spin system.

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[2]See also ref. [19] for an extension to two spatial dimensions.

2 Review of Previous Results

We first review the definition of the negativity. For a state $|\Psi\rangle$ in a quantum system with bipartite Hilbert space $\mathcal{H} = \mathcal{H}_A \bigotimes \mathcal{H}_B$ and density matrix $\rho = |\Psi\rangle\langle\Psi|$, the reduced density matrix is defined as $\rho_A = \mathrm{tr}_B\, \rho$. If $\mathcal{H}_A$ is factored further into $\mathcal{H}_A = \mathcal{H}_{A_1} \bigotimes \mathcal{H}_{A_2}$, one can define the partial transpose of the reduced density matrix $\rho_A^{T_2}$ as the operator such that the following identity holds for any $e_i^{(1)}$, $e_k^{(1)} \in \mathcal{H}_{A_1}$ and $e_j^{(2)}$, $e_l^{(2)} \in \mathcal{H}_{A_2}$: $\left\langle e_i^{(1)} e_j^{(2)} \middle| \rho_A^{T_2} \middle| e_k^{(1)} e_l^{(2)} \right\rangle = \left\langle e_i^{(1)} e_l^{(2)} | \rho_A | e_k^{(1)} e_j^{(2)} \right\rangle$. Negativity is defined as the trace norm[3] of $\rho_A^{T_2}$. Since $\rho_A^{T_2}$ is Hermitian, its trace norm can be written as the following limit

$$\mathcal{E} \equiv |\rho_A^{T_2}| = \lim_{N_e \to 1} \mathrm{tr}\left(\rho_A^{T_2}\right)^{N_e} \tag{2}$$

where N_e is an even integer. This analytic continuation suggests the utility of also defining higher moments of the partial transpose:

$$\mathcal{E}(N) \equiv \mathrm{tr}\left[(\rho_A^{T_2})^N\right] . \tag{3}$$

We are interested in systems in one time and one spatial dimension. We will assume a factorization of the Hilbert space corresponding to a partition of the real line with A_1 and A_2 each being the union of a collection of disjoint intervals: $A_1 = \cup_{i=1}^p (s_i, t_i)$ and $A_2 = \cup_{i=1}^q (u_i, v_i)$.

In this paper, we are particularly interested in the case of free, massless fermions in 1+1 dimension with the continuum Hamiltonian

$$H = \mp i \int_0^L \Psi^\dagger(t,x)\partial_x \Psi(t,x)\, \mathrm{x} \tag{4}$$

where $\{\Psi^\dagger(t,x), \Psi(t,x')\} = \delta(x - x')$. The sign determines whether the fermions are left moving or right moving. We will take one copy of each to reassemble a Dirac fermion. It will often be convenient to consider the lattice version of this Hamiltonian as well

$$H = \mp \frac{i}{2} \sum_j \left(\Psi_j^\dagger \Psi_{j+1} - \Psi_{j+1}^\dagger \Psi_j\right) , \tag{5}$$

and anticommutation relation $\{\Psi_j^\dagger, \Psi_k\} = \delta_{jk}$, which suffers the usual fermion doubling problem. We choose as our vacuum the state annihilated by all of the Ψ_j.

The authors of ref. [12] were able to give a relatively simple expression for the negativity in the discrete case by working instead with Majorana fermions $a_{2j-1} = \frac{1}{2}(\Psi_j^\dagger + \Psi_j)$ and $a_{2j} = \frac{1}{2i}(\Psi_j^\dagger - \Psi_j)$. Re-indexing, we can write the reduced density matrix as a sum over words made of the a_j:

$$\rho_A = \sum_\tau c_\tau \prod_{j=1}^{2n} a_j^{\tau_j} \tag{6}$$

where τ_j is either zero or one, depending on whether the word τ contains the Majorana fermion a_j, and n is the length of region A. Consider now instead the matrices $O_\pm$ constructed from ρ_A by

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[3]The trace norm of a matrix M is defined as the sum of its singular values: $|M| \equiv \mathrm{tr}\left[(M^\dagger M)^{1/2}\right]$. For Hermitian matrices, singular values are absolute values of the eigenvalues.

multiplying all the a_j in region A_2 by $\pm i$:

$$O_\pm = \sum_{\tau,\sigma} c_{\tau,\sigma} \left(\prod_{j=1}^{2n_1} a_j^{\tau_j} \right) \left(\prod_{j=2n_1+1}^{2n_1+2n_2} (\pm i a_j)^{\sigma_j} \right) . \tag{7}$$

Here n_j is the length of region A_j, and we have broken the sum into words τ involving region A_1 and words σ involving region A_2. As we already described in eq. (??), the central result of ref. [12] is that the partial transpose of the reduced density matrix can be written in terms of $O_\pm$.

While the spectrum of ρ_A is not simply related to the spectra of $O_\pm$, it is true that O_+ and O_- are not only Hermitian conjugates but are also related by a similarity transformation and so have the same eigenvalue spectrum. Consider a product of all of the Majorana fermions in A_2,

$$S = i^{n_2} \prod_{j=2n_1+1}^{2(n_1+n_2)} a_j , \tag{8}$$

which squares to one, $S^2 = 1$. This operator provides the similarity transformation between O_+ and O_-, i.e. $O_+ = SO_-S$. This similarity transformation means, along with cyclicity of the trace, that if we have a trace over a word constructed from a product of O_+ and O_-, the trace is invariant under the swap $O_+ \leftrightarrow O_-$. Employing this similarity transformation, the negativity for the first few even N can be written thus

$$\mathrm{tr}[(\rho_A^{T_2})^2] = \mathrm{tr}(O_+O_-) , \tag{9}$$

$$\mathrm{tr}[(\rho_A^{T_2})^4] = -\frac{1}{2}\mathrm{tr}(O_+^4) + \mathrm{tr}(O_+^2 O_-^2) + \frac{1}{2}\mathrm{tr}[(O_+O_-)^2] , \tag{10}$$

$$\mathrm{tr}[(\rho_A^{T_2})^6] = -\frac{3}{2}\mathrm{tr}(O_+O_-^5) + \frac{1}{4}\mathrm{tr}[(O_+O_-)^3] + \frac{3}{4}\mathrm{tr}(O_+^3 O_-^3) + \frac{3}{2}\mathrm{tr}(O_+O_-O_+^2 O_-^2) . \tag{11}$$

To obtain analytic expressions for $\mathrm{tr}[(\rho_A^{T_2})^N]$ from the decomposition (??) of $\rho_A^{T_2}$, a key step [13] is the relation between matrix elements of ρ_A and matrix elements of O_+. Consider arbitrary coherent states $\langle\zeta(x)|$ and $|\eta(x)\rangle$ that further break up into $\langle\zeta_1(x_1),\zeta_2(x_2)|$ and $|\eta_1(x_1),\eta_2(x_2)\rangle$ according to the decomposition of A into A_1 and A_2. Then the matrix elements of ρ_A and O_+ are related via

$$\langle\zeta(x)|O_+|\eta(x)\rangle = \langle\zeta_1(x_1),\eta^*(x_2)|U_2^\dagger \rho_A U_2|\eta_1(x_1),-\zeta_2^*(x_2)\rangle , \tag{12}$$

where U_2 is a unitary operator (whose precise form [13] does not concern us) that acts only on the part of the state in region A_2.

In pursuit of an analytic expression, let us move now to a path integral interpretation of $\mathrm{tr}[\rho_A^N]$ and $\mathrm{tr}[(\rho_A^{T_2})^N]$. The trace over ρ_A^N becomes a path integral over an N sheeted cover of the plane, branched over A. Now consider instead $\mathrm{tr}\,O_+^N$ given the relation (12). Performing a change of variables, we can replace U_2 acting on ζ_2^* and η_2^* with ζ_2 and η_2 inside the trace, and we see that $\mathrm{tr}(\rho_A^N)$ is related to $\mathrm{tr}\,O_+^N$ by an orientation reversal of region A_2. In terms of the N sheets, fixing a direction, passing through an interval in A_1, we move up a sheet while passing through an interval in A_2 we move down a sheet. Indeed, the trace of any word constructed from the O_+ and O_- has a similar path integral interpretation. Given the sign flip relation $O_- = SO_+S$ however, replacing some of the O_+ by O_- in the word will change the spin structure of the N sheeted cover.

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For simplicity, consider the case where A_1 is a single interval bounded by $s < t$ and A_2 a single interval bounded by $u < v$. The trace of a word constructed from $O_\pm$, up to an undetermined over-all normalization c_N, can be written in terms of a Riemann-Siegel theta function [13]

$$\mathrm{tr}\left[\prod_{i=1}^N O_{s_i}\right] = c_N^2 \left(\frac{1-x}{(t-s)(v-u)}\right)^{2\Delta_N} \left|\frac{\Theta[\mathbf{e}](\tilde\tau(x))}{\Theta(\tilde\tau(x))}\right|^2 , \quad \mathbf{e} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\delta} \end{pmatrix} , \tag{13}$$

where $\mathbf{0}$ is a vector of $N-1$ zeros and $\boldsymbol{\delta}$ is fixed by the word $\prod_{i=1}^N O_{s_i}$. In particular, if $s_i \neq s_{i+1}$, then $\delta_i = 1/2$ and $\delta = 0$ otherwise. The exponent

$$\Delta_N = \frac{c}{12}\left(N - \frac{1}{N}\right) \tag{14}$$

is the dimension of a twist operator field with $c = 1$ for a Dirac fermion. The cross ratio is defined to be

$$x \equiv \frac{(s-t)(u-v)}{(s-u)(t-v)} \in (0,1) . \tag{15}$$

(The limit in which the intervals become adjacent corresponds to $x \to 1$.) The Riemann-Siegel theta function is defined as

$$\Theta[\mathbf{e}](\mathbf{z}|M) \equiv \sum_{\mathbf{m}\in\mathbb{Z}^{N-1}} e^{i\pi(\mathbf{m}+\boldsymbol{\epsilon})^t \cdot M \cdot (\mathbf{m}+\boldsymbol{\epsilon}) + 2\pi i(\mathbf{m}+\mathbf{e})^t \cdot (\mathbf{z}+\boldsymbol{\delta})} , \quad \mathbf{e} \equiv \begin{pmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\delta} \end{pmatrix} , \tag{16}$$

and further $\Theta(\mathbf{z}|M) \equiv \Theta[\mathbf{0}](\mathbf{z}|M)$. The $(N-1) \times (N-1)$ period matrix is then [9, 20]

$$\tau_{i,j} = i\frac{2}{N} \sum_{k=1}^{N-1} \sin(\pi k/n) \frac{{}_2F_1(k/N, 1-k/N; 1; 1-x)}{{}_2F_1(k/N, 1-k/N; 1; x)} \cos[2\pi(k/N)(i-j)] , \tag{17}$$

and further $\tilde\tau(x) = \tau(x/(x-1))$. There are Riemann-Siegel theta functions that one can write down for multiple interval cases as well, but we shall not need their explicit form.

Among the words that enter in the binomial expansion of $\mathrm{tr}[(\rho_A^{T_2})^N]$, the traces $\mathrm{tr}(O_+^N) = \mathrm{tr}(O_-^N)$ and $\mathrm{tr}[(O_+O_-)^{N/2}]$ are special. Even in the multiple interval case, these two traces can be expressed as rational functions of the endpoints of the intervals. Although we have no proof in general, observationally it seems to be true that among the words of a fixed length $\mathrm{tr}(O_+^N)$ is the smallest in magnitude while $\mathrm{tr}[(O_+O_-)^{N/2}]$ is the largest. These two considerations suggest the utility of trying to bound the negativity using the rational functions $\mathrm{tr}(O_+)^N$ and $\mathrm{tr}[(O_+O_-)^{N/2}]$, as we pursue in section 4

In the two interval case, it follows from the result (13) that $\mathrm{tr}(O_+^N)$ and $\mathrm{tr}[(O_+O_-)^N]$ are rational functions. That $\mathrm{tr}(O_+^N)$ reduces to a rational function is obvious since $\boldsymbol{\delta} = \mathbf{0}$. That $\mathrm{tr}[(O_+O_-)^{N/2}]$ reduces as well follows from Thomae's formula [21, 22] that when $\delta_i = 1/2$ for all i,

$$\left|\frac{\Theta[\mathbf{e}](\tilde\tau)}{\Theta(\tilde\tau)}\right|^2 = |1-x|^{-N/4} . \tag{18}$$

To see more generally that these words are rational functions of the endpoints, in the next section we employ bosonization.[4]

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[4]For an application of Thomae's formula to a multiple interval Rényi entropy computation, see ref. [23].

3   Bosonization and Rationality

Consider the normalized partition function of the free Dirac field on the $\mathbb{Z}_N$-curve defined by the following set:

$$X_N = \left\{ (z,y) \,\bigg|\, y^N = \prod_{i=1}^{p} \frac{z - s_i}{z - t_i} \prod_{i=1}^{q} \frac{z - v_i}{z - u_i}, (z,y) \in \mathbb{C}^2 \right\} . \tag{19}$$

One can see that X_N, as the set of all points in $\mathbb{C}^2$ satisfying the equation in the set, has N sheets corresponding to N different roots of a nonzero complex number. These N copies of $\mathbb{C}$ are cut open along intervals in A on the real axis. As we choose the ordering $s_i < t_i$ and $u_i < v_i$, such open cuts are glued cyclicly if in A_1 and anti-cyclicly if in A_2.

While the Riemann surface (19) has an explicit $\mathbb{Z}_N$ symmetry, to specify a partition function, we also have to give the spin structure. The spin structure can generically break this symmetry, i.e. we can associate relative factors of minus one to cycles that would otherwise be related by the $\mathbb{Z}_N$ shift symmetry. A generic word $\prod_i O_{s_i}$ will generically have a spin structure that does not respect this symmetry. However, a few words do, namely $\mathrm{tr}(O_+^N) = \mathrm{tr}(O_-^N)$ and $\mathrm{tr}[(O_+O_-)^{N/2}]$. The word $\mathrm{tr}(O_+)^N$ associates a $+1$ to all the fundamental cycles constructed from region A_2, while the word $\mathrm{tr}[(O_+O_-)^{N/2}]$ associates a -1.

If we assume the $\mathbb{Z}_N$ symmetry is preserved by the spin structure, then the bosonization procedure is especially simple. Denote the partition function on X_N by $Z[N]$. Rather than a path integral of a single Dirac field on X_N in (19), $Z[N]$ can be considered as a path integral of a vector valued Dirac field $\vec{\Psi}(z)$ on $\mathbb{C}$: $\boldsymbol{\Psi}(x) = (\Psi_1(z), \cdots, \Psi_N(z))$. $\Psi_i(x)$ is the value of the original field Ψ at coordinate (z, y_i) on X_N. When going anti-clockwise around a branch point w by a small enough circle C_w, $\boldsymbol{\Psi}(x)$ gets multiplied by a monodromy matrix $T(w)$.

Define the matrix

$$T \equiv \begin{pmatrix} 0 & \omega & & & \\ & & \omega & & \\ & & & \cdot & \cdot \\ & & & & 0 & \omega \\ \omega & & & & & 0 \end{pmatrix} \tag{20}$$

where $\omega = e^{2\pi i \frac{N-1}{N}}$. This value of ω is chosen so that T satises the overall boundary condition $T^N = (-1)^{N-1}$ id where id is the $N \times N$ identity matrix. The reason for the factor $(-1)^{N-1}$ comes from considering a closed loop that circles one of the branch points N times. Such a loop should be a trivial closed loop in the y coordinate and come with an overall factor of -1, standard from performing a 2π rotation of a fermion.[5]

The matrix T is not the only $\mathbb{Z}_N$ symmetric matrix satisfying $T^N = (-1)^{N-1}$ id. A relative phase $e^{i2\pi k/N}$, $k = 1, 2, \ldots, N-1$, between monodromy matrices at different branch points is also

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[5]In order to preserve an explicit $\mathbb{Z}_N$ symmetry, we have chosen a slightly different matrix than in ref. [24].

allowed. Choose the basis of $\mathbf{\Psi}(x)$ so that $T(s_1) = T$ and take into account the constraint that $T(t_i) T(s_{i+1}) = \mathrm{id}$, $T(v_i) T(u_{i+1}) = \mathrm{id}$. Then, the monodromy matrices are fixed to be

$$T(s_i) = T \ , \qquad\qquad\qquad\qquad T(t_i) = T^{-1} \ , \tag{21}$$

$$T(u_i) = \exp\left(2\pi i \left(N - k\right)/N\right) T^{-1} \ , \qquad\qquad T(v_i) = \exp\left(2\pi i k/N\right) T \ , \tag{22}$$

If we insist on the usual spin structure for fermions, that they can only pick up an overall factor of ± 1 around any closed cycle, then two values of k are singled out, $k = 0$ for all N and $k = N/2$ for even N. The choice $k = 0$ will produce a partition function that computes $\mathrm{tr}(O_+^N)$, while the choice $k = N/2$ will produce a partition function that computes $\mathrm{tr}[(O_+ O_-)^{N/2}]$. As we will discuss below, there are a pair of additional special choices, $k = (N \pm 1)/2$ for odd N, which do not have an interpretation as a $\mathrm{tr}[\prod_i O_{s_i}]$, but which nevertheless have some nice properties. For now, we will keep the dependence on k arbitrary.

As introduced in refs. [9, 24, 25, 26], a twist operator $\sigma_R^k(w)$ is defined as the field that simulates the following monodromy behavior: $\vec{\Psi}(x) \sigma_R^k(w) \to \exp\left(2\pi k/N\right) T^R \vec{\Psi}(x) \sigma_R^k(w)$ when x is rotated counter-clockwise around w. Then $Z[N]$ can be expressed as a correlation function of twist operators on a single copy of $\mathbb{C}$ rather than as a partition function on X_N,

$$Z[N] \sim \left\langle \left(\prod_{i=1}^{p} \sigma_1^0(s_i) \sigma_{-1}^0(t_i) \prod_{j=1}^{q} \sigma_{-1}^k(u_j) \sigma_1^k(v_j) \right)_{\mathcal{AO}} \right\rangle \ . \tag{23}$$

The subscript $\mathcal{AO}$ means the operators are in ascending order of coordinates. Such correlation functions can be calculated through bosonization (see e.g. ref. [24]). Diagonalization of T leads to N decoupled fields, $\widetilde{\Psi}_l$. Each $\widetilde{\Psi}_l$ is multivalued, picking up a phase $e^{-i\frac{l}{N}2\pi}$, $e^{i\frac{l}{N}2\pi}$, $e^{i\frac{l-k}{N}2\pi}$ or $e^{-i\frac{l-k}{N}2\pi}$ when rotated counter-clockwise around s_i, t_i, u_i, or v_i respectively. Then one can factorize each multi-valued field $\widetilde{\Psi}_l$ into a gauge factor that describes this multi-valuedness and a single valued free Dirac field: $\Psi^l = e^{i\int_{x_0}^{x} dx'^\mu A_\mu^l(x)} \psi^l(x)$. The gauge field dependent part of the partition function contains the branch point dependence of $Z[N]$ and is moreover straightforward to evaluate. With the notation [25],

$$q_l(R, k) \equiv \frac{1-N}{2N} + \left\{ \frac{lR + k + (N-1)/2}{N} \right\} \ , \tag{24}$$

where the curly braces denote the fractional part of a number and $l \in \ell = \left\{ -\frac{N-1}{2}, -\frac{N-1}{2} + 1, ..., \frac{N-1}{2} \right\}$, the gauge field $A_\mu^l(x)$ satisfies the contour integrals

$$\oint_{C_{s_i}} dx^\mu A_\mu^l(x) = -\frac{2\pi l}{N} \ , \qquad\qquad \oint_{C_{s_i}} dx^\mu A_\mu^l(x) = \frac{2\pi l}{N} \ , \tag{25}$$

$$\oint_{C_{u_i}} dx^\mu A_\mu^l(x) = 2\pi q_l(1, N-k) \ , \qquad\qquad \oint_{C_{v_i}} dx^\mu A_\mu^l(x) = 2\pi q_l(-1, k) \ . \tag{26}$$

The Lagrangian density[6] in terms of $\psi^l(x)$ becomes $\mathcal{L} = \sum_{l=1}^{N} \bar{\psi}^l \gamma^\mu \left(\partial_\mu + i A_\mu^l \right) \psi^l$. From eqs. (??)

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[6]Our conventions for the Clifford algebra are that $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$. For example, we could choose $\gamma^x = \sigma^3$ and $\gamma^t = \sigma^1$

and (**??**) and Green's theorem we have:

$$\epsilon^{\mu\nu}\partial_\nu A_\mu^l(x) = 2\pi \sum_{i=1}^{p}\sum_{j=1}^{q}\left[\frac{l}{N}\left(\delta\left(x-s_i\right)-\delta\left(x-t_i\right)\right)-q_l\left(1,N-k\right)\left(\delta\left(x-u_i\right)-\delta\left(x-v_i\right)\right)\right].$$

(27)

Since the ψ^l's are decoupled, the partition function becomes a product of expectation values of operators that depend on the gauge field A_μ:

$$\mathcal{T}[N] \equiv \frac{Z[N]}{(Z[1])^N} = \prod_{l\in\ell}\langle e^{i\int A_\mu^l j_l^\mu d^2 x}\rangle,$$

(28)

where j_l^μ is the Dirac current $\bar{\psi}^l\gamma^\mu\psi^l$. After bosonization, it becomes $j_l^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi^l$. Then $\mathcal{T}[N]$ can be written as a correlation function of free boson vertex operators $V_e(w) = e^{-i\frac{e}{2}\phi_l(w)}$,

$$\prod_{l=-\frac{N-1}{2}}^{\frac{N-1}{2}}\langle e^{i\int A_\mu^l j_l^\mu d^2 x}\rangle = \left\langle \prod_{i=1}^{p}\prod_{j=1}^{q} V_{2l/N}(s_i)\, V_{-2l/N}(t_i)\, V_{2q_l(-1,k)}(u_j)\, V_{2q_l(1,N-k)}(v_j)\right\rangle.$$

(29)

To evaluate the correlation function of twist operators, we use

$$\left\langle \prod_{l_i=1}^{m} V_{e_i}(w_i)\right\rangle = \prod_{i\neq j}|w_i-w_j|^{-e_i e_j}\,\epsilon^{-m}$$

(30)

where ϵ is a UV cut-off to take into account the effect of coincident points in the correlation function. We also need the sums

$$\sum_{l=-\frac{N-1}{2}}^{\frac{N-1}{2}}\frac{l^2}{N^2} = \frac{N^2-1}{12N},$$

(31)

$$\sum_{l=-\frac{N-1}{2}}^{\frac{N-1}{2}}\frac{l q_l(1,N-k)}{N^2} = \frac{N^2-1}{12N}-\frac{(N-k)k}{2N}.$$

(32)

to get an explicit expression for $\mathcal{T}[N]$.

To shorten the expressions, we adopt the following notation: $\{s_i\}=S;\{t_i\}=T;\{u_i\}=U;\{v_i\}=V$ along with

$$[Y,Z]=\left|\prod_{y\in Y, z\in Z}(y-z)\right|,\quad [Y,Y]=\left|\prod_{y_1,y_2\in Y, y_1\neq y_2}(y_1-y_2)\right|.$$

(33)

Then $\mathcal{T}[N]$ can be written as:

$$\mathcal{T}[N] = L^{-\frac{N^2-1}{6N}} X^{\frac{N^2-1}{6N}-\frac{(N-k)k}{N}},$$

(34)

where we have defined

$$L \equiv \frac{[S,T][U,V]}{[S,S][T,T][U,U][V,V]\,\epsilon^{p+q}},\quad X \equiv \frac{[S,V][T,U]}{[S,U][T,V]}.$$

(35)

Fixing the appropriate spin structures, we claim then that

$$\mathrm{tr}(O_+^N) = \mathrm{tr}(O_-^N) = \left(\frac{L}{X}\right)^{-\frac{N^2-1}{6N}},$$

(36)

$$\mathrm{tr}[(O_+O_-)^{N/2}] = \left(\frac{L}{X}\right)^{-\frac{N^2-1}{6N}} X^{-N/4}.$$

(37)

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Comparing with the two interval case (13), we can absorb c_N into the ϵ dependence of L. A nice feature of these expressions is that it is straightforward to take the $N \to 1$ limit.

3.1 Adjacent Limits

Let us consider adjacent limits of the two-interval negativity. We call the single-interval negativity the case when $s = v$ and $t = u$, and there is only one length scale, say $l = t - s$. We call the two-adjacent-interval negativity the case where $t = u$ and we have two length scales, $l_1 = t - s$ and $l_2 = v - u$. The single-interval and two-adjacent-interval negativities are given by a two point function and a three point function of twist fields respectively. They are therefore fully determined by conformal symmetry [8, 9]:

$$\mathcal{E}\left(N_o\right) \sim l^{-\frac{N_o^2-1}{6}} , \qquad \mathcal{E}\left(N_e\right) \sim l^{-\frac{N_e^2-4}{6}} , \tag{38}$$

$$\mathcal{E}\left(N_o\right) \sim \left(l_1 l_2 \left(l_1 + l_2\right)\right)^{-\frac{N_o^2-1}{12}} , \qquad \mathcal{E}\left(N_e\right) \sim \left(l_1 l_2\right)^{-\frac{N_e^2-4}{12}} \left(l_1 + l_2\right)^{-\frac{N_e^2+2}{12}} . \tag{39}$$

While $\mathrm{tr}(O_+^N)$ simply vanishes in these coincident limits, we claim that $\mathrm{tr}[(O_+ O_-)^{N/2}]$ reproduces $\mathcal{E}(N_e)$ for even N, in both the single-interval and two-adjacent-interval cases. This agreement provokes the question is there a choice of k for odd N for which $\mathcal{T}[N]$ has the correct adjacent interval limits? The answer is yes. If we choose $k = (N_o \pm 1)/2$, then

$$\mathcal{T}\left[N_o\right] = L^{-\frac{N_o^2-1}{6N_o}} X^{-\frac{N_o^2-1}{12N_o}} , \tag{40}$$

and this expression reproduces $\mathcal{E}(N_o)$ in the adjacent interval limits.

To see why the values $k = N_e/2$ and $k = (N_o \pm 1)/2$ are singled out, we consider the merging of twist operators $\sigma_1^k(w_i)\sigma_1^0(w_{i+1}) \to \sigma_2^k(w_i)$. The corresponding constraint on the correlation function is

$$\lim_{w_{i+1} \to w_i} \left\langle \sigma_{R_1}^{k_1}\left(w_1\right) \cdots \sigma_1^k\left(w_i\right) \sigma_1^0\left(w_{i+1}\right) \cdots \right\rangle |w_i - w_{i+1}|^{-\gamma_{i(i+1)}} = \left\langle \sigma_{R_1}^{k_1}\left(w_1\right) \cdots \sigma_2^k\left(w_i\right) \cdots \right\rangle \tag{41}$$

along with a corresponding constraint from considering $\sigma_{-1}^0\left(w_i\right)\sigma_{-1}^k\left(w_{i+1}\right)$. We have defined

$$\gamma_{ij} \equiv \sum_{l \in \ell} q_l\left(R_i, k_i\right) q_l\left(R_j, k_j\right) . \tag{42}$$

These constraints can only be satisfied if the following identities holds for all $l \in \ell$:

$$q_l\left(-2, k\right) = q_l\left(-1, 0\right) + q_l\left(-1, k\right) , \qquad q_l\left(2, k\right) = q_l\left(1, 0\right) + q_l\left(1, k\right) . \tag{43}$$

The k values $(N_o - 1)/2$, $N_e/2$ and $(N_o + 1)/2$ are the only solutions.

4 Bounds on the Negativity

We discuss three types of bounds on $\mathcal{E}\left(N\right)$ in the following subsections. The first, which follows from a triangle inequality on the Schatten p-norm, is an upper bound on the moments of the partially

transposed density matrix. The second two are conjectural. We are able to demonstrate these conjectured bounds only for small $N > 1$.

The Schatten p-norm, defined as

$$\|M\|_p \equiv \left(\text{tr} \left((M^\dagger M)^{p/2} \right) \right)^{1/p} , \quad p \in [1, \infty) , \tag{44}$$

is a generalization of the trace norm. Indeed, the Schatten 1-norm is the trace norm.

Because $\text{tr}[(\rho_A^{T_2})^N]^{1/N}$ is the Schatten N-norm of $\rho_A^{T_2}$, for all even N we have by the triangle inequality that

$$\text{tr}[(\rho_A^{T/2})^N] = \left(\left\| \rho_A^{T_2} \right\|_N \right)^N \leq 2^{-N/2} \left(\left\| e^{i\pi/4} O_+ \right\|_N + \left\| e^{-i\pi/4} O_- \right\|_N \right)^N = 2^{N/2} \text{tr}[(O_+ O_-)^{N/2}] . \tag{45}$$

The $N \to 1$ limit of (??) leads to an upper bound on the negativity in terms of $\text{tr}[(O_+ O_-)^{1/2}]$

$$\mathcal{E} = \left\| \rho_A^{T_2} \right\|_T \leq \left\| \frac{1+i}{2} O_+ \right\|_T + \left\| \frac{1-i}{2} O_- \right\|_T = \sqrt{2} \, \text{tr}[(O_+ O_-)^{1/2}] = \sqrt{2} X^{-1/4} . \tag{46}$$

We have thus established that $\text{tr}[(O_+ O_-)^{N/2}]$ provides a rigorous upper bound on the negativity and its Nth moments, for free fermions.

4.1 Conjecture 1: Bounds from Word Order

As we discussed briefly above, for words of a fixed, even length, we conjecture that $\text{tr}(O_\pm^N)$ is the smallest and $\text{tr}[(O_+ O_-)^{N/2}]$ is the largest among the traces. In the notation of the previous section, we expect that the trace of an arbitrary word $O_+^{n_1} O_-^{n_2} \cdots$ of length N is bounded above and below by

$$\text{tr}(O_+^N) = \text{tr}[(O_+ O_-)^{N/2}] X^{N/4} \leq \text{tr}(O_+^{n_1} O_-^{n_2} \cdots) \leq \text{tr}[(O_+ O_-)^{N/2}] . \tag{47}$$

We can refine this conjecture on word order further. Define $s = |n_+ - n_-|$ to be the difference between the number of times n_+ that O_+ appears in a word and the times n_- that O_- appears in a word. For two words W_1 and W_2, we conjecture that if $s(W_1) > s(W_2)$, then $\text{tr}(W_1) < \text{tr}(W_2)$. Indeed, we have checked this conjecture in the two interval case for small N, using the explicit representation of these traces in terms of Riemann-Siegel theta functions. See figure 1.

Given this refined conjecture on word order, we can obtain upper and lower bounds on the negativity. For an upper bound, we first consider all the terms in the binomial expansion of $\text{tr}[(\rho_A^{T_2})^{N/2}]$ that appear with a positive sign such that $s \neq 0$. We replace every word with charge s with a word of charge $s - 4$ and hence larger trace. (Note there will be no traces of words of charge $s - 2$ with nonzero coefficient.) Because the number of words grows as the charge decreases, we will still have a net negative contribution from words of charge $s - 4$. We then replace all the traces of words with negative coefficient by the yet smaller trace $\text{tr}(O_+)^N$. For the words of charge $s = 0$, we simply replace all of them by the larger $\text{tr}[(O_+ O_-)^{N/2}]$. At the end of this procedure, we find the following upper bound

$$\text{tr}[(\rho_A^{T_2})^{N/2}] \leq \left[1 - \frac{1}{2^{N/2}} \binom{N}{\frac{N}{2}} \right] \text{tr}(O_+^N) + \frac{1}{2^{N/2}} \binom{N}{\frac{N}{2}} \text{tr}[(O_+ O_-)^{N/2}] . \tag{48}$$

10

In the large N limit, the right hand side of this expression approaches

$$\sqrt{\frac{2^{N+1}}{\pi N}} \left(\mathrm{tr}[(O_+O_-)^{N/2}] - \mathrm{tr}(O_+^N) \right) \ , \tag{49}$$

which appears to be a somewhat more stringent condition than our rigorous upper bound (**??**).

We can obtain a lower bound in a similar fashion, reversing the procedure. We consider all the terms in the binomial expansion of $\mathrm{tr}[(\rho_A^{T_2})^{N/2}]$ that appear with negative coefficient. We replace every word with charge s by a word of charge $s-4$. All the traces will then have positive coefficient. Next, except for $\mathrm{tr}[(O_+O_-)^{N/2}]$ itself, we replace all the traces of words with the smaller $\mathrm{tr}(O_+)^N$. In this case, we find the lower bound

$$\left(1 - \frac{1}{2^{N/2-1}}\right) \mathrm{tr}(O_+^N) + \frac{1}{2^{N/2-1}} \mathrm{tr}[(O_+O_-)^{N/2}] \leq \mathrm{tr}[(\rho_A^{T_2})^{N/2}] \ . \tag{50}$$

In comparison with the conjecture we discuss next, this lower bound is not particularly stringent in the large N limit.

We can establish these bounds rigorously only for small N. Note that for $N = 2$, the upper and lower bound reduce to the known equality (9). For $N = 4$ and $N = 6$, we obtain the constraints

$$\frac{1}{2}(1 + X) \, \mathrm{tr}[(O_+O_-)^2] \leq \mathrm{tr}[(\rho_A^{T_2})^4] \leq \frac{1}{2}(3 - X) \, \mathrm{tr}[(O_+O_-)^2] \ , \tag{51}$$

$$\frac{1}{4}(1 + 3X^{3/2}) \, \mathrm{tr}[(O_+O_-)^3] \leq \mathrm{tr}[(\rho_A^{T_2})^6] \leq \frac{1}{2}(5 - 3X^{3/2}) \, \mathrm{tr}[(O_+O_-)^3] \ . \tag{52}$$

Indeed, in the two interval case, using the explicit representation of the negativity in terms of Riemann-Siegel theta functions, we can verify that these bounds are indeed satisfied. See the insets in figure 2.
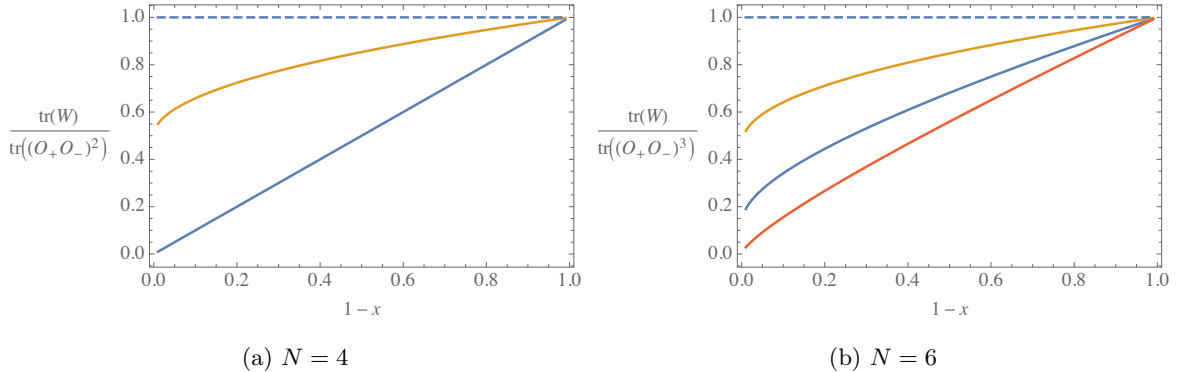

(a) $N = 4$ (b) $N = 6$

Figure 1: Plots of ratios of traces of words versus the four point ratio $1 - x$ for the two disjoint interval system. In the $N = 4$ case, we compare $\mathrm{tr}(O_+^4)$ and $\mathrm{tr}(O_+^2 O_-^2)$ to $\mathrm{tr}[(O_+O_-)^2]$. The lowest curve is the ratio of $\mathrm{tr}(O_+^4)$ to $\mathrm{tr}[(O_+O_-)^2]$. In the $N = 6$, we compare $\mathrm{tr}(O_+^5 O_-)$, $\mathrm{tr}(O_+O_-O_+^2 O_-^2)$ and $\mathrm{tr}(O_+^3 O_-^3)$ to $\mathrm{tr}[(O_+O_-)^3]$. The curve at the bottom corresponds to the ratio of $\mathrm{tr}(O_+^5 O_-)$ to $\mathrm{tr}[(O_+O_-)^3]$ and establishes that $\mathrm{tr}(O_+^5 O_-)$ is the smallest among the words that appears in the negativity. The dashed line is included as a guide to the eye.

For $N = 4$, we can do better and prove the inequalities in general. That $\mathrm{tr}[|O_+^2 - O_-^2|^2] \geq 0$ implies that $\mathrm{tr}(O_+^4) \leq \mathrm{tr}(O_+^2 O_-^2)$. Similarly, that $\mathrm{tr}[|O_+O_- - O_-O_+|^2] \geq 0$ implies that $\mathrm{tr}(O_+^2 O_-^2) \leq$

$\text{tr}[(O_+O_-)^2]$ and the desired inequalities on $\text{tr}[(\rho_A^{T_2})^4]$ follows directly.[7] It is tempting to apply these inequalities to the case $N = 1$.

Conjecture 2: A Lower Bound from Extremization

The plot of the two disjoint interval system suggests another possible type of lower bound on $\mathcal{E}(N)$. At least for $N = 2$, 4 and 6, and conjecturally for all even N, we find that

$$\text{tr}[(O_+O_-)^{N/2}] \le \mathcal{E}(N) = \text{tr}[(\rho_A^{T_2})^N] \, . \tag{53}$$

Figure 2 is a comparison of the ratio $\text{tr}[(\rho_A^{T_2})^N]/\text{tr}[(O_+O_-)^{N/2}]$ as a function of the four point ratio x to the constant function one. We consider $N = 4$ and $N = 6$ for the two interval case only. For $N = 2$, the inequality is saturated given (9). Given the saturation, we further conjecture that the negativity itself is bounded above,

$$\mathcal{E} = |\rho_A^{T_2}| \le \text{tr}[(O_+O_-)^{1/2}] \, , \tag{54}$$

further tightening the triangle inequality (46). In the appendix, we compute the Nth moments $\text{tr}[(\rho^{T_2})^N]$ and $\text{tr}[(O_+O_-)^{N/2}]$ explicitly for a two-spin system in a Gaussian state. We are able to show that the bounds (53) and (54) are satisfied in this simple case.

We can try to put more structure behind this conjecture. We begin by introducing some notation. Recalling that $O_+^\dagger = O_-$ and that $O_+ = SO_-S$, we can assume without loss of generality the following block structure for $O_\pm$:

$$O_\pm = \begin{pmatrix} A & \pm B \\ \mp B^\dagger & C \end{pmatrix} \tag{55}$$

where A and C are Hermitian. It will be useful in what follows to consider

$$\alpha \equiv \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \, , \quad \beta \equiv \begin{pmatrix} 0 & B \\ -B^\dagger & 0 \end{pmatrix} \, , \tag{56}$$

such that $O_\pm = \alpha \pm \beta$ and, from (??), $\rho_A^{T_2} = \alpha + i\beta$. Finally, we introduce

$$\gamma \equiv S\beta = \begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix} \, , \quad \eta_\pm \equiv \begin{pmatrix} A & \pm B \\ \pm B^\dagger & -C \end{pmatrix} \, . \tag{57}$$

Note that the $\eta_\pm$ are Hermitian and that $\eta_+ = SO_+ = O_-S$ while $\eta_- = O_+S = SO_-$.

Define the function

$$f_N(\theta) \equiv \text{tr}[((\alpha + e^{i\theta}\beta)(\alpha - e^{-i\theta}\beta))^{N/2}] \, . \tag{58}$$

From this definition, it follows that $f_N(\frac{\pi}{2}) = \text{tr}[(\rho_A^{T_2})^N]$ and $f_N(0) = \text{tr}[(O_+O_-)^{N/2}]$. This function has a few other useful properties. It is periodic, with period 2π: $f_N(\theta) = f_N(\theta + 2\pi)$. It also has

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[7]Alternately, one can employ von Neumann's trace inequality.

two reflection symmetries. The first, $f_N(\theta) = f_N(\pi - \theta)$, follows from cyclicity of the trace:

$$
\begin{aligned}
f(\pi - \theta) &= \operatorname{tr}[((\alpha - e^{-i\theta}\beta)(\alpha + e^{i\theta}\beta))^{N/2}] \\
&= \operatorname{tr}[((\alpha + e^{i\theta}\beta)(\alpha - e^{-i\theta}\beta))^{N/2}] \\
&= f(\theta) \ .
\end{aligned}
$$

The second, $f_N(\theta) = f_N(-\theta)$, is more subtle. Consider expanding out the product of matrices inside the trace. A generic term in the product will involve n_+ factors of $e^{i\theta}\beta$ and n_- factors $-e^{-i\theta}\beta$. If $n_+ = n_-$, then the θ dependence drops out, and such terms are irrelevant for the argument that follows. Let us therefore assume $n_+ \neq n_-$. Because β is off diagonal, any term that contributes to the trace must have an even number of factors of β. Thus either n_+ and n_- are both odd or both even. For every such term, there will also be a term with n_+ factors of $-e^{-i\theta}\beta$ and n_- factors of $e^{i\theta}\beta$. This second term will always have the same sign and coefficient as the first and the same cyclic ordering of operators. Thus, we can re-express the θ dependence of the combined terms as $\cos((n_+ - n_-)\theta)$, which is an even function of θ.

The two reflection symmetries, $f(\theta) = f(-\theta)$ and $f(\theta) = f(\pi - \theta)$ along with periodicity imply that $f(\pi/2) = f(3\pi/2)$ are extrema of $f(\theta)$ as are $f(0) = f(\pi)$. If we can show that these four extrema are the only extrema in the domain $0 \leq \theta < 2\pi$, and that $f(\pi/2)$ is a local maximum (or alternatively that $f(0)$ is a local minimum), then our conjecture is proven since $f(\theta)$ is a smooth bounded function on this domain.

For even N, the difference between the first few $\mathcal{E}(N)$ and $\operatorname{tr}[(O_+ O_-)^{N/2}]$ can be written in terms of α and γ:

$$
\begin{aligned}
\mathcal{E}(2) - \mathcal{T}[2] &= 0 \ , &&(59) \\
\mathcal{E}(4) - \mathcal{T}[4] &= 4\operatorname{tr}\left((\alpha\gamma)^2\right) \ , &&(60) \\
\mathcal{E}(6) - \mathcal{T}[6] &= 6\operatorname{tr}\left((\alpha^2 + \gamma^2)\left((\alpha\gamma)^2 + (\gamma\alpha)^2\right)\right) \ . &&(61)
\end{aligned}
$$

A sufficient condition for $\operatorname{tr}[(O_+ O_-)^{N/2}] \leq \mathcal{E}(N)$ to hold for $N = 4$ and $N = 6$ is that $(\alpha\gamma)^2 + (\gamma\alpha)^2$ be positive definite.

5   Comments and Future Directions

While a determination of the negativity $\mathcal{E}$ for massless free fermions in 1+1 dimensions remains an open problem, we have argued in this paper that $\operatorname{tr}[(O_+ O_-)^{N/2}]$ and $\operatorname{tr}(O_+^N)$, which have simple closed form expressions for all real N, can be used to bound $\mathcal{E}$ as well as higher moments of $\rho_A^{T_2}$. One of our main results is that

$$
\mathcal{E} \leq \sqrt{2} \operatorname{tr}[(O_+ O_-)^{1/2}] \ , \tag{62}
$$

which follows from the triangle inequality. Part of our Conjecture 2 is that the bound can be tightened by removing the $\sqrt{2}$. Also, in the appendix, we demonstrated this tighter upper bound for a two-spin system in a Gaussian state.
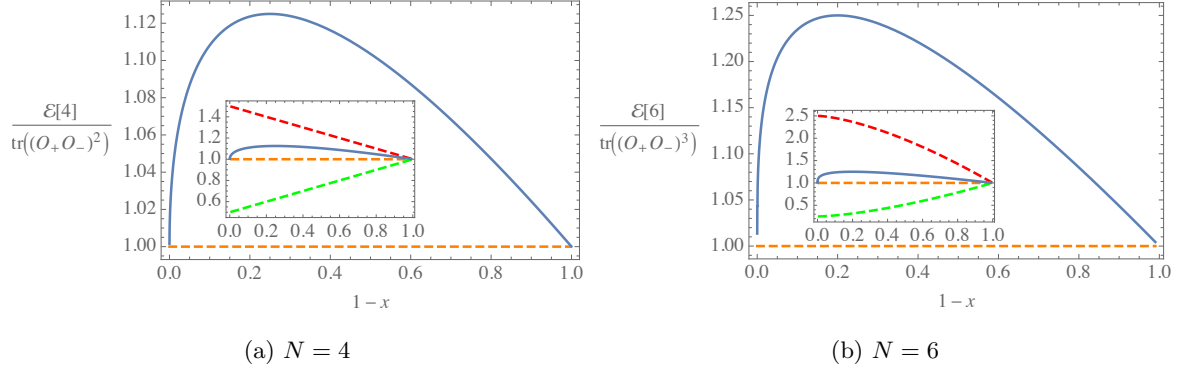
(a) $N = 4$

(b) $N = 6$

Figure 2: Proposed bounds on the negativity. The solid blue line is $\mathrm{tr}[(\rho_A^{T_2})^N]/\mathrm{tr}[(O_+O_-)^{N/2}]$. The dashed line is the constant function 1. In the insets, the upper and lower bounds (**??**) and (**??**) are included. The horizontal axis is the four point ratio $1 - x$.

For $N > 2$, we have both upper and lower bounds on the moments of $\rho_A^{T_2}$. In their strongest form, our conjectures state that

$$\mathrm{tr}[(O_+O_-)^{N/2}] \leq \mathrm{tr}[(\rho_A^{T_2})^N] \leq \left[1 - \frac{1}{2^{N/2}}\binom{N}{\frac{N}{2}}\right]\mathrm{tr}(O_+)^N + \frac{1}{2^{N/2}}\binom{N}{\frac{N}{2}}\mathrm{tr}[(O_+O_-)^{N/2}] \,. \tag{63}$$

Using the triangle inequality, we were also able to argue rigorously for a somewhat weaker upper bound (**??**).

An advantage of working with $\mathrm{tr}[(O_+O_-)^{N/2}]$ and $\mathrm{tr}(O_+^N)$ instead of with $\mathrm{tr}[(\rho_A^{T_2})^N]$ is that they are much simpler quantities. In the paper, we discussed how to compute the multiple interval case on the plane. It is straightforward to consider the torus instead, i.e. finite volume and nonzero temperature.[8] One can even introduce a chemical potential. These generalizations require the use of the appropriate torus correlation function in place of eq. (**??**). See for example refs. [28, 29].

There are many interesting questions that could be asked regarding $\mathrm{tr}[(O_+O_-)^{N/2}]$. What can we deduce about the eigenvalues of $(O_+O_-)^{1/2}$ from the relation $\mathrm{tr}[(O_+O_-)^{N/2}] = L^{-\frac{N^2-1}{6N}}X^{-\frac{N^2+2}{12}}$? Can we prove the two conjectures involving $\mathrm{tr}[(O_+O_-)^{N/2}]$ discussed in the text? Among all such open questions, the most important and intriguing one is whether we can construct both an upper bound and lower bound for $\mathcal{E}(N)$ using $\mathrm{tr}[(O_+O_-)^{N/2}]$ that have the same $N \to 1$ limit. If so, then we can extract the value of the negativity $\mathcal{E}$ from these bounds.

Acknowledgments

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[8]See ref. [27] for a discussion of subtleties associated with thermal effects on negativity.

A Two Bit System

Consider a two spin system in a Gaussian state with density matrix

$$\rho = \exp\left(\sum_{i=1}^{2} M_{ij} c_i^\dagger c_j\right), \tag{64}$$

where M is a 2×2 Hermitian matrix and c_j and $c_j^\dagger$ satisfy the usual anti-commutation relations, $\{c_i, c_j^\dagger\} = \delta_{ij}$. In the basis $(1, c_2^\dagger, c_1^\dagger, c_1^\dagger c_2^\dagger)|0\rangle$, such a density matrix takes the explicit form

$$\rho = \frac{1}{1 + \operatorname{tr} e^M + \det e^M} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^M & 0 \\ 0 & 0 & \det e^M \end{pmatrix} = \frac{1}{1 + x + y + w} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & z & 0 \\ 0 & \bar{z} & y & 0 \\ 0 & 0 & 0 & w \end{pmatrix} \tag{65}$$

where $x \geq 0$ and $y \geq 0$ while $z \in \mathbb{C}$. Note that $w = xy - |z|^2 \geq 0$.

The usual partial transpose of this density matrix with respect to the second spin is

$$\rho^{\widetilde{T_2}} = \frac{1}{1 + x + y + w} \begin{pmatrix} 1 & 0 & 0 & z \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ \bar{z} & 0 & 0 & w \end{pmatrix}. \tag{66}$$

However, defining Majorana fermions $a_{2j-1} = \frac{1}{2}(c_j^\dagger + c_j)$ and $a_{2j} = \frac{1}{2i}(c_j^\dagger - c_j)$, this naive partial transpose is related to the one in the body of the paper by a similarity transformation: $\rho^{T_2} = (\sigma_x \otimes I)(\rho^{\widetilde{T_2}})^T (\sigma_x \otimes I)$. More explicitly,

$$\rho^{T_2} = \frac{1}{1 + x + y + w} \begin{pmatrix} y & 0 & 0 & 0 \\ 0 & w & z & 0 \\ 0 & \bar{z} & 1 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}. \tag{67}$$

Both will thus have the same spectrum. In particular, we find the eigenvalues

$$\frac{1}{1 + x + y + w}\left(x, y, \frac{1}{2}\left(1 + w \pm \sqrt{1 - 6w + w^2 + 4xy}\right)\right). \tag{68}$$

A sufficient condition for this density matrix to possess quantum entanglement is a negative eigenvalue. We thus require $2w < xy$.

In the body of the paper, we also introduced the matrices $O_\pm$, which for this simple system take the explicit form

$$O_\pm = \frac{1}{1 + x + y + w} \begin{pmatrix} y & 0 & 0 & 0 \\ 0 & w & \pm iz & 0 \\ 0 & \pm i\bar{z} & 1 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}. \tag{69}$$

As we did in the body of the paper in a more complicated case, we would like to compare the $\mathrm{tr}[(\rho^{T_2})^{2N}]$ with $\mathrm{tr}[(O_+O_-)^N]$. We need first the eigenvalues of $(\rho^{T_2})^2$ and O_+O_-. We find that

$$\mathrm{tr}[(\rho^{T_2})^N] - \mathrm{tr}[(O_+O_-)^{N/2}] = \frac{4^n}{(1+x+y+w)^N} \Big[(A+1+w)^N + (A-1-w)^N$$
$$-(B+1-w)^N - (B-1+w)^N \Big] \tag{70}$$

where we have defined $A^2 \equiv (1-w)^2 - 4w + 4xy$ and $B^2 \equiv (1-w)^2 + 4xy$. We have used the fact that for $2w < xy$, $A > w+1$. Note that the right hand side vanishes when $N = 2$ as expected. It also vanishes when $w = 0$ and (provided N is even) when $w = xy$. For $N = 1$, the difference is negative and proportional to $A - B$, indicating that $\mathrm{tr}[(O_+O_-)^{1/2}]$ is an upper bound for the negativity. Meanwhile for $N > 2$, the right hand side is always positive over the region $0 < 2w < xy$, indicating that $\mathrm{tr}[(O_+O_-)^{N/2}]$ is a lower bound on $\mathrm{tr}[(\rho^{T_2})^{N/2}]$. We prove this last statement below.

Proof of the lower bound

We can make the further redefinitions

$$A + 1 + w = R\cos\alpha , \tag{71}$$
$$A - 1 - w = R\sin\alpha , \tag{72}$$
$$\max\{B+1-w, B-1+w\} = R\cos\beta , \tag{73}$$
$$\min\{B+1-w, B-1+w\} = R\sin\beta , \tag{74}$$

where $\alpha, \beta \in (0, \frac{\pi}{4})$ and $R = \sqrt{A^2 + (1+w)^2} = \sqrt{B^2 + (1-w)^2}$. Recalling the $N = 1$ case, because $\cos\theta + \sin\theta$ is an increasing function in the domain $(0, \frac{\pi}{4})$, the fact that $A < B$ implies that $\alpha < \beta$. We need then to establish that for $n > 2$ that the following difference is positive:

$$\mathrm{tr}[(\rho^{T_2})^N] - \mathrm{tr}[(O_+O_-)^{N/2}] = \frac{(4R)^N}{(1+w+y+w)^N} \Big[\cos^N\alpha + \sin^N\alpha - \cos^N\beta - \sin^N\beta \Big] . \tag{75}$$

For $N > 2$ in the domain $\theta \in (0, \frac{\pi}{4})$, $\sin^N\theta + \cos^N\theta$ is a decreasing function:

$$\frac{d}{d\theta}\left(\cos^N\theta + \sin^N\theta\right) = \frac{N}{2}\sin 2\theta \left(\sin^{N-2}\theta - \cos^{N-2}\theta\right) < 0 . \tag{76}$$

Since $0 < \alpha < \beta < \frac{\pi}{4}$, it follows then that

$$\cos^N\alpha + \sin^N\alpha - \cos^N\beta - \sin^N\beta > 0 , \tag{77}$$

and the difference in question, $\mathrm{tr}[(\rho^{T_2})^N] - \mathrm{tr}[(O_+O_-)^{N/2}] > 0$, is positive.

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