Irreducible Continuous Representations of the Simple Linearly Compact n-Lie Superalgebra of type W

Carina Boyallian and Vanesa Meinardi*

Abstract

In the present paper we classify all irreducible continuous representations of the simple linearly compact n-Lie superalgebra of type W. The classification is based on a bijective correspondence between the continuous representations of the n-Lie algebras W^n and continuous representations of the Lie algebra of Cartan type W_{n-1}, on which some two-sided ideal acts trivially.

*Ciem - FAMAF, Universidad Nacional de Córdoba - (5000) Córdoba, Argentina <boyallia@mate.uncor.edu - meinardi@mate.uncor.edu>.

1 Introduction

In 1985 Filippov [F] introduced a generalization of a Lie algebra, which he called an n-Lie algebra. The Lie bracket is taken between n elements of the algebra instead of two. This new bracket is n-linear, anti-symmetric and satisfies a generalization of the Jacobi identity.
In [F] and several subsequent papers, [F1],[K], [K1],[L] a structure theory of finite dimensional n-Lie algebras over a field $\mathbb{F}$ of characteristic 0 was developed. In [L], W. Ling proved that for every $n \geq 3$ there is, up to isomorphism only one finite dimensional simple n-Lie algebra, namely $\mathbb{C}^{n+1}$ where the n-ary operation is given by the generalized vector product, namely, if $e_1, \cdots, e_{n+1}$ is the standard basis of $\mathbb{C}^{n+1}$, the n-ary bracket is given by

$$[e_1, \cdots, \hat{e}_i, \cdots e_{n+1}] = (-1)^{n+i-1}\, e_i,$$

where i ranges from 1 to $n+1$ and the hat means that e_i does not appear in the bracket.

A. Dzhumadildaev studied in [D1] the finite dimensional irreducible representations of the simple n-Lie algebra $\mathbb{C}^{n+1}$. D.Balibanu and J. van de Leur in [BL] classified both, finite and infinite-dimensional irreducible highest weight representations of this algebra. Another examples of n-Lie algebras appeared earlier in Nambu's generalization of Hamiltonian dynamics [N]. A more recent important example of an n-Lie algebra structure on $C^\infty(M)$, where M is a finite-dimensional manifold, was given by Dzhumadildaev in [D], and it is associated to $n-1$ commuting vector fields $D_1, \cdots, D_{n-1}$ on M. More precisely, it is the space $C^\infty(M)$ of C^∞-functions on M, endowed with a n-ary bracket, associated to $n-1$ commuting vector fields $D_1, \cdots, D_{n-1}$ on M:

$$[f_1, \ldots, f_n] = \det \begin{pmatrix} f_1 & \ldots & f_n \\ D_1(f_1) & \ldots & D_1(f_n) \\ \ldots & \ldots & \ldots \\ D_{n-1}(f_1) & \ldots & D_{n-1}(f_n) \end{pmatrix}. \tag{1}$$

A linearly compact algebra is a topological algebra, whose underlying vector space is linearly compact, namely is a topological product of finite-dimensional vector spaces, endowed with discrete topology (and it is assumed that the algebra product is continuous in this topology). In 2010, N. Cantirini and V. Kac, ([CK]), classified simple linearly compact n-Lie superalgebras with $n > 2$ over a field $\mathbb{F}$ of characteristic 0. The list consists in four examples, one of them being $n+1$-dimensional vector product n-Lie algebra, and the remaining three are infinite-dimensional n-Lie algebras. More precisely,

Theorem 1. [CK]

(a) Any simple linearly compact n-Lie algebra with $n > 2$, over an algebraically closed field $\mathbb{F}$ of characteristic 0, is isomorphic to one of the following four examples:

(i) the $n+1$-dimensional vector product n-Lie algebra $\mathbb{C}^{n+1}$;

(ii) the n-Lie algebra, denoted by S^n, which is the linearly compact vector space of formal power series $\mathbb{F}[[x_1, \dots, x_n]]$, endowed with the n-ary bracket

$$[f_1, \dots, f_n] = \det \begin{pmatrix} D_1(f_1) & \dots & D_1(f_n) \\ \multicolumn{3}{c}{\dotfill} \\ D_n(f_1) & \dots & D_n(f_n) \end{pmatrix}.$$

where $D_i = \frac{\partial}{\partial x_i}$;

(iii) the n-Lie algebra, denoted by W^n, which is the linearly compact vector space of formal power series $\mathbb{F}[[x_1, \dots, x_{n-1}]]$, endowed with the n-ary bracket,

$$[f_1, \dots, f_n] = \det \begin{pmatrix} f_1 & \dots & f_n \\ D_1(f_1) & \dots & D_1(f_n) \\ \multicolumn{3}{c}{\dotfill} \\ D_{n-1}(f_1) & \dots & D_{n-1}(f_n) \end{pmatrix}.$$

where $D_i = \frac{\partial}{\partial x_i}$;

(iv) the n-Lie algebra, denoted by SW^n, which is the direct sum of $n-1$ copies of $\mathbb{F}[[x]]$, endowed with the following n-ary bracket, where $f^{\langle j \rangle}$ is an element of the j^{th} copy and $f' = \frac{df}{dx}$:

$$[f_1^{\langle j_1 \rangle}, \dots f_n^{\langle j_n \rangle}] = 0, \quad \text{unless } \{j_1, \dots, j_n\} \supset \{1, \dots, n-1\},$$
$$[f_1^{\langle 1 \rangle}, \dots, f_{k-1}^{\langle k-1 \rangle}, f_k^{\langle k \rangle}, f_{k+1}^{\langle k \rangle}, f_{k+2}^{\langle k+1 \rangle}, \dots, f_n^{\langle n-1 \rangle}]$$
$$= (-1)^{k+n} (f_1 \dots f_{k-1} (f'_k f_{k+1} - f'_{k+1} f_k) f_{k+2} \dots f_n)^{\langle k \rangle}.$$

(b) There are no simple linearly compact n-Lie superalgebras over $\mathbb{F}$, which are not n-Lie algebras, if $n > 2$.

In the present paper, we aim to classify all irreducible continuous representation of the simple linearly compact n-Lie algebra W^n. In the same way that D.Balibanu and J. van de Leur did the classification of irreducible modules in [BL], we reduced the problem to find irreducible continuous representations of simple linearly compact n-Lie (super) algebra W^n to find irreducible continuous representations of its associated basic Lie algebra on which some two-sided ideal acts trivially, plus some minor extra condition on the representation which is related with the definition of the associated basic Lie algebra of $\mathfrak{g}$. This minor condition is relevant for the representation theory of the remaining infinite dimensional linearly compact n- Lie (super) algebras classified in [CK], which is part of a work in progress.
The paper is organized as follow: In Section 2 we give the basic definitions and results related with n-lie algebras and state the relationship between representation of n- Lie algebras and representations of its associated Lie algebra. In Section 3, we introduce the simple linearly compact n- Lie algebra W^n, we identify its associated Lie algebra with the Lie algebra of its inner derivations which is nothing but W_{n-1}, the Lie algebra of Cartan type W and finally we relate representation of the n-Lie algebra W^n with representations of W_{n-1}. In Section 4, we present some general results of the representation theory of W_{n-1}, prove some technical lemmas and we describe some generators of the two sided ideal that must act trivially in our representations. Finally in Section 5, we state and prove the main result of the paper.

2 n-Lie algebras and n-Lie modules

We will give an introduction to n-Lie algebras and n-Lie modules. We will also introduce some useful results over the correspondence between representations of n-Lie algebra and representations of its basic associated Lie algebra.

From now on, $\mathbb{F}$ is a field of characteristic zero. As mentioned before, we are interested in studying irreducible representations of the linearly compact n-Lie superalgebra W^n. N. Cantarini and V. Kac stated in [CK] that there are no simple linearly compact n-Lie superalgebras over $\mathbb{F}$, which are not n-Lie algebras. Then we will use the representation theory of n-Lie algebras to give the representation theory of simple linearly compact n-Lie superalgebras. Given an integer $n \geq 2$, an *n-Lie algebra* V is a vector space over a field $\mathbb{F}$, endowed with an n-ary anti-commutative product

$$\begin{aligned} \wedge^n V &\longrightarrow V \\ a_1 \wedge \cdots \wedge a_n &\mapsto [a_1, \cdots, a_n], \end{aligned}$$

subject to the following Filippov-Jacobi identity:

$$\begin{aligned} &[a_1, \ldots, a_{n-1}, [b_1, \ldots, b_n]] = [[a_1, \ldots, a_{n-1}, b_1], b_2, \ldots, b_n] + \\ &[b_1, [a_1, \ldots, a_{n-1}, b_2], b_3, \ldots, b_n] + \cdots + [b_1, \ldots, b_{n-1}, [a_1, \ldots, a_{n-1}, b_n]]. \end{aligned} \tag{2}$$

A *derivation* D of an n-Lie algebra V is an endomorphism of the vector space V such that:

$$D([a_1, \cdots, a_n]) = [D(a_1), a_2, \cdots, a_n] + [a_1, D(a_2), \cdots, a_n] + \cdots + [a_1, \cdots, D(a_n)].$$

As in the Lie algebra case ($n = 2$), the meaning of the Filippov- Jacobi identity is that all endomorphisms $D_{a_1,\ldots,a_{n-1}}$ of V ($a_1, \ldots a_{n-1} \in V$), defined by

$$D_{a_1,\ldots a_{n-1}}(a) = [a_1, \ldots, a_{n-1}, a]$$

are derivations of V. These derivations are called *inner*.

A subspace $W \subset V$ is called a *n-Lie subalgebra* of the n-Lie algebra V if $[W, \cdots, W] \subset W$. An n-Lie subalgebra $I \subset V$ of an n-Lie algebra is called an *ideal* if $[I, V, \cdots, V] \subset I$. An n-Lie algebra is called *simple* if it has not proper ideal besides 0.

Let V be an n-Lie algebra, $n \geq 3$. We will associate to V a Lie algebra called *the basic Lie algebra*, following the presentation given in [BL]. Consider $ad : \wedge^{n-1} V \to \mathrm{End}(V)$ given by $ad(a_1 \wedge \ldots \wedge a_{n-1})(b) := D_{a_1,\ldots a_{n-1}}(b) = [a_1, \ldots, a_{n-1}, b]$. One can easily see that we could have chosen the codomain of ad to be $\mathrm{Der}(V)$ (the set of derivations of V) instead of $\mathrm{End}(V)$. ad induces a map $\widetilde{ad} : \wedge^{n-1} V \to \mathrm{End}(\wedge^\bullet V)$ defined as $\widetilde{ad}(a_1 \wedge \ldots \wedge a_{n-1})(b_1 \wedge \ldots \wedge b_m) = \sum_{i=1}^m b_1 \wedge \ldots \wedge [a_1, \ldots, a_{n-1}, b_i] \wedge \ldots \wedge b_m$. Denote by $\mathrm{Inder}(V)$ the set of inner derivations of V, i.e. endomorphisms of the form $D_{a_1,\ldots a_{n-1}} = ad(a_1 \wedge \ldots \wedge a_{n-1})$.

Let $a := a_1 \wedge \ldots \wedge a_{n-1}$ and $b := b_1 \wedge \ldots \wedge b_{n-1}$ be elements of $\wedge^{n-1} V$. Define

$$[a, b] = \frac{1}{2}(\widetilde{ad}(a)(b) - \widetilde{ad}(b)(a)). \tag{3}$$

The following proposition was proved in [BL].

Proposition 1. $[\cdot,\cdot]$ *defines a Lie algebra structure on* $\wedge^{n-1}V$ *and* $ad : \wedge^{n-1}V \to Inder(V)$ *is a surjective Lie algebra homomorphism.*

Consider

$$\mathrm{Ker}(\mathrm{ad}) = \{a_1\wedge\cdots\wedge a_{n-1} \in \wedge^{n-1}V : \mathrm{ad}(a_1\wedge\cdots\wedge a_{n-1})(b) = 0 \text{ for all}\, b \in V\},$$

and

$$\mathrm{Ker}(\widetilde{\mathrm{ad}}) = \{a_1\wedge\cdots\wedge a_{n-1} \in \wedge^{n-1}V : \widetilde{\mathrm{ad}}(a_1\wedge\cdots\wedge a_{n-1})(b) = 0 \text{ for all}\, b \in \wedge^{\bullet}V\}$$

It is straightforward to check that $\mathrm{Ker}(\mathrm{ad})$ is an abelian ideal of $\wedge^{n-1}V$ and $\mathrm{Ker}(\mathrm{ad}) \subseteq \mathrm{Ker}(\widetilde{\mathrm{ad}})$. Thus

$$\wedge^{n-1} V/\mathrm{Ker}(\mathrm{ad}) \simeq \mathrm{Inder}(V), \tag{4}$$

as Lie algebras. Thus,

$$\wedge^{n-1} V \simeq \mathrm{Ker}(\mathrm{ad}) \rtimes \mathrm{Inder}(V). \tag{5}$$

A vector space M is called an *n-Lie module* for the n-Lie algebra V, if on the direct sum $V \oplus M$ there is a structure of n-Lie algebra, such that the following conditions are satisfied:

- V is a subalgebra;
- M is an abelian ideal, i.e. when at least two slots of the n-bracket are occupied by elements in M, the result is 0.

We have the following results that establish some relations between representations of $\wedge^{n-1}V$ and n-Lie modules.

Theorem 2. *1) Let M be an n-Lie module of the n-Lie algebra V and define $\rho : \wedge^{n-1}V \to End(M)$ given by*

$$\rho(a_1 \wedge \cdots \wedge a_{n-1})(m) := [a_1, \cdots, a_{n-1}, m]$$

for all $m \in M$, where this n-Lie bracket corresponds to the n-Lie structure of $V \oplus M$. Then ρ is an homomorphism of Lie algebras.

2) Given (M, ρ) a representation of $\wedge^{n-1}V$ such that

$$\rho([a, b]) = \rho(\widetilde{ad}(a)(b)) \tag{6}$$

is satisfied for all $a := a_1 \wedge \cdots \wedge a_{n-1}$ *and* $b := b_1 \wedge \cdots \wedge b_{n-1}$ *in* $\wedge^{n-1}V$ *and the two sided ideal* $Q(V)$ *of the universal enveloping algebra of* $\wedge^{n-1}V$, *generated by the elements*

$$x_{a_1,\cdots,a_{2n-2}} = [a_1, \cdots, a_n] \wedge a_{n+1} \wedge \cdots a_{2n-2} -$$

$$- \sum_{i=1}^{n} (-1)^{i+n} (a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_n)(a_i \wedge a_{n+1} \wedge \cdots \wedge a_{2n-2}) \qquad (7)$$

acts trivially on M, *then* M *is an* n*-Lie module.*

Proof. Part 1) is direct from the definition of the Lie bracket in $\wedge^{n-1}V$ and the Filippov-Jacobi identity of the n-Lie bracket corresponding to the n-Lie structure of the semidirect product of V and M.

Let's prove part 2). Consider the n-ary map $[[\,,\,]] : \wedge^{n-1}(V \ltimes M) \to V \ltimes M$ such that M is an abelian ideal and V is a subalgebra with its own n-Lie bracket and define

$$[[a_1, \cdots, a_{n-1}, m]] := \rho(a_1 \wedge \cdots \wedge a_{n-1})(m) \qquad (8)$$

where $a_i \in V$, $m \in M$. We need to show the Filippov-Jacobi identity holds for the n-ary bracket defined above. It is enough to show that

$$[[a_1, \cdots, a_{n-1}, [[b_1, \cdots, b_{n-1}, m]]]] - [[b_1, \cdots, b_{n-1}, [[a_1, \cdots, a_{n-1}, m]]]] =$$

$$\sum_{i=1}^{n-1} [[b_1, \cdots [a_1, \cdots, a_{n-1}, b_i], \cdots, b_{n-1}, m]] \qquad (9)$$

and

$$[[[a_1, \cdots, a_n], a_{n+1}, \cdots, a_{2n-2}, m]] =$$

$$\sum_{i=1}^{n-1} (-1)^{n+i+1} [[a_1, \cdots [a_{n+1}, \cdots, a_{2n-2}, a_i, m], \cdots, a_{2n-2}]] \qquad (10)$$

hold for a_i and $b_i \in V$ and $m \in M$. Let's prove the identity (9). Since ρ is a representation of $\wedge^{n-1}V$ we have that

$$\begin{aligned} &\rho\Big(\frac{1}{2}\sum_{i=1}^{n-1} b_1 \wedge \ldots \wedge [a_1, \ldots, a_{n-1}, b_i] \wedge \ldots \wedge b_{n-1} \\ &\qquad - \frac{1}{2}\sum_{i=1}^{n-1} a_1 \wedge \ldots \wedge [b_1, \ldots, b_{n-1}, a_i] \wedge \ldots \wedge a_{n-1}\Big)(m) \\ &= \rho(a)\rho(b)(m) - \rho(b)\rho(a)(m). \end{aligned}$$

Using (8) this identity means

$$\frac{1}{2}\sum_{i=1}^{n-1}[[b_1,\cdots[a_1,\cdots,a_{n-1},b_i],\cdots,b_{n-1},m]] \\ -\frac{1}{2}\sum_{i=1}^{n-1}[[a_1,\cdots[b_1,\cdots,b_{n-1},a_i],\cdots,a_{n-1},m]] \\ =[[a_1,\cdots,a_{n-1},[[b_1,\cdots,b_{n-1},m]]]]-[[b_1,\cdots,b_{n-1},[[a_1,\cdots,a_{n-1},m]]]]. \tag{11}$$

By hypothesis ρ satisfies (6). Writing (6) evaluated in m and using (8), we get

$$\frac{1}{2}\sum_{i=1}^{n-1}[[b_1,\cdots[a_1,\cdots,a_{n-1},b_i],\cdots,b_{n-1},m]] \\ -\frac{1}{2}\sum_{i=1}^{n-1}[[a_1,\cdots[b_1,\cdots,b_{n-1},a_i],\cdots,a_{n-1},m]] \tag{12}$$

$$=\sum_{i=1}^{n-1}[[b_1,\cdots[a_1,\cdots,a_{n-1},b_i],\cdots,b_{n-1},m]]. \tag{13}$$

Thus, equating (11) and (13), we have (9).

Let's prove the identity (10). Writing the identity (10) using (8) we have that

$$\rho([a_1,\ldots,a_n]\wedge a_{n+1}\wedge\ldots\wedge a_{2n-2})(m) \\ =\sum_{i=1}^{n}(-1)^{i+n}\rho(a_1\wedge\ldots\wedge\hat{a}_i\wedge\ldots\wedge a_n)\rho(a_i\wedge a_{n+1}\wedge\ldots\wedge a_{2n-2})(m). \tag{14}$$

Therefore (14) is equivalent to the fact that the ideal $Q(V)$ acts trivially on M, finishing our proof. □

Proposition 2. *Let M be a n-Lie module over an n-Lie algebra V. Then any submodule, any factor-module and dual module of M are also n-Lie modules. If M_1 and M_2 are n-Lie modules over V, then their direct sum $M_1 \oplus M_2$ is also n-Lie module.*

Proof. Due to condition (6), the proof of Proposition 2.1 in [D1] remains the same in our case. □

As in [D1] we deduce the following Corollary.

Corollary 1. *Let M be a n-Lie module over n-Lie algebra V. Then*

a) M is irreducible if and only if M is irreducible as a Lie module over Lie algebra $\wedge^{n-1}V$.

b) M is completely reducible, if only if M is completely reducible as a Lie module over Lie algebra $\wedge^{n-1}V$.

Since we are aiming the study of the representation theory of V as an n-Lie algebra, Theorem 2 shows that it is closely related to the representation theory of the Lie algebra $\wedge^{n-1}V$. But first, due to (5), we need to characterize the ideal Ker(ad). We have the following Lemma.

Lemma 1. *If $a \in Ker(ad)$ and ρ is a representation of $\wedge^{n-1}V$ satisfying (6), then $\rho(a)$ commutes with $\rho(b)$ for any $b \in \wedge^{n-1}V$.*

Proof. Consider $a \in \mathrm{Ker}(\mathrm{ad}) \subseteq \mathrm{Ker}(\widetilde{\mathrm{ad}})$. Thus by condition (6) we have

$$\rho(a)\rho(b) - \rho(b)\rho(a) = \rho[a, b] = \rho(\widetilde{\mathrm{ad}}(a)(b)) = 0.$$

□

Thus, we have the following Proposition.

Proposition 3. *Let ρ be an irreducible representation of $\wedge^{n-1}V$ in M with countable dimension that satisfies (6). Then $Ker(ad)$ acts by scalars in M.*

Proof. Immediate from the Lemma above and Schur Lemma. □

Theorem 3. *Let (M, ρ) be an irreducible representation of $\wedge^{n-1}V$ that satisfies (6) and the ideal $Q(V)$ acts trivially on M. Then*

a) $\rho|_{Ker(ad)} := \lambda\, Id$ with Id the identity map in $End(M)$ and $\lambda \in (Ker(ad))^*$ is an $Inder(\mathfrak{g})$-module homomorphism (where $\mathbb{F}$ is thought as a trivial $Inder(V)$-module),

b) $\rho|_{Inder(V)}$ is an irreducible representation of $Inder(V)$ that satisfies (6) and such that the ideal $Q(V)$ acts trivially on M.

c) $\rho = \rho|_{Inder(V)} \oplus \lambda\, Id$.

Proof. Let's prove part a). If $l \in \mathrm{Inder}(V)$ and $a \in \mathrm{Ker}(\mathrm{ad})$, since $\mathrm{Ker}(\mathrm{ad})$ is an abelian ideal, by Lemma 1 we have $0 = \rho([l,a])(m) = \lambda([l,a])\, Id(m)$. Thus λ is an $\mathrm{Inder}(V)$-module homomorphism.

Let's prove part b). Consider $N \subsetneq M$ a non-trivial $\mathrm{Inder}(V)$-subrepresentation of M and take $0 \neq m \in M$ such that $0 \neq \widetilde{N} := \rho(\mathrm{Inder}(V))(m) \subseteq N$. Note if $a \in \mathrm{Ker}(\mathrm{ad})$, due to Lemma 1 and Proposition 3, $\rho(a)\widetilde{N} = \rho(a)\rho(\mathrm{Inder}(\mathfrak{g}))(m) = \rho(\mathrm{Inder}(V)\rho(a)(m) = \lambda(a)\rho(\mathrm{Inder}(V))(m) = \widetilde{N}$. Using (5), we can conclude that $0 \neq \widetilde{N}$ is a subrepresentation of M as a $\wedge^{n-1}V$-module but M was irreducible by hypothesis which is a contradiction. Part c) is an immediate consequence of (5) and Lemma 1. $\square$

3 The simple linearly compact n-Lie algebra W^n

We denote by W^n the simple infinite-dimensional linearly compact n-Lie superalgebra , whose underlying vector space is the linearly compact vector space of formal power series $\mathbb{F}[[x_1, \cdots, x_{n-1}]]$ endowed with the following n-ary bracket:

$$[f_1, \cdots, f_n] = \det \begin{pmatrix} f_1 & \cdots & f_n \\ D_1(f_1) & \cdots & D_1(f_n) \\ \cdots & \cdots & \cdots \\ D_{n-1}(f_1) & \cdots & D_{n-1}(f_n) \end{pmatrix} \tag{15}$$

where $D_i = \frac{\partial}{\partial x_i}$.

Remark 1. (a) Consider the n-Lie algebra W^n endowed with the n-bracket (15). Then, the map $\mathrm{ad} : \wedge^{n-1}W^n \to \mathrm{Inder}(W^n)$, which sends $f_1 \wedge \cdots \wedge f_{n-1} \to \mathrm{ad}(f_1 \wedge \cdots \wedge f_{n-1})$ is an isomorphism of Lie algebras. Due to Proposition 1 we only need to show that $\mathrm{Ker}(\mathrm{ad}) = \{0\}$. Let $f_1 \wedge \cdots \wedge f_{n-1} \in \mathrm{Ker}(\mathrm{ad})$, then

$$\mathrm{ad}(f_1 \wedge \cdots \wedge f_{n-1})(f) = \det \begin{pmatrix} f_1 & \cdots & f \\ D_1(f_1) & \cdots & D_1(f) \\ \cdots & \cdots & \cdots \\ D_{n-1}(f_1) & \cdots & D_{n-1}(f) \end{pmatrix} = 0, \tag{16}$$

for all $f \in \wedge^{n-1}W^n$. But, $\mathbb{F}[[x_1, \cdots, x_{n-1}]]$ is infinite dimensional, we have that at least two f_i's are linearly dependent, which means that $f_1 \wedge \cdots \wedge f_{n-1} = 0$.

(b) $\mathrm{Inder}(W^n)$ is a Lie algebra with the bracket given by the commutator. Thus, by the Filippov-Jacobi identity it follows that

$$[\mathrm{ad}(f_1 \wedge \cdots \wedge f_{n-1}),\, \mathrm{ad}(g_1 \wedge \cdots \wedge g_{n-1})] = \mathrm{ad}(\widetilde{\mathrm{ad}}(f))(g),$$

for all $f = f_1 \wedge \cdots \wedge f_{n-1}$ and $g = g_1 \wedge \cdots g_{n-1} \in \mathrm{Inder}(W^n)$. This is condition (6) of Theorem 2. Thus this condition always holds for any representation ρ of $\mathrm{Inder}(W^n)$.

Denote $W(m,n)$ the Lie superalgebra of continuous derivations of the tensor product $\mathbb{F}(m,n)$ of the algebra of formal power series in m even commuting variables $x_1, \ldots, x_m$ and the Grassmann algebra in n anti-commuting odd variables $\xi_1, \ldots, \xi_n$. Elements of $W(m,n)$ can be viewed as linear differential operators of the form

$$X = \sum_{i=1}^{m} P_i(x,\xi)\frac{\partial}{\partial x_i} + \sum_{j=1}^{n} Q_j(x,\xi)\frac{\partial}{\partial \xi_j},\ P_i, Q_j \in \mathbb{F}(m,n).$$

The Lie superalgebra $W(m,n)$ is simple linearly compact (and it is finite-dimensional if and only if $m = 0$). From now on, we will denoted the Lie algebras $W(n-1,0)$ by W_{n-1}.

Proposition 5.1 in [CK], gives us the description of the Lie algebra of continuous derivation of each simple linearly compact n-Lie algebra. Moreover, they state in particular, that the Lie algebra of continuous derivations of the n-Lie algebra W^n is isomorphic to W_{n-1} and in the proof of this Proposition, they show that the Lie algebra of continuous derivations of the n-Lie algebra W^n coincides with the Lie algebra of its inner derivations. Thus,

$$\mathrm{Inder}(W^n) \simeq W_{n-1}. \tag{17}$$

Therefore, Theorems 2 and 3 and Remark 1 gives us the following.

Theorem 4. *Irreducible representations of the n-Lie algebra W^n are in $1-1$ correspondence with irreducible representations of the universal enveloping algebra $U(W_{n-1})$, on which the two sided ideal $Q(W^n)$, generated by the elements*

$$x_{a_1,\cdots,a_{2n-2}} = ad\ ([a_1, \cdots, a_n] \wedge a_{n+1} \wedge \cdots \wedge a_{2n-2})$$

$$-\sum_{i=1}^{n} (-1)^{i+n} ad\ (a_1 \wedge \cdots \wedge \widehat{a_i} \cdots \wedge a_n)\ ad\ (a_i \wedge a_{n+1} \wedge \cdots \wedge a_{2n-2})$$

acts trivially.

4 Representations of simple linearly compact Lie superalgebra W_{n-1}.

In this section we present the approach given by A. Rudakov in [R] for the representation theory of the infinite-dimensional simple linearly compact Lie algebra W_{n-1}.

Recall that any $D \in W_{n-1}$ has the form $D = \sum_{i=1}^{n-1} f_i \partial/\partial x_i$ with $f_i \in \mathbb{F}[[x_1, \cdots, x_{n-1}]]$. Consider the filtration

$$(W_{n-1})_{(j)} = \{D, \ \deg f_i \geq j+1\}$$

of W_{n-1}, such that the subspaces $(W_{n-1})_{(j)}$ form a fundamental system of neighborhood of zero. The corresponding gradation is

$$(W_{n-1})_j = \{D, \ \deg f_i = j+1\}.$$

This gives a triangular decomposition

$$W_{n-1} = (W_{n-1})_- \oplus (W_{n-1})_0 \oplus (W_{n-1})_+,$$

with $(W_{n-1})_\pm = \oplus_{\pm m>0}(W_{n-1})_m$. We shall consider continuous representations in spaces with discrete topology. The continuity of a representation of a linearly compact Lie superalgebra W_{n-1} in a vector space V with discrete topology means that the stabilizer $(W_{n-1})_v = \{g \in W_{n-1} | \, gv = 0\}$ of any $v \in V$ is an open (hence of finite codimension) subalgebra of W_{n-1}. Let $(W_{n-1})_{\geq 0} = (W_{n-1})_{>0} \oplus (W_{n-1})_0$. Denote by $P(W_{n-1}, (W_{n-1})_{\geq 0})$ the category of all continuous W_{n-1}-modules V, where V is a vector space with discrete topology, that are $(W_{n-1})_0$-locally finite, that is any $v \in V$ is contained in a finite-dimensional $(W_{n-1})_0$-invariant subspace. Given an $(W_{n-1})_{\geq 0}$-module F, we may consider the associated induced W_{n-1}-module

$$M(F) = \operatorname{Ind}_{(W_{n-1})_{\geq 0}}^{W_{n-1}} F = U(W_{n-1}) \otimes_{U((W_{n-1})_{\geq 0})} F$$

called the *generalized Verma module* associated to F.

Let V be an W_{n-1}-module. The elements of the subspace

$$\operatorname{Sing}(V) := \{v \in V | \, (W_{n-1})_{>0} v = 0\}$$

are called *singular vectors*. When $V = M(F)$, the $(W_{n-1})_{\geq 0}$-module F is canonically an $(W_{n-1})_{\geq 0}$-submodule of $M(F)$, and Sing(F) is a subspace of $\mathrm{Sing}(M(F))$, called the *subspace of trivial singular vectors*. Observe that $M(F) = F \oplus F_+$, where $F_+ = U_+((W_{n-1})_-) \otimes F$ and $U_+((W_{n-1})_-)$ is the augmentation ideal in the symmetric algebra $U((W_{n-1})_-)$. Then

$$\mathrm{Sing}_+(M(F)) := \ \mathrm{Sing}(M(F)) \cap F_+$$

are called the *non-trivial singular vectors*.

Theorem 5. *[KR][R] (a) If F is a finite-dimensional $(W_{n-1})_{\geq 0}$-module, then $M(F)$ is in $P(W_{n-1},(W_{n-1})_{\geq 0})$.*

(b) In any irreducible finite-dimensional $(W_{n-1})_{\geq 0}$-module F the subalgebra $(W_{n-1})_+$ acts trivially.

(c) If F is an irreducible finite-dimensional $(W_{n-1})_{\geq 0}$-module, then $M(F)$ has a unique maximal submodule.

(d) Denote by I(F) the quotient by the unique maximal submodule of $M(F)$. Then the map $F \mapsto I(F)$ defines a bijective correspondence between irreducible finite-dimensional $(W_{n-1})_{\geq 0}$-modules and irreducible (W_{n-1})-modules in $P((W_{n-1}),(W_{n-1})_{\geq 0})$, the inverse map being $V \mapsto Sing(V)$.

(e) An $(W_{n-1})_{\geq 0}$-module $M(F)$ is irreducible if and only if the $(W_{n-1})_{\geq 0}$-module F is irreducible and $M(F)$ has no non-trivial singular vectors.

Remark 2. (a) Note that

$$(W_{n-1})_0 \cong \mathfrak{g}l_{n-1}(\mathbb{F}), \tag{18}$$

the isomorphism is given by the map that sends $x_i\partial/\partial x_j \to E_{i,j}$ where $E_{i,j}$ denote as is usual the matrix whose (i,j) entry is 1 and all the other entries are 0 for $i,j = 1,\cdots,n-1$.

(b) Due to Theorem 5 part b) any irreducible finite dimensional $(W_{n-1})_{\geq 0}$-module F will be obtained extending by zero the irreducible finite dimensional $\mathfrak{gl}_{n-1}$- module.

In the Lie algebra $\mathfrak{gl}_{n-1}(\mathbb{F})$ we choose the Borel subalgebra $\mathfrak{b} = \{x_i\partial/\partial x_j : i < j,\, i,j = 1,\cdots,n-1\}$. We denote by

$$\mathfrak{h} = \{x_i\partial/\partial x_i,\, i = 1,\cdots,n-1\}$$

the Cartan subalgebra corresponding to $\mathfrak{b}$.

Let $F^1,\cdots F^{n-1}$ be the irreducible $(W_{n-1})_{\geq 0}$-modules irreducibles obtained by extending trivially the irreducible $\mathfrak{gl}_{n-1}$-modules with highest weight $\lambda^1 = (0,\cdots,-1)$, $\lambda^2 = (0,\cdots,-1,-1),\cdots,\lambda^{n-1} = (-1,\cdots,-1,-1)$ respectively. We will call them *exceptional* $(W_{n-1})_{\geq 0}$-modules.

Theorem 6. *[R] If F is an irreducible finite dimensional $\mathfrak{gl}_{n-1}$-module which coincides with none of the exceptional modules $F^1, \cdots F^{n-1}$ then the W_{n-1}-module $M(F)$ is irreducible. Each module $N^p = M(F^p)$ contains a unique irreducible submodule K^p which is generated by all its non-trivial singular vectors.*

Corollary 2. *[R] If the W_{n-1}-module E is irreducible, then the $\mathfrak{gl}_{n-1}(\mathbb{F})$-module $F := Sing(E)$ is also irreducible. If F coincides with none of the modules $F^1, \cdots, F^{n-1}$, then $E = M(F)$. If $F = F^p$, then E is isomorphic to $J(F^p) = N^p / \ Sing_+(M(F^p))$.*

4.1 Some useful lemmas

Let F be an irreducible finite dimensional $\mathfrak{gl}_{n-1}$-module with highest weight vector v_λ and highest weight λ. Let $J(F) = M(F)/ \ \mathrm{Sing}_+(M(F))$.

Our main goal is to find those irreducible finite dimensional $\mathfrak{gl}_{n-1}(\mathbb{F})$-modules F for which $J(F)$ is an irreducible module over the n-Lie algebra W^n, more precisely, we are looking for those $J(F)$ where the ideal $Q(W^n)$ acts trivially.

Lemma 2. *$Q(W^n) \otimes_{U(W_{n-1})_{\geq 0}} F \subset \ Sing_+(M(F))$ if and only if $Q(W^n)$ acts trivially on $J(F)$.*

Proof. Suppose $Q(W^n)$ acts trivially on $J(F)$. Note that by Theorem 6 and Corollary 2 this means that $Q(W^n) \cdot (U(W_{n-1}) \otimes_{U(W_{n-1})_{\geq 0}} F) \subset \ \mathrm{Sing}_+(M(F))$.

Since $Q(W^n)$ is a two sided ideal, we have that $Q(W^n) \otimes_{(W_{n-1})_{\geq 0}} F \subset U(W_{n-1}) Q(W^n) \otimes_{(W_{n-1})_{\geq 0}} F + Q(W^n) \otimes_{(W_{n-1})_{\geq 0}} F = Q(W^n)(U(W_{n-1}) \otimes_{U(W_{n-1})_{\geq 0}} F) \subseteq \mathrm{Sing}_+ M(F)$.

Reciprocally if $Q(W^n) \otimes_{U(W_{n-1})_{\geq 0}} F \subset \ \mathrm{Sing}_+(M(F))$, it is enough to show that $U(W_{n-1}) Q(W^n) \otimes_{U(W_{n-1})_{\geq 0}} F \subset \mathrm{Sing}_+(M(F))$.

Note that $U(W_{n-1})(Q(W^n) \otimes_{U(W_{n-1})_{\geq 0}} F)$ is the submodule generated by $Q(W^n) \otimes_{U(W_{n-1})_{\geq 0}} F$ and $Q(W^n) \otimes_{U(W_{n-1})_{\geq 0}} F \subset \ \mathrm{Sing}_+(M(F))$ by hypothesis, thus we have $U(W_{n-1}) Q(W^n) \otimes_{U(W_{n-1})_{\geq 0}} F$ is a subset of the unique irreducible submodule $\mathrm{Sing}_+(M(F))$ of $M(F)$, therefore $Q(W^n)$ acts trivially on $J(F)$. $\square$

Lemma 3. *$Q(W^n) \otimes_{U(W_{n-1})_{\geq 0}} v_\lambda \subset \ Sing_+(M(F))$ if and only if $Q(W^n)$ acts trivially on $J(F)$.*

Proof. Due to Lemma 2 we only need to show that $Q(W^n) \otimes_{U(W_{n-1})_{\geq 0}} v_\lambda \subset \ \mathrm{Sing}_+(M(F))$ implies that $Q(W^n)$ acts trivially on $J(F)$. It is immediate from the definition of generalized Verma module and the facts that F is a highest weight $\mathfrak{gl}_{n-1}(\mathbb{F})$-module and $\mathfrak{gl}_{n-1}(\mathbb{F}) \subseteq U(W_{n-1})_{\geq 0}$. $\square$

4.2 Description of the ideal $Q(W^n)$

Recall that $\mathrm{Inder}(W^n) \simeq W_{n-1}$, where the isomorphism is given explicitly by

$$\mathrm{ad}(f_1 \wedge \cdots \wedge f_{n-1}) \longrightarrow \sum_{i=1}^{n-1} (-1)^{n+1-i} \det \begin{pmatrix} f_1 & \cdots & f_{n-1} \\ D_1(f_1) & \cdots & D_1(f_{n-1}) \\ \cdots & \cdots & \cdots \\ \hat{D}_i(f_1) & \cdots & \hat{D}_i(f_{n-1}) \\ \cdots & \cdots & \cdots \\ D_{n-1}(f_1) & \cdots & D_{n-1}(f_{n-1}) \end{pmatrix} D_i. \tag{19}$$

for any $f_1, \cdots, f_{n-1} \in \mathbb{F}[[x_1, \cdots, x_{n-1}]]$, $D_j = \frac{\partial}{\partial x_j}$ and the hat means that the row i does no appear in the matrix. Consider the subset

$$A = \{D = \sum_{i=1}^{n-1} f_i D_i : \; f_i \in \mathbb{F}[x_1, \cdots, x_{n-1}]\}.$$

It is dense in W_{n-1}. Since we are classifying continuous representations it is enough to characterize a set of generator of $Q_A(W^n) := Q(W^n) \bigcap A$. Take $f_1, \cdots f_{2n-2} \in \mathbb{F}[x_1, \cdots, x_{n-1}]$, where $f_l = X^{I_l}$ with

$$X^{I_l} := x_1^{i_1^l} x_2^{i_2^l} \cdots x_{n-1}^{i_{n-1}^l},$$

where $I_l := (i_1^l, \cdots, i_{n-1}^l)$ for any $i_1^l, \cdots, i_{n-1}^l \in \mathbb{Z}_{\geq 0}$ and $l \in \{1 \cdots, 2n-2\}$. Then the generators of $Q_A(W^n)$ are given by

$$x_{f_1, \cdots, f_{2n-2}} = \left(\sum_{k=1}^{n-1} \alpha(k) \, D_k \right) - \sum_{i=1}^{n} (-1)^{i+n} \left(\sum_{q=1}^{n-1} \beta(i,q) \, D_q \right) \left(\sum_{s=1}^{n-1} \gamma(i,s) \, D_s \right), \tag{20}$$

where

$$\alpha(k) = (-1)^{n+1+k} \frac{f_1 \cdots f_{2n-2}}{x_1^2 \cdots x_k \cdots x_{n-1}^2} \;\; \det A \; \det B_{k+1}, \quad k = 1 \cdots, n-1,$$

$$\beta(i,q) = (-1)^{n+1+q} \frac{f_1 \cdots \hat{f_i} \cdots f_{n-1}}{x_1 \cdots \hat{x_q} \cdots x_{n-1}} \; \det A_{q+1,i}, \quad q = 1 \cdots, n-1,$$

$$\gamma(i,s) = (-1)^{n+1+s} \frac{f_i \cdots\cdots f_{2n-2}}{x_1 \cdots \hat{x_s} \cdots x_{n-1}} \ \mathrm{det} C^{(i)}_{s+1}, \quad s = 1 \cdots, n-1,$$

with $i = 1, \cdots, n$ and the matrices A, B and C's are defined as follows:

$$A = \begin{pmatrix} 1 & \cdots & 1 \\ i_1^1 & \cdots & i_{n-1}^1 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ i_1^n & \cdots & i_{n-1}^n \end{pmatrix},$$

$A_{q+1,i}$ is the matrix A with the $q+1$-row and the i-column removed,

$$B_{k+1} = \begin{pmatrix} 1 & \cdots & 1 & \cdots & 1 \\ \sum_{r=1}^{n-1}(i_1^r - 1) & \cdots & i_1^{n+1} & \cdots & i_1^{2n-2} \\ \cdots & \cdots & \cdots & \cdots & \\ \widehat{\sum_{r=1}^{n-1}(i_k^r - 1)} & \cdots & \widehat{i_k^{n+1}} & \cdots & \widehat{i_k^{2n-2}} \\ \cdots & \cdots & \cdots & \cdots & \\ \sum_{r=1}^{n-1}(i_{n-1}^r - 1) & \cdots & i_{n-1}^{n+1} & \cdots & i_{n-1}^{2n-2} \end{pmatrix}$$

and

$$C^{(i)}_{s+1} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ i_1^i & i_1^{n+1} & \cdots & i_1^{2n-2} \\ \cdots & \cdots & \cdots & \cdots \\ \widehat{i_s^i} & \widehat{i_s^{n+1}} & \cdots & \widehat{i_s^{2n-2}} \\ \cdots & \cdots & \cdots & \cdots \\ i_{n-1}^i & i_{n-1}^{n+1} & \cdots & i_{n-1}^{2n-2} \end{pmatrix}$$

where the hats mean that the corresponding row is removed.

5 Main theorems and their proofs

In this section we will state the main result of this paper. Recall that the inner derivations of the simple linearly compact n-Lie algebra W^n are isomorphic to W_{n-1} and denote by $\mathfrak{h}$ the Cartan subalgebra of the Lie algebra $\mathfrak{gl}_{n-1}(\mathbb{F})$ chosen above Theorem 6. Let F be a finite dimensional irreducible highest weight $\mathfrak{gl}_{n-1}(\mathbb{F})$-module, with highest weight $\lambda \in \mathfrak{h}^*$ and highest weight vector v_λ. Recall that our goal is to determine for which $\lambda \in \mathfrak{h}^*$, the two sided ideal $Q(W^n)$ acts trivially on the irreducible highest weight module $J(F) = M(F)/ \operatorname{Sing}_+(M(F))$, This will ensure us that $J(F)$ is an n-Lie module of W^n. Let's denote by $\lambda_i = \lambda(E_{i,i})$ for $i = 1, \cdots, n-1$ and introduce the following useful notation for the proof of the theorem,

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i \geq j \\ 0 & \text{otherwise,} \end{cases} \tag{21}$$

with $i, j \in \{1, \cdots, n-1\}$.

Theorem 7. *(1) Let $n = 3$.*

(a) If F is a highest weight irreducible finite dimensional $\mathfrak{gl}_2(\mathbb{F})$-module, with highest weigh $\lambda \in \mathfrak{h}^*$ which coincides with none of the exceptional modules F^p with $p = 1, 2$, then the irreducible continuous W_2-module $M(F)$ is an irreducible continuous representation of the simple linearly compact 3-Lie algebra W^3 if and only if $\lambda \in \mathfrak{h}^*$ is such that $\lambda_1 = \lambda_2$, $\lambda_i \neq -1$, $i = 1, 2$ or $\lambda_1 = -1 - \lambda_2$, $\lambda_2 \neq 0$.

(b) If F coincides with some of the exceptional $\mathfrak{gl}_2(\mathbb{F})$-modules, then the irreducible continuous representation $J(F)$ of W_2 is an irreducible continuous representation of the simple linearly compact n-Lie algebra W^3.

(2) Let $n \geq 4$.

(a) If F is a highest weight irreducible finite dimensional $\mathfrak{gl}_{n-1}(\mathbb{F})$-module, with highest weigh $\lambda \in \mathfrak{h}^*$ which coincides with none of the exceptional modules $F^1, \cdots F^{n-1}$, then the irreducible continuous representation $M(F)$ of W_{n-1} is an irreducible continuous representation of the simple linearly compact n-Lie algebra W^n if and only if $\lambda \in \mathfrak{h}^*$ is such that $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$ with $\lambda_i \neq -1$ for $i = 1, \cdots n-1$.

(b) If F coincides with some of the exceptional $\mathfrak{gl}_{n-1}$-modules, then the irreducible continuous representation $J(F)$ of W_{n-1} is an irreducible continuous representation of the simple linearly compact n-Lie algebra W^n if and only if $\lambda \in \mathfrak{h}^*$ is such that $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = -1$.

Proof. Let F be a highest weight irreducible finite dimensional $\mathfrak{gl}_{n-1}(\mathbb{F})$-module, with highest weigh $\lambda \in \mathfrak{h}^*$ and highest weigh vector $v_{\lambda.}$ Recall that $\mathfrak{h} := \oplus_{i=1}^{n-1} \mathbb{F}\, E_{i,i}$ is the chosen Cartan subalgebra of the Lie algebra $\mathfrak{gl}_{n-1}(\mathbb{F})$.

Here we are identifying the subalgebra $\mathfrak{h}$ with the subalgebra of W_{n-1} generated by the elements $x_i\frac{\partial}{\partial x_i}$, $i = 1, \cdots, n-1$. Consider F as a $(W_{n-1})_{\geq 0}$-module and take the induced module $M(F) = U(W_{n-1}) \otimes_{U((W_{n-1})_{\geq 0})} F$. We will use Lemma 3 and the general look of the generators of $Q_A(W^n)$ to find out for which λ's, $Q_A(W^n)$ acts trivially in $J(F)$. Let $w_\lambda = 1 \otimes_{U((W_{n-1})_{\geq 0})} v_\lambda = 1 \otimes v_\lambda$.

According to the description of the generators given in (20) and taking into account that $(W_{n-1})_+$ acts by zero on w_λ, it is enough to consider the subset of generators $Q_A(W^n)$ and ask them to either act trivially w_λ if F is non-exceptional or $Q_A(W^n) \otimes v_\lambda \subseteq \mathrm{Sing}(M(F))$ otherwise. It is enough to consider $x_{f_1,\cdots,f_{2n-2}}$ with monomials $f_i \in \mathbb{F}[x_1, \cdots, x_{n-1}]$ as in (20) such that,

(1) $\deg(f_1 \cdots f_{2n-2}) = 2n-2$ and there exist $i \in \{1, \cdots, n\}$ such that
   - (a) $\deg(f_i f_{n+1} \cdots f_{2n-2}) = n-2$ or
   - (b) $\deg(f_i f_{n+1} \cdots f_{2n-2}) = n-1$,

(2) $\deg(f_1 \cdots f_{2n-2}) = 2n-3$ and there exist $i \in \{1, \cdots, n\}$ such that
   - (a) $\deg(f_i f_{n+1} \cdots f_{2n-2}) = n-2$ or
   - (b) $\deg(f_i f_{n+1} \cdots f_{2n-2}) = n-1$,

(3) $\deg(f_1 f_2 \cdots f_{2n-2}) = 2n-4$, and there exist $i \in \{1, \cdots, n\}$ such that $\deg(f_i f_{n+1} \cdots f_{2n-2}) = n-2$

since the remaining ones are either zero or act trivially any way. Here, we are assuming by simplicity that $i = n$. Let's analyze each possible case.

<u>Case 1</u>:

Here, $\deg(f_1 \cdots f_{2n-2}) = 2n-2$. We have four possible expressions for $f_1 f_2 \cdots f_{2n-2}$ such that $x_{f_1,\cdots,f_{2n-2}} \neq 0$. Namely, there exist $j,\ k \in \{1, \cdots, n-1\}$ $(j < k)$, such that

$$f_1 \cdots f_{2n-2} = x_1^2 \cdots x_j \cdots x_k^3 \cdots x_{n-1}^2, \tag{22}$$

$$f_1 \cdots f_{2n-2} = x_1^2 \cdots \hat{x_j}^2 \cdots x_k^4 \cdots x_{n-1}^2, \tag{23}$$

$$f_1 \cdots f_{2n-2} = x_1^2 \cdots x_j^2 \cdots x_k^2 \cdots x_{n-1}^2, \tag{24}$$

or for some $l < j < k \in \{1, \cdots, n-1\}$,

$$f_1 \cdots f_{2n-2} = x_1^2 \cdots x_l \cdots x_j \cdots x_k^4 \cdots x_{n-1}^2. \tag{25}$$

Case 1 (a):

If $f_1 \cdots f_{2n-2} = x_1^2 \cdots x_j \cdots x_k^3 \cdots x_{n-1}^2$, since $\deg(f_n f_{n+1} \cdots f_{2n-2}) = n-2$, its follows that $f_n f_{n+1} \cdots f_{2n-2} = x_1 \cdots \hat{x}_l \cdots x_{n-1}$ for some $l \in \{1, \cdots, n-1\}$ and $f_1 \cdots f_{n-1} = x_1 \cdots x_k^2 \cdots x_{n-1}$ or $f_1 \cdots f_{n-1} = x_1 \cdots \hat{x}_j \cdots x_k^3 \cdots x_{n-1}$ for some $j < k \in \{1, \cdots, n-1\}$.

Suppose $f_1 \cdots f_{n-1} = x_1 \cdots x_k^2 \cdots x_{n-1}$. Then $l = j$ and $f_n \cdots f_{2n-2} = x_1 \cdots \hat{x}_j \cdots x_{n-1}$. Therefore, we can consider the monomials as follows.

(i) Let $n \geq 3$ and $j, k \in \{1, \cdots n-1\}$ with $j < k$.

$$\begin{aligned} & f_s = x_s && s = 1, \cdots n-1, \ s \neq k, \\ & f_k = x_k^2, \quad f_n = 1, && \\ & f_{n+s} = x_s && s = 1, \cdots, j-1, \\ & f_{n+s} = x_{s+1} && s = j, \cdots n-2. \end{aligned}$$

Thus, using (20) for these f_i's, it follows that,

$$x_{f_1, \cdots, f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{j+1} \, 2 \, (\sum_{s=1}^{n-1} \lambda_s + 1)(1 \otimes E_{k,j} v_\lambda). \tag{26}$$

(ii) Let $n \geq 4$ and $l, j, k \in \{1, \cdots n-1\}$. Here, to set the monomials $f_{n+1}, \cdots f_{2n-2}$, we assume that $l < j$, otherwise we can interchange those indexes in the definition of $f_{n+1}, \cdots f_{2n-2}$. Set,

$$\begin{aligned} & f_s = x_s && s = 1, \cdots, j-1, \ s \neq l - \delta_{l,j} \text{ or } k - \delta_{k,j}, \\ & f_{l-\delta_{l,j}} = x_l x_j, \quad f_{k-\delta_{k,j}} = x_k^2, && \\ & f_s = x_{s+1} && s = j, \cdots, n-2, \\ & \quad f_{n-1} = 1, && f_n = x_l, \\ & f_{n+s} = x_s && s = 1, \cdots, l-1, \\ & f_{n+s} = x_{s+1} && s = l, \cdots j-2, \\ & f_{n+s} = x_{s+2} && s = j-1, \cdots n-3, \quad f_{2n-2} = 1. \end{aligned}$$

By (20) we have, if $l < k < j$.

$$x_{f_1, \cdots, f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{n+l+k} 2(\lambda_l - \lambda_k)(1 \otimes v_\lambda). \tag{27}$$

(iii) Let $n \geq 5$ and $l, j, k, m \in \{1, \cdots n-1\}$ with $j < k$. Here, we set

$$\begin{aligned}
&f_s = x_s \qquad s = 1, \cdots, k-1, \quad s \neq m - \delta_{m,k} \text{ or } l - \delta_{l,k}, \\
&f_{m-\delta_{m,k}} = x_m x_k, \quad f_{l-\delta_{l,k}} = x_l x_k, \\
&f_s = x_{s+1} \quad s = k, \cdots, n-2, \quad f_{n-1} = 1, \quad f_n = x_l,
\end{aligned}$$

The monomials $f_{n+1}, \cdots, f_{2n-2}$ are the same as in the case above, then by (20), we get ,

$$x_{f_1,\cdots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{l+k+j}(\lambda_k - \lambda_l + 1)(1 \otimes E_{k,j} v_\lambda). \tag{28}$$

(iv) Let $n \geq 4$ and take $j = m$ in the definition of $f_1, \cdots, f_{n-1}$ and keep the same definitions for $f_n, \cdots, f_{2n-2}$ we took in (iii). By (20) its follows: If $l < j < k$,

$$x_{f_1,\cdots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{j+l+k+1}(\lambda_k - \lambda_j + 1)(1 \otimes E_{k,j} v_\lambda); \tag{29}$$

if $j < l < k$,

$$x_{f_1,\cdots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{j+l+k+1}((\lambda_k - \lambda_j + 1) 1 \otimes E_{k,j} v_\lambda - 1 \otimes E_{k,l} E_{l,j} v_\lambda); \tag{30}$$

and if $j < k < l$,

$$x_{f_1,\cdots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{j+l+k+1}(\lambda_j - \lambda_k)(1 \otimes E_{k,j} v_\lambda). \tag{31}$$

(v) Let $n \geq 5$ and $l, j, k, m \in \{1, \cdots n-1\}$.

$$\begin{aligned}
&f_s = x_s \qquad s = 1, \cdots, m-1, \ s \neq k - \delta_{k,m} \text{ or } l - \delta_{l,m}, \\
&f_{k-\delta_{k,m}} = x_m x_k, \quad f_{l-\delta_{l,k}} = x_l x_k, \\
&f_s = x_{s+1} \qquad s = m, \cdots, n-2 \quad f_{n-1} = 1, \quad f_n = x_m, \\
&f_{n+s} = x_s \qquad s = 1, \cdots, j-1, \\
&f_{n+s} = x_{s+1} \quad s = j, \cdots m-2, \\
&f_{n+s} = x_{s+2} \quad s = m-1, \cdots n-3, \quad f_{2n-2} = 1.
\end{aligned}$$

Again, if $l < j < k < m$, $l < j < m < k$ or $j < m < k < l$, by (20) we have

$$x_{f_1,\cdots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^j(1 \otimes E_{k,j} v_\lambda). \tag{32}$$

(vi) Let $n \geq 5$ and $l, j, k, m \in \{1, \cdots n-1\}$.

$$\begin{array}{ll}
f_s = x_s & s = 1, \cdots, m-1,\ s \neq k - \delta_{k,m} \text{ or } l - \delta_{l,m}, \\
f_{l-\delta_{l,m}} = x_m x_l & f_{k-\delta_{k,m}} = x_k^2, \\
f_s = x_{s+1} & s = m, \cdots, n-2, \quad f_{n-1} = 1, \quad f_n = x_k, \\
f_{n+s} = x_s & s = 1, \cdots, j-1, \\
f_{n+s} = x_{s+1} & s = j, \cdots k-2, \\
fn + s = x_{s+2} & s = k-1, \cdots n-3, \quad f_{2n-2} = 1.
\end{array}$$

Thus for $l < j < m < k$, $l < j < k < m$ or $j < k < m < l$, we have,

$$x_{f_1, \cdots, f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{j+m+k} 2(\lambda_m - \lambda_l)(1 \otimes E_{k,j} v_\lambda). \tag{33}$$

(vii) Let $n \geq 6$ and $l, j, k, m, t \in \{1, \cdots n-1\}$ with $j < k$.

$$\begin{array}{ll}
f_s = x_s & s = 1, \cdots, t-1,\ s \neq m - \delta_{m,t} \text{ or } l - \delta_{l,t}, \\
f_{m-\delta_{m,t}} = x_m x_t, & f_{l-\delta_{l,t}} = x_l x_k, \\
f_s = x_{s+1} & s = t, \cdots, n-2, \quad f_{n-1} = 1, \quad f_n = x_m, \\
f_{n+s} = x_s & s = 1, \cdots, m-1, \\
f_{n+s} = x_{s+1} & s = m, \cdots j-2, \\
f_{n+s} = x_{s+2} & s = j-1, \cdots n-3, \quad f_{2n-2} = 1.
\end{array}$$

Again, by (20),

$$x_{f_1, \cdots, f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^j (1 \otimes E_{k,j} v_\lambda). \tag{34}$$

Now if, $f_1 \cdots f_n = x_1 \cdots \hat{x_j} \cdots x_k^3 \cdots x_{n-1}$ and $f_n \cdots f_{2n-2} = x_1 \cdots \hat{x_l} \cdots x_{n-1}$, then $l = k$. Therefore, we have the following possibilities.

(viii) Let $n \geq 4$ and $l, j, k \in \{1, \cdots n-1\}$. Note that to define the monomials $f_{n+1}, \cdots f_{2n-2}$ we are assuming that $j < k$. Otherwise, we can interchange those indexes in the definition of $f_{n+1},\cdots f_{2n-2}$.

$$\begin{aligned}
&f_s = x_s && s=1,\cdots,j-1,\ s\neq l-\delta_{l,j} \text{ or } k-\delta_{k,j},\\
&f_{l-\delta_{l,j}} = x_l x_k, \quad f_{k-\delta_{k,j}} = x_k^2, &&\\
&f_s = x_{s+1} && s=j,\cdots,n-2, \quad f_{n-1}=1, \quad f_n = x_j,\\
&f_{n+s} = x_s && s=1,\cdots,j-1,\\
&f_{n+s} = x_{s+1} && s=j,\cdots k-2,\\
&f_{n+s} = x_{s+2} && s=k-1,\cdots n-3, \quad f_{2n-2}=1.
\end{aligned}$$

By (20) we have: If $j<l<k$,

$$x_{f_1,\cdots,f_{2n-2}}\cdot(1\otimes v_\lambda) = (-1)^{j+1}2((\lambda_k-\lambda_l+2)(1\otimes E_{k,j}v_\lambda)+1\otimes E_{l,j}E_{k,l}v_\lambda); \tag{35}$$

if $j<k<l$,

$$x_{f_1,\cdots,f_{2n-2}}\cdot(1\otimes v_\lambda) = (-1)^{j}2(\lambda_k-\lambda_l+2)(1\otimes E_{k,j}v_\lambda); \tag{36}$$

if $l<j<k$,

$$x_{f_1,\cdots,f_{2n-2}}\cdot(1\otimes v_\lambda) = (-1)^{j+1}2(\lambda_k-\lambda_l+1)(1\otimes E_{k,j}v_\lambda). \tag{37}$$

Now consider (25), namely suppose $f_1\cdots f_{n-1} = x_1\cdots\hat{x}_j\cdots x_k^3\cdots x_{n-1}$ and $f_n\cdots f_{2n-2} = x_1\cdots\hat{x}_l\cdots x_{n-1}$.

(ix) Let $n\geq 4$ and $l,j,k\in\{1,\cdots n-1\}$ and set

$$\begin{aligned}
&f_s = x_s && s=1,\cdots,j-1,\ s\neq k-\delta_{k,j} \text{ or } l-\delta_{l,j},\\
&f_{l-\delta_{l,j}} = x_l x_k, \quad f_{k-\delta_{k,j}} = x_k^2, &&\\
&f_s = x_{s+1} && s=j,\cdots,n-2, \quad f_{n-1}=1, \quad f_n = x_k,\\
&f_{n+s} = x_s && s=1,\cdots,l-1,\\
&f_{n+s} = x_{s+1} && s=l,\cdots k-2,\\
&f_{n+s} = x_{s+2} && s=k-1,\cdots n-3, \quad f_{2n-2}=1.
\end{aligned}$$

Then by (20) if $l<j<k$ or $j<l<k$ we have.

$$x_{f_1,\cdots,f_{2n-2}}\cdot(1\otimes v_\lambda) = (-1)^{l+1}(1\otimes 2E_{k,j}E_{k,l}v_\lambda). \tag{38}$$

(x) Let $n \geq 4$, $l, j, k \in \{1, \cdots n-1\}$ and

$$\begin{aligned}
&f_s = x_s && s = 1, \cdots, j-1,\ s \neq k - \delta_{k,j}, \quad f_{k-\delta_{k,j}} = x_k^2, \\
&f_s = x_{s+1} && s = j, \cdots, n-2, \quad f_{n-1} = 1, \quad f_n = x_j, \\
&f_{n+s} = x_s && s = 1, \cdots, l-1, \\
&f_{n+s} = x_{s+1} && s = l, \cdots j-2, \\
&f_{n+s} = x_{s+2} && s = j-1, \cdots n-3, \quad f_{2n-2} = 1.
\end{aligned}$$

By (20) we have: If $j < l < k$ or $l < j < k$,

$$x_{f_1, \cdots, f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^j 2(D_j \otimes E_{k,l} v_\lambda - D_l \otimes E_{k,j} v_\lambda); \tag{39}$$

if $l < k < j$,

$$x_{f_1, \cdots, f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^l 2(D_j \otimes E_{k,l} v_\lambda); \tag{40}$$

and if $j < k < l$,

$$x_{f_1, \cdots, f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^l 2(D_l \otimes E_{k,j} v_\lambda). \tag{41}$$

If $l = j$, we have that $f_1 \cdots f_{n-1} = x_1 \cdots \hat{x_j} \cdots x_k^3 \cdots x_{n-1}$ and $f_n \cdots f_{2n-2} = x_1 \cdots \hat{x_j} \cdots x_{n-1}$ and (23) holds. Thus, we have the following cases.

(xi) Let $n \geq 4$ and $l, j, k \in \{1, \cdots n-1\}$ with $j < k$. We consider the same $f_1, \cdots f_{n-1}$ of the previous example and $f_n, \cdots f_{2n-2}$ are defined as follows.

$$\begin{aligned}
&f_n = x_l, \quad f_{n+s} = x_s \quad s = 1, \cdots, l-1, \\
&f_{n+s} = x_{s+1} \qquad s = l, \cdots j-2, \\
&f_{n+s} = x_{s+2} \quad s = j-1, \cdots n-3, \quad f_{2n-2} = 1.
\end{aligned}$$

Then by (20),

$$x_{f_1, \cdots, f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{l+1} (\lambda_j - \lambda_l)(1 \otimes E_{k,j} v_\lambda). \tag{42}$$

(xii) Let $n \geq 4$ and $l, j, k \in \{1, \cdots n-1\}$ with $j < k$. We consider the same $f_1, \cdots f_{n-1}$ as in (xi) and $f_n, \cdots f_{2n-2}$ are defined as follows:

$$\begin{aligned}
&f_n = x_k, \quad f_{n+s} = x_s \quad s = 1, \cdots j-1, \\
&f_{n+s} = x_{s+1} \quad s = j, \cdots k-2, \\
&f_{n+s} = x_{s+2} \quad s = k-1, \cdots n-3, \quad f_{2n-2} = 1.
\end{aligned}$$

Then by (20) we have,

$$x_{f_1,\cdots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{j+1} 2(1 \otimes E_{k,j} E_{k,j} v_\lambda). \tag{43}$$

Now, consider (24), namely $f_1 \cdots f_{2n-2} = x_1^2 \cdots x_{n-1}^2$ with $\deg(f_n \cdots f_{2n-2}) = n-2$. Then, there exist $k \in \{1, \cdots, n-1\}$ such that $f_n \cdots f_{2n-2} = x_1 \cdots \hat{x_k} \cdots x_{n-1}$ and $f_1 \cdots f_{n-1} = x_1 \cdots x_k^2 \cdots x_{n-1}$.

(xiii) Let $n \geq 3$ and $j, k \in \{1, \cdots n-1\}$ with $j < k$.

$$\begin{aligned}
&f_s = x_s && s = 1, \cdots n-1,\ s \neq j, \quad f_j = x_j x_k, \quad f_n = 1,\\
&f_{n+s} = x_s && s = 1, \cdots, k-1,\\
&f_{n+s} = x_{s+1} && s = k, \cdots n-2.
\end{aligned}$$

Again, by (20) we have,

$$x_{f_1,\cdots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^k (\lambda_j - \lambda_k)(\sum_{s=1}^{n-1} \lambda_s + 1)(1 \otimes v_\lambda). \tag{44}$$

(xiv) Let $n \geq 4$ and $l, j, k \in \{1, \cdots n-1\}$. Take the same definition of $f_1, \cdots f_{n-1}$ given in (xiii) and define the remaining polynomials as follows,

$$\begin{aligned}
&f_{n+s} = x_s && s = 1, \cdots, j-1,\ s \neq n + l - \delta_{l,j} - \delta_{l,k}\\
&f_{n+l-\delta_{l,j}-\delta_{l,k}} = x_l x_j, &&\\
&f_{n+s} = x_{s+1} && s = j, \cdots k-2,\\
&f_{n+s} = x_{s+2} && s = k-1, \cdots n-3, \quad f_{2n-2} = 1.
\end{aligned}$$

Then by (20), it follows:
If $j < k < l$,

$$x_{f_1,\cdots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{n+l+k} (\lambda_j - \lambda_k)(1 + \lambda_l - \lambda_j)(1 \otimes v_\lambda); \tag{45}$$

if $l < j < k$,

$$x_{f_1,\cdots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{n+l+k} (\lambda_l - \lambda_j)(1 + \lambda_j - \lambda_k)(1 \otimes v_\lambda); \tag{46}$$

and if $k < l < j$,

$$x_{f_1,\cdots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{n+l+k} (\lambda_j - \lambda_k)(1 + \lambda_l - \lambda_j)(1 \otimes v_\lambda). \tag{47}$$

Case 1(b):

Equations (23) and (25) don't give us new equations. Thus, consider $f_1, \cdots f_{2n-2}$ such that (22) holds. If $f_n \cdots f_{2n-2} = x_1 \cdots x_{n-1}$ then $f_1 \cdots f_{n-1} = x_1 \cdots \hat{x}_j \cdots x_k^2 \cdots x_{n-1}$ for some $j, k \in \{1, \cdots, n-1\}$.

Then we have.

(i) Let $n \geq 4$ and $l, j, k, m \in \{1, \cdots n-1\}$.

$$\begin{array}{ll} f_s = x_s & s = 1, \cdots j-1, \quad s \neq l - \delta_{l,j} \text{ and } f_{l-\delta_{l,j}} = x_l x_k, \\ f_s = x_{s+1} & s = j, \cdots n-2, \quad f_{n-1} = 1, \quad f_n = x_l x_m, \\ f_{n+s} = x_s & s = 1, \cdots, l-1, \\ f_{n+s} = x_{s+1} & s = l, \cdots m-2, \\ f_{n+s} = x_{s+2} & s = m-1, \cdots n-3, \quad f_{2n-2} = 1. \end{array}$$

Again, by (20), it follows:

If $j < l < m < k$, $j < m < k < l$, $l < j < m < k$ or $j < m < l < k$,

$$x_{f_1, \cdots, f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{j+l+m}((1 \otimes E_{k,m} E_{m,j} v_\lambda) - (\lambda_l - \lambda_m)(1 \otimes E_{k,j} v_\lambda)); \tag{48}$$

if $j < k < l < m$,$j < k < m < l$, $j < l < k < m$ or $l < j < k < m$

$$x_{f_1, \cdots, f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{j+l+m}(1 + \lambda_l - \lambda_m)(1 \otimes E_{k,j} v_\lambda); \tag{49}$$

and if $l < m < j < k$, $m < l < j < k$, $m < j < k < l$, $m < j < l < k$,

$$x_{f_1, \cdots, f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{j+l+m+1}(\lambda_l - \lambda_m)(1 \otimes E_{k,j} v_\lambda). \tag{50}$$

Suppose (24) holds, then $f_1 \cdots f_{n-1} = x_1 \cdots x_{n-1} = f_n \cdots f_{2n-2}$. Therefore the polynomials $f_1 \cdots, f_{2n-2}$ are defined as follows.

(ii) Let $n \geq 5$ and $l, j, k, m \in \{1, \cdots n-1\}$.

$$\begin{array}{ll} f_s = x_s & s = 1, \cdots, m-1, \ s \neq l - \delta_{l,m} \text{ and } f_{l-\delta_{l,m}} = x_m x_l, \\ f_s = x_{s+1} & s = m, \cdots, n-2, \quad f_{n-1} = 1, \quad f_n = x_j x_k, \\ f_{n+s} = x_s & s = 1, \cdots, j-1, \\ f_{n+s} = x_{s+1} & s = j, \cdots k-2, \\ f_{n+s} = x_{s+2} & s = k-1, \cdots n-3, \quad f_{2n-2} = 1. \end{array}$$

Then, by (20),

$$x_{f_1,\cdots,f_{2n-2}}\cdot(1\otimes v_\lambda)=(-1)^{m+j+k+1}(\lambda_m-\lambda_l)(\lambda_j-\lambda_k)(1\otimes v_\lambda). \quad (51)$$

Case 2(a):

Here we have that $\deg(f_1\cdots f_{2n-2})=2n-3$ and $\deg(f_nf_{n+1}\cdots f_{2n-2})=n-2$. We have two possible expressions for $f_1f_2\cdots f_{2n-2}$. There exist $j\in\{1,\cdots,n-1\}$ such that

$$f_1\cdots f_{2n-2}=x_1^2\cdots x_j\cdots x_{n-1}^2 \quad (52)$$

$$f_1\cdots f_{2n-2}=x_1^2\cdots x_j\cdots x_l\cdots x_k^3\cdots x_{n-1}^2 \quad (53)$$

for some $l,j,k\in\{1,\cdots,n-1\}$.

If $f_1\cdots f_{2n-2}=x_1^2\cdots x_j\cdots x_{n-1}^2$, since $\deg(f_nf_{n+1}\cdots f_{2n-2})=n-2$, its follows that $f_nf_{n+1}\cdots f_{2n-2}=x_1\cdots\hat{x}_j\cdots x_{n-1}$ and $f_1\cdots f_{n-1}=x_1\cdots x_{n-1}$. Then we have.

(i) Let $n\geq 4$ and $l,j,k\in\{1,\cdots n-1\}$. Note that to define the monomials $f_{n+1},\cdots f_{2n-2}$ we are assuming $l<j$, otherwise we can interchange those indexes in the definition of $f_{n+1},\cdots f_{2n-2}$.

$$\begin{array}{lll}
f_s=x_s & s=1,\cdots,k-1, & s\neq l-\delta_{l,j},\ \text{and}\ f_{l-\delta_{l,j}}=x_lx_k,\\
f_s=x_{s+1} & s=k,\cdots,n-2, & f_{n-1}=1,\quad f_n=x_l,\\
f_{n+s}=x_s & s=1,\cdots,l-1, & \\
f_{n+s}=x_{s+1} & s=l,\cdots j-2, & \\
f_{n+s}=x_{s+2} & s=j-1,\cdots n-3, & f_{2n-2}=1.
\end{array}$$

Then, we have:
If $j<l<k$ or $j<k<l$,

$$x_{f_1,\cdots,f_{2n-2}}\cdot(1\otimes v_\lambda)=(-1)^{j+l+1+k}(D_l\otimes E_{l,j}v_\lambda-D_k\otimes E_{k,j}v_\lambda-(\lambda_k-\lambda_l)(D_j\otimes v_\lambda)); \quad (54)$$

if $k<j<l$,

$$x_{f_1,\cdots,f_{2n-2}}\cdot(1\otimes v_\lambda)=(-1)^{j+k+l+1}(D_l\otimes E_{l,j}v_\lambda-(\lambda_k-\lambda_l)(D_j\otimes v_\lambda)); \quad (55)$$

if $l<j<k$,

$$x_{f_1,\cdots,f_{2n-2}}\cdot(1\otimes v_\lambda)=(-1)^{j+l+k+1}(D_k\otimes E_{k,j}v_\lambda-(\lambda_k-\lambda_l)(D_j\otimes v_\lambda)); \quad (56)$$

and if $l < k < j$ or $k < l < j$,

$$x_{f_1,\cdots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{j+l+k+1}(\lambda_k - \lambda_l)(D_j \otimes v_\lambda). \tag{57}$$

The equation (53) doesn't provide new information.
<u>Case 2 (b) and 3:</u>

After doing the same analysis, these cases don't provide new equations.

Observe that in all the equations (26) to (50), except for equations (39) to (41) and (50) to (57), their right hand side belongs to $1 \otimes_{U((W_{n-1}))_{\geq 0}} F$, therefore they are trivial singular vectors. Due Lemma 3, we need to insure that all the equations (26) to (50), except the equations (39) to (41) and (50) to (57), are equal to zero. Since different equations hold for $n = 3$ and $n \geq 4$, we will study theses cases separately.

If $n = 3$, equations (26) and (44) hold and they have to be zero. Thus,

$$(\lambda_1 + \lambda_2 + 1)(1 \otimes E_{2,1} v_\lambda) = 0 \tag{58}$$

$$(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 + 1)(1 \otimes v_\lambda) = 0 \tag{59}$$

Equation (59) implies $\lambda_1 = \lambda_2$ or $\lambda_2 = -1 - \lambda_1$. Suppose $\lambda_2 = -1 - \lambda_1$ then the equation (58) holds, hence $M(F)$ is a continuous representation of the 3-Lie algebra W^3. Due Theorem 6, $M(F)$ will be irreducible if $\lambda_1 \neq 0$. Otherwise if $\lambda = (0, -1)$, then F coincides with the exceptional module F^1 and we need to take the quotient of $M(F^1)$ by the submodule generated by all its non-trivial singular vectors to make the module irreducible. In the other hand if $\lambda_1 = \lambda_2$, we will show that $E_{2,1} v_\lambda = 0$, using the Freudental's formula.

Now, if $n \geq 4$, equations (27), (44) and (47) equate to zero implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$ or $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-2} = 0$ and $\lambda_{n-1} = -1$.

Therefore we will apply the Freudenthal's formula to calculate the dimensions of the weight spaces and check wether the remaining equations are satisfied. To give the root basis of $\mathfrak{gl}_{n-1}(\mathbb{F})$ it is convenient to have consider another basis for $\mathfrak{gl}_{n-1}(\mathbb{F})$. For do this we need to take into account that $\mathfrak{gl}_{n-1}(\mathbb{F}) = \mathfrak{sl}_{n-1}(\mathbb{F}) \oplus \mathfrak{s}_n(\mathbb{F})$, where $\mathfrak{sl}_{n-1}(\mathbb{F})$ are the traceless matrices and $\mathfrak{s}_n(\mathbb{F})$ denote the subspace of scalar multiple of the identity. We define a basis of $\mathfrak{gl}_{n-1}(\mathbb{F})$ as a basis of $\mathfrak{sl}_{n-1}(\mathbb{F})$ and the identity matrix.

As usual let $E_{i,j}$ be the matrix with a 1 in the (i, j) position and 0's everywhere else, $D_{i,j} = E_{i,i} - E_{j,j}$ and $h_i = D_{i,i+1}$. A basis for $\mathfrak{sl}_{n-1}(\mathbb{F})$ is given by $h_1, \cdots, h_{n-2}$. Let $\widetilde{\mathfrak{h}}$ be the subalgebra of diagonal traceless matrices which is a Cartan subalgebra of $\mathfrak{sl}_{n-1}(\mathbb{F})$. Let $\epsilon_j : \widetilde{\mathfrak{h}} \to \mathbb{F}$ be defined by $\epsilon_j\left(\sum_{i=1}^{n-1} a_i E_{i,i}\right) = a_j$. We define the set of root

$$\phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n-1\}$$

and the $\epsilon_i - \epsilon_j$ root space is generated by $E_{i,j}$ and a basis for this set of root is given by

$$\Delta = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \cdots, \epsilon_{n-2} - \epsilon_{n-1}\}.$$

Let Λ^+ be the set of all dominant weights and $\delta = \frac{1}{2}\sum_{\alpha \succ 0} \alpha$. If $\alpha_i := \epsilon_i - \epsilon_{i+1}$, the fundamental dominant weights relatives to Δ of $\mathfrak{sl}_{n-1}(\mathbb{F})$ are given by,

$$\begin{aligned} \pi_i = \ & \tfrac{1}{n-1}[(n-1-i)\alpha_1 + 2(n-1-i)\alpha_2 + \cdots (i-1)(n-1-i)\alpha_{i-1} \\ & +i(n-1-i)\alpha_i + i(n-2-i)\alpha_{i+1} + \cdots + i\alpha_{n-2}] \end{aligned} \tag{60}$$

Therefore Λ is a lattice with basis $(\pi_i, i = 1, \cdots, n-2)$. Let $n \geq 3$, $L = \mathfrak{sl}_{n-1}(\mathbb{F})$, $\alpha_i = \epsilon_i - \epsilon_{i+1}$ and π_i as (60). Require $(\alpha_i, \alpha_i) = 1$, so that $(\alpha_i, \alpha_j) = -1/2$ if $\mid i - j \mid = 1$ and $(\alpha_i, \alpha_j) = 0$ if $\mid i - j \mid \geq 2$. Rewriting $\lambda = (\lambda_1, \cdots, \lambda_{n-1})$ with $\lambda_1 = \cdots = \lambda_{n-1}$ in this new basis we have that $\lambda := (0, \cdots, 0)$. Since $(\lambda+\delta, \lambda+\delta) - (\mu+\delta, \mu+\delta) = 0$ for $\mu = -\alpha_{k-1}$ with $k \in \{1, \cdots, n-1\}$ then Freudenthal's formula gives that the multiplicities for $\mu = -\alpha_{k-1}$ is equal to zero. Besides, it follows from Freudenthals formula too, that the multiplicities for $\mu = -\sum_{k=j}^{i-1} \alpha_k$ are also equal to zero for all $i, j \in \{1, \cdots, n-1\}$, $i < j$. Thus $E_{i,j} v_\lambda = 0$, for all $i, j \in 1, \cdots n-1, i < j$. In particular all the equations from (26) to (57) are equal to zero and M(F) results a continuous representation of the n-Lie algebra W^n with $n \geq 3$. Due Theorem 6, $M(F)$ will be irreducible if $\lambda_i \neq -1$ with $i = 1, \cdots n-1$. Otherwise, if $\lambda = (-1, \cdots, -1)$, then F coincides with the exceptional module F^{n-1} and we have to take the quotient of $M(F^{n-1})$ by the submodule generated by all its non-trivial singular vectors, to make the module irreducible.

Finally, if $\lambda = (0, 0, \cdots, -1)$, rewriting it in the new basis we have that $\lambda = \pi_{n-2}$. The Freudenthal's formula gives that the multiplicities for $\mu = -\alpha_{n-3} - \alpha_{n-2}$ are equal to one. This implies that $E_{n-1,n-3} v_\lambda \neq 0$, therefore equation (32) is non zero and the induce representation $M(F)$ is not a representation of the n-Lie algebra W^n, finishing our proof. $\square$

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