ON CHARACTERIZATIONS OF BISTOCHASTIC KADISON-SCHWARZ OPERATORS ON $M_2(\mathbb{C})$

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Abstract. In this paper we describe bistochastic Kadison-Schawrz operators acting on $M_2(\mathbb{C})$. Such a description allows us to find positive, but not Kadison-Schwarz operators. Moreover, by means of that characterization we construct Kadison-Schawrz operators, which are not completely positive.

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1. Introduction

It is known that entanglement is one of the essential features of quantum physics and is fundamental in modern quantum technologies [34]. One of the central problems in the entanglement theory is the discrimination between separable and entangled states. There are several tools which can be used for this purpose. There are many papers devoted to find a given state is separable (see [16]). The most general approach to characterize quantum entanglement uses a notion of an entanglement witness [17, 7, 44]. One of the big advantages of entanglement witness is that they provide an economic method of detection which does not need the full information about the quantum state (see for recent review [10]). Interestingly, the entanglement witnesses are deeply connected to a theory of positive maps in operator algebras [6, 9, 15]. Therefore, it would interesting to find some conditions for the positivity of given mappings. In this direction there are several papers [4, 6, 8, 9, 15, 21, 40, 41]. Therefore, it would interesting to find some conditions for the positivity of given mappings (see [21]-[25]). In the literature the most tractable maps, the completely positive mapping, have proved to be of great importance in the study of quantum system (see [11, 35, 36, 37, 42]). It is therefore of interest to study conditions stronger than positivity, but weaker than complete positivity. Such a condition is called *Kadison-Schwarz (KS) property*. Note that KS-operators no need to be completely positive. In [39] relations between n-positivity of a map ϕ and the KS property of certain map is established (see also [2]). Some ergodic properties of the Kadison-Schwarz maps were investigated in [20, 14, 38]. Unfortunately, like completely positive maps, the description of Kadison-Schwarz maps is not provided. Very recently, one of the authors of this paper in [28] has described bistochastic KS-operators from $M_2(\mathbb{C})$ to itself. But, in general, the problem still remains open.

In [13] it was proposed to study positive operators P from a von Neumann algebra M to its tensor square $M \otimes M$ (we refer a reader to [12, 33] for recent review on quadratic operators). It turns out that this kind of mappings have some applications to quantum information theory. One of such an application is to detect entangled states. For example, let P be a block positive, then a state ϕ on the algebra $M \otimes M$ is separable, then the state $P_*\phi$ is positive. If ϕ is entangled, then $P_*\phi$ may not be positive. This observation leads to more investigation of operators from M to $M \otimes M$. In general, description of this kind of mappings was fully not studied yet. Some positivity conditions were found in [21, 24]. In [30, 27] it was considered trace preserving mappings from $M_2(\mathbb{C})$ to $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$, and each such kind of mappings can be written as a sum of two "linear" and "nonlinear" operators (see (2.7)-(2.9)). In [29] mappings of the form

(2.8) have been studied. Namely, some sufficient conditions for positivity (resp. Kadison-Schwarz property) of the mentioned mappings were found.

In the present paper we are going to describe or characterize operators of the form (2.9). To do it, we first in Section 3 we provide a characterization of KS-operators form $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$ which improves the main result of [28]. In section 4, we give a sufficient condition for a class of bistochastic mappings from $M_2(\mathbb{C})$ to $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ to be KS-operator. Note that this class of operators are totally different from the operators studied in [29]. Such a description allows us to find positive, but not Kadison-Schwarz operators. Moreover, by means of that conditions one can construct KS-operators, which are not completely positive. Note that some parts of this section have been announced in [31]. Moreover, our results allow to produce higher dimensional examples of positive, but completely positive maps. The proposed approach can be extended to a more general setting rather that $M_2(\mathbb{C})$, and will produce non trivial examples of positive mappings.

2. Preliminaries

In this section we recall some definitions and notations.

Let $M_n(\mathbb{C})$ be the algebra of $n \times n$ matrices over the complex field $\mathbb{C}$. Recall that a linear mapping $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is called

(i) *positive* if $\Phi(x) \geq 0$ whenever $x \geq 0$;
(ii) *unital* if $\Phi(\mathbb{1}) = \mathbb{1}$;
(iii) *trace preserving* if $\tau(\Phi(x)) = \tau(x)$, where τ is the normalized trace;
(iv) *bistochastic* if Φ is unital and trace preserving;
(v) *n-positive* if the mapping $\Phi_n : M_n(A) \to M_n(B)$ defined by $\Phi_n(a_{ij}) = (\Phi(a_{ij}))$ is positive. Here $M_n(A)$ denotes the algebra of $n \times n$ matrices with A-valued entries;
(vi) *completely positive* if it is n-positive for all $n \in \mathbb{N}$;
(vii) *Kadison-Schwarz operator (KS-operator)*, if one has

$$\Phi(x)^*\Phi(x) \leq \Phi(x^*x) \quad \text{for all} \quad x \in A. \tag{2.1}$$

It is clear that any KS-operator is positive. Note that every unital 2- positive map is KS-operator, and a famous result of Kadison states that any positive unital map satisfies the inequality (2.1) for all self-adjoint elements $x \in A$.

By $\mathcal{KS}(M_n, M_m)$ we denote the set of all KS-operators mapping from $M_n(\mathbb{C})$ to $M_m(\mathbb{C})$.

Theorem 2.1. [28] *The following assertions hold true:*

(i) *Let $\Phi, \Psi \in \mathcal{KS}(M_n, M_m)$, then for any $\lambda \in [0, 1]$ the mapping $\Gamma = \lambda\Phi + (1 - \lambda)\Psi$ belongs to $\mathcal{KS}(M_n, M_m)$. This means $\mathcal{KS}(M_n, M_m)$ is convex;*
(ii) *Let U, V be unitaries in $M_n(\mathbb{C})$ and $M_m(\mathbb{C})$, respectively, then for any $\Phi \in \mathcal{KS}(M_n, M_m)$ the mapping $\Psi_{U,V}(x) = U\Phi(VxV^*)U^*$ belongs to $\mathcal{KS}(M_n, M_m)$.*

By $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ we mean tensor product of $M_2(\mathbb{C})$ into itself. We note that such a product can be considered as an algebra of 4×4 matrices $M_4(\mathbb{C})$ over $\mathbb{C}$. By $S(M_2(\mathbb{C}))$ we denote the set of all states (i.e. linear positive functionals which take value 1 at $\mathbb{1}$) defined on $M_2(\mathbb{C})$.

Recall that a linear operator $\Delta : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ is said to be *quantum quadratic operator (q.q.o.)* if it is unital and positive.

A state $h \in S(M_2(\mathbb{C}))$ is called *a Haar state* for a q.q.o. Δ if for every $x \in M_2(\mathbb{C})$ one has

$$(h \otimes id) \circ \Delta(x) = (id \otimes h) \circ \Delta(x) = h(x)\mathbb{1}. \tag{2.2}$$

Remark 2.2. Let $U : M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \to M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ be a linear operator such that $U(x \otimes y) = y \otimes x$ for all $x, y \in M_2(\mathbb{C})$. If a q.q.o. Δ satisfies $U\Delta = \Delta$, then Δ is called a *quantum quadratic stochastic operator* or *symmetric q.q.o.* Recent reviews on this kind of operators can be found in [12, 33]).

Recall [5] that the identity and Pauli matrices $\{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}$ form a basis for $M_2(\mathbb{C})$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this basis every matrix $x \in M_2(\mathbb{C})$ can be written as $x = w_0\mathbb{1} + \mathbf{w}\sigma$ with $w_0 \in \mathbb{C}$, $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{C}^3$, here $\mathbf{w}\sigma = w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3$.

Lemma 2.3. [40] *The following assertions hold true:*

(a) *x is self-adjoint iff $w_0, \mathbf{w}$ are reals;*
(b) *$\mathrm{Tr}(x) = 1$ iff $w_0 = 0.5$, here Tr is the trace of a matrix x;*
(c) *$x > 0$ iff $\|\mathbf{w}\| \leq w_0$, where $\|\mathbf{w}\| = \sqrt{|w_1|^2 + |w_2|^2 + |w_3|^2}$.*

Note that any state $\varphi \in S(M_2(\mathbb{C}))$ can be represented by

$$\varphi(w_0\mathbb{1} + \mathbf{w}\sigma) = w_0 + \langle \mathbf{w}, \mathbf{f} \rangle, \tag{2.3}$$

where $\mathbf{f} = (f_1, f_2, f_3) \in \mathbb{R}^3$ with $\|\mathbf{f}\| \leq 1$. Here as before $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{C}^3$. Therefore, in the sequel we will identify a state φ with a vector $\mathbf{f} \in \mathbb{R}^3$.

In what follows by τ we denote a normalized trace, i.e. $\tau(x) = \frac{1}{2}\mathrm{Tr}(x)$, $x \in M_2(\mathbb{C})$,

Let $\Delta : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ be a q.q.o. We write the operator Δ in terms of a basis in $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ formed by the Pauli matrices. Namely,

$$\Delta\mathbb{1} = \mathbb{1} \otimes \mathbb{1}; \tag{2.4}$$

$$\Delta(\sigma_i) = b_i(\mathbb{1} \otimes \mathbb{1}) + \sum_{j=1}^{3} b_{ij}^{(1)}(\mathbb{1} \otimes \sigma_j) + \sum_{j=1}^{3} b_{ij}^{(2)}(\sigma_j \otimes \mathbb{1}) + \sum_{m,l=1}^{3} b_{ml,i}(\sigma_m \otimes \sigma_l), \tag{2.5}$$

where $i = 1, 2, 3$.

In general, a description of positive operators is one of the main problems of quantum information. In the literature most tractable maps are positive and trace-preserving ones, since such maps arise naturally in quantum information theory (see [18, 19, 34, 40]). Therefore, in the sequel we shall restrict ourselves to the trace preserving q.q.o. Hence, from (2.4),(2.5) one finds

$$\Delta(x) = w_0\mathbb{1} \otimes \mathbb{1} + \mathbf{B}^{(1)}\mathbf{w} \cdot \sigma \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{B}^{(2)}\mathbf{w} \cdot \sigma + \sum_{m,l=1}^{3} \langle \mathbf{b}_{ml}, \overline{\mathbf{w}} \rangle \sigma_m \otimes \sigma_l, \tag{2.6}$$

where $x = w_0 + \mathbf{w}\sigma$, $\mathbf{b}_{ml} = (b_{ml,1}, b_{ml,2}, b_{ml,3})$, and $\mathbf{B}^{(k)} = (b_{ij}^{(k)})_{i,j=1}^{3}$, $k = 1, 2$.

In general, to find some conditions for Δ to be KS-operator, is a tricky job. Therefore, one can rewrite (2.6) as follows

$$\Delta(x) = \lambda\Delta_1(x) + (1 - \lambda)\Delta_2(x), \tag{2.7}$$

where

$$\Delta_1(x) = w_0\mathbb{1} \otimes \mathbb{1} + \frac{1}{\lambda} \sum_{m,l=1}^{3} \langle \mathbf{b}_{ml}, \overline{\mathbf{w}} \rangle \sigma_m \otimes \sigma_l, \tag{2.8}$$

$$\Delta_2(x) = w_0\mathbb{1} \otimes \mathbb{1} + \frac{1}{1 - \lambda}\left(\mathbf{B}^{(1)}\mathbf{w} \cdot \sigma \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{B}^{(2)}\mathbf{w} \cdot \sigma\right). \tag{2.9}$$

In [29, 32] we have studied q.q.o. of the form (2.8). It is found necessary conditions for (2.8) kind of operators to be KS-operator. But operators of the form (2.9) has not been studied yet. Therefore, main aim of this paper to find some conditions on operators (2.9) to be Kadison-Schwarz. Then using Theorem 2.1 and our findings with the results of [32], we can find sufficient conditions for (2.7) to be KS-operator.

3. Kadison-Schwarz operators from $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$

To investigate operators of the form (2.9) (see section 4) we need some preliminary facts from [28]. In this section we collect some of them, and improve a main result of [28].

It is known that every $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ linear mapping can also be represented in this basis by a unique 4×4 matrix $\mathbf{F}$. It is trace preserving if and only if $\mathbf{F} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{t} & T \end{pmatrix}$ where T is a 3×3 matrix and $\mathbf{0}$ and $\mathbf{t}$ are row and column vectors, respectively, so that

$$\Phi(w_0 \mathbb{1} + \mathbf{w} \cdot \sigma) = w_0 \mathbb{1} + (w_0 \mathbf{t} + T\mathbf{w}) \cdot \sigma. \tag{3.1}$$

When Φ is also positive then it maps the subspace of self-adjoint matrices of $M_2(\mathbb{C})$ into itself, which implies that T is real. A linear mapping Φ is unital if and only if $t = 0$. So, in this case we have

$$\Phi(w_0 \mathbb{1} + \mathbf{w} \cdot \sigma) = w_0 \mathbb{1} + (T\mathbf{w}) \cdot \sigma. \tag{3.2}$$

Hence, any bistochastic mapping $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ has a form (3.2). In [28] it has been given a characterization bistochastic KS-operators, i.e. the following

Theorem 3.1. [28] *Any bistochastic mapping $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is KS-operator if and only if one has*

$$\|T\mathbf{w}\| \leq \|\mathbf{w}\|, \quad T\overline{\mathbf{w}} = \overline{T\mathbf{w}} \tag{3.3}$$

$$\left\| T[\mathbf{w}, \overline{\mathbf{w}}] - \big[T\mathbf{w}, \overline{T\mathbf{w}}\big] \right\| \leq \|\mathbf{w}\|^2 - \|T\mathbf{w}\|^2 \tag{3.4}$$

for all $\mathbf{w} \in \mathbb{C}^3$.

Let Φ be a bistochastic KS-operator on $M_2(\mathbb{C})$, then it can be represented by (3.2). Following [18] let us decompose the matrix T as follows $T = RS$, here R is a rotation and S is a self-adjoint matrix (see [18]). Define a mapping Φ_S as follows

$$\Phi_S(w_0 \mathbb{1} + \mathbf{w} \cdot \sigma) = w_0 \mathbb{1} + (S\mathbf{w}) \cdot \sigma. \tag{3.5}$$

Every rotation is implemented by a unitary matrix in $M_2(\mathbb{C})$, therefore there is a unitary $U \in M_2(\mathbb{C})$ such that

$$\Phi(x) = U \Phi_S(x) U^*, \quad x \in M_2(\mathbb{C}). \tag{3.6}$$

On the other hand, every self-adjoint operator S can be diagonalized by some unitary operator, i.e. there is a unitary $V \in M_2(\mathbb{C})$ such that $S = V D_{\lambda_1,\lambda_2,\lambda_3} V^*$, where

$$D_{\lambda_1,\lambda_2,\lambda_3} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \tag{3.7}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$.

Consequently, the mapping Φ can be represented by

$$\Phi(x) = \tilde{U} \Phi_{D_{\lambda_1,\lambda_2,\lambda_3}}(x) \tilde{U}^*, \quad x \in M_2(\mathbb{C}) \tag{3.8}$$

for some unitary $\tilde{U}$. Due to Theorem 2.1 the mapping $\Phi_{D_{\lambda_1,\lambda_2,\lambda_3}}$ is also KS-operator. Hence, all bistochastic KS-operators can be characterized by $\Phi_{D_{\lambda_1,\lambda_2,\lambda_3}}$ and unitaries. In what follows, for the sake of shortness by $\Phi_{(\lambda_1,\lambda_2,\lambda_3)}$ we denote the mapping $\Phi_{D_{\lambda_1,\lambda_2,\lambda_3}}$. It is clear to observe from (3.3) that $|\lambda_k| \leq 1, k = 1, 2, 3$.

In [40] it has been given a characterization of completely positivity of $\Phi_{(\lambda_1,\lambda_2,\lambda_3)}$.

Using Theorem 2.1 we are going to characterize KS-operators of the form $\Phi_{(\lambda_1,\lambda_2,\lambda_3)}$.

Theorem 3.2. *$\Phi_{(\lambda_1,\lambda_2,\lambda_3)}$ is a KS-operator if and only if the following inequalities are satisfied:*

$$(1+\lambda_1^2)(3+\lambda_2^2+\lambda_3^2-\lambda_1^2) \le 4(1+\lambda_1\lambda_2\lambda_3); \tag{3.9}$$

$$(1+\lambda_2^2)(3+\lambda_1^2+\lambda_3^2-\lambda_2^2) \le 4(1+\lambda_1\lambda_2\lambda_3); \tag{3.10}$$

$$(1+\lambda_3^2)(3+\lambda_1^2+\lambda_2^2-\lambda_3^2) \le 4(1+\lambda_1\lambda_2\lambda_3). \tag{3.11}$$

where $\lambda_1, \lambda_2, \lambda_3 \in [-1,1]$.

Proof. 'only if' part. Using simple calculation from (3.4) of Theorem 3.1 with $T = D_{\lambda_1,\lambda_2,\lambda_3}$ we obtain

$$\begin{aligned} A|w_2\overline{w}_3 - w_3\overline{w}_2|^2 \quad &+ \quad B|w_1\overline{w}_3 - w_3\overline{w}_1|^2 \\ &+ \quad C|w_1\overline{w}_2 - w_2\overline{w}_1|^2 \le \big(\alpha|w_1|^2+\beta|w_2|^2+\gamma|w_3|^2\big)^2, \end{aligned} \tag{3.12}$$

where $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{C}^3$ and

$$\alpha = |1-\lambda_1^2|, \quad \beta = |1-\lambda_2^2|, \quad \gamma = |1-\lambda_3^2| \tag{3.13}$$

$$A = |\lambda_1 - \lambda_2\lambda_3|^2, \quad B = |\lambda_2 - \lambda_1\lambda_3|^2, \quad C = |\lambda_3 - \lambda_1\lambda_2|^2. \tag{3.14}$$

Due to the inequality $|2\Im(uv)| \le |u|^2 + |v|^2$, one has

$$|w_i\overline{w}_j - w_j\overline{w}_i|^2 = |2\Im(w_iw_j)|^2 \le |w_i|^4 + 2|w_i|^2|w_j|^2 + |w_j|^4 \quad (i \ne j) \tag{3.15}$$

Note that this inequality is reachable by appropriate choosing of values w_i and w_j.
Hence, we estimate LHS of (3.12) by

$$A(|w_2|^4+2|w_2|^2|w_3|^2+|w_3|^4) + B(|w_1|^4+2|w_1|^2|w_3|^2+|w_3|^4) + C(|w_1|^4+2|w_1|^2|w_2|^2+|w_2|^4)$$

Consequently, from (3.12) we derive the following one

$$\begin{aligned} &|w_1|^4(\alpha^2 - B - C) + |w_2|^4(\beta^2 - A - C) + |w_3|^4(\gamma^2 - A - B) \\ &+2|w_1|^2|w_2|^2(\alpha\beta - C) + 2|w_1|^2|w_3|^2(\alpha\gamma - B) + 2|w_2|^2|w_3|^2(\beta\gamma - A) \ge 0 \end{aligned} \tag{3.16}$$

for all $(w_1, w_2, w_3) \in \mathbb{C}^3$. It is easy to see that (3.16) is satisfied if one has

$$\begin{aligned} &\alpha^2 \ge B + C, \quad \beta^2 \ge A + C, \quad \gamma^2 \ge A + B, \\ &\alpha\beta \ge C, \quad \alpha\gamma \ge B, \quad \beta\gamma \ge A. \end{aligned}$$

Substituting above denotations (3.13),(3.14) to the last inequalities, and doing simple calculation one derives

$$(1+\lambda_1^2)(3+\lambda_2^2+\lambda_3^2-\lambda_1^2) \le 4(1+\lambda_1\lambda_2\lambda_3); \tag{3.17}$$

$$(1+\lambda_2^2)(3+\lambda_1^2+\lambda_3^2-\lambda_2^2) \le 4(1+\lambda_1\lambda_2\lambda_3); \tag{3.18}$$

$$(1+\lambda_3^2)(3+\lambda_1^2+\lambda_2^2-\lambda_3^2) \le 4(1+\lambda_1\lambda_2\lambda_3); \tag{3.19}$$

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \le 1 + 2\lambda_1\lambda_2\lambda_3. \tag{3.20}$$

where $\lambda_1, \lambda_2, \lambda_3 \in [-1,1]$.

Now we would like to show that (3.20) is an extra condition, i.e. the inequality (3.20) always satisfies when (3.17), (3.18) and (3.19) are true. Suppose that

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1 + 2\lambda_1\lambda_2\lambda_3 \tag{3.21}$$

is true. We will show that the elements of the surface do not satisfy the inequalities (3.17), (3.18) and (3.19) except for $(0,0,0)$, $(\pm1,\pm1,\pm1)$. Using simple algebra from (3.17), (3.18) and

(3.19) with (3.21) we obtain the followings

$$\begin{aligned}(1-\lambda_1^2)(\lambda_1^2-\lambda_1\lambda_2\lambda_3)\leq 0;\\ (1-\lambda_2^2)(\lambda_2^2-\lambda_1\lambda_2\lambda_3)\leq 0;\\ (1-\lambda_3^2)(\lambda_3^2-\lambda_1\lambda_2\lambda_3)\leq 0,\end{aligned}$$

where $\lambda_1,\lambda_2,\lambda_3\in[-1,1]$. Due to our assumption $\lambda_1\neq\pm1,\lambda_2\neq\pm1,\lambda_3\neq\pm1$ from the last inequalities we infer that

$$\lambda_1(\lambda_1-\lambda_2\lambda_3)\leq 0 \tag{3.22}$$

$$\lambda_2(\lambda_2-\lambda_1\lambda_3)\leq 0 \tag{3.23}$$

$$\lambda_3(\lambda_3-\lambda_1\lambda_2)\leq 0, \tag{3.24}$$

where $\lambda_1,\lambda_2,\lambda_3\in(-1,1)$. Let $\lambda_1>0$, then one gets $\lambda_1\leq\lambda_2\lambda_3$. It implies $\lambda_2>0,\lambda_3>0$ or $\lambda_2<0,\lambda_3<0$. Now assume $\lambda_2>0,\lambda_3>0$, then from (3.23) and (3.24) one gets

$$\lambda_2\leq\lambda_1\lambda_3,\quad \lambda_3\leq\lambda_1\lambda_2.$$

From $\lambda_1\leq\lambda_2\lambda_3$ and $\lambda_2\leq\lambda_1\lambda_3$ one has $\lambda_2\leq\lambda_2\lambda_3^2$. This means $1\leq\lambda_3^2$. This contradicts to our assumption.

Now let $\lambda_1>0,\lambda_2<0$ and $\lambda_3<0$, then from (3.23) and (3.24) one finds

$$\lambda_2\geq\lambda_1\lambda_3,\qquad \lambda_3\geq\lambda_1\lambda_2. \tag{3.25}$$

From (3.25) one finds $\lambda_3\geq\lambda_1^2\lambda_3$. This implies that $\lambda_1^2\geq 1$. It is again a contradiction. In case $\lambda_1<0$, using the similar argument we will get again contradiction. This implies the required assertion.

'if' part. Let (3.9)-(3.11) be satisfied. Then it implies that (3.20) is always true. This means (3.16) is satisfied. This yields (3.12), hence $\Phi_{(\lambda_1,\lambda_2,\lambda_3)}$ is a KS-operator. This completes the proof. □

Note that the proved theorem provided necessary and sufficient conditions for the mapping $\Phi_{(\lambda_1,\lambda_2,\lambda_3)}$ to be KS-operator. In [28] it was proved only sufficient conditions to be KS-operators. Therefore, the last theorem essentially improves a main result of [28]. Moreover, the last theorem allows us to construct lots of KS-operators, which are not completely positive.

4. A class of Kadison-Schwarz operators from $M_2(\mathbb{C})$ to $M_2(\mathbb{C})\otimes M_2(\mathbb{C})$

In this section we are going to provide description of operators of the form (2.9). First we need the following auxiliary

Lemma 4.1. *Let $x=w_0\mathbb{1}\otimes\mathbb{1}+\mathbf{w}\cdot\sigma\otimes\mathbb{1}+\mathbb{1}\otimes\mathbf{r}\cdot\sigma$. Then the following statements hold true:*

(i) *x is self-adjoint if and only if $w_0\in\mathbb{R}$ and $\mathbf{w},\mathbf{r}\in\mathbb{R}^3$;*

(ii) *x is positive if and only if $w_0>0$ and $\|\mathbf{w}\|+\|\mathbf{r}\|\leq w_0$.*

Proof. (i). One can see that

$$x^*=\overline{w_0}\mathbb{1}\otimes\mathbb{1}+\overline{\mathbf{w}}\cdot\sigma\otimes\mathbb{1}+\mathbb{1}\otimes\overline{\mathbf{r}}\cdot\sigma$$

So, self adjointness x implies $\overline{w_0}=w_0$, $\overline{\mathbf{w}}=\mathbf{w}$, $\overline{\mathbf{r}}=\mathbf{r}$.

(ii). Let x be self-adjoint. Then from the definition of Pauli matrices one finds

$$x=\begin{pmatrix} w_0+w_3+r_3 & w_1-iw_2 & r_1-ir_2 & 0\\ w_1+iw_2 & w_0-w_3+r_3 & 0 & r_1-ir_2\\ r_1+ir_2 & 0 & w_0+w_3-r_3 & w_1-iw_2\\ 0 & r_1+ir_2 & w_1+iw_2 & w_0-w_3-r_3\end{pmatrix}$$

It is easy to calculate that eigenvalues of last matrix are the followings

$$\lambda_1 = w_0 - \|\mathbf{r}\| + \|\mathbf{w}\|, \quad \lambda_2 = w_0 - \|\mathbf{r}\| - \|\mathbf{w}\|,$$

$$\lambda_3 = w_0 + \|\mathbf{r}\| + \|\mathbf{w}\|, \quad \lambda_4 = w_0 + \|\mathbf{r}\| - \|\mathbf{w}\|$$

So, we can conclude that x is positive if and only if the smallest eigenvalue is positive. This means $w_0 - \|\mathbf{r}\| - \|\mathbf{w}\| \geq 0$, which completes the proof. □

Now we rewrite operator (2.9) as $T : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ given by

$$T(w_0 \mathbb{1} + \mathbf{w} \cdot \sigma) = w_0 \mathbb{1} \otimes \mathbb{1} + \mathbf{A}\mathbf{w} \cdot \sigma \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{C}\mathbf{w} \cdot \sigma \tag{4.1}$$

where $\mathbf{A}, \mathbf{C}$ are linear operators on $\mathbb{C}^3$.

We first find conditions when T is positive. This is given by the following

Theorem 4.2. *The mapping T given by* (4.1) *is positive if and only if*

$$\|\mathbf{A}\mathbf{w}\| + \|\mathbf{C}\mathbf{w}\| \leq 1,$$

for all $\mathbf{w} \in \mathbb{R}^3$ with $\|\mathbf{w}\| = 1$.

Proof. Let $x = w_0 \mathbb{1} + \mathbf{w} \cdot \sigma$ be positive, i.e. $w_0 > 0$, $\|\mathbf{w}\| \leq w_0$. Without lost of generality we may assume $w_0 = 1$. Now Lemma 4.1 yields that $T(x)$ is positive if and only if $\|\mathbf{A}\mathbf{w}\| + \|\mathbf{C}\mathbf{w}\| \leq 1$. This competes the proof. □

Corollary 4.3. *Let $\mathbf{A} = \mathbf{C}$ then T is positive if and only if $\|\mathbf{A}\| \leq \frac{1}{2}$.*

Now let us turn to the Kadison-Schwarz property.

Define the following mappings

$$\Phi(x) = w_0 \mathbb{1} + 2\mathbf{A}\mathbf{w} \cdot \sigma \tag{4.2}$$

$$\Psi(x) = w_0 \mathbb{1} + 2\mathbf{C}\mathbf{w} \cdot \sigma \tag{4.3}$$

Then one finds

$$T(x) = \frac{1}{2}\bigg(\Phi(x) \otimes \mathbb{1} + \mathbb{1} \otimes \Psi(x)\bigg). \tag{4.4}$$

Theorem 4.4. *Let T be a mapping given by* (4.4). *If one has*

$$\|\mathbf{w}\|^2 - 2\|\mathbf{A}\mathbf{w}\|^2 - 2\|\mathbf{C}\mathbf{w}\|^2 \geq 0 \tag{4.5}$$

$$\|\mathbf{A}[\mathbf{w}, \overline{\mathbf{w}}] - 2[\mathbf{A}\mathbf{w}, \mathbf{A}\overline{\mathbf{w}}]\| + \|\mathbf{C}[\mathbf{w}, \overline{\mathbf{w}}] - 2[\mathbf{C}\mathbf{w}, \mathbf{C}\overline{\mathbf{w}}]\| \leq \|\mathbf{w}\|^2 - 2\|\mathbf{A}\mathbf{w}\|^2 - 2\|\mathbf{C}\mathbf{w}\|^2 \tag{4.6}$$

Then T is a Kadison-Schwarz operator.

Proof. From (4.4) one finds that

$$\begin{aligned} T(x^*x) - T(x)^*T(x) &= \frac{1}{2}\bigg(\big(\Phi(x^*x) - \Phi(x)^*\Phi(x)\big) \otimes \mathbb{1} \\ &\quad + \mathbb{1} \otimes \big(\Psi(x^*x) - \Psi(x)^*\Psi(x)\big)\bigg) \\ &\quad + \frac{1}{4}\bigg(\mathbb{1} \otimes \Psi(x) - \Phi(x) \otimes \mathbb{1}\bigg)^*\bigg(\mathbb{1} \otimes \Psi(x) - \Phi(x) \otimes \mathbb{1}\bigg). \end{aligned} \tag{4.7}$$

Now taking into account the following formula

$$x^*x = \big(|w_0|^2 + \|\mathbf{w}\|^2\big)\mathbb{1} + \big(w_0\overline{\mathbf{w}} + \overline{w_0}\mathbf{w} - i[\mathbf{w}, \overline{\mathbf{w}}]\big) \cdot \sigma$$

from (4.2) and (4.3) we have

$$\Phi(x^*x)-\Phi(x)^*\Phi(x)=\big(\|\mathbf{w}\|^2-\|2\mathbf{A}\mathbf{w}\|^2\big)\mathbb{1}-2i\big(\mathbf{A}[\mathbf{w},\overline{\mathbf{w}}]-2[\mathbf{A}\mathbf{w},\mathbf{A}\overline{\mathbf{w}}]\big)\sigma,$$
$$\Psi(x^*x)-\Psi(x)^*\Psi(x)=\big(\|\mathbf{w}\|^2-\|2\mathbf{C}\mathbf{w}\|^2\big)\mathbb{1}-2i\big(\mathbf{C}[\mathbf{w},\overline{\mathbf{w}}]-2[\mathbf{C}\mathbf{w},\mathbf{C}\overline{\mathbf{w}}]\big)\sigma.$$

Therefore, one gets

$$\begin{aligned}
&\Big(\Phi(x^*x)-\Phi(x)^*\Phi(x)\Big)\otimes\mathbb{1}+\mathbb{1}\otimes\Big(\Psi(x^*x)-\Psi(x)^*\Psi(x)\Big)\\
=\ &\Big(\big(\|\mathbf{w}\|^2-4\|\mathbf{A}\mathbf{w}\|^2\big)\mathbb{1}-2i\big(\mathbf{A}[\mathbf{w},\overline{\mathbf{w}}]-2[\mathbf{A}\mathbf{w},\mathbf{A}\overline{\mathbf{w}}]\big)\sigma\Big)\otimes\mathbb{1}\\
&+\mathbb{1}\otimes\Big(\big(\|\mathbf{w}\|^2-4\|\mathbf{C}\mathbf{w}\|^2\big)\mathbb{1}-2i\big(\mathbf{C}[\mathbf{w},\overline{\mathbf{w}}]-2[\mathbf{C}\mathbf{w},\mathbf{C}\overline{\mathbf{w}}]\big)\sigma\Big)\\
=\ &\big(2\|\mathbf{w}\|^2-4\|\mathbf{A}\mathbf{w}\|^2-4\|\mathbf{C}\mathbf{w}\|^2\big)\mathbb{1}\otimes\mathbb{1}\\
&-2i\big(\mathbf{A}[\mathbf{w},\overline{\mathbf{w}}]-2[\mathbf{A}\mathbf{w},\mathbf{A}\overline{\mathbf{w}}]\big)\sigma\otimes\mathbb{1}-\mathbb{1}\otimes 2i\big(\mathbf{C}[\mathbf{w},\overline{\mathbf{w}}]-2[\mathbf{C}\mathbf{w},\mathbf{C}\overline{\mathbf{w}}]\big)\sigma
\end{aligned}$$

According to Lemma 4.1 we conclude that the last expression is positive if and only if (4.5) and (4.6) are satisfied. Consequently, from (4.7) we infer that under the last conditions the mapping T is a KS operator. This completes the proof. □

We should stress that the conditions (4.5),(4.6) are sufficient to be KS-operator.

Corollary 4.5. *If the mappings Φ and Ψ are KS operators, then T is also KS operator.*

The proof immediately follows from (4.7).

Remark 4.6. We have to stress that if T is KS operator, then the mappings Φ and Ψ no need to be KS.

4.1. **Case: $\mathbf{C}=\mathbf{A}$.** Now let us study the operator T given by (4.1) when $\mathbf{C}=\mathbf{A}$. Consequently from (4.1) one finds

$$T_A(w_0\mathbb{1}+\mathbf{w}\cdot\sigma)=w_0\mathbb{1}\otimes\mathbb{1}+\mathbf{A}\mathbf{w}\cdot\sigma\otimes\mathbb{1}+\mathbb{1}\otimes\mathbf{A}\mathbf{w}\cdot\sigma. \tag{4.8}$$

From Theorem 4.4 we immediately have the following

Corollary 4.7. *Let T_A be a mapping given by* (4.8). *If one has*

$$\|\mathbf{w}\|^2-4\|\mathbf{A}\mathbf{w}\|^2\geq 0$$

$$2\|\mathbf{A}[\mathbf{w},\overline{\mathbf{w}}]-2[\mathbf{A}\mathbf{w},\mathbf{A}\overline{\mathbf{w}}]\|\leq\|\mathbf{w}\|^2-4\|\mathbf{A}\mathbf{w}\|^2. \tag{4.9}$$

Then T_A is a Kadison-Schwarz operator.

Now using the same argument as in section 3, we can write

$$T_A(x)=\tilde{U}T_{D_{\lambda_1,\lambda_2,\lambda_3}}(x)\tilde{U}^*,\quad x\in M_2(\mathbb{C}) \tag{4.10}$$

for some unitary $\tilde{U}$. Due to Theorem 2.1 all bistochastic KS-operators can be characterized by $T_{D_{\lambda_1,\lambda_2,\lambda_3}}$ and unitaries. In what follows, for the sake of shortness by $T_{(\lambda_1,\lambda_2,\lambda_3)}$ we denote the mapping $T_{D_{\lambda_1,\lambda_2,\lambda_3}}$.

Next we want to characterize KS operators of the form $T_{(\lambda_1,\lambda_2,\lambda_3)}$.

Theorem 4.8. *If*

$$4(1+8\lambda_1\lambda_2\lambda_3)\geq(1+4\lambda_1^2)(3+4\lambda_2^2+4\lambda_3^2-4\lambda_1^2),$$
$$4(1+8\lambda_1\lambda_2\lambda_3)\geq(1+4\lambda_2^2)(3+4\lambda_1^2+4\lambda_3^2-4\lambda_2^2),$$
$$4(1+8\lambda_1\lambda_2\lambda_3)\geq(1+4\lambda_3^2)(3+4\lambda_1^2+4\lambda_2^2-4\lambda_3^2)$$

are satisfied, then $T_{(\lambda_1,\lambda_2,\lambda_3)}$ is a KS operator.

Proof. Taking $\mathbf{A} = D_{\lambda_1,\lambda_2,\lambda_3}$ in (4.9), we obtain

$$4A_1|w_2\overline{w_3} - \overline{w_2}w_3|^2 + 4A_2|\overline{w_1}w_3 - w_1\overline{w_3}|^2 + 4A_3|w_1\overline{w_2} - \overline{w_1}w_2|^2 \leq \Big(B_1|w_1|^2 + B_2|w_2|^2 + B_3|w_3|^2\Big)^2, \tag{4.11}$$

where $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{C}^3$ and

$$A_1 = |\lambda_1 - 2\lambda_2\lambda_3|^2,\ A_2 = |\lambda_2 - 2\lambda_1\lambda_3|^2,\ A_3 = |\lambda_3 - 2\lambda_1\lambda_2|^2, \tag{4.12}$$

$$B_1 = (1 - 4\lambda_1^2),\ B_2 = (1 - 4\lambda_2^2),\ B_3 = (1 - 4\lambda_3^2). \tag{4.13}$$

By (3.15) LHS of (4.11) can be evaluated as follows

$$\begin{aligned} 4A_1\Big(|w_2|^4 + 2|w_2|^2|w_3|^2 + |w_3|^4\Big) &+ 4A_2\Big(|w_1|^4 + 2|w_1|^2|w_3|^2 + |w_3|^4\Big) \\ &+ 4A_3\Big(|w_1|^4 + 2|w_1|^2|w_2|^2 + |w_2|^4\Big). \end{aligned}$$

Therefore, from (4.11) one gets

$$\begin{aligned} &\Big(B_1^2 - 4A_2 - 4A_3\Big)|w_1|^4 + \Big(B_2^2 - 4A_1 - 4A_3\Big)|w_2|^4 + \Big(B_3^2 - 4A_1 - 4A_2\Big)|w_3|^4 \\ &+2|w_2|^2|w_3|^2(B_2B_3 - 4A_1) + 2|w_1|^2|w_3|^2(B_1B_3 - 4A_2) + 2|w_1|^2|w_2|^2(B_1B_2 - 4A_3) \geq 0 \end{aligned}$$

It is obvious that above inequality is satisfied if one has

$$\begin{aligned} &B_1^2 \geq 4A_2 + 4A_3, \quad B_2^2 \geq 4A_1 + 4A_3, \quad B_3^2 \geq 4A_1 + 4A_2, \\ &B_2B_3 \geq 4A_1, \quad B_1B_3 \geq 4A_2, \quad B_1B_2 \geq 4A_3. \end{aligned}$$

Substituting above denotations (4.12), (4.13) to the last inequalities, and doing some calculations one derives

$$4(1 + 8\lambda_1\lambda_2\lambda_3) \geq (1 + 4\lambda_1^2)(3 + 4\lambda_2^2 + 4\lambda_3^2 - 4\lambda_1^2), \tag{4.14}$$

$$4(1 + 8\lambda_1\lambda_2\lambda_3) \geq (1 + 4\lambda_2^2)(3 + 4\lambda_1^2 + 4\lambda_3^2 - 4\lambda_2^2), \tag{4.15}$$

$$4(1 + 8\lambda_1\lambda_2\lambda_3) \geq (1 + 4\lambda_3^2)(3 + 4\lambda_1^2 + 4\lambda_2^2 - 4\lambda_3^2), \tag{4.16}$$

$$1 + 16\lambda_1\lambda_2\lambda_3 \geq 4\lambda_1^2 + 4\lambda_2^2 + 4\lambda_3^2, \tag{4.17}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \Big[-\frac{1}{2}, \frac{1}{2}\Big]$.

Now using the same argument as in the proof of Theorem 3.2 one can show that (4.17) is an extra condition. This completes the proof. □

It is interesting to study when the operator $T_{(\lambda_1,\lambda_2,\lambda_3)}$ is complete positive. Let us characterize completely positivity of $T_{(\lambda_1,\lambda_2,\lambda_3)}$.

Theorem 4.9. *A map $T_{(\lambda_1,\lambda_2,\lambda_3)}$ is complete positive if and only if the followings inequalities are satisfied*

(1) $|\lambda_3| < \frac{1}{2}$;
$4\lambda_1^2 + 4\lambda_2^2 + 4\lambda_3^2 \leq 1 + 16\lambda_1\lambda_2\lambda_3$;
$\lambda_1^2 + \lambda_2^2 + \sqrt{\Big(\lambda_1^2 + \lambda_2^2\Big)^2 - 4\lambda_1\lambda_2\lambda_3 + \lambda_3^2} \leq \frac{1}{2}$;

(2) $\lambda_3 = \frac{1}{2}, \quad \lambda_1, \lambda_2 \in \Big[-\frac{1}{2}, \frac{1}{2}\Big]$

(3) $\lambda_3 = -\frac{1}{2}, \quad \lambda_1 = \pm\frac{1}{2}, \quad \lambda_2 = \mp\frac{1}{2}$

Proof. From [11] we know that the complete positivity of $T_{(\lambda_1,\lambda_2,\lambda_3)}$ is equivalent to the positivity of the following matrix

$$\widehat{T}_{(\lambda_1,\lambda_2,\lambda_3)} = \begin{pmatrix} T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{11}) & T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{12}) \\ T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{21}) & T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{22}) \end{pmatrix}.$$

It is clear that

$$T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{11}) = \frac{1}{2}\begin{pmatrix} 1+2\lambda_3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-2\lambda_3 \end{pmatrix},$$

$$T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{12}) = \frac{1}{2}\begin{pmatrix} 0 & \lambda_1+\lambda_2 & \lambda_1+\lambda_2 & 0 \\ \lambda_1-\lambda_2 & 0 & 0 & \lambda_1+\lambda_2 \\ \lambda_1-\lambda_2 & 0 & 0 & \lambda_1+\lambda_2 \\ 0 & \lambda_1-\lambda_2 & \lambda_1-\lambda_2 & 0 \end{pmatrix}$$

and $T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{22}) = \mathbb{1}\otimes\mathbb{1} - T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{11})$, $T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{21}) = T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{12})^*$.

(1). According to [3, Theorem 1.3.3] the matrix $\widehat{T}_{(\lambda_1,\lambda_2,\lambda_3)}$ is positive if and only if

$$T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{11}) - T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{12})T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{22})^{-1}T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{21}) \geq 0, \tag{4.18}$$

where $T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{11})$ and $T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{22})$ are positive matrices.

It is easy to see that $T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{11})$ and $T_{(\lambda_1,\lambda_2,\lambda_3)}(e_{22})$ are positive if and only if

$$|\lambda_3| \leq \frac{1}{2}. \tag{4.19}$$

One can calculate that (4.18) is equivalent to

$$\begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_4 \\ 0 & 1+\alpha_3 & \alpha_3 & 0 \\ 0 & \alpha_3 & 1+\alpha_3 & 0 \\ \alpha_4 & 0 & 0 & \alpha_2 \end{pmatrix} \geq 0$$

where

$$\alpha_1 = 1+2\lambda_3 - 2(\lambda_1+\lambda_2)^2, \quad \alpha_2 = 1-2\lambda_3-2(\lambda_1-\lambda_2)^2,$$

$$\alpha_3 = \frac{(\lambda_1-\lambda_2)^2}{2\lambda_3-1} - \frac{(\lambda_1+\lambda_2)^2}{2\lambda_3+1}, \quad \alpha_4 = -2\Big(\lambda_1^2-\lambda_2^2\Big).$$

It is known that the matrix is positive if and only if the eigenvalues are positive. The eigenvalues of the last matrix can be calculated as follows

$$s_1 = 1, \quad s_2 = \frac{4\lambda_1^2+4\lambda_2^2+4\lambda_3^2-16\lambda_1\lambda_2\lambda_3-1}{4\lambda_3^2-1},$$

$$s_3 = 1-2\lambda_1^2-2\lambda_2^2+2\sqrt{\Big(\lambda_1^2+\lambda_2^2\Big)^2-4\lambda_1\lambda_2\lambda_3+\lambda_3^2},$$

$$s_4 = 1-2\lambda_1^2-2\lambda_2^2-2\sqrt{\Big(\lambda_1^2+\lambda_2^2\Big)^2-4\lambda_1\lambda_2\lambda_3+\lambda_3^2}.$$

To check the their positivity, it is enough to have $s_2 \geq 0$ and $s_4 \geq 0$. These mean

$$\lambda_3 \neq \frac{1}{2}; \tag{4.20}$$

$$4\lambda_1^2+4\lambda_2^2+4\lambda_3^2 \leq 1+16\lambda_1\lambda_2\lambda_3; \tag{4.21}$$

$$\lambda_1^2+\lambda_2^2+\sqrt{\Big(\lambda_1^2+\lambda_2^2\Big)^2-4\lambda_1\lambda_2\lambda_3+\lambda_3^2} \leq \frac{1}{2}; \tag{4.22}$$

$$\Big(\lambda_1^2+\lambda_2^2\Big)^2+\lambda_3^2 \geq 4\lambda_1\lambda_2\lambda_3. \tag{4.23}$$

Note that the expression standing inside the square root is always positive, indeed, we have

$$\left(\lambda_1^2+\lambda_2^2\right)^2+\lambda_3^2\geq 2\left(\lambda_1^2+\lambda_2^2\right)\lambda_3\geq 2(2\lambda_1\lambda_2)\lambda_3=4\lambda_1\lambda_2\lambda_3.$$

Therefore, from (4.19), (4.20), (4.21) and (4.22) one has

$$\begin{aligned}&|\lambda_3|<\frac{1}{2};\\ &4\lambda_1^2+4\lambda_2^2+4\lambda_3^2\leq 1+16\lambda_1\lambda_2\lambda_3;\\ &\lambda_1^2+\lambda_2^2+\sqrt{\left(\lambda_1^2+\lambda_2^2\right)^2-4\lambda_1\lambda_2\lambda_3+\lambda_3^2}\leq\frac{1}{2}.\end{aligned}$$

(2). Let $\lambda_3=\frac{1}{2}$, then $\widehat{T}_{(\lambda_1,\lambda_2,\lambda_3)}$ has the following form

$$\widehat{T}_{(\lambda_1,\lambda_2,\frac{1}{2})}=\begin{pmatrix}2&0&0&0&0&\beta_1&\beta_1&0\\0&1&0&0&\beta_2&0&0&\beta_1\\0&0&1&0&\beta_2&0&0&\beta_1\\0&0&0&0&0&\beta_2&\beta_2&0\\0&\beta_2&\beta_2&0&0&0&0&0\\\beta_1&0&0&\beta_2&0&1&0&0\\\beta_1&0&0&\beta_2&0&0&1&0\\0&\beta_1&\beta_1&0&0&0&0&2\end{pmatrix},$$

where where $\beta_1=\lambda_1+\lambda_2$, $\beta_2=\lambda_1-\lambda_2$, $\lambda_1,\lambda_2\in\left[-\frac{1}{2},\frac{1}{2}\right]$. According to the Silvester's criterion, the matrix given above is positive if and only if the leading principal minors are positive. Let $D_n,(n=\overline{1,8})$ be the leading principal minor of $\widehat{T}_{(\lambda_1,\lambda_2,\frac{1}{2})}$. One can see that for each $n\in\{1,\dots,8\}$, the minor D_n is positive. Hence, if $\lambda_3=\frac{1}{2}$ then $\widehat{T}_{(\lambda_1,\lambda_2,\frac{1}{2})}$ is positive.

(3). Now assume $\lambda_3=-\frac{1}{2}$, then one finds

$$\widehat{T}_{(\lambda_1,\lambda_2,-\frac{1}{2})}=\begin{pmatrix}0&0&0&0&0&\beta_1&\beta_1&0\\0&1&0&0&\beta_2&0&0&\beta_2\\0&0&1&0&\beta_2&0&0&\beta_1\\0&0&0&2&0&\beta_2&\beta_2&0\\0&\beta_2&\beta_1&0&2&0&0&0\\\beta_1&0&0&\beta_2&0&1&0&0\\\beta_1&0&0&\beta_1&0&0&1&0\\0&\beta_1&\beta_1&0&0&0&0&0\end{pmatrix},$$

where as before $\beta_1=\lambda_1+\lambda_2$, $\beta_2=\lambda_1-\lambda_2$, $\lambda_1,\lambda_2\in\left[-\frac{1}{2},\frac{1}{2}\right]$. One can calculate that principal minors of the last matrix are

$$\begin{aligned}&D_n=0\ (n=\overline{1,5}),\\ &D_6=(\lambda_1+\lambda_2)^2\Big(4(\lambda_1-\lambda_2)^2-4\Big),\\ &D_7=(\lambda_1+\lambda_2)^2\Big(8(\lambda_1-\lambda_2)^2-8\Big),\\ &D_8=16(\lambda_1+\lambda_2)^4,\end{aligned}$$

It is easy to see that $\widehat{T}_{(\lambda_1,\lambda_2,-\frac{1}{2})}$ is positive if $D_6\geq 0$ and $D_7\geq 0$. It implies that $\lambda_1=\pm\frac{1}{2}$, $\lambda_2=\mp\frac{1}{2}$. This completes the proof. □

In [40] a characterization of completely positivity of $\Phi_{(\lambda_1,\lambda_2,\lambda_3)}$ has been given. Namely, the following result holds.

Theorem 4.10. *A mapping $\Phi_{(\lambda_1,\lambda_2,\lambda_3)}$ is complete positive if and only if the following inequalities are satisfied*

$$(\lambda_1+\lambda_2)^2 \leq (1+\lambda_3)^2, \tag{4.24}$$

$$(\lambda_1-\lambda_2)^2 \leq (1-\lambda_3)^2, \tag{4.25}$$

$$\Big(1-\Big(\lambda_1^2+\lambda_2^2+\lambda_3^2\Big)\Big)^2 \geq 4\Big(\lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_1^2\lambda_3^2-2\lambda_1\lambda_2\lambda_3\Big). \tag{4.26}$$

Example. Let us consider a mapping $T_{(a,a,b)}$, where $a,b \in \Big[-\frac{1}{2},\frac{1}{2}\Big]$. Then one can see that $\Phi_{(2a,2a,2b)}$ is the corresponding operator. Now let us check conditions of Theorem 4.9 and Theorem 4.10. From conditions of Theorem 4.9 one finds

$$|b| < \frac{1}{2}; \tag{4.27}$$

$$a^2 \leq \frac{1+2b}{8}; \tag{4.28}$$

$$1-4a^2-2\sqrt{\Big(2a^2-b\Big)^2} \geq 0. \tag{4.29}$$

Now we would like to show that (4.29) is extra condition. It means that the left hand side of (4.29) is always positive if (4.27) and (4.28) are satisfied. Let (4.27) and (4.28) be true. Then

$$\begin{aligned} 1-4a^2-2\sqrt{\Big(2a^2-b\Big)^2} &\geq 1-4\cdot\frac{1+2b}{8}-2\sqrt{\Big(2\cdot\frac{1+2b}{8}-b\Big)^2} \\ &= 1-\frac{1+2b}{2}-2\sqrt{\Big(\frac{1-2b}{4}\Big)^2}=0. \end{aligned}$$

Now from conditions of Theorem 4.10 one has

$$a^2 \leq \frac{(1+2b)^2}{16}. \tag{4.30}$$

The graphics of the inequalities (4.27), (4.28) and (4.30) are given in the following figure.

From the graph we can see that the class of CP operators corresponding to $T_{(a,a,b)}$ are much bigger then the class of CP operators corresponding to $\Phi_{(2a,2a,2b)}$

4.2. **Case: $\mathbf{Aw}=\lambda\mathbf{w}$, $\mathbf{Cw}=\mu\mathbf{w}$.** In this subsection we consider a more concrete case, namely, $\mathbf{Aw}=\lambda\mathbf{w}$ and $\mathbf{Cw}=\mu\mathbf{w}$. By $T_{\lambda,\mu}$ we denote the corresponding operator (see (4.1)). Then one can see that $\Phi_{(2\lambda,2\lambda,2\lambda)}$ and $\Psi_{(2\mu,2\mu,2\mu)}$ are the corresponding mappings (see (4.2)-(4.4)). Due to Theorem 3.1 one can find that $\Phi_{(2\lambda,2\lambda,2\lambda)}$ is a KS-operator if and only if

$$2|\lambda||1-2\lambda|\|[\mathbf{w},\overline{\mathbf{w}}]\| \leq \big(1-4\lambda^2\big)\|\mathbf{w}\|^2.$$

From $\|[\mathbf{w},\overline{\mathbf{w}}]\| \leq \|\mathbf{w}\|^2$ (if we choose $\mathbf{w}=(0,1,i)$, then one gets $\|[\mathbf{w},\overline{\mathbf{w}}]\|=\|\mathbf{w}\|^2$) one finds

$$2|\lambda|(1-2\lambda) \leq 1-4\lambda^2.$$

The solution of the last inequality is $\lambda \in \big[-\frac{1}{4};\frac{1}{2}\big]$.

Similarly, one finds that $\Psi_{(2\mu,2\mu,2\mu)}$ is a KS-operator if and only if $\mu \in \big[-\frac{1}{4};\frac{1}{2}\big]$.

From Corollary 4.5 we immediately conclude that if $\lambda,\mu \in \big[-\frac{1}{4};\frac{1}{2}\big]$ then $T_{\lambda,\mu}$ is a KS- operator.

Next we want to provide other values of λ and μ for which $T_{\lambda,\mu}$ is Kadison-Schwarz.

Theorem 4.11. *Let $T_{\lambda,\mu}: M_2(\mathbb{C}) \to M_2(\mathbb{C})\otimes M_2(\mathbb{C})$ be given by (4.1). If*

$$|\lambda||1-2\lambda|+|\mu||1-2\mu| \leq 1-2\lambda^2-2\mu^2$$

is satisfied, then the map $T_{\lambda,\mu}$ is KS-operator.

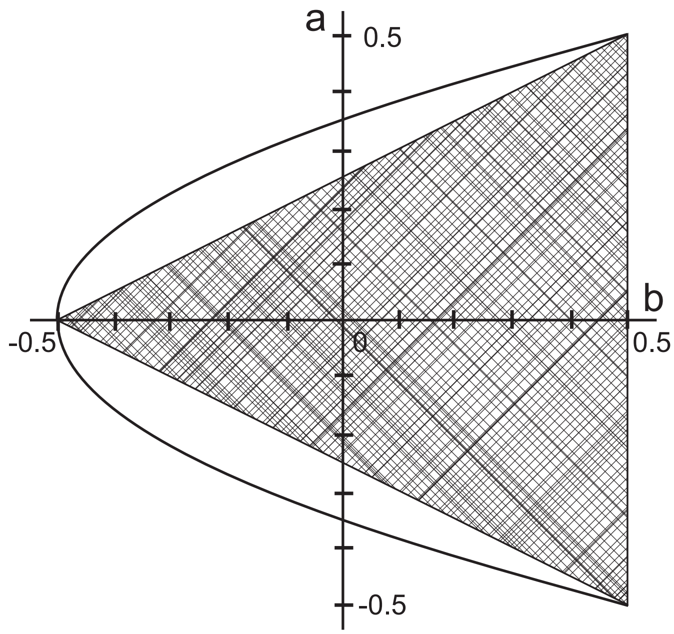

FIGURE 1. Shaded region is CP operators corresponding to $\Phi_{(2a,2a,2b)}$. White region indicates CP operators corresponding to $T_{(a,a,b)}$.

Proof. From (4.5),(4.6) one has

$$\begin{aligned} &\lambda^2 + \mu^2 \leq \frac{1}{2} \\ &\big(|\lambda||1-2\lambda| + |\mu||1-2\mu|\big)\big\|[\mathbf{w}, \overline{\mathbf{w}}]\big\| \leq \big(1 - 2\lambda^2 - 2\mu^2\big)\|\mathbf{w}\|^2. \end{aligned}$$

From the arbitrariness of $\mathbf{w}$ with $\big\|[\mathbf{w}, \overline{\mathbf{w}}]\big\| \leq \|\mathbf{w}\|^2$ we find

$$|\lambda||1-2\lambda| + |\mu||1-2\mu| \leq 1 - 2\lambda^2 - 2\mu^2,$$

which is the required assertion. □

From the figure 2, we conclude that if the pair (λ, μ) belongs to the outside of the yellow and red regions, then the mappings $\Phi_{(2\lambda,2\lambda,2\lambda)}$ and $\Psi_{(2\mu,2\mu,2\mu)}$ are not Kadison-Schwarz, but the mapping $T_{\lambda,\mu}$ is Kadison-Schwarz.

Now we are interested when the operator $T_{\lambda,\mu}$ is complete positive.

Theorem 4.12. *Let* $T_{\lambda,\mu} : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ *be given by* (4.1). *Then* $T_{\lambda,\mu}$ *is completely positive if and only if*

$$\begin{aligned} &\lambda + \mu + 1 - 2\sqrt{\lambda^2 - \lambda\mu + \mu^2} \geq 0 \\ &\lambda + \mu \leq 1 \end{aligned}$$

Proof. It is know [11] that the complete positivity of $T_{\lambda,\mu}$ is equivalent to the positivity of the following matrix

$$\widehat{T}_{\lambda,\mu} = \begin{pmatrix} T_{\lambda,\mu}(e_{11}) & T_{\lambda,\mu}(e_{12}) \\ T_{\lambda,\mu}(e_{21}) & T_{\lambda,\mu}(e_{22}) \end{pmatrix}.$$

One can calculate that

$$T_{\lambda,\mu}(e_{11}) = \frac{1}{2}\begin{pmatrix} 1+M_1 & 0 & 0 & 0 \\ 0 & 1-M_2 & 0 & 0 \\ 0 & 0 & 1+M_2 & 0 \\ 0 & 0 & 0 & 1-M_1 \end{pmatrix},$$

$$T_{\lambda,\mu}(e_{12}) = \frac{1}{2}\begin{pmatrix} 0 & 2\lambda & 2\mu & 0 \\ 0 & 0 & 0 & 2\mu \\ 0 & 0 & 0 & 2\lambda \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $T_{\lambda,\mu}(e_{22}) = \mathbb{1}\otimes\mathbb{1} - T_{\lambda,\mu}(e_{11})$, $T_{\lambda,\mu}(e_{21}) = T_{\lambda,\mu}(e_{12})^*$. Where $M_1 = \lambda+\mu$, $M_2 = \lambda-\mu$. Therefore, we obtain

$$\widehat{T}_{\lambda,\mu} = \frac{1}{2}\begin{pmatrix} 1+M_1 & 0 & 0 & 0 & 0 & 2\lambda & 2\mu & 0 \\ 0 & 1-M_2 & 0 & 0 & 0 & 0 & 0 & 2\mu \\ 0 & 0 & 1+M_2 & 0 & 0 & 0 & 0 & 2\lambda \\ 0 & 0 & 0 & 1-M_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-M_1 & 0 & 0 & 0 \\ 2\lambda & 0 & 0 & 0 & 0 & 1+M_2 & 0 & 0 \\ 2\mu & 0 & 0 & 0 & 0 & 0 & 1-M_2 & 0 \\ 0 & 2\mu & 2\lambda & 0 & 0 & 0 & 0 & 1+M_1 \end{pmatrix}$$

One can calculate the the eigenvalues of $\widehat{T}_{\lambda,\mu}$ are the followings

$$\begin{aligned} &\lambda+\mu+1+2\sqrt{\lambda^2-\lambda\mu+\mu^2}, \\ &\lambda+\mu+1-2\sqrt{\lambda^2-\lambda\mu+\mu^2}, \\ &1-\lambda-\mu. \end{aligned}$$

Hence, $\widehat{T}_{\lambda,\mu}$ is positive if and only if the the eigenvalues are positive, which implies the assertion. □

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References

[1] G.G. Amosov, A.S. Holevo, and R.F. Werner, On the additivity hypothesis in quantum information theory, *Probl. Inf. Transm.* **36** (2000), 305–313

[2] S.J. Bhatt, Stinespring representability and Kadison's Schwarz inequality in non-unital Banach star algebras and applications, *Proc. Indian Acad. Sci. (Math. Sci.)*, **108** (1998), 283–303.

[3] R. Bhatia, *Positive definite matrices.* Princeton University Press, 2009.

[4] I. Bengtsson, K. Zyczkowski, *Geometry of quantum states*, Cambridge Univ. Press, 2006.

[5] O. Bratteli and D. W. Robertson, Operator algebras and quantum statistical mechanics. *I, Springer, New YorkHeidelbergBerlin* (1979).

[6] D. Chruscinski, Quantum-correlation breaking channels, quantum conditional probability and PerronFrobenius theory, *Phys. Lett A* **377**(2013), 606–611.

[7] D. Chruscinski, A class of symmetric Bell diagonal entanglement witnesses - a geometric perspective, *J. Phys. A: Math. Theor.* **47**(2014), 424033.

[8] D. Chruscinski, On Kossakowski construction of positive maps on matrix algebras, *Open Systems & Infor. Dynam.* **21** (2014), 1450001.

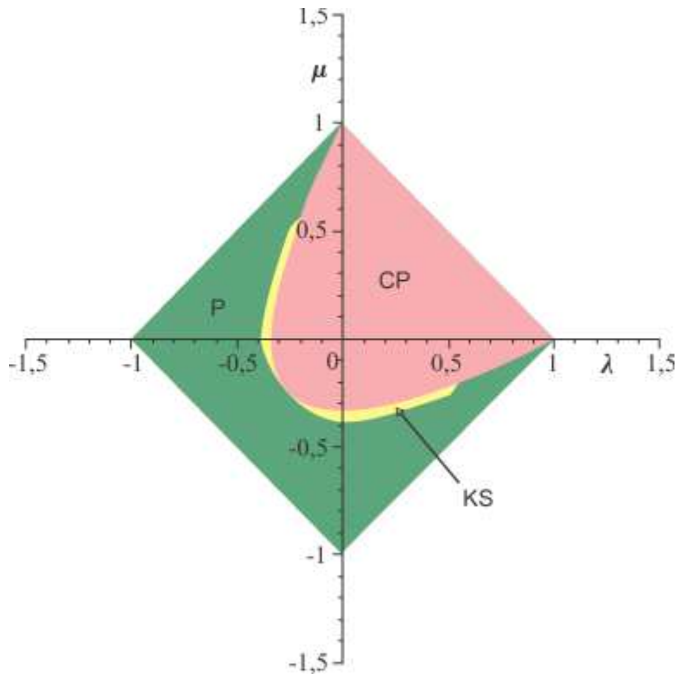

FIGURE 2. If the pair (λ, μ) does not belong to the red region, then the corresponding mapping $T_{\lambda,\mu}$ is not CP.

[9] D. Chruscinski, G. Sarbicki, Exposed positive maps in $M_4(C)$, *Open Systems & Infor. Dynam.* **19** (2012), 1250017.

[10] D. Chruscinski, G. Sarbicki, Entanglement witnesses: construction, analysis and classification, *J. Phys. A.: Math. Theor.* **47** (2014), 4830001.

[11] M-D. Choi, Completely positive linear maps on complex matrices, *Lin. Alg. Appl.* **10**(1975), 285–290.

[12] R. Ganikhodzhaev, F. Mukhamedov, U. Rozikov, Quadratic stochastic operators and processes: results and open problems, *Inf. Dim. Anal. Quantum Probab. and Related Topics* **14**(2011) 279-335.

[13] N. N. Ganikhodjaev, F,M. Mukhamedov, Ergodic properties of quantum quadratic stochastic processes defined on von Neumann algebras. *Russian Math. Surveys* **53**(1998), 1350–1351.

[14] U. Groh, Uniform ergodic theorems for identity preserving Schwarz maps on W^*-algebras, *J. Operator Theory* **11** (1984), 395–404.

[15] K.-C. Ha, Entangled states with strong positive partial transpose, *Phys. Rev. A* **81**(2010), 064101.

[16] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, Quantum entanglement, *Rev. Mod. Phys.* **81**(2009), 865.

[17] M. Horodecki, P. Horodecki, R. Horodecki, Separability of mixed states: Necessary and sufficient conditions, *Phys. Lett. A* **223** (1996), 1–8.

[18] C. King, M.B. Ruskai, Minimal entropy of states emerging from noisy quantum channels. *IEEE Trans. Info. Theory* **47**, (2001) 192-209 .

[19] A. Kossakowski, A Class of Linear Positive Maps in Matrix Algebras, *Open Sys. & Information Dyn.* **10**(2003) 213–220.

[20] L.E. Labuschagne, W.A. Majewski, M. Marciniak, On k-decomposability of positive maps. *Expo. Math.* **24** (2006), 103–125.

[21] W.A. Majewski, M. Marciniak, On a characterization of positive maps, *J. Phys. A: Math. Gen.* **34** (2001) 5863-5874.

[22] W.A. Majewski, On non-completely positive quantum dynamical maps on spin chains, *J. Phys. A: Math. Gen.* **40** (2007) 11539-11545.

[23] W.A. Majewski, On positive decomposable maps. *Rep. Math. Phys.* **59** (2007), 289–298.

[24] W.A. Majewski, On the structure of positive maps. I. Finite dimensional case, *J. Math. Phys.* **53** (2012) 023515.

[25] W.A. Majewski, T.I. Tylec, On the structure of positive maps. II. Low dimensional matrix algebras, *J. Math. Phys.* **54** (2013) 073508.

[26] F.M. Mukhamedov, On decomposition of quantum quadratic stochastic processes into layer-Markov processes defned on von Neumann algebras, *Izvestiya Math.* **68**(2004), No.5. 1009–1024.
[27] F. Mukhamedov, On pure quasi-quantum quadratic operators of $M_2(\mathbb{C})$ II, *Open Sys. & Infor. Dyn.* **22** (2015) 1550024.
[28] F. Mukhamedov, A. Abduganiev, On description of bistochastic Kadison-Schwarz operators on $M_2(\mathbb{C})$, *Open Sys. & Inform. Dynam.* **17**(2010), 245-253.
[29] F. Mukhamedov, A. Abduganiev, On Kadison-Schwarz type quantum quadratic operators on $M_2(C)$, *Abst. Appl. Anal.* **2013**(2013), Article ID 278606, 9 p.
[30] F. Mukhamedov, A. Abduganiev, On pure quasi-quantum quadratic operators of $M_2(\mathbb{C})$, *Open Systems & Infor. Dynam.* **20**(2013), 1350018
[31] F. Mukhamedov, A. Abduganiev, On bistochastic Kadison-Schwarz operators on $M_2(C)$, *J. Phys.: Conf. Ser.* **435** (2013), 012018.
[32] F. Mukhamedov, H. Akin, S. Temir, A. Abduganiev, On quantum quadratic operators on $M_2(\mathbb{C})$ and their dynamics, *Jour. Math. Anal. Appl.* **376**(2011), 641–655.
[33] F. Mukhamedov, N. Ganikhodjaev, *Quantum quadratic operators and processes*, Lect. Notes Math. Ser. Vol. 2133, Springer, 2015.
[34] M.A. Nielsen, I.L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, 2000.
[35] M. Ohya, D. Petz, *Quantum Entropy and Its Use*, Springer, Berlin 1993.
[36] M. Ohya, I. Volovich, *Mathematical foundations of quantum information and computation and its applications to nano- and bio-systems*, Springer, New York, 2011.
[37] V. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge University Press, 2003.
[38] A. G. Robertson, A Korovkin theorem for Schwarz maps on C^*-algebras, *Math. Z.* **156**(1977), 205–206.
[39] A. G. Robertson, Schwarz inequalities and the decomposition of positive maps on C^*-algebras *Math. Proc. Camb. Philos. Soc.* **94**(1983), 291–296.
[40] M.B. Ruskai, S. Szarek, E. Werner, An analysis of completely positive trace-preserving maps on M_2, *Lin. Alg. Appl.* **347** (2002) 159-187.
[41] A. Rutkowski, G. Sarbicki, D. Chruscinski, A class of bistochastic positive optimal maps in $M_d(C)$, *Open Systems & Infor. Dynam.* **22** (2015), 1550016.
[42] E. Stormer, Positive linear maps of operator algebras, *Acta Math.* **110**(1963), 233–278.
[43] S. L. Woronowicz, Positive maps of low dimensional matrix algebras, *Rep. Math. Phys.* **10** (1976), 165–183.
[44] B.M. Terhal, Bell inequlities and seperability conditions, *Phys.Lett. A* **271** (2000), 319–326.

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