DEGENERATION OF TORSORS OVER FAMILIES OF DEL PEZZO SURFACES

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ABSTRACT. Let S be a split family of del Pezzo surfaces over a discrete valuation ring R such that the general fiber is smooth and the special fiber has **ADE**-singularities. Let G be the reductive group given by the root system of these singularities. We construct a G-torsor over S whose restriction to the generic fiber is the extension of structure group of the universal torsor. This extends a construction of Friedman and Morgan for individual singular del Pezzo surfaces.

CONTENTS

1. INTRODUCTION

Let S be a split cubic surface over a field k. Assume that S is singular and that its singularities are rational double points. Then its minimal desingularization $\widetilde{S}$ has a Picard group $\mathrm{Pic}(\widetilde{S})$ of rank 7, and we have a *universal torsor* $\mathcal{T}$ over $\widetilde{S}$ (see [CTS87]), which is a torsor under the Néron–Severi torus T associated to $\mathrm{Pic}(\widetilde{S})$. As $\widetilde{S}$ is split, $\mathcal{T}$ is unique up to isomorphism.

If k is the field $\mathbb{C}$ of complex numbers, Friedman and Morgan [FM02] observed that the universal torsor $\mathcal{T}$ over $\widetilde{S}$ does not descend to S. It is well-known that there is a natural root system Ψ of type $\mathbf{E}_6$ in the lattice $\mathrm{Pic}(\widetilde{S})$. It contains a subsystem Φ corresponding to the singularities of S. Let $G \subset H$ be the split reductive groups associated to these root systems. Let $B \subset C$ be Borel subgroups of $G \subset H$ which contain T. Friedman and Morgan show that it is possible to lift $\mathcal{T}$ to a C-torsor over $\widetilde{S}$ such that the induced H-torsor descends to S [FM02, Theorem 3.1].

Now let R be a discrete valuation ring. We consider a family S of split cubic surfaces over R, whose generic fiber is smooth over the quotient field of R, and whose special fiber is singular over the residue field of R. Our main result is:

Theorem. *The universal torsor $\mathcal{T}$ over $\widetilde{S}$ can be lifted to a B-torsor $\mathcal{B}$ over $\widetilde{S}$ such that the induced G-torsor $\mathcal{G}$ descends to S.*

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In the special fiber, we recover the construction of Friedman and Morgan via extension of the structure group. Hence our work reduces the structure group in the work of Friedman and Morgan, generalizes it to the split case over arbitrary fields, and shows that it can be put into families containing also smooth cubic surfaces. For more details and the proof, see Theorem 3.5 below.

Actually, all this works not only for cubic surfaces, but also for arbitrary del Pezzo surfaces of degree at most 6.

For a related construction of vector bundles over families of rational surfaces over $\mathbb{C}$, using a Fourier-Mukai transform, see [DW15].

2. Degenerating del Pezzo surfaces

Let R be a discrete valuation ring with quotient field K, maximal ideal $\mathfrak{m} \subset R$ and residue field $k = R/\mathfrak{m}$. Choose $d \in \{1, \dots, 6\}$, and let

$$x_1, \dots, x_{9-d} \in \mathbb{P}^2(K)$$

be points in general position. By the latter, we mean distinct points such that no line in $\mathbb{P}^2$ contains more than two of them, no quadric curve in $\mathbb{P}^2$ contains more than five of them, and no singular cubic curve in $\mathbb{P}^2$ contains eight with one lying in its singularity. We consider the chain of blow-ups

$$\widetilde{S} = \widetilde{S}_{9-d} \xrightarrow{p_{9-d}} \widetilde{S}_{8-d} \longrightarrow \dots \longrightarrow \widetilde{S}_2 \xrightarrow{p_2} \widetilde{S}_1 \xrightarrow{p_1} \widetilde{S}_0 = \mathbb{P}^2_R$$

where $p_i : \widetilde{S}_i \to \widetilde{S}_{i-1}$ is the blow-up in the closure $\bar{x}_i \in \widetilde{S}_{i-1}(R)$ of the preimage of x_i in $\widetilde{S}_{i-1}(K)$. The generic fiber $\widetilde{S}_K$ is the blow-up of $\mathbb{P}^2_K$ in $x_1, \dots, x_{9-d}$, and therefore a del Pezzo surface of degree d over K.

We assume that the images of $\bar{x}_i$ in $\widetilde{S}_{i-1}(k)$ are in almost general position, by which we mean that the image of $\bar{x}_i$ does not lie on a (-2)-curve in $\widetilde{S}_{i-1,k}$. Then the special fiber $\widetilde{S}_k$ is a *generalized* (or: *weak*) del Pezzo surface over k [Dem80], or in other words a smooth rational surface whose anticanonical class is nef and big.

Lemma 2.1. *The canonical bundle $\omega_{\widetilde{S}_k}$ of the special fiber $\widetilde{S}_k$ is isomorphic to the restriction of the canonical bundle $\omega_{\widetilde{S}}$ of the total space $\widetilde{S}$.*

Proof. The two differ by the normal bundle of $\widetilde{S}_k$ in $\widetilde{S}$, which is the pullback of the normal bundle of $\operatorname{Spec}(k)$ in $\operatorname{Spec}(R)$, and therefore trivial. □

Lemma 2.2. *The R-module $H^0(\widetilde{S}, \omega_{\widetilde{S}}^{-m})$ is free, and the natural map*

$$H^0(\widetilde{S}, \omega_{\widetilde{S}}^{-m}) \otimes_R k \to H^0(\widetilde{S}_k, \omega_{\widetilde{S}_k}^{-m})$$

is an isomorphism, for each integer $m \geqslant 0$.

Proof. We carry some arguments from [Kol96, §III.3] over to the generalized del Pezzo surface $\widetilde{S}_k$. We have

$$H^1(\widetilde{S}_k, \mathcal{O}_{\widetilde{S}_k}) = 0$$

since this is a birational invariant [Har77, Proposition V.3.4]. Let D be a general member of the anticanonical linear system on $\widetilde{S}_k$. Then D does not contain any (-2)-curve on $\widetilde{S}_k$, since $\omega_{\widetilde{S}_k}^{-1}$ is globally generated. Therefore,

$$H^0(D, \omega_{\widetilde{S}_k}^m \otimes \mathcal{O}_D) = 0$$

for $m \geqslant 1$. Being a local complete intersection, D has dualizing sheaf

$$\omega_D = \det(I_{D\subset\widetilde{S}_k}/I^2_{D\subset\widetilde{S}_k})^\vee \otimes \omega_{\widetilde{S}_k} \cong \mathcal{O}_{\widetilde{S}_k}(-D)^\vee_{|D} \otimes \mathcal{O}_{\widetilde{S}_k}(D) = \mathcal{O}_D$$

according to [Liu02, Definition 6.4.7]. Therefore, Serre duality on D implies

$$H^1(D, \omega_{\widetilde{S}_k}^{-m} \otimes \mathcal{O}_D) = 0$$

for $m \geqslant 1$. By means of the exact sequence

$$H^1(\widetilde{S}_k, \omega_{\widetilde{S}_k}^{-(m-1)}) \to H^1(\widetilde{S}_k, \omega_{\widetilde{S}_k}^{-m}) \to H^1(D, \omega_{\widetilde{S}_k}^{-m} \otimes \mathcal{O}_D) = 0,$$

and induction over m, we conclude that

$$H^1(\widetilde{S}_k, \omega_{\widetilde{S}_k}^{-m}) = 0$$

for $m \geqslant 0$. Using Cohomology and Base Change [Har77, Theorem III.12.11] together with Lemma 2.1, the claim follows. $\square$

Choosing a sufficiently large integer m and a basis of $H^0(\widetilde{S}, \omega_{\widetilde{S}}^{-m})$, we get an anticanonical map

$$\phi : \widetilde{S} \twoheadrightarrow S \subset \mathbb{P}^N_R.$$

Up to isomorphism over R, the scheme S does not depend in the choices made. As S is integral and R is a discrete valuation ring, S is flat over R by [Har77, Proposition III.9.7]. Lemma 2.2 implies that the special fiber S_k of S is the anticanonical image of the generalized del Pezzo surface $\widetilde{S}_k$.

In particular, S_k is a del Pezzo surface with at most rational double points, and ϕ contracts precisely the (-2)-curves on $\widetilde{S}_k$.

Proposition 2.3. $\phi_*\mathcal{O}_{\widetilde{S}} = \mathcal{O}_S$.

Proof. Since ϕ_* commutes with flat base change, and the completion of R is flat over R, we may assume without loss of generality that R is complete.

We show by induction that $\phi_*\mathcal{O}_{n\widetilde{S}_k} = \mathcal{O}_{nS_k}$ and $R^1\phi_*\mathcal{O}_{n\widetilde{S}_k} = 0$. For $n = 1$, this holds by [Dem80, Théorème V.2]. The induction step follows from the short exact sequence

$$0 \to \mathcal{O}_{(n-1)\widetilde{S}_k} \xrightarrow{\cdot\pi} \mathcal{O}_{n\widetilde{S}_k} \to \mathcal{O}_{\widetilde{S}_k} \to 0,$$

where $\pi \in R$ is a generator of $\mathfrak{m}$, and its analog for S_k.

It follows that the formal completions $(\widetilde{S})^\wedge$ of $\widetilde{S}$ along $\widetilde{S}_k$ and $\widehat{S}$ of S along S_k satisfy $\phi_*\mathcal{O}_{(\widetilde{S})^\wedge} = \mathcal{O}_{\widehat{S}}$. By [Gro61, Corollaire 5.1.6], the claim follows. $\square$

For the rest of this section, we fix one singular point x on S_k. Let $D_1, \dots, D_r$ be the (-2)-curves on $\widetilde{S}_k$ that map to x. Let

$$Z = n_1 D_1 + \dots + n_r D_r$$

with $n_1, \dots, n_r \geqslant 1$ denote the *fundamental cycle* on $\widetilde{S}_k$ over x (see [Art66]). It has the property that $(Z, D_i) \leqslant 0$ for all $i = 1, \dots, r$, and is minimal with this property. Here $(\cdot, \cdot)$ denotes the intersection number of divisors on $\widetilde{S}_k$. Put $N := n_1 + \dots + n_r$, and $Z^{\mathrm{red}} := D_1 + \dots + D_r$.

Lemma 2.4. *There is a sequence of effective divisors*

$$0 = Z_0 < Z_1 < Z_2 < \cdots < Z_r = Z^{\mathrm{red}} < \cdots < Z_N = Z$$

on $\widetilde{S}_k$ *such that*

- $Z_j - Z_{j-1}$ *is a* (-2)*-curve* D_{i_j} *for all* $j = 1, \dots, N$*, and*
- $(Z_j, D_i) \leqslant 1$ *for all* $i = 1, \dots, r$ *and* $j = 1, \dots, N$.

Proof. The classes of $D_1, \dots, D_r$ are the simple roots of an irreducible root system of type $\mathbf{A}_n$ ($n \geqslant 1$) or $\mathbf{D}_n$ ($n \geqslant 4$) or $\mathbf{E}_n$ ($n = 6, 7, 8$).

For $\mathbf{A}_r$, we have $Z = Z^{\mathrm{red}} = D_1 + \cdots + D_r$, with $N = r$. We may assume that the configuration of $D_1, \dots, D_r$ is given by the Dynkin diagram

$$D_1 — D_2 — \dots — D_r$$

Then we can clearly choose $Z_j := D_1 + \cdots + D_j$ for $j = 1, \dots, r$.

For $\mathbf{D}_r$, we may assume that the Dynkin diagram is

$$\begin{array}{ccccccc} D_1 & — & D_2 & — & D_3 & — \dots — & D_{r-1} \\ & & | & & & & \\ & & D_r & & & & \end{array}$$

Then $Z = D_1 + 2(D_2 + \cdots + D_{r-2}) + D_{r-1} + D_r$, with $N = 2r - 3$. We can again start with $Z_j := D_1 + \cdots + D_j$ for $j = 1, \dots, r$, and then we may continue with $Z_j := Z^{\mathrm{red}} + D_2 + \cdots + D_{j-r+1}$ for $j = r + 1, \dots, 2r - 3$.

For $\mathbf{E}_r$, we may assume that the Dynkin diagram is

$$\begin{array}{ccccccccc} D_1 & — & D_2 & — & D_3 & — & D_4 & — \dots — & D_{r-1} \\ & & & & | & & & & \\ & & & & D_r & & & & \end{array}$$

Here the divisor Z and the number N are given by the following table.

type	fundamental cycle Z	N
$\mathbf{E}_6$	$D_1 + 2D_2 + 3D_3 + 2D_4 + D_5 + 2D_6$	11
$\mathbf{E}_7$	$2D_1 + 3D_2 + 4D_3 + 3D_4 + 2D_5 + D_6 + 2D_7$	17
$\mathbf{E}_8$	$2D_1 + 4D_2 + 6D_3 + 5D_4 + 4D_5 + 3D_6 + 2D_7 + 3D_8$	29

We take $Z_j := D_1 + \cdots + D_j$ for $j = 1, \dots, r$. For $j = r + 1, \dots, N$, we choose i_j as any index with $(Z_{j-1}, D_{i_j}) = 1$, and define $Z_j := Z_{j-1} + D_{i_j}$. It is easy to check that this gives a sequence with the desired properties. $\square$

Lemma 2.5. *Let* $Z_j \subset \widetilde{S}_k$ *be the closed subschemes given by Lemma 2.4.*

(i) $H^1(D_{i_j}, \mathcal{I}_{Z_{j-1} \subset Z_j}) = 0$ *for* $j = 1, \dots, N$.

(ii) $H^1(Z, \mathcal{I}^n_{Z \subset \widetilde{S}_k} / \mathcal{I}^{n+1}_{Z \subset \widetilde{S}_k}) = 0$ *for* $n \geqslant 0$.

Proof. The ideal sheaf of the effective divisor Z on the smooth projective surface $\widetilde{S}_k$ is the line bundle $\mathcal{O}(-Z) := \mathcal{O}_{\widetilde{S}_k}(-Z)$. Therefore, we have

$$\frac{\mathcal{I}^n_{Z \subset \widetilde{S}_k}}{\mathcal{I}^{n+1}_{Z \subset \widetilde{S}_k}} \cong \frac{\mathcal{O}(-nZ)}{\mathcal{O}(-(n+1)Z)} \cong \frac{\mathcal{O}_{\widetilde{S}_k}}{\mathcal{I}_{Z \subset \widetilde{S}_k}} \otimes \mathcal{O}(-nZ) \cong \mathcal{O}_Z \otimes \mathcal{O}(-nZ). \tag{1}$$

Since $Z_j - Z_{j-1} = D_{i_j}$ according to Lemma 2.4, we similarly have

$$\mathcal{I}_{Z_{j-1}\subset Z_j} \cong \frac{\mathcal{O}(-Z_{j-1})}{\mathcal{O}(-Z_j)} \cong \frac{\mathcal{O}_{\widetilde{S}_k}}{\mathcal{I}_{D_{i_j}\subset \widetilde{S}_k}} \otimes \mathcal{O}(-Z_{j-1}) \cong \mathcal{O}_{D_{i_j}} \otimes \mathcal{O}(-Z_{j-1}), \tag{2}$$

which is a line bundle of degree $(-Z_{j-1}, D_{i_j}) \geqslant -1$ on $D_{i_j} \cong \mathbb{P}^1$. But the first cohomology of any such line bundle vanishes. This proves part (i).

Twisting the isomorphism (2) by the line bundle $\mathcal{O}(-nZ)$ on $\widetilde{S}_k$, we get

$$\mathcal{I}_{Z_{j-1}\subset Z_j} \otimes \mathcal{O}(-nZ) \cong \mathcal{O}_{D_{i_j}} \otimes \mathcal{O}(-Z_{j-1} - nZ),$$

which is now a line bundle of degree $(-Z_{j-1}-nZ, D_{i_j})$ on $D_{i_j} \cong \mathbb{P}^1$. But this degree is still $\geqslant -1$, because the fundamental cycle Z satisfies $(Z, D_{i_j}) \leqslant 0$ by definition, and $n \geqslant 0$ by assumption. Hence we have more generally

$$H^1(D_{i_j}, \mathcal{I}_{Z_{j-1}\subset Z_j} \otimes \mathcal{O}(-nZ)) = 0.$$

Using induction over j, and the short exact sequences

$$0 \to \mathcal{I}_{Z_{j-1}\subset Z_j} \to \mathcal{O}_{Z_j} \to \mathcal{O}_{Z_{j-1}} \to 0$$

twisted by the line bundle $\mathcal{O}(-nZ)$ on $\widetilde{S}_k$, we conclude that

$$H^1(Z, \mathcal{O}_Z \otimes \mathcal{O}(-nZ)) = 0.$$

Because of the isomorphism (1), this proves part (ii) of the lemma. □

Proposition 2.6. *$H^1(Z, \mathcal{I}^n_{Z\subset \widetilde{S}}/\mathcal{I}^{n+1}_{Z\subset \widetilde{S}}) = 0$ for $n \geqslant 0$.*

Proof. Let $\pi \in R$ be a generator of $\mathfrak{m}$. We first claim that the inclusion

$$\pi \mathcal{I}^n_{Z\subset \widetilde{S}} \subset \mathcal{I}^{n+1}_{Z\subset \widetilde{S}} \cap \pi \mathcal{O}_{\widetilde{S}} \tag{3}$$

is an equality. It suffices to check this over the local ring $\mathcal{O}_{\widetilde{S},z}$ of each point $z \in Z$. We choose a local function $f \in \mathcal{O}_{\widetilde{S},z}$ whose residue class

$$\overline{f} \in \mathcal{O}_{\widetilde{S},z}/\pi\mathcal{O}_{\widetilde{S},z} = \mathcal{O}_{\widetilde{S}_k,z}$$

is a local equation for the divisor $Z \subset \widetilde{S}_k$. Then π and f generate $\mathcal{I}_{Z\subset \widetilde{S}}$ in z. Hence $\pi \mathcal{I}^n_{Z\subset \widetilde{S}}$ and f^{n+1} generate $\mathcal{I}^{n+1}_{Z\subset \widetilde{S}}$ in z. Suppose that

$$f^{n+1} g \in \pi \mathcal{O}_{\widetilde{S},z}$$

for some $g \in \mathcal{O}_{\widetilde{S},z}$. Then its residue class $\overline{g} \in \mathcal{O}_{\widetilde{S}_k,z}$ satisfies

$$\overline{f}^{n+1} \overline{g} = 0 \in \mathcal{O}_{\widetilde{S}_k,z}.$$

Since $\widetilde{S}_k$ is integral and $\overline{f} \neq 0$, this implies $\overline{g} = 0$, and hence $g \in \pi \mathcal{O}_{\widetilde{S},z}$. In particular, $f^{n+1}g$ lies in $\pi \mathcal{I}^n_{Z\subset \widetilde{S}}$. Therefore, (3) is indeed an equality.

Because of the natural short exact sequence

$$0 \to \mathcal{O}_{\widetilde{S}} \xrightarrow{\cdot \pi} \mathcal{I}_{Z\subset \widetilde{S}} \to \mathcal{I}_{Z\subset \widetilde{S}_k} \to 0,$$

the induced map $\mathcal{I}^n_{Z\subset \widetilde{S}}/\mathcal{I}^{n+1}_{Z\subset \widetilde{S}} \to \mathcal{I}^n_{Z\subset \widetilde{S}_k}/\mathcal{I}^{n+1}_{Z\subset \widetilde{S}_k}$ is surjective with kernel

$$(\mathcal{I}^n_{Z\subset \widetilde{S}} \cap \pi \mathcal{O}_{\widetilde{S}})/(\mathcal{I}^{n+1}_{Z\subset \widetilde{S}} \cap \pi \mathcal{O}_{\widetilde{S}}).$$

As (3) is an equality, this kernel is $\pi\mathcal{I}^{n-1}_{Z\subset\widetilde{S}}/\pi\mathcal{I}^{n}_{Z\subset\widetilde{S}}$. Thus the sequence

$$0 \to \mathcal{I}^{n-1}_{Z\subset\widetilde{S}}/\mathcal{I}^{n}_{Z\subset\widetilde{S}} \xrightarrow{\cdot\pi} \mathcal{I}^{n}_{Z\subset\widetilde{S}}/\mathcal{I}^{n+1}_{Z\subset\widetilde{S}} \to \mathcal{I}^{n}_{Z\subset\widetilde{S}_k}/\mathcal{I}^{n+1}_{Z\subset\widetilde{S}_k} \to 0$$

is exact. The proposition follows from this by induction over n, using part (ii) of Lemma 2.5 for the case $n = 0$ and for the induction step. □

3. Reductive groups and universal torsors

We continue in the setting of Section 2 and construct certain algebraic groups naturally associated to the Picard group of $\widetilde{S}_k$.

Since $\widetilde{S}$, $\widetilde{S}_K$ and $\widetilde{S}_k$ are obtained by the same sequence of blow-ups of a $\mathbb{P}^2$, the canonical restriction maps

$$\mathrm{Pic}(\widetilde{S}_K) \leftarrow \mathrm{Pic}(\widetilde{S}) \to \mathrm{Pic}(\widetilde{S}_k)$$

are isomorphisms; we denote this abelian group by Λ. Given $\lambda \in \Lambda$, we denote by L_λ the corresponding line bundle on $\widetilde{S}$, and by $L_{\lambda,K}$ and $L_{\lambda,k}$ its restrictions to $\widetilde{S}_K$ and $\widetilde{S}_k$, respectively. The canonical bundles of $\widetilde{S}$, $\widetilde{S}_K$ and $\widetilde{S}_k$ define the same class in Λ due to Lemma 2.1; we denote it by $K_{\widetilde{S}} \in \Lambda$.

The intersection forms on $\widetilde{S}_K$ and on $\widetilde{S}_k$ define the same bilinear form $(\cdot,\cdot)$ on Λ. Let $\Lambda^\vee$ be the dual of Λ, and denote the canonical pairing between $\Lambda^\vee$ and Λ by $\langle\cdot,\cdot\rangle$. Let

$$\Psi := \{\alpha \in \Lambda \mid (\alpha,\alpha) = -2,\ (\alpha, -K_{\widetilde{S}}) = 0\}$$

be the set of roots. For $\alpha \in \Psi$, define $\alpha^\vee \in \Lambda^\vee$ by $\langle\alpha^\vee, x\rangle := (-\alpha, x)$. Put

$$\Psi^\vee := \{\alpha^\vee \in \Lambda^\vee \mid \alpha \in \Psi\}.$$

Then a simple computation shows that $(\Lambda, \Psi, \Lambda^\vee, \Psi^\vee)$ is a reduced root datum in the sense of [DG70, Exposé XXI, Définition 1.1.1, 2.1.3]. Let H be the associated split reductive group over R [DG70, Exposé XXV, Corollaire 1.2]. Note that H corresponds to $\widetilde{\mathbf{E}}_{9-d}$ in [FM02, Section 2].

Let $\Phi \subset \Psi$ be the set of roots that are effective or anti-effective on $\widetilde{S}_k$, and let $\Phi^\vee$ be the associated subset of $\Lambda^\vee$. Let G be the split reductive group over R associated to the root datum $(\Lambda, \Phi, \Lambda^\vee, \Phi^\vee)$.

Then H contains G. The maximal torus T of G is also a maximal torus of H, and its character group is Λ. Therefore, T, T_K and T_k are the Néron-Severi tori of $\widetilde{S}$, $\widetilde{S}_K$ and $\widetilde{S}_k$, respectively. Let B be the Borel subgroup of G containing T such that the associated set Δ of simple roots in Φ is the set of classes of the (-2)-curves on $\widetilde{S}_k$. The corresponding set Φ^+ of positive roots consists precisely of the effective (-2)-classes on $\widetilde{S}_k$.

Lemma 3.1. *For $\alpha \in \Phi^+$, the R-module $H^1(\widetilde{S}, L_\alpha)$ is non-zero, cyclic and torsion (hence isomorphic to $R/\mathfrak{m}^{n_\alpha}$ for some $n_\alpha \geqslant 1$), the canonical map*

$$H^1(\widetilde{S}, L_\alpha) \otimes_R k \to H^1(\widetilde{S}_k, L_{\alpha,k}) \tag{4}$$

is an isomorphism, and $H^0(\widetilde{S}, L_\alpha) = H^2(\widetilde{S}, L_\alpha) = 0$.

Proof. Since α is effective on $\widetilde{S}_k$, we know that

$$\dim H^i(\widetilde{S}_k, L_{\alpha,k}) = \begin{cases} 1 & \text{for } i = 0, 1 \\ 0 & \text{for } i = 2 \end{cases}$$

Indeed, $p_a(\widetilde{S}_k) = p_a(\mathbb{P}^2_k) = 0$ since the arithmetic genus is a birational invariant, and hence the Riemann–Roch formula gives

$$\chi(L_{\alpha,k}) = 0.$$

The class $K_{\widetilde{S}} - \alpha$ has intersection number $-d < 0$ with the nef class $-K_{\widetilde{S}}$, and is therefore not effective. Consequently, Serre duality gives

$$H^2(\widetilde{S}_k, L_{\alpha,k}) = 0.$$

Since the anticanonical morphism $\widetilde{S}_k \to S_k$ is birational, there are only finitely many curves on $\widetilde{S}_k$ whose intersection number with $-K_{\widetilde{S}}$ is 0. But every curve of class α has this property, which implies

$$\dim H^0(\widetilde{S}_k, L_{\alpha,k}) \leqslant 1.$$

As α is effective on $\widetilde{S}_k$, we get $H^0(\widetilde{S}_k, L_{\alpha,k}) \cong k$, and hence also

$$H^1(\widetilde{S}_k, L_{\alpha,k}) \cong k. \tag{5}$$

Over K instead of k, the same arguments apply, but α is not effective over $\widetilde{S}_K$, and therefore $H^i(\widetilde{S}_K, L_{\alpha,K}) = 0$ for $i = 0, 1, 2$.

This implies that $H^1(\widetilde{S}, L_\alpha)$ is torsion, and $H^2(\widetilde{S}, L_\alpha) = 0$ by Grauert's Theorem [Har77, Corollary III.12.9]. Each section of the line bundle L_α vanishes on the generic fiber $\widetilde{S}_K$, and hence on $\widetilde{S}$. Therefore, $H^0(\widetilde{S}, L_\alpha) = 0$.

Applying cohomology and base change, we consider the natural maps

$$\varphi^i : H^i(\widetilde{S}, L_\alpha) \otimes_R k \to H^i(\widetilde{S}_k, L_{\alpha,k}).$$

For $i = 2$, both sides vanish. Using [Har77, Theorem III.12.11] twice, we conclude first that φ^1 is surjective, and then that φ^1 is an isomorphism. Due to (5), this implies that $H^1(\widetilde{S}, L_\alpha)$ is non-zero and cyclic. □

Let $\mathcal{T}$ be the universal T-torsor over over $\widetilde{S}$. The next step is to lift $\mathcal{T}$ to a B-torsor $\mathcal{B}$ over $\widetilde{S}$. We construct $\mathcal{B}$ as follows.

For $\alpha \in \Phi^+$, let $U_\alpha \cong \mathbb{G}_{\mathrm{a},R}$ be the associated root group in B. Let $U_{\geqslant 2}$ be the subgroup of B generated by all U_α with $\alpha \notin \Delta$, and put $B_{\leqslant 1} := B/U_{\geqslant 2}$. We have the exact sequence

$$0 \to U_{=1} := \bigoplus_{\alpha\in\Delta} U_\alpha \to B_{\leqslant 1} \to T \to 0. \tag{6}$$

Here T acts on $U_{=1}$ by conjugation. Associated to the T-torsor $\mathcal{T}$ over $\widetilde{S}$, we thus obtain a fibration over $\widetilde{S}$ with fiber $U_{=1}$. This group scheme over $\widetilde{S}$ is by construction the underlying additive group scheme of $\bigoplus_{\alpha\in\Delta} L_\alpha$.

We will first lift $\mathcal{T}$ to a $B_{\leqslant 1}$-torsor $\mathcal{B}_{\leqslant 1}$ over $\widetilde{S}$. This is possible because (6) admits a splitting $T \to B_{\leqslant 1}$. Such lifts are parameterized by

$$H^1(\widetilde{S}, \bigoplus_{\alpha\in\Delta} L_\alpha)$$

due to the previous paragraph (see [Hof10, Proposition 3.1.ii], for example). We choose a lift $\mathcal{B}_{\leqslant 1}$ such that, for each $\alpha \in \Delta$, the component

$$c_\alpha \in H^1(\widetilde{S}, L_\alpha) \tag{7}$$

of the class of $\mathcal{B}_{\leqslant 1}$ generates this R-module. This is possible by Lemma 3.1.

Lemma 3.2. *This $B_{\leqslant 1}$-torsor $\mathcal{B}_{\leqslant 1}$ can be lifted to a B-torsor $\mathcal{B}$ over $\widetilde{S}$.*

Proof. Let $\Phi^+_{=n}$ (resp. $\Phi^+_{\geqslant n}$) be the set of all $\alpha \in \Phi^+$ that are sums of precisely (resp. at least) n not necessarily distinct simple roots. Generalizing the above notation, we let $U_{\geqslant n}$ be the subgroup of B generated by all U_α with $\alpha \in \Phi^+_{\geqslant n}$, and put $B_{\leqslant n} := B/U_{\geqslant n+1}$. We have the exact sequences

$$0 \to U_{=n} := \bigoplus_{\alpha \in \Phi^+_{=n}} U_\alpha \to B_{\leqslant n} \to B_{\leqslant n-1} \to 0, \tag{8}$$

in which $B_{\leqslant n} = B/U_{\geqslant n+1}$ acts on $U_{=n}$ by conjugation. Here $U_{\geqslant 1}/U_{\geqslant n+1}$ acts trivially, because $[U_{\geqslant 1}, U_{\geqslant n}] \subset U_{\geqslant n+1}$. Therefore, the action descends to an action of $B/U_{\geqslant 1} = T$ on $U_{=n}$. Associated to the T-torsor $\mathcal{T}$ over $\widetilde{S}$, we thus obtain a fibration over $\widetilde{S}$, with fiber $U_{=n}$. This group scheme over $\widetilde{S}$ is by construction the underlying additive group scheme of $\bigoplus_{\alpha\in\Phi^+_{=n}} L_\alpha$.

Using induction, we assume that $\mathcal{B}_{\leqslant 1}$ can be lifted to a $B_{\leqslant n-1}$-torsor $\mathcal{B}_{\leqslant n-1}$ for some $n \geqslant 2$. We try to lift $\mathcal{B}_{\leqslant n-1}$ to a $B_{\leqslant n}$-torsor $\mathcal{B}_{\leqslant n}$ along the exact sequence (8). The obstruction against such a lift is an element in

$$H^2(\widetilde{S}, \bigoplus_{\alpha\in\Phi^+_{=n}} L_\alpha)$$

(see [Hof10, Proposition 3.1.i]). This cohomology vanishes by Lemma 3.1.

For sufficiently large n, we have $B_{\geqslant n} = B$, and $\mathcal{B} := \mathcal{B}_{\geqslant n}$ is the required lift of $\mathcal{B}_{\geqslant 1}$. □

Lemma 3.2 allows us to lift $\mathcal{B}_{\leqslant 1}$ to a B-torsor. We choose such a lift $\mathcal{B}$. Extending its structure group to G, we obtain a G-torsor $\mathcal{G}$ over $\widetilde{S}$.

Proposition 3.3. *This G-torsor $\mathcal{G}$ is trivial on every (-2)-curve $D \subset \widetilde{S}_k$.*

Proof. Let $\alpha \in \Delta$ be the class of D. The restriction $L_{\alpha|D}$ is a line bundle of degree $(\alpha, \alpha) = -2$ on $D \cong \mathbb{P}^1_k$, which implies

$$H^1(D, L_{\alpha|D}) \cong H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-2)) \cong k. \tag{9}$$

Tensoring the short exact sequence

$$0 \to \mathcal{O}_{\widetilde{S}_k}(-D) \to \mathcal{O}_{\widetilde{S}_k} \to \mathcal{O}_D \to 0$$

with the line bundle $L_{\alpha,k} \cong \mathcal{O}_{\widetilde{S}_k}(D)$, we get a short exact sequence

$$0 \to \mathcal{O}_{\widetilde{S}_k} \to L_{\alpha,k} \to L_{\alpha|D} \to 0$$

of coherent sheaves on $\widetilde{S}_k$. Since $H^i(\widetilde{S}_k, \mathcal{O}_{\widetilde{S}_k})$ vanishes for $i = 1, 2$ by their birational invariance [Har77, Proposition V.3.4], the associated long exact cohomology sequence shows that the restriction homomorphism

$$H^1(\widetilde{S}_k, L_{\alpha,k}) \to H^1(D, L_{\alpha|D}) \tag{10}$$

is bijective. For $\beta \in \Phi^+$ with $\beta \neq \alpha$, the degree of $L_{\beta|D}$ on $D \cong \mathbb{P}^1_k$ is

$$(\alpha, \beta) = -\langle \beta^\vee, \alpha \rangle =: n \in \{-1, 0, 1\},$$

because $\alpha \neq \beta$ are roots in the simply laced root system Φ. This implies

$$H^1(D, L_{\beta|D}) \cong H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(n)) = 0. \tag{11}$$

Let $G_\alpha \subset G$ be the split reductive subgroup with the same maximal torus T and only the two roots $\pm\alpha$. Then $B_\alpha := B \cap G_\alpha$ sits in an exact sequence

$$0 \to U_\alpha \to B_\alpha \to T \to 0.$$

Let the B_α-torsor $\mathcal{B}_\alpha$ on $\widetilde{S}$ be the lift of the T-torsor $\mathcal{T}$ corresponding to the class c_α chosen in (7). Let $\mathcal{G}_\alpha$ be the G_α-torsor over $\widetilde{S}$ induced by $\mathcal{B}_\alpha$.

The B-torsor induced by $\mathcal{B}_\alpha$ becomes isomorphic to $\mathcal{B}$ when both are restricted to D, because there the lifting over each U_β with $\beta \in \Phi^+ \setminus \{\alpha\}$ is unique by (11). Hence it suffices to prove that $\mathcal{G}_\alpha$ is trivial on D.

By [Dem80, II.2(6)], we have $\alpha = e_1 - e_2$ for two classes $e_i \in \Lambda$ satisfying $(e_i, e_i) = -1$ and $(e_1, e_2) = 0$. Let Σ be the subgroup of Λ generated by e_1 and e_2. Let $\Sigma^\perp$ be its orthogonal complement in Λ with respect to the intersection form. Then $\Lambda = \Sigma \oplus \Sigma^\perp$, since

$$\lambda + (\lambda, e_1)e_1 + (\lambda, e_2)e_2 \in \Sigma^\perp$$

for any $\lambda \in \Lambda$. Here $\alpha \in \Sigma$, and also $\alpha^\vee \in \Sigma^\vee$ because

$$\langle \alpha^\vee, \Sigma^\perp \rangle = (-\alpha, \Sigma^\perp) = 0.$$

Choosing a basis $e_3, \dots, e_{10-d}$ of $\Sigma^\perp$, we thus obtain

$$(\Lambda, \{\pm\alpha\}, \Lambda^\vee, \{\pm\alpha^\vee\}) \cong (\Sigma, \{\pm\alpha\}, \Sigma^\vee, \{\pm\alpha^\vee\}) \oplus (\mathbb{Z}, \emptyset, \mathbb{Z}^\vee, \emptyset)^{8-d}$$

as root data. This corresponds to a decomposition

$$G_\alpha \cong \mathrm{GL}_2 \times \mathbb{G}_{\mathrm{m}}^{8-d}$$

which induces a decomposition of B_α, and the decomposition $T \cong \mathbb{G}_{\mathrm{m}}^{10-d}$ given by $e_1, \dots, e_{10-d}$. Under these decompositions, the B_α-torsor $\mathcal{B}_\alpha$ corresponds to the $10-d$ line bundles L_{e_i} over $\widetilde{S}$ and the vector bundle extension

$$0 \to L_{e_1} \to E \to L_{e_2} \to 0$$

of class $c_\alpha \in \mathrm{Ext}^1(L_{e_2}, L_{e_1}) \cong H^1(\widetilde{S}, L_\alpha)$, and the G_α-torsor $\mathcal{G}_\alpha$ corresponds to the vector bundle E and the line bundles $L_{e_3}, \dots, L_{e_{10-d}}$ over $\widetilde{S}$.

The restriction of L_{e_i} to $D \cong \mathbb{P}^1_k$ is a line bundle of degree (α, e_i). For $i \geqslant 3$, we have $(\alpha, e_i) = 0$, and therefore $L_{e_i|D}$ is trivial. Since $(\alpha, e_1) = -1$ and $(\alpha, e_2) = 1$, the restriction of E to $D \cong \mathbb{P}^1_k$ is given as an extension

$$0 \to \mathcal{O}_{\mathbb{P}^1_k}(-1) \to E_{|D} \to \mathcal{O}_{\mathbb{P}^1_k}(1) \to 0, \tag{12}$$

whose class in $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-2)) \cong k$ corresponds to the restricted class

$$c_{\alpha|D} \in H^1(D, L_{\alpha|D})$$

under the isomorphism in (9). This class is nontrivial since the class

$$c_{\alpha|\widetilde{S}_k} \in H^1(\widetilde{S}_k, L_{\alpha,k})$$

is nontrivial by the choice of c_α in (7) together with Lemma 3.1, and the restriction map from $\widetilde{S}_k$ to D in (10) is bijective. Therefore, the extension (12) does not split. This implies that the vector bundle $E_{|D}$ over $D \cong \mathbb{P}^1_k$ is trivial. Hence the G_α-torsor $\mathcal{G}_{\alpha|D}$ over D is also trivial, as required. $\square$

Corollary 3.4. *Let x be a singular point on S_k. The G-torsor $\mathcal{G}$ over $\widetilde{S}$ constructed above becomes trivial over the following fiber product $\widetilde{S}_x$:*

$$\begin{array}{ccc} \widetilde{S}_x & \longrightarrow & \widetilde{S} \\ \big\downarrow{\scriptstyle \phi_x} & & \big\downarrow{\scriptstyle \phi} \\ \mathrm{Spec}(\widehat{\mathcal{O}}_{S,x}) & \longrightarrow & S \end{array} \tag{13}$$

Proof. We work with the sequence of effective divisors on $\widetilde{S}_k$

$$0 = Z_0 < Z_1 < \cdots < Z_r = Z^{\mathrm{red}} < \cdots < Z_N = Z$$

from Lemma 2.4, where Z is still the fundamental cycle on $\widetilde{S}_k$ over x.

First, we show that $\mathcal{G}$ is trivial on Z_j for $j = 1, \dots, r$. Indeed, by induction, we can find a trivialization of $\mathcal{G}$ on Z_{j-1}. Then

$$Z_j = Z_{j-1} \cup D_{i_j}$$

where D_{i_j} meets Z_{j-1} in at most one point. Proposition 3.3 states that $\mathcal{G}$ is trivial on D_{i_j}. We can trivialize on D_{i_j} in such a way that both trivializations agree on $Z_{j-1} \cap D_{i_j}$. Then they define a trivialization of $\mathcal{G}$ on Z_j.

Next, we show by induction that $\mathcal{G}$ is trivial on Z_j for all $j = r+1, \dots, N$. Since $\mathcal{G}$ is trivial on D_{i_j} by Proposition 3.3, its adjoint vector bundle

$$\mathrm{ad}(\mathcal{G}) \to \widetilde{S}$$

is also trivial on D_{i_j}. Therefore, Lemma 2.5 implies that

$$H^1(D_{i_j}, \mathcal{I}_{Z_{j-1} \subset Z_j} \otimes \mathrm{ad}(\mathcal{G})_{|D_{i_j}}) = 0. \tag{14}$$

Assuming by induction that $\mathcal{G}$ is trivial on Z_{j-1}, the vanishing of (14) means that $\mathcal{G}$ is also trivial on Z_j [Ill72, Théorème VII.2.4.4].

In particular, $\mathcal{G}$ is trivial on Z. Therefore, Proposition 2.6 implies that

$$H^1(Z, \mathcal{I}^n_{Z \subset \widetilde{S}} / \mathcal{I}^{n+1}_{Z \subset \widetilde{S}} \otimes \mathrm{ad}(\mathcal{G})_{|Z}) = 0. \tag{15}$$

Let $\mathfrak{m}_x \subset \mathcal{O}_S$ denote the ideal sheaf of x. We have

$$\mathcal{I}_{Z \subset \widetilde{S}} = \phi^*(\mathfrak{m}_x)$$

according to [Art66, Theorem 4], and therefore

$$\mathcal{I}^n_{Z \subset \widetilde{S}} = \phi^*(\mathfrak{m}_x^n).$$

Let $Z^{(n)}$ denote the closed subscheme in $\widetilde{S}$ with this ideal sheaf. Assuming by induction that we have a section of $\mathcal{G}$ over $Z^{(n)}$, the vanishing of (15) means that this section can be extended to a section of $\mathcal{G}$ over $Z^{(n+1)}$.

These compatible sections induce a section of $\mathcal{G}$ over $\widetilde{S}_x$ by Grothendieck's Existence Theorem [Gro61, Scholie 5.4.2], since $\widetilde{S}$ is proper over S. $\square$

Theorem 3.5. *Let the B-torsor $\mathcal{B}$ over $\widetilde{S}$ be a lift of the universal torsor $\mathcal{T}$ as given by Lemma 3.2. Then the induced G-torsor $\mathcal{G}$ descends to S.*

Proof. Since $\mathcal{G}$ is an affine scheme over $\widetilde{S}$, we have

$$\mathcal{G} \cong \mathbf{Spec}_{\mathcal{O}_{\widetilde{S}}}(\mathcal{A})$$

for some quasicoherent $\mathcal{O}_{\widetilde{S}}$-algebra $\mathcal{A}$. We define

$$\mathcal{G}' := \mathbf{Spec}_{\mathcal{O}_S}(\phi_*\mathcal{A}).$$

The adjunction morphism $\phi^*\phi_*\mathcal{A} \to \mathcal{A}$ induces a natural map

$$\mathcal{G} \to \mathcal{G}' \times_S \widetilde{S}. \tag{16}$$

Assume that G is the spectrum of the R-algebra A. The group action

$$G \times_R \mathcal{G} \to \mathcal{G}$$

induces a morphism $\mathcal{A} \to A \otimes_R \mathcal{A}$ of $\mathcal{O}_{\widetilde{S}}$-algebras, and hence a morphism

$$\phi_*\mathcal{A} \to \phi_*(A \otimes_R \mathcal{A}) = A \otimes_R (\phi_*\mathcal{A})$$

of $\mathcal{O}_S$-algebras. Here, the last equality holds because G, and hence also A, is flat over R. This morphism of $\mathcal{O}_S$-algebras induces a morphism

$$G \times_R \mathcal{G}' \to \mathcal{G}' \tag{17}$$

over S. We claim that the following statements hold:

- The morphism (17) is a group action of G on $\mathcal{G}'$ over S.
- This group action turns $\mathcal{G}'$ into a G-torsor over S.
- The natural map (16) is an isomorphism of G-torsors.

According to [Gro65, Propositions 2.5.1 and 2.7.1], all this can be tested locally in the fpqc-topology on S. We use the fpqc-covering

$$(S \setminus S_k^{\text{sing}}) \amalg \coprod_{x \in S_k^{\text{sing}}} \operatorname{Spec}(\widehat{\mathcal{O}}_{S,x}) \to S$$

where $S_k^{\text{sing}} \subset S_k \subset S$ denotes the singular locus of S_k.

All our claims hold over $S \setminus S_k^{\text{sing}}$ because ϕ is an isomorphism there. They also hold over each $\operatorname{Spec}(\widehat{\mathcal{O}}_{S,x})$ because $\mathcal{G}$ is trivial there, and

$$(\phi_x)_*\mathcal{O}_{\widetilde{S}_x} = \mathcal{O}_{\operatorname{Spec}(\widehat{\mathcal{O}}_{S,x})}$$

by Proposition 2.3 and flat base change in the diagram (13). □

Remark 3.6. The restriction of $\mathcal{G}$ to the generic fiber S_K is induced by the T-torsor $\mathcal{T}$. But over the special fiber S_k, the restriction of $\mathcal{G}$ does not come from a T-torsor. The universal T-torsor over the desingularization $\widetilde{S}_k$ is nontrivial on the (-2)-curves, and therefore does not descend to S_k.

Corollary 3.7. *The universal torsor $\mathcal{T}$ over $\widetilde{S}$ can be lifted to a torsor $\mathcal{C}$ under the Borel subgroup $C \subset H$ such that the induced H-torsor $\mathcal{H}$ over $\widetilde{S}$ descends to an H-torsor $\mathcal{H}'$ over S.*

Proof. The torsors $\mathcal{C}$, $\mathcal{H}$ and $\mathcal{H}'$ can be obtained by extension of the structure group from the torsors $\mathcal{B}$, $\mathcal{G}$ and $\mathcal{G}'$ constructed above. □

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