Asymptotics of Prolate Spheroidal Wave Functions

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Abstract. Uniform asymptotic approximations are obtained for the prolate spheroidal functions $\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ $\left(-1<x<1\right)$ and $Ps_n^m\left(x,\gamma^2\right)$ $\left(1<x<\infty\right)$. Here $\gamma\to\infty$, and the results are uniformly valid in the stated intervals, and where *m* and *n* are integers, with *m* fixed and $0\le m\le n=o\left(\gamma\right)$. The results are obtained by an application of certain existing asymptotic solutions of differential equations, and involve elementary, modified Bessel, and parabolic cylinder functions. An asymptotic relationship between the prolate spheroidal equation separation parameter and the other parameters is also obtained, and error bounds are available for all approximations.

1. Introduction

Separation of the wave equation in prolate spheroidal coordinates leads to the prolate spheroidal wave equation (PSWE)

$$\left(1-z^2\right)\frac{d^2y}{dz^2}-2z\frac{dy}{dz}+\left(\lambda-\frac{\mu^2}{1-z^2}+\gamma^2\left(1-z^2\right)\right)y=0\,, \tag{1.1}$$

where λ and μ are separation constants, and γ is proportional to the frequency. The solution of the PSWE has long been a subject of interest in many areas of mathematics and physics. The most classical is the scalar and electromagnetic scattering by spheroidal obstacles using the method of separation of variables. Other related applications:

- signal processing and telecommunications
- light scattering in optics
- astrophysics
- nuclear shell models
- theoretical cosmological models
- atomic and molecular physics
- biophysics (radiation scattered by nonspherical particles)

In this paper we obtain asymptotic approximations to certain solutions of (1.1) for the high-frequency case, i.e. $\gamma \to \infty$. Explicit error bounds are available for all approximations. In [2] uniform asymptotic approximations for the high frequency case were also obtained, complete with error bounds, but for a different parameter regime then in the present study.

For single-valued solutions it is well known [1, 4] that the necessary condition is $\mu = m = \text{integer}$, and we shall assume this throughout. We shall write $z = x$ when z is real. We will primarily be concerned with $z = x$ real, but will use complex-valued argument results as needed to obtain our final results.

The PSWE has regular singularities at $z = \pm 1$, each with exponents $\pm\frac{1}{2}m$. When $\gamma = 0$ the PSWE degenerates into the associated Legendre equation (regular singularities at $z = \pm 1$ and $z = \infty$), which for $-1 < x < 1$ has solutions the Ferrers functions $\mathrm{P}_\nu^\mu(x)$, and for complex z the associated Legendre functions $P_\nu^\mu(z)$.

The significant difference is if $\gamma \neq 0$ the PSWE has an irregular singularity at infinity. In fact one can show (from an algebraic form of Floquet's Theorem) that there exists a parameter ν (say) that depends on the other parameters, and a solution $S_\nu^{\mu(1)}(z,\gamma)$ (say), with the property

$$S_\nu^{\mu(1)}\left(ze^{p\pi i},\gamma\right) = e^{p\nu\pi i} S_\nu^{\mu(1)}(z,\gamma), \tag{1.2}$$

for any integer p. The parameter ν is called the characteristic exponent of the PSWE. $S_\nu^{\mu(1)}(z,\gamma)$ can be expressed as an infinite series of Bessel functions [1]. An important property is that

$$S_\nu^{\mu(1)}(z,\gamma) = \frac{\cos\left\{\gamma z - \frac{1}{2}\pi(\nu+1)\right\}}{\gamma z}\left\{1 + O\left(\frac{1}{z}\right)\right\} \quad (z \to \infty), \tag{1.3}$$

for $\left|\arg(z)\right| < \pi$. We shall assume that $\nu = n = \text{integer}$, so that all solutions are single-valued in the z-plane having a cut along the interval $[-1,1]$.

Other fundamental solutions at infinity are given by $S_n^{m(j)}(z,\gamma)$ ($j=3,4$). These are defined by

$$S_n^{m(j)}(z,\gamma)=\left(\frac{\pi}{2\gamma z}\right)^{1/2}\frac{\left(z^2-1\right)^{-m/2}z^m}{A_n^m\left(\gamma^2\right)}\sum_{j=-\infty}^{\infty}a_{n,j}^m\left(\gamma^2\right)H_{n+2j}^{(j-2)}(\gamma z), \tag{1.4}$$

and have the fundamental properties

$$S_n^{m(3)}(z,\gamma)=i^{-n-1}\frac{\mathrm{e}^{i\gamma z}}{\gamma z}\left\{1+O\left(\frac{1}{z}\right)\right\} \quad (z\to\infty), \tag{1.5}$$

$$S_n^{m(4)}(z,\gamma)=i^{n+1}\frac{\mathrm{e}^{-i\gamma z}}{\gamma z}\left\{1+O\left(\frac{1}{z}\right)\right\} \quad (z\to\infty), \tag{1.6}$$

for $|\arg(z)|<\pi$. In particular $S_n^{m(3)}(z,\gamma)$ is the unique solution that is recessive in the upper half plane, and $S_n^{m(4)}(z,\gamma)$ is the unique solution that is recessive in the lower half plane.

An important connection formula is

$$S_n^{m(1)}(z,\gamma)=\tfrac{1}{2}\left\{S_n^{m(3)}(z,\gamma)+S_n^{m(4)}(z,\gamma)\right\}. \tag{1.7}$$

For $n\geq m\geq 0$ and $-1<x<1$ there is a solution $\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ defined by

$$\mathrm{Ps}_n^m\left(x,\gamma^2\right)=\sum_{k=-k_0}^{\infty}(-1)^k a_{n,k}^m\left(\gamma^2\right)\mathrm{P}_{n+2k}^m(x), \tag{1.8}$$

where

$$k_0=\left\lfloor\tfrac{1}{2}(n-m)\right\rfloor. \tag{1.9}$$

The coefficients are defined by the three-term recurrence relation

$$A_{n,k}^m\left(\gamma^2\right)a_{n,k-1}^m\left(\gamma^2\right)+\left\{\lambda_n^m\left(\gamma^2\right)+B_{n,k}^m\left(\gamma^2\right)\right\}a_{n,k}^m\left(\gamma^2\right)+C_{n,k}^m a_{n,k+1}^m\left(\gamma^2\right)=0, \tag{1.10}$$

where

$$A_{n,k}^{m}\left(\gamma^2\right)=\frac{(n-m+2k-1)(n-m+2k)}{(2n+4k-3)(2n+4k-1)}\gamma^2\,, \tag{1.11}$$

$$B_{n,k}^{m}\left(\gamma^2\right)=\frac{2\left[(n+2k)(n+2k+1)+m^2-1\right]}{(2n+4k-1)(2n+4k+3)}\gamma^2-(n+2k)(n+2k+1)\,, \tag{1.12}$$

and

$$C_{n,k}^{m}\left(\gamma^2\right)=\frac{(n+m+2k+1)(n+m+2k+2)}{(2n+4k+3)(2n+4k+5)}\gamma^2\,. \tag{1.13}$$

The coefficients $a_{n,k}^{m}\left(\gamma^2\right)$ vanish for $k\le -1-k_0$, where k_0 is defined by (1.9).

$\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ is the unique solution with the property of being recessive at $x=1$, namely

$$\mathrm{Ps}_n^m\left(x,\gamma^2\right)=K_n^m\left(\gamma^2\right)(1-x)^{m/2}\left\{1+O(1-x)\right\}\quad\left(x\to 1^-\right), \tag{1.14}$$

where

$$K_n^m\left(\gamma^2\right)=\frac{(-1)^m}{2^{m/2}m!}\sum_{k=-k_0}^{\infty}(-1)^k\frac{(n+2k+m)!}{(n+2k-m)!}a_{n,k}^{m}\left(\gamma^2\right). \tag{1.15}$$

Using connection formulas, it can be shown that $\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ is also bounded at $x=-1$; this is consequence of the characteristic exponent $\nu=n$ being an integer. In this case the function satisfies the normalization condition

$$\int_{-1}^{1}\left\{\mathrm{Ps}_n^m\left(x,\gamma^2\right)\right\}^2dx=\frac{2(n+m)!}{(2n+1)(n-m)!}\,. \tag{1.16}$$

The separation constant $\lambda=\lambda_n^m\left(\gamma^2\right)$ is now regarded as an eigenvalue that makes *m* and *n* integers, and hence admits an eigensolution $\mathrm{Ps}_n^m\left(z,\gamma^2\right)$, which is bounded at both $x=\pm 1$. $\mathrm{Ps}_n^m\left(z,\gamma^2\right)$ is entire if *m* is even. In the case *m* odd, $\left(1-z^2\right)^{1/2}\mathrm{Ps}_n^m\left(z,\gamma^2\right)$ is entire. We write $z=x$ for *x* real and lying in the interval $(-1,1)$ (the angular case).

For our purposes the PSWE therefore takes the form

$$\left(1-z^2\right)\frac{d^2y}{dz^2}-2z\frac{dy}{dz}+\left(\lambda_n^m-\frac{m^2}{1-z^2}+\gamma^2\left(1-z^2\right)\right)y=0\,. \tag{1.17}$$

In this paper γ is positive and large, m is fixed, and $0\le m\le n=o(\gamma)$.

For x real and lying in $(1,\infty)$ (the radial case) we have a solution

$$Ps_n^m\left(x,\gamma^2\right)=\sum_{k=-k_0}^{\infty}(-1)^k\,a_{n,k}^m\left(\gamma^2\right)P_{n+2k}^m(x)\,, \tag{1.18}$$

and also extend this to define $Ps_n^m\left(z,\gamma^2\right)$ for complex z.

Now since m and n are integers, it can be shown that

$$Ps_n^m\left(ze^{\pi i},\gamma^2\right)=(-1)^n\,Ps_n^m\left(z,\gamma^2\right), \tag{1.19}$$

which is (unique) property of the Floquet solution $S_n^{m(1)}(z,\gamma)$. Hence

$$S_n^{m(1)}(z,\gamma)=(-1)^n(n-m)!V_n^m(\gamma)Ps_n^m\left(z,\gamma^2\right), \tag{1.20}$$

for some constant $V_n^m(\gamma)$, and hence from the known behavior of $S_n^{m(1)}(z,\gamma)$ at infinity

$$Ps_n^m\left(z,\gamma^2\right)=V_n^m(\gamma)\frac{\cos\left\{\gamma z-\frac{1}{2}\pi(\nu+1)\right\}}{\gamma z}\left\{1+O\left(\frac{1}{z}\right)\right\}\quad(z\to\infty). \tag{1.21}$$

From [1, p. 171] we also have

$$Ps_n^m\left(z,\gamma^2\right)=K_n^m\left(\gamma^2\right)(z-1)^{m/2}\left\{1+O(z-1)\right\}\quad(z\to 1), \tag{1.22}$$

where $K_n^m\left(\gamma^2\right)$ is given by (1.15).

2. Liouville-Green Asymptotics: radial case.

Making the transformation $w=\left(z^2-1\right)^{1/2}y$ in (1.17) we obtain

$$\frac{d^2w}{dz^2}=\left\{-\gamma^2+\frac{\lambda_n^m}{z^2-1}+\frac{m^2-1}{\left(z^2-1\right)^2}\right\}w\,. \tag{2.1}$$

From [1, p. 186] it is known that for fixed *m* and *n* and large γ that

$$\lambda_n^m\left(\gamma^2\right)=-\gamma^2+2\left(n-m+\tfrac{1}{2}\right)\gamma+O(1)\,. \tag{2.2}$$

With this in mind we define

$$\lambda_n^m\left(\gamma^2\right)=-\gamma^2\left(1-\sigma^2\right), \tag{2.3}$$

and we assume

$$0\le\sigma=\sqrt{1+\frac{\lambda_n^m\left(\gamma^2\right)}{\gamma^2}}\le\sigma_0<1\,, \tag{2.4}$$

where σ_0 is an arbitrary positive constant. Thus we can express (2.1) in the form

$$\frac{d^2w}{dz^2}=\left[\gamma^2 f(\sigma,z)+g(z)\right]w\,, \tag{2.5}$$

where

$$f(\sigma,z)=\frac{\sigma^2-z^2}{z^2-1},\quad g(z)=\frac{m^2-1}{\left(z^2-1\right)^2}\,. \tag{2.6}$$

We construct Liouville-Green approximations for $Ps_n^m\left(z,\gamma^2\right)$, using the theory of [6, Chap. 10]. Firstly, define a new independent variable

$$\xi = \int_1^z \{-f(\sigma,t)\}^{1/2} dt = \int_1^z \left(\frac{t^2-\sigma^2}{t^2-1} \right)^{1/2} dt \,. \tag{2.7}$$

Branch cuts are suitably chosen, so that $0 \le \xi < \infty$ for $1 \le z < \infty$. The RHS of (2.7) can be expressed in terms of the elliptic integral of the second kind [7, eq. (19.2.5)]

$$E(a;b) = \int_0^a \left(\frac{1-b^2t^2}{1-t^2} \right)^{1/2} dt = b\int_0^a \left(\frac{b^{-2}-t^2}{1-t^2} \right)^{1/2} dt \,. \tag{2.8}$$

Here $b = \sigma^{-1} > 1$, and the branches of the square roots are such that integrand is positive for $0 \le t < b^{-1}$ and negative for $1 < t < \infty$, and continuous elsewhere in the complex t-plane having a cut along the interval $\left[b^{-1},1\right]$. We thus have

$$\xi = \sigma E\left(1;\sigma^{-1}\right) - \sigma E\left(z;\sigma^{-1}\right). \tag{2.9}$$

Then with the new dependent variable $W = \{-f\}^{1/4} w$ we obtain

$$\frac{d^2W}{d\xi^2} = \left[-\gamma^2 + \psi(\xi)\right] W \,, \tag{2.10}$$

where

$$\psi(\xi) = \frac{m^2-1}{\left(z^2-1\right)\left(z^2-\sigma^2\right)} + \frac{\left(1-\sigma^2\right)\left(6z^4-\left(3+\sigma^2\right)z^2-2\sigma^2\right)}{4\left(z^2-1\right)\left(z^2-\sigma^2\right)^3}. \tag{2.11}$$

We observe that $\psi(\xi) = O\left(\xi^{-2}\right)$ as $\xi \to \infty$, but is unbounded at $z = \pm 1$ and also $z = \pm\sigma$.

From the definition of ξ we find that

$$\xi = z - J(\sigma) + O\left(z^{-1}\right) \quad (z \to \infty), \tag{2.12}$$

where

$$J(\sigma)=1-\int_1^\infty\left[\left(\frac{t^2-\sigma^2}{t^2-1}\right)^{1/2}-1\right]dt\,. \tag{2.13}$$

Note $J(0)=0$ and $J(1)=1$. Now by Cauchy's theorem

$$0=\Re\int_{-\infty}^\infty\left[\left(\frac{t^2-\sigma^2}{t^2-1}\right)^{1/2}-1\right]dt=2\Re\int_0^\infty\left[\left(\frac{t^2-\sigma^2}{t^2-1}\right)^{1/2}-1\right]dt\,. \tag{2.14}$$

Hence

$$\int_1^\infty\left[\left(\frac{t^2-\sigma^2}{t^2-1}\right)^{1/2}-1\right]dt=-\Re\int_0^1\left[\left(\frac{t^2-\sigma^2}{t^2-1}\right)^{1/2}-1\right]dt\,, \tag{2.15}$$

and consequently from (2.13)

$$J(\sigma)=1+\Re\int_0^1\left[\left(\frac{t^2-\sigma^2}{t^2-1}\right)^{1/2}-1\right]dt=\int_0^\sigma\left(\frac{\sigma^2-t^2}{1-t^2}\right)^{1/2}dt\,; \tag{2.16}$$

i.e.

$$J(\sigma)=\sigma E\left(\sigma;\sigma^{-1}\right), \tag{2.17}$$

for $\sigma>0$, in which E is the Elliptic integral of the second kind given by (2.8). Thus

$$\xi=z-\sigma E\left(\sigma;\sigma^{-1}\right)+O\left(z^{-1}\right)\quad\left(z\to\infty\right). \tag{2.18}$$

We now apply theorem 3.1 of [5], with u replaced by γ, and with ξ replaced by $i\xi$. Then, by matching solutions that are recessive at $z=\pm i\infty$, we have from (1.5), (1.6) and (2.18)

$$S_n^{m(3)}(z,\gamma)=i^{-1-n}\gamma^{-1}\left[\left(z^2-1\right)\left(z^2-\sigma^2\right)\right]^{-1/4}e^{i\gamma J(\sigma)}\left[e^{i\gamma\xi}\sum_{s=0}^{p-1}(-i)^s\frac{A_s(\xi)}{\gamma^s}+\varepsilon_{p,1}(\gamma,\xi)\right], \tag{2.19}$$

and

$$S_n^{m(4)}(z,\gamma)=i^{1+n}\gamma^{-1}\left[\left(z^2-1\right)\left(z^2-\sigma^2\right)\right]^{-1/4}e^{-i\gamma J(\sigma)}\left[e^{-i\gamma\xi}\sum_{s=0}^{p-1}i^s\frac{A_s(\xi)}{\gamma^s}+\varepsilon_{p,2}(\gamma,\xi)\right]. \tag{2.20}$$

The error terms $\varepsilon_{p,j}(\gamma,\xi)$ ($j=1,2$) are bounded by Olver's theorem, and are $O(\gamma^{-p})$ in unbounded domains containing the real interval $1+\delta \le z < \infty$ ($\delta > 0$). Here the coefficients are defined recursively by $A_0(\xi)=1$ and

$$A_{s+1}(\xi) = -\tfrac{1}{2}A_s'(\xi) + \tfrac{1}{2}\int \psi(\xi) A_s(\xi)\, d\xi \quad (s=0,1,2,\cdots). \tag{2.21}$$

Then from (1.7), (1.20), (2.17), (2.19) and (2.20) we obtain the desired L-G approximation for $Ps_n^m(x,\gamma^2)$. In particular we have

$$Ps_n^m(x,\gamma^2) = \frac{(-1)^n \sin\left(\gamma\xi + \gamma\sigma E(\sigma;\sigma^{-1}) - \tfrac{1}{2}n\pi\right) + O(\gamma^{-1})}{\gamma(n-m)!V_n^m(\gamma)\left[(x^2-1)(x^2-\sigma^2)\right]^{1/4}}, \tag{2.22}$$

as $\gamma \to \infty$, uniformly for $1+\delta \le x < \infty$ ($\delta > 0$). We note that this breaks down at the simple pole $x=1$.

3. Bessel function approximations: radial case.

We obtain approximations valid at the simple pole of $f(\sigma,z)$ at $z=1$, using the asymptotic theory of [6, Chap. 12]. We consider $z=x$ real and positive. Then the appropriate Liouville transformation is now given by

$$\eta = \xi^2 = \left[\int_1^x \left\{-f(\sigma,t)\right\}^{1/2} dt\right]^2, \tag{3.1}$$

along with

$$\hat{W} = \left\{\frac{\eta(x^2-\sigma^2)}{x^2-1}\right\}^{1/4} w, \tag{3.2}$$

which gives the new equation

$$\frac{\mathrm{d}^2\hat{W}}{\mathrm{d}\eta^2} = \left[-\frac{\gamma^2}{4\eta} + \frac{m^2-1}{4\eta^2} + \frac{\hat{\psi}(\eta)}{\eta}\right]\hat{W}. \tag{3.3}$$

Here

$$\hat{\psi}(\eta)=\frac{1-4m^2}{16\eta}+\frac{m^2-1}{4(x^2-1)(x^2-\sigma^2)}+\frac{(1-\sigma^2)(6x^4-(3+\sigma^2)x^2-2\sigma^2)}{16(x^2-1)(x^2-\sigma^2)^3}. \tag{3.4}$$

This has the same main features of (2.5), namely a dominating simple pole and a not-so-dominating double pole: note $x=1$ corresponds to $\eta=0$.

The difference here is that non-dominant term $\hat{\psi}(\eta)$ is now analytic at $\eta=0$, i.e. $x=1$. Neglecting $\hat{\psi}(\eta)$ in (3.3) gives an equation solvable in terms of Bessel functions. We then find (by matching recessive solutions at $x=1$) and applying theorem 4.1 of [6, Chap. 12] (with u replaced by γ and ζ replaced by η)

$$Ps_n^m(x,\gamma^2)=\hat{c}_n^m(\gamma)\left\{\frac{\eta}{(x^2-1)(x^2-\sigma^2)}\right\}^{1/4}\left[J_m(\gamma\eta^{1/2})+O(\gamma^{-1})\mathrm{env}\,J_m(\gamma\eta^{1/2})\right], \tag{3.5}$$

as $\gamma\to\infty$, uniformly for $1<x<\infty$. Olver's theorem provides an asymptotic expansion in inverse powers of γ, but we present just the leading term here. The so-called envelope env of the J Bessel function is defined by [7, §2.8].

The constant of proportionality $c_n^m(\gamma)$ can be found by comparing both sides of (3.5) as $x\to 1$ ($\eta\to 0$). Using

$$\eta=2(1-\sigma^2)(x-1)+O\{(x-1)^2\}\quad (x\to 1), \tag{3.6}$$

along with (1.22), (2.3) and the behavior of the J Bessel function at the origin (e.g. [6, Chap. 12, §1]) we find that

$$c_n^m(\gamma)=\left(-\frac{2}{\lambda_n^m(\gamma^2)}\right)^{m/2} m!K_n^m(\gamma^2)\ . \tag{3.7}$$

Next, from the well-known behavior of $J_m(z)$ at infinity (e.g. see [6, Chap. 12, §1]), we find from (3.5) that

$$\begin{aligned} &Ps_n^m\left(x,\gamma^2\right) \sim \text{constant} \times \left\{\left(x^2-1\right)\left(x^2-\sigma^2\right)\right\}^{-1/4} \\ &\times \left\{\cos\left(\gamma\xi - \frac{1}{2}m\pi - \frac{1}{4}\pi\right) + O\left(\frac{1}{\xi}\right)\right\} \quad \left(\eta = \xi^2 \to \infty\right) \end{aligned}. \tag{3.8}$$

But we found out in the previous section the behavior (2.22). Comparing these gives an approximation of the form

$$\gamma\sigma E\left(\sigma;\sigma^{-1}\right) = \left(2N + \tfrac{1}{2}n - \tfrac{1}{2}m + \tfrac{1}{4}\right)\pi + O\left(\gamma^{-1}\right), \tag{3.9}$$

for some integer *N*. It can be shown that $N = 0$. We can extend this to asymptotic expansions, and we arrive at

$$\gamma \int_0^\sigma \left(\frac{\sigma^2 - t^2}{1 - t^2}\right)^{1/2} dt \sim \frac{1}{2}\left(n - m + \frac{1}{2}\right)\pi + \sum_{s=0}^{\infty} \frac{\kappa_s}{\gamma^{2s+1}}. \tag{3.10}$$

Eqs. (2.4) and (3.10) provide an means of computing the eigenvalue $\lambda = \lambda_n^m\left(\gamma^2\right)$ asymptotically in terms of *m* and *n* as $\gamma \to \infty$.

4. Angular prolate spheroidal wave functions: $0 \le x < 1$**.**

Recall $\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ is the unique solution with the property of being recessive at $x = \pm 1$. It is also uniquely determined by the property

$$\mathrm{Ps}_n^m\left(-x,\gamma^2\right) = \left(-1\right)^{m+n} \mathrm{Ps}_n^m\left(x,\gamma^2\right). \tag{4.1}$$

Thus, it suffices to approximate $\mathrm{P}s_n^m\left(x,\gamma^2\right)$ in the interval $0 \le x < 1$. We consider the subintervals $0 \le x \le 1 - \delta$ and $1 - \delta \le x < 1$ separately, where $\delta \in \left(0, 1 - \sigma_0\right)$ is arbitrary, where σ_0 is defined by (2.4).

For $1-\delta \le x < 1$ we apply theorem 3.1 of [6, Chap.12]. It can then be shown that by utilizing the recessive behavior at $x=1$ ($\eta = 0$)

$$\mathrm{Ps}_n^m\left(x,\gamma^2\right) = C\left\{\frac{|\eta|}{\left(1-x^2\right)\left(x^2-\sigma^2\right)}\right\}^{1/4} I_m\left(\gamma|\eta|^{1/2}\right)\left[1+O\left(\frac{1}{\gamma}\right)\right], \tag{4.2}$$

where C is a constant that can be determined by comparing both sides as $x \to 1$ ($\eta \to 0$): we shall determine C by an alternative method below. Expansions and error bounds are obtainable from Olver's theorem.

The interval $0 \le x \le 1-\delta$ is trickier. From (2.5) and (2.6) we observe that equation has two turning points in this interval, which coalesce when $\alpha \to 0$. The appropriate asymptotic theory is provided by [5], and from eq. (2.3) of this reference the appropriate transformation is given by

$$\frac{d\zeta}{dx} = \left(\frac{\sigma^2 - x^2}{\left(1-x^2\right)\left(\alpha^2-\zeta^2\right)}\right)^{1/2}, \tag{4.3}$$

which, upon integration, yields

$$\int_{-\alpha}^{\zeta}\left(\alpha^2-\tau^2\right)^{1/2} d\tau = \int_{-\sigma}^{x}\left\{-f(\sigma,t)\right\}^{1/2} dt = \int_{-\sigma}^{x}\left(\frac{\sigma^2-t^2}{1-t^2}\right)^{1/2} dt\,. \tag{4.4}$$

The lower limits ensure that the turning point $x=-\sigma$ is mapped to a new turning point at $\zeta = -\alpha$. From [5, eq. (2.5)] α is defined by

$$\alpha^2 = \frac{2}{\pi}\int_{-\sigma}^{\sigma}\left(\frac{\sigma^2-t^2}{1-t^2}\right)^{1/2} dt = \frac{4}{\pi}J(\sigma), \tag{4.5}$$

which ensures that the other old turning point $x=\sigma$ is mapped to the new one at $\zeta=\alpha$. By symmetry $x=0$ is mapped to $\zeta = 0$, and so the lower limits in the integrals of (4.4) can be replaced by 0. Thus we have

$$\frac{1}{2}\alpha^2 \arcsin\left(\frac{\zeta}{\alpha}\right)+\frac{1}{2}\zeta\left(\alpha^2-\zeta^2\right)^{1/2}=\sigma E\left(x;\sigma^{-1}\right), \tag{4.6}$$

for $0 \le x \le \sigma$ $(0 \le \zeta \le \alpha)$. For $\sigma \le x \le 1-\delta$ we have

$$\int_\alpha^\zeta \left(\tau^2-\alpha^2\right)^{1/2} d\tau = \int_\sigma^x \left\{f(\sigma,t)\right\}^{1/2} dt = \int_\sigma^x \left(\frac{t^2-\sigma^2}{1-t^2}\right)^{1/2} dt \,. \tag{4.7}$$

Thus

$$-\frac{1}{2}\alpha^2 \operatorname{arccosh}\left(\frac{\zeta}{\alpha}\right)+\frac{1}{2}\zeta\left(\zeta^2-\alpha^2\right)^{1/2}=\left|\operatorname{Im}\left\{\sigma E\left(x;\sigma^{-1}\right)\right\}\right|. \tag{4.8}$$

With

$$W=\left\{\frac{\sigma^2-x^2}{\left(\alpha^2-\zeta^2\right)\left(1-x^2\right)}\right\}^{1/4} w\,, \tag{4.9}$$

we transform (2.5) to the form

$$\frac{d^2W}{d\zeta^2}=\left\{\gamma^2\left(\zeta^2-\alpha^2\right)+\psi(\gamma,\alpha,\zeta)\right\}W\,, \tag{4.10}$$

where

$$\begin{aligned}\psi(\gamma,\alpha,\zeta)=&\frac{\left(1-m^2\right)\left(\alpha^2-\zeta^2\right)}{\left(1-x^2\right)\left(\sigma^2-x^2\right)}+\frac{2\alpha^2+3\zeta^2}{4\left(\alpha^2-\zeta^2\right)^2}\\&-\frac{\left(1-\sigma^2\right)\left(\alpha^2-\zeta^2\right)\left\{6x^4-\left(\sigma^2+3\right)x^2-2\sigma^2\right\}}{4\left(1-x^2\right)\left(\sigma^2-x^2\right)^3}.\end{aligned} \tag{4.11}$$

To sharpen the subsequent error bounds we could perturb the parameter $\alpha^2=\omega^2+\psi(\gamma,\alpha,0)\gamma^{-2}$, but we won't pursue this here.

From theorem I of [6], with *u* replaced by γ, we obtain two independent solutions of (4.10) given by

$$w_1(\gamma,\alpha,\zeta) = U\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right) + \varepsilon_1(\gamma,\alpha,\zeta), \tag{4.12}$$

and

$$w_2(\gamma,\alpha,\zeta) = \overline{U}\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right) + \varepsilon_2(\gamma,\alpha,\zeta). \tag{4.13}$$

The error terms are bounded by [5], and in particular these show that

$$\varepsilon_1(\gamma,\alpha,\zeta) = O\left(\ln(\gamma)\gamma^{-2/3}\right)\mathrm{env}U\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right), \tag{4.14}$$

and

$$\varepsilon_2(\gamma,\alpha,\zeta) = O\left(\ln(\gamma)\gamma^{-2/3}\right)\mathrm{env}\overline{U}\left(-\tfrac{1}{2}\gamma\alpha^2, \zeta\sqrt{2\gamma}\right), \tag{4.15}$$

uniformly for $0 \le x \le 1-\delta$. Here the envelope function env is defined for the parabolic cylinder functions by [7, eq. (14.15.23)].

The parabolic cylinder function U has the unique recessive property

$$U\left(-\tfrac{1}{2}a, x\right) \sim x^{(a-1)/2} e^{-x^2/4} \quad (x \to \infty), \tag{4.16}$$

whereas $\overline{U}$ is dominant, with the behavior

$$\overline{U}\left(-\tfrac{1}{2}a, x\right) \sim (2/\pi)^{1/2}\,\Gamma\left(\tfrac{1}{2}a + \tfrac{1}{2}\right) x^{-(a+1)/2} e^{x^2/4} \quad (x \to \infty). \tag{4.17}$$

Now, for negative x and ζ, we will also need the solution given in [5]

$$w_4(\gamma,\alpha,\zeta) = \overline{U}\left(-\tfrac{1}{2}\gamma\alpha^2, -\zeta\sqrt{2\gamma}\right) + \varepsilon_4(\gamma,\alpha,\zeta). \tag{4.18}$$

We remark that

$$\varepsilon_j(\gamma,\alpha,0) = \partial\varepsilon_j(\gamma,\alpha,0)/\partial\zeta = 0 \quad (j = 2,4), \tag{4.19}$$

and hence

$$w_2(\gamma,\alpha,0)=w_4(\gamma,\alpha,0)=\bar{U}\left(-\tfrac{1}{2}\gamma\alpha^2,0\right), \tag{4.20}$$

as well as

$$\partial w_2(\gamma,\alpha,0)/\partial\zeta=-\partial w_4(\gamma,\alpha,0)/\partial\zeta=\sqrt{2\gamma}\,\bar{U}'\left(-\tfrac{1}{2}\gamma\alpha^2,0\right). \tag{4.21}$$

The error bounds for $\varepsilon_4(\gamma,\alpha,\zeta)$ only apply for non-positive ζ. In order to extend the solution to positive values of ζ we use (6.23) and (6.24) of [5] to obtain the connection formula

$$\begin{aligned} w_4(\gamma,\alpha,\zeta)=&-\left\{\sin\left(\tfrac{1}{2}\pi\gamma\alpha^2\right)+O\left(\gamma^{-2/3}\right)\right\}w_2(\gamma,\alpha,\zeta)\\ &+\left\{\cos\left(\tfrac{1}{2}\pi\gamma\alpha^2\right)+O\left(\gamma^{-2/3}\right)\right\}w_1(\gamma,\alpha,\zeta). \end{aligned} \tag{4.22}$$

Now from (2.17), (3.9) and (4.5)

$$\tfrac{1}{2}\pi\gamma\alpha^2=\left(n-m+\tfrac{1}{2}\right)\pi+O\left(\gamma^{-1}\right). \tag{4.23}$$

Bearing in mind that $w_1(\gamma,\alpha,\zeta)$ is exponentially small compared to $w_2(\gamma,\alpha,\zeta)$ in $0\le x\le 1-\delta$ (except near its zeros) we deduce from (4.22) and (4.23) that

$$w_2(\gamma,\alpha,\zeta)-(-1)^{m+n}w_4(\gamma,\alpha,\zeta)=2w_2(\gamma,\alpha,\zeta)+O\left(\gamma^{-2/3}\right)\left\{w_1(\gamma,\alpha,\zeta)+w_2(\gamma,\alpha,\zeta)\right\}. \tag{4.24}$$

We thus express

$$\mathrm{Ps}_n^m\left(x,\gamma^2\right)=\left\{\frac{\alpha^2-\zeta^2}{\left(\sigma^2-x^2\right)\left(1-x^2\right)}\right\}^{1/4}\left[Aw_1(\gamma,\alpha,\zeta)+\frac{1}{2}B\left\{w_2(\gamma,\alpha,\zeta)-(-1)^{m+n}w_4(\gamma,\alpha,\zeta)\right\}\right]. \tag{4.25}$$

Firstly we assume that $\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ is even, so that $m+n=2N$ for some integer *N*. Then, setting $x=\zeta=0$ in (4.25), and invoking (4.20), yields

$$A=\left(\frac{\sigma}{\alpha}\right)^{1/2}\frac{\mathrm{Ps}_n^m\left(0,\gamma^2\right)}{w_1(\gamma,\alpha,0)}. \tag{4.26}$$

If we try to find B from the property $\mathrm{Ps}_n^{m\prime}\left(0,\gamma^2\right)=0$ we only get $B=O\left(\gamma^{-1}\right)$ which is not sharp enough. Instead we match the parabolic cylinder and Bessel function approximations, and their derivatives, at an arbitrary fixed point x lying in $\left(1-\frac{1}{2}\delta,1\right)$. We then get using (4.2) (4.24) and (4.25)

$$B\sim -A\frac{\mathscr{W}\left\{\left(\zeta^2-\alpha^2\right)^{1/4}U\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right),|\eta|^{1/4}I_m\left(\gamma|\eta|^{1/2}\right)\right\}}{\mathscr{W}\left\{\left(\zeta^2-\alpha^2\right)^{1/4}\overline{U}\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right),|\eta|^{1/4}I_m\left(\gamma|\eta|^{1/2}\right)\right\}}, \tag{4.27}$$

and

$$C\sim A\frac{\mathscr{W}\left\{\left(\zeta^2-\alpha^2\right)^{1/4}\overline{U}\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right),\left(\zeta^2-\alpha^2\right)^{1/4}U\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right)\right\}}{\mathscr{W}\left\{\left(\zeta^2-\alpha^2\right)^{1/4}\overline{U}\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right),|\eta|^{1/4}I_m\left(\gamma|\eta|^{1/2}\right)\right\}}. \tag{4.28}$$

Here the Wronskians are with respect to x, and evaluated at the arbitrary point in $\left(1-\frac{1}{2}\delta,1\right)$.

From [7, eqs. (12.10.3) – (12.10.6)] we have the asymptotic approximations for fixed $\zeta\in(\alpha,\infty)$ and fixed $\alpha>0$

$$U\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right)\sim\left(\frac{\gamma\alpha^2}{2e}\right)^{\gamma\alpha^2/4}\frac{\exp\left\{-\gamma\int_\sigma^x\{f(\sigma,t)\}^{1/2}dt\right\}}{\left\{2\gamma\left(\zeta^2-\alpha^2\right)\right\}^{1/4}}, \tag{4.29}$$

$$U'\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right)\sim-\frac{1}{2}\left(\frac{\gamma\alpha^2}{2e}\right)^{\gamma\alpha^2/4}\left\{2\gamma\left(\zeta^2-\alpha^2\right)\right\}^{1/4}\exp\left\{-\gamma\int_\sigma^x\{f(\sigma,t)\}^{1/2}dt\right\}, \tag{4.30}$$

$$\overline{U}\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right)\sim 2\left(\frac{\gamma\alpha^2}{2e}\right)^{\gamma\alpha^2/4}\frac{\exp\left\{\gamma\int_\sigma^x\{f(\sigma,t)\}^{1/2}dt\right\}}{\left\{2\gamma\left(\zeta^2-\alpha^2\right)\right\}^{1/4}}, \tag{4.31}$$

and

$$\overline{U}'\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right)\sim\left(\frac{\gamma\alpha^2}{2e}\right)^{\gamma\alpha^2/4}\left\{2\gamma\left(\zeta^2-\alpha^2\right)\right\}^{1/4}\exp\left\{\gamma\int_\sigma^x\{f(\sigma,t)\}^{1/2}dt\right\}. \tag{4.32}$$

These, along with

$$\frac{d\zeta}{dx}=\left(\frac{\sigma^2-x^2}{\left(1-x^2\right)\left(\alpha^2-\zeta^2\right)}\right)^{1/2}, \quad \frac{d\eta}{dx}=2\left(\frac{\eta\left(\sigma^2-x^2\right)}{1-x^2}\right)^{1/2}, \tag{4.33}$$

and

$$I_m(x)\sim\frac{e^x}{(2\pi x)^{1/2}} \quad (x\to\infty), \tag{4.34}$$

can be used to simplify (4.27) and (4.28). In particular, we find that

$$B\left\{w_2(\gamma,\alpha,\zeta)-w_4(\gamma,\alpha,\zeta)\right\}=o(1)A\,\mathrm{env}U\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right), \tag{4.35}$$

where $o(1)$ is exponentially small for $0\le x\le 1-\delta$ as $\gamma\to\infty$. We arrive at our desired result, for $m+n$ even,

$$\begin{aligned}\mathrm{Ps}_n^m\left(x,\gamma^2\right)&=\frac{\mathrm{Ps}_n^m\left(0,\gamma^2\right)}{U\left(-\frac{1}{2}\gamma\alpha^2,0\right)}\left\{\frac{\sigma^2\left(\alpha^2-\zeta^2\right)}{\alpha^2\left(\sigma^2-x^2\right)\left(1-x^2\right)}\right\}^{1/4}\\ &\times\left\{U\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right)+O\left(\ln(\gamma)\gamma^{-2/3}\right)\mathrm{env}U\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right)\right\},\end{aligned} \tag{4.36}$$

as $\gamma\to\infty$, uniformly for $0\le x\le 1-\delta$.

For the case $\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ odd, so that $m+n=2N+1$, we differentiate both sides of

$$\left\{\frac{\left(\sigma^2-x^2\right)\left(1-x^2\right)}{\alpha^2-\zeta^2}\right\}^{1/4}\mathrm{Ps}_n^m\left(x,\gamma^2\right)=\left[Aw_1(\gamma,\alpha,\zeta)+\frac{1}{2}B\left\{w_2(\gamma,\alpha,\zeta)+w_4(\gamma,\alpha,\zeta)\right\}\right], \tag{4.37}$$

with respect to ζ, and then set $x=\zeta=0$. As a result, on referring to (4.3) and (4.21), along with the fact that $\mathrm{Ps}_n^m\left(0,\gamma^2\right)=0$, we obtain

$$A=\left(\frac{\alpha}{\sigma}\right)^{1/2}\frac{\mathrm{Ps}_n^{m\prime}\left(0,\gamma^2\right)}{\partial w_1(\gamma,\alpha,0)/\partial\zeta}. \tag{4.38}$$

We then similarly find that (4.28) again is valid, and that the term $\frac{1}{2}B\left\{w_2(\gamma,\alpha,\zeta)+w_4(\gamma,\alpha,\zeta)\right\}$ in (4.37) can be neglected, and hence for $m+n$ odd

$$\begin{aligned}\mathrm{Ps}_n^m\left(x,\gamma^2\right)=&\frac{\mathrm{Ps}_n^{m\prime}\left(0,\gamma^2\right)}{U'\left(-\frac{1}{2}\gamma\alpha^2,0\right)}\left\{\frac{\alpha^2\left(\alpha^2-\zeta^2\right)}{4\gamma^2\sigma^2\left(\sigma^2-x^2\right)\left(1-x^2\right)}\right\}^{1/4}\\&\times\left\{U\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right)+O\left(\ln(\gamma)\gamma^{-2/3}\right)\mathrm{env}U\left(-\tfrac{1}{2}\gamma\alpha^2,\zeta\sqrt{2\gamma}\right)\right\},\end{aligned} \tag{4.39}$$

as $\gamma\to\infty$, uniformly for $0\le x\le 1-\delta$.

For fixed *m and n* we can simplify the above results, by applying the theory of [3]. To this end we observe that (2.1) can be expressed in the form

$$\frac{d^2w}{dx^2}=\left[\frac{\gamma^2x^2}{1-x^2}-\frac{a\gamma}{1-x^2}+\frac{m^2-1}{\left(1-x^2\right)^2}\right]w, \tag{4.40}$$

where

$$a=\lambda\gamma^{-1}+\gamma=2\left(n-m+\tfrac{1}{2}\right)+O\left(\gamma^{-1}\right), \tag{4.41}$$

the $O\left(\gamma^{-1}\right)$ term being valid for fixed *m* and *n* and $\gamma\to\infty$. In particular, *a* is bounded.

Equation (4.40) is characterized as having a pair of almost coalescent turning points near $x=0$. We make the Liouville transformation

$$\frac{1}{2}\zeta^2=\int_0^x\frac{t}{\left(1-t^2\right)^{1/2}}dt=1-\left(1-x^2\right)^{1/2}. \tag{4.42}$$

Here ζ differs from the variable defined by (4.3). Note $x=0$ corresponds to $\zeta=0$, and $x=1$ corresponds to $\zeta=\sqrt{2}$. Then with

$$W=\frac{x^{1/2}}{\zeta^{1/2}\left(1-x^2\right)^{1/4}}w, \tag{4.43}$$

we get

$$\frac{d^2W}{d\zeta^2} = \left[\gamma^2\zeta^2 - \gamma a + \gamma\zeta\phi(\zeta) + \psi(\zeta)\right]W \ , \tag{4.44}$$

where

$$\phi(\zeta) = -\frac{a\zeta}{4-\zeta^2} \ , \tag{4.45}$$

and

$$\psi(\zeta) = \frac{\zeta^2\left(4m^2-1\right)}{\left(2-\zeta^2\right)^2} + \frac{7\zeta^2-40}{4\left(4-\zeta^2\right)^2} + \frac{4m^2}{\left(4-\zeta^2\right)} , \tag{4.46}$$

with $\psi(\zeta) = O(1)$ as $\gamma \to \infty$. Remark: $\psi(\zeta)$ is analytic at $\zeta = 0$ ($x = 0$), but not at $\zeta = \sqrt{2}$ $(x = 1)$.

Our approximants are the numerically satisfactory pair of parabolic cylinder functions $U\left(-\tfrac{1}{2}a, (2\gamma)^{1/2}\zeta\right)$ and $\bar{U}\left(-\tfrac{1}{2}a, (2\gamma)^{1/2}\zeta\right)$, which are solutions of

$$\frac{d^2W}{d\zeta^2} = \left[\gamma^2\zeta^2 - \gamma a\right]W \ . \tag{4.47}$$

Note the extra $\gamma\zeta\phi(\zeta)$ term in (4.44) as compared to (4.47).

On account of the pesky $\gamma\zeta\phi(\zeta)$ term in (4.44) we perturb the independent variable, thus taking as approximants

$$U_1 = \left\{1 + \gamma^{-1}\Phi'(\zeta)\right\}^{-1/2} U\left(-\tfrac{1}{2}a, (2\gamma)^{1/2}\hat{\zeta}\right), \tag{4.48}$$

and

$$U_2 = \left\{1 + \gamma^{-1}\Phi'(\zeta)\right\}^{-1/2} \bar{U}\left(-\tfrac{1}{2}a, (2\gamma)^{1/2}\hat{\zeta}\right), \tag{4.49}$$

where

$$\hat{\zeta} = \zeta + \gamma^{-1}\Phi(\zeta), \tag{4.50}$$

in which $\Phi(\zeta)$ is at our disposal. Then U_j satisfy

$$\frac{d^2U}{d\zeta^2} = \left\{\gamma^2\zeta^2 - \gamma a + 2\gamma\zeta\left\{\zeta\Phi' + \Phi\right\} + \hat{\psi}(\gamma,\zeta)\right\}U\ , \tag{4.51}$$

where $\hat{\psi}(\gamma,\zeta) = O(1)$ as $\gamma \to \infty$.

We want (4.51) to match the large terms in the transformed equation

$$\frac{d^2W}{d\zeta^2} = \left[\gamma^2\zeta^2 - \gamma a + \gamma\zeta\phi(\zeta) + \psi(\zeta)\right]W\ , \tag{4.52}$$

We thus choose $2\zeta\left\{\zeta\Phi' + \Phi\right\} = \zeta\phi$, which gives

$$\Phi(\zeta) = \frac{1}{2\zeta}\int_0^\zeta \phi(v)\,\mathrm{d}v = \frac{a\ln\left(1 - \frac{1}{4}\zeta^2\right)}{4\zeta}. \tag{4.53}$$

Following [3] we now define

$$w_j(\gamma,\zeta) = U_j(\gamma,\zeta) + \varepsilon_j(\gamma,\zeta), \tag{4.54}$$

as exact solutions of (4.52). From [3] we have

$$\varepsilon_1(\gamma,\zeta) = O\left(\ln(\gamma)\gamma^{-1}\right)\mathrm{env}U\left(-\tfrac{1}{2}a,(2\gamma)^{1/2}\hat{\zeta}\right), \tag{4.55}$$

uniformly for $0 \le x \le 1 - \delta$, and similarly for $\varepsilon_2(\gamma,\zeta)$.

Let us assume that $\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ is even (the case odd is done similarly), so that $m + n = 2N$ for some integer *N*. Similarly to (4.25) we write

$$\mathrm{Ps}_n^m\left(x,\gamma^2\right)=\zeta^{1/2}x^{-1/2}\left(1-x^2\right)^{-1/4}\left[Aw_1(\gamma,\zeta)+B\left\{w_2(\gamma,\zeta)-w_4(\gamma,\zeta)\right\}\right], \tag{4.56}$$

where $w_4(\gamma,\zeta)$ is the solution (involving $\bar{U}$) given by eq. (110) of [3]. By matching at $x=0$ we find

$$A=\frac{\mathrm{Ps}_n^m\left(0,\gamma^2\right)}{w_1(\gamma,0)}. \tag{4.57}$$

Again, if we try to find *B* from the property $\mathrm{Ps}_n^{m\prime}\left(0,\gamma^2\right)=0$ we only get $B=O\left(\gamma^{-1}\right)$ which is too crude. Instead we match the parabolic cylinder and Bessel function approximations, and their derivatives, at an arbitrary fixed *x* value in $(1-\delta,1)$. We then get using (4.2)

$$B\sim -A\frac{\mathscr{W}\left\{U\left(-\tfrac{1}{2}a,(2\gamma)^{1/2}\hat{\zeta}\right),\eta^{1/4}I_m\left(\gamma|\eta|^{1/2}\right)\right\}}{\mathscr{W}\left\{\bar{U}\left(-\tfrac{1}{2}a,(2\gamma)^{1/2}\hat{\zeta}\right),\eta^{1/4}I_m\left(\gamma|\eta|^{1/2}\right)\right\}}, \tag{4.58}$$

where the Wronskians are with respect to *x*. From this we find that

$$B\left\{w_2(\gamma,\zeta)-w_4(\gamma,\zeta)\right\}=o(1)A\,\mathrm{env}U\left(-\tfrac{1}{2}a,(2\gamma)^{1/2}\hat{\zeta}\right), \tag{4.59}$$

where $o(1)$ is exponentially small for $0\le x\le 1-\delta$ as $\gamma\to\infty$.

We arrive at our desired result

$$\begin{aligned}\mathrm{Ps}_n^m\left(x,\gamma^2\right)&=\frac{\mathrm{Ps}_n^m\left(0,\gamma^2\right)}{U\left(-\tfrac{1}{2}a,0\right)}\zeta^{1/2}x^{-1/2}\left(1-x^2\right)^{-1/4}\left(\frac{d\hat{\zeta}}{d\zeta}\right)^{-1/2}\\&\times\left[U\left(-\tfrac{1}{2}a,(2\gamma)^{1/2}\hat{\zeta}\right)+O\left(\gamma^{-1}\right)\mathrm{env}U\left(-\tfrac{1}{2}a,(2\gamma)^{1/2}\hat{\zeta}\right)\right],\end{aligned} \tag{4.60}$$

as $\gamma\to\infty$, uniformly for $0\le x\le 1-\delta$.

For the case $\mathrm{Ps}_n^m\left(x,\gamma^2\right)$ odd we similarly obtain, under the same conditions,

$$\mathrm{Ps}_n^m\left(x,\gamma^2\right)=\frac{\mathrm{Ps}_n^{m\prime}\left(0,\gamma^2\right)}{U'\left(-\tfrac{1}{2}a,0\right)}\zeta^{1/2}x^{-1/2}\left(1-x^2\right)^{-1/4}\left(\frac{d\hat{\zeta}}{d\zeta}\right)^{-1/2}$$
$$\times\left[U\left(-\tfrac{1}{2}a,(2\gamma)^{1/2}\hat{\zeta}\right)+O\left(\ln(\gamma)\gamma^{-1}\right)\mathrm{env}U\left(-\tfrac{1}{2}a,(2\gamma)^{1/2}\hat{\zeta}\right)\right]. \tag{4.61}$$

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