EXPLICIT FORMULAE FOR CHERN-SIMONS INVARIANTS OF THE HYPERBOLIC ORBIFOLDS OF THE KNOT WITH CONWAY'S NOTATION $C(2n, 3)$

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Abstract. We calculate the Chern-Simons invariants of the hyperbolic orbifolds of the knot with Conway's notation $C(2n, 3)$ using the Schläfli formula for the generalized Chern-Simons function on the family of $C(2n, 3)$ cone-manifold structures. We present the concrete and explicit formula of them. We apply the general instructions of Hilden, Lozano, and Montesinos-Amilibia and extend the Ham and Lee's methods. As an application, we calculate the Chern-Simons invariants of cyclic coverings of the hyperbolic $C(2n, 3)$ orbifolds.

1. Introduction

Chern-Simons invariant [1, 20] was defined to be a geometric invariant and became a topological invariant after the Mostow Rigidity Theorem [22]. Various methods of finding Chern-Simons invariant using ideal triangulations have been introduced [23, 24, 32, 4, 3, 2] and implemented [6, 9]. But, for orbifolds, to our knowledge, there does not exist a single convenient program which computes Chern-Simons invariant.

Instead of working on complicated combinatorics of 3-dimensional ideal tetrahedra to find the Chern-Simons invariants of the hyperbolic orbifolds of the knot with Conway's notation $C(2n, 3)$, we deal with simple one dimensional singular loci. Similar methods for volumes can be found in [11, 12]. We use the Schläfli formula for the generalized Chern-Simons function on the family of $C(2n, 3)$ cone-manifold structures [14]. In [15] a method of calculating the Chern-Simons invariants of two-bridge knot orbifolds were introduced but without explicit formulae. In [10], the Chern-Simons invariants of the twist knot orbifolds are computed. Similar approaches for $SU(2)$-connections can be found in [18] and for $\mathrm{SL}(2, C)$-connections in [17]. For explanations of cone-manifolds, you can refer to [5, 30, 19, 25, 13, 26, 12].

The main purpose of the paper is to find the explicit and efficient formulae for Chern-Simons invariants of the hyperbolic orbifolds of the knot with Conway's notation $C(2n, 3)$. For a two-bridge hyperbolic link, there exists an angle $\alpha_0 \in [\frac{2\pi}{3}, \pi)$ for each link K such that the cone-manifold $K(\alpha)$ is hyperbolic for $\alpha \in (0, \alpha_0)$, Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha \in (\alpha_0, \pi]$ [25, 13, 19, 26]. We will use the Chern-Simons invariant of the lens space $L(6n + 1, 4n + 1)$ calculated in [15]. The following theorem gives the formulae for T_{2n}. Note that if $2n$ of T_{2n} is replaced by an odd integer, then T_{2n} becomes a link with two components. Also, note that the Chern-Simons invariant of hyperbolic cone-manifolds of the knot

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with Conway's notation $C(-2n,-3)$ is the same as that of the knot with Conway's notation $C(2n,3)$ up to sign. For the Chern-Simons invariant formula, since the knot T_{2n} has to be hyperbolic, we exclude the case when $n=0$.

Theorem 1.1. *Let $X_{2n}(\alpha)$, $0 \leq \alpha < \alpha_0$ be the hyperbolic cone-manifold with underlying space S^3 and with singular set T_{2n} of cone-angle α. Let k be a positive integer such that k-fold cyclic covering of $X_{2n}(\frac{2\pi}{k})$ is hyperbolic. Then the Chern-Simons invariant of $X_{2n}(\frac{2\pi}{k})$ (mod $\frac{1}{k}$ if k is even or mod $\frac{1}{2k}$ if k is odd) is given by the following formula:*

$$\begin{aligned}
&\mathrm{cs}\left(X_{2n}\left(\frac{2\pi}{k}\right)\right) \equiv \frac{1}{2}\mathrm{cs}\left(L(6n+1,4n+1)\right)\\
&+\frac{1}{4\pi^2}\int_{\frac{2\pi}{k}}^{\alpha_0} Im\left(2*\log\left(-M^{-4n-2}\frac{M^{-2}+t}{M^2+t}\right)\right)\, d\alpha\\
&+\frac{1}{4\pi^2}\int_{\alpha_0}^{\pi} Im\left(\log\left(-M^{-4n-2}\frac{M^{-2}+t_1}{M^2+t_1}\right)+\log\left(-M^{-4n-2}\frac{M^{-2}+t_2}{M^2+t_2}\right)\right)\, d\alpha,
\end{aligned}$$

where for $M=e^{\frac{\alpha}{2}}$, t $(Im(t)\leq 0)$, t_1, and t_2 are zeroes of Riley-Mednykh polynomial $P_{2n}=P_{2n}(t,M)$ which is given recursively by

$$P_{2n}=\begin{cases} QP_{2(n-1)}-M^8P_{2(n-2)} & \text{if } n>1,\\ QP_{2(n+1)}-M^8P_{2(n+2)} & \text{if } n<-1,\end{cases}$$

with initial conditions

$P_{-2}=M^2t^2+\left(M^4-M^2+1\right)t+M^2,$

$P_0=M^6$ for $n\leq 0$ and $P_0=M^8$ for $n\geq 0$,

$P_2=-M^4t^3+\left(-2M^6+M^4-2M^2\right)t^2+\left(-M^8+M^6-2M^4+M^2-1\right)t+M^4,$

and $M=e^{\frac{\alpha}{2}}$ and

$Q=-M^4t^3+\left(-2M^6+2M^4-2M^2\right)t^2+\left(-M^8+2M^6-3M^4+2M^2-1\right)t+2M^4,$

where $M=e^{\frac{\alpha}{2}}$ and t_1 and t_2 approach common t as α decreases to α_0 and they come from the components of t and $\overline{t}$.

2. Two bridge knots with Conway's notation $C(2n,3)$

A knot K is a two bridge knot with Conway's notation $C(2n,3)$ if K has a regular two-dimensional projection of the form in Figure 1. For example, Figure 2 is knot $C(4,3)$. K has 3 left-handed horizontal crossings and $2n$ right-handed vertical crossings. We will denote it by T_{2n}. One can easily check that the slope of T_{2n} is $3/(6n+1)$ which is equivalent to the knot with slope $(4n+1)/(6n+1)$ [29]. For example, Figure 2 shows the regular projections of knot 7_3 with slope $3/13$ which is equivalent to the knot with slope $9/13$ (right).

We will use the following fundamental group of the knot with Conway's notation $C(2n,3)$ [11, 16, 27]. The following theorem can also be obtained by reading off the fundamental group from the Schubert normal form of T_{2n} with slope $\frac{4n+1}{6n+1}$ [29, 27].

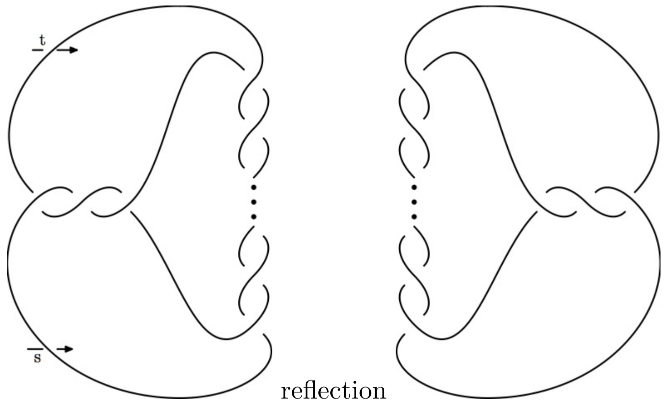

FIGURE 1. A two bridge knot with Conway's notation C[2n,3] (left) and its mirror image C[-2n,-3](right)

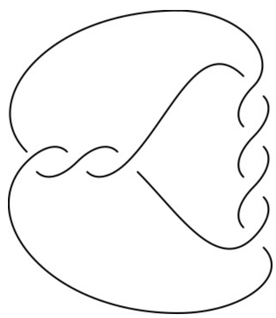

FIGURE 2. The knot 7_3

Proposition 2.1.

$$\pi_1(X_{2n}) = \left\langle s, t \mid swt^{-1}w^{-1} = 1 \right\rangle,$$

where $w = (ts^{-1}tst^{-1}s)^n$.

3. The Riley-Mednykh polynomial

Given a set of generators, $\{s, t\}$, of the fundamental group for $\pi_1(X_{2n})$, we define a representation $\rho : \pi_1(X_{2n}) \to \mathrm{SL}(2, \mathbb{C})$ by

$$\rho(s) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix}, \quad \rho(t) = \begin{bmatrix} M & 0 \\ 2 - M^2 - M^{-2} - t & M^{-1} \end{bmatrix}.$$

Then ρ can be identified with the point $(M, t) \in \mathbb{C}^2$. By [11], when M varies we have an algebraic set whose defining equation is the following Riley-Mednykh polynomial.

Theorem 3.1. *t is a root of the following Riley-Mednykh polynomial $P_{2n} = P_{2n}(t, M)$ which is given recursively by*

$$P_{2n} = \begin{cases} QP_{2(n-1)} - M^8 P_{2(n-2)} & \text{if } n > 1, \\ QP_{2(n+1)} - M^8 P_{2(n+2)} & \text{if } n < -1, \end{cases}$$

with initial conditions

$P_{-2} = M^2t^2 + \left(M^4 - M^2 + 1\right)t + M^2$,

$P_0 = M^6$ for $n \leq 0$ and $P_0 = M^8$ for $n \geq 0$,

$P_2(t, M) = -M^4t^3 + \left(-2M^6 + M^4 - 2M^2\right)t^2 + \left(-M^8 + M^6 - 2M^4 + M^2 - 1\right)t + M^4$,

and

$Q = -M^4t^3 + \left(-2M^6 + 2M^4 - 2M^2\right)t^2 + \left(-M^8 + 2M^6 - 3M^4 + 2M^2 - 1\right)t + 2M^4$.

3.1. **Longitude.** Let $l = ww^*M^{-4n}$, where w^* is the word obtained by reversing w. Let $L = \rho(l)_{11}$. Then l is the longitude which is null-homologus in X_{2n}. And we have

Theorem 3.2. [11]

$$L = -M^{-4n-2}\frac{M^{-2} + t}{M^2 + t}.$$

4. Schläfli formula for the generalized Chern-Simons function

The general references for this section are [14, 15, 31, 21] and [10]. We introduce the generalized Chern-Simons function on the family of $C(2n, 3)$ cone-manifold structures. For the oriented knot T_{2n}, we orient a chosen meridian s such that the orientation of s followed by orientation of T_{2n} coincides with orientation of S^3. Hence, we use the definition of Lens space in [15] so that we can have the right orientation when the definition of Lens space is combined with the following frame field. On the Riemannian manifold $S^3 - T_{2n} - s$ we choose a special frame field Γ. A *special* frame field $\Gamma = (e_1, e_2, e_3)$ is an orthonomal frame field such that for each point x near T_{2n}, $e_1(x)$ has the knot direction, $e_2(x)$ has the tangent direction of a meridian curve, and $e_3(x)$ has the knot to point direction. A special frame field always exists by Proposition 3.1 of [14]. From Γ we obtain an orthonomal frame field Γ_α on $X_{2n}(\alpha) - s$ by the Schmidt orthonormalization process with respect to the geometric structure of the cone manifold $X_{2n}(\alpha)$. Moreover it can be made special by deforming it in a neighborhood of the singular set and s if necessary. Γ' is an extention of Γ to $S^3 - T_{2n}$. For each cone-manifold $X_{2n}(\alpha)$, we assign the real number:

$$I\left(X_{2n}(\alpha)\right) = \frac{1}{2}\int_{\Gamma(S^3 - T_{2n} - s)} Q - \frac{1}{4\pi}\tau(s, \Gamma') - \frac{1}{4\pi}\left(\frac{\beta\alpha}{2\pi}\right),$$

where $-2\pi \leq \beta \leq 2\pi$, Q is the Chern-Simons form:

$$Q = \frac{1}{4\pi^2}\left(\theta_{12} \wedge \theta_{13} \wedge \theta_{23} + \theta_{12} \wedge \Omega_{12} + \theta_{13} \wedge \Omega_{13} + \theta_{23} \wedge \Omega_{23}\right),$$

and

$$\tau(s,\Gamma') = -\int_{\Gamma'(s)} \theta_{23},$$

where (θ_{ij}) is the connection 1-form, (Ω_{ij}) is the curvature 2-form of the Riemannian connection on $X_{2n}(\alpha)$ and the integral is over the orthonomalizations of the same frame field. When $\alpha = \frac{2\pi}{k}$ for some positive integer, $I\left(X_{2n}\left(\frac{2\pi}{k}\right)\right)$ (mod $\frac{1}{k}$ if k is even or mod $\frac{1}{2k}$ if k is odd) is independent of the frame field Γ and of the representative in the equivalence class $\overline{\beta}$ and hence an invariant of the orbifold $X_{2n}\left(\frac{2\pi}{k}\right)$. $I\left(X_{2n}\left(\frac{2\pi}{k}\right)\right)$ (mod $\frac{1}{k}$ if k is even or mod $\frac{1}{2k}$ if k is odd) is called *the Chern-Simons invariant of the orbifold* and is denoted by $\text{cs}\left(X_{2n}\left(\frac{2\pi}{k}\right)\right)$.

On the generalized Chern-Simons function on the family of $C(2n,3)$ cone-manifold structures we have the following Schläfli formula.

Theorem 4.1. *(Theorem 1.2 of* [15]*) For a family of geometric cone-manifold structures, $X_{2n}(\alpha)$, and differentiable functions $\alpha(t)$ and $\beta(t)$ of t we have*

$$dI\left(X_{2n}(\alpha)\right) = -\frac{1}{4\pi^2}\beta d\alpha.$$

5. Proof of the theorem 1.1

For $n \geq 1$ and $M = e^{i\frac{\alpha}{2}}$, $P_{2n}(t, M)$ have $3n$ component zeros, and for $n \leq -1$, $-(3n+1)$ component zeros. The component which gives the maximal volume is the geometric component [7, 8, 11] and in [11] it is identified. For each T_{2n}, there exists an angle $\alpha_0 \in [\frac{2\pi}{3}, \pi)$ such that T_{2n} is hyperbolic for $\alpha \in (0, \alpha_0)$, Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha \in (\alpha_0, \pi]$ [25, 13, 19, 26]. Denote by $D(X_{2n}(\alpha))$ be the set of zeros of the discriminant of $P_{2n}(t, e^{i\frac{\alpha}{2}})$ over t. Then α_0 will be one of $D(X_{2n}(\alpha))$.

On the geometric component we can calculate the Chern-Simons invariant of an orbifold $X_{2n}(\frac{2\pi}{k})$ (mod $\frac{1}{k}$ if k is even or mod $\frac{1}{2k}$ if k is odd), where k is a positive integer such that k-fold cyclic covering of $X_{2n}(\frac{2\pi}{k})$ is hyperbolic:

$$\begin{aligned}
&\text{cs}\left(X_{2n}\left(\frac{2\pi}{k}\right)\right) \equiv I\left(X_{2n}\left(\frac{2\pi}{k}\right)\right) \qquad\qquad \left(\text{mod } \frac{1}{k}\right)\\
&\equiv I\left(X_{2n}(\pi)\right) + \frac{1}{4\pi^2}\int_{\frac{2\pi}{k}}^{\pi} \beta\, d\alpha \qquad\qquad \left(\text{mod } \frac{1}{k}\right)\\
&\equiv \frac{1}{2}\text{cs}\left(L(6n+1, 4n+1)\right) + \frac{1}{4\pi^2}\int_{\frac{2\pi}{k}}^{\alpha_0} Im\left(2 * \log\left(-M^{-4n-2}\frac{M^{-2}+t}{M^2+t}\right)\right) d\alpha\\
&+ \frac{1}{4\pi^2}\int_{\alpha_0}^{\pi} Im\left(\log\left(-M^{-4n-2}\frac{M^{-2}+t_1}{M^2+t_1}\right) + \log\left(-M^{-4n-2}\frac{M^{-2}+t_2}{M^2+t_2}\right)\right) d\alpha\\
&\left(\text{mod } \frac{1}{k} \text{ if } k \text{ is even or mod } \frac{1}{2k} \text{ if } k \text{ is odd}\right)
\end{aligned}$$

where the second equivalence comes from Theorem 4.1 and the third equivalence comes from the fact that $I\left(X_{2n}(\pi)\right) \equiv \frac{1}{2}\text{cs}\left(L(6n+1, 4n+1)\right)$ $\left(\text{mod } \frac{1}{2}\right)$, Theorem 3.2, and geometric interpretations of hyperbolic and spherical holonomy representations.

The following theorem gives the Chern-Simons invariant of the Lens space $L(6n+1, 4n+1)$.

Theorem 5.1. *(Theorem 1.3 of* [15]*)*

$$\text{cs}\left(L\left(6n+1,4n+1\right)\right) \equiv \frac{4n+4}{12n+2} \qquad (mod\ 1).$$

6. Chern-Simons invariants of the hyperbolic orbifolds of the knot with Conway's notation $C(2n,3)$ and of its cyclic coverings

The table 1 (resp. the table 2) gives the approximate Chern-Simons invariant of the hyperbolic orbifold, $\text{cs}\left(X_{2n}(\frac{2\pi}{k})\right)$ for n between 2 and 9 (resp. for n between -9 and -2) and for k between 3 and 10, and of its cyclic covering, $\text{cs}\left(M_k(X_{2n})\right)$. We used Simpson's rule for the approximation with 10^4 (5×10^3 in Simpson's rule) intervals from $\frac{2\pi}{k}$ to α_0 and 10^4 (5×10^3 in Simpson's rule) intervals from α_0 to π. The table 3 gives the approximate Chern-Simons invariant of T_{2n} for each n between -9 and 9 except the unknot and the amphicheiral knot. We again used Simpson's rule for the approximation with 10^4 (5×10^3 in Simpson's rule) intervals from 0 to α_0 and 10^4 (5×10^3 in Simpson's rule) intervals from α_0 to π. We used Mathematica for the calculations. We record here that our data in table 3 and those obtained from SnapPy match up up to six decimal points.

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Table 1: Chern-Simons invariant of the hyperbolic orbifold, $\mathrm{cs}\left(X_{2n}(\frac{2\pi}{k})\right)$ for n between 1 and 9 and for k between 3 and 10, and of its cyclic covering, $\mathrm{cs}\left(M_k(X_{2n})\right)$.

k	$\mathrm{cs}\left(X_2(\frac{2\pi}{k})\right)$	$\mathrm{cs}\left(M_k(X_2)\right)$
3	0.0200137	0.0600411
4	0.186810	0.747239
5	0.00166425	0.00832123
6	0.0504594	0.302756
7	0.0163411	0.114387
8	0.116987	0.935894
9	0.0292866	0.263580
10	0.0595395	0.595395

k	$\mathrm{cs}\left(X_4(\frac{2\pi}{k})\right)$	$\mathrm{cs}\left(M_k(X_4)\right)$
3	0.163905	0.491714
4	0.207480	0.829920
5	0.0602662	0.301331
6	0.140577	0.843464
7	0.0610011	0.427008
8	0.00457501	0.0366000
9	0.0181733	0.163560
10	0.0302655	0.302655

k	$\mathrm{cs}\left(X_6(\frac{2\pi}{k})\right)$	$\mathrm{cs}\left(M_k(X_6)\right)$
3	0.0117308	0.0351925
4	0.0254160	0.101664
5	0.0770172	0.385086
6	0.130155	0.780930
7	0.0343996	0.240797
8	0.0925471	0.740377
9	0.0295838	0.266254
10	0.0810442	0.810442

k	$\mathrm{cs}\left(X_8(\frac{2\pi}{k})\right)$	$\mathrm{cs}\left(M_k(X_8)\right)$
3	0.0392668	0.117800
4	0.115898	0.463593
5	0.0209964	0.104982
6	0.149082	0.894495
7	0.0382671	0.267870
8	0.0866540	0.693232
9	0.0170042	0.153038
10	0.0636841	0.636841

k	$\mathrm{cs}\left(X_{10}(\frac{2\pi}{k})\right)$	$\mathrm{cs}\left(M_k(X_{10})\right)$
3	0.0749335	0.224800
4	0.218720	0.874878
5	0.0783315	0.391658
6	0.0150995	0.0905970
7	0.0560983	0.392688
8	0.0948488	0.758790
9	0.0185935	0.167341
10	0.0605490	0.605490

k	$\mathrm{cs}\left(X_{12}(\frac{2\pi}{k})\right)$	$\mathrm{cs}\left(M_k(X_{12})\right)$
3	0.116132	0.348396
4	0.0784470	0.313788
5	0.0428520	0.214260
6	0.0550832	0.330499
7	0.00986235	0.0690364
8	0.110442	0.883540
9	0.0276064	0.248458
10	0.0648550	0.648550

k	$\mathrm{cs}\left(X_{14}(\frac{2\pi}{k})\right)$	$\mathrm{cs}\left(M_k(X_{14})\right)$
3	0.161005	0.483014
4	0.192332	0.769328
5	0.0116320	0.0581602
6	0.0993703	0.596222
7	0.0393825	0.275677
8	0.00537856	0.0430285
9	0.0409719	0.368747
10	0.0735205	0.735205

k	$\mathrm{cs}\left(X_{16}(\frac{2\pi}{k})\right)$	$\mathrm{cs}\left(M_k(X_{16})\right)$
3	0.0416866	0.125060
4	0.0588936	0.235574
5	0.0831339	0.415670
6	0.146399	0.878396
7	0.000227239	0.00159067
8	0.0280750	0.224600
9	0.00154689	0.0139220
10	0.0849545	0.849545

k	$\mathrm{cs}\left(X_{18}(\frac{2\pi}{k})\right)$	$\mathrm{cs}\left(M_k(X_{18})\right)$
3	0.0907588	0.272277
4	0.177274	0.709096
5	0.0564774	0.282387
6	0.0286139	0.171683
7	0.0343586	0.240510
8	0.0526332	0.421066
9	0.0195418	0.175876
10	0.0982547	0.982547

Table 2: Chern-Simons invariant of the hyperbolic orbifold, $\mathrm{cs}\left(X_{2n}(\frac{2\pi}{k})\right)$ for n between -9 and -2 and for k between 3 and 10, and of its cyclic covering, $\mathrm{cs}\left(M_k(X_{2n})\right)$.

k	$\mathrm{cs}\left(_{-4}(\frac{2\pi}{k})\right)$	$\mathrm{cs}\left(M_k(T_{-4})\right)$
3	0.0578105	0.173431
4	0.0141698	0.0566791
5	0.0771122	0.385561
6	0.113440	0.680638
7	0.0647357	0.453150
8	0.0262590	0.210072
9	0.0506565	0.455908
10	0.0693643	0.693643

k	$\mathrm{cs}\left(T_{-6}(\frac{2\pi}{k})\right)$	$\mathrm{cs}\left(M_k(T_{-6})\right)$
3	0.0502767	0.150830
4	0.206063	0.824252
5	0.0724185	0.362092
6	0.136957	0.821740
7	0.0334583	0.234208
8	0.0770408	0.616327
9	0.0530941	0.477846
10	0.0324771	0.324771

k	$\mathrm{cs}\left(T_{-8}(\frac{2\pi}{k})\right)$	$\mathrm{cs}\left(M_k(T_{-8})\right)$
3	0.0260938	0.0782813
4	0.121024	0.484097
5	0.0343014	0.171507
6	0.123924	0.743545
7	0.0354455	0.248118
8	0.0887397	0.709918
9	0.0158804	0.142923
10	0.0555635	0.555635

k	$\mathrm{cs}\left(T_{-10}(\frac{2\pi}{k})\right)$	$\mathrm{cs}\left(M_k(T_{-10})\right)$
3	0.159369	0.478108
4	0.0211627	0.0846509
5	0.0799373	0.399686
6	0.0941609	0.564965
7	0.0204861	0.143403
8	0.0833782	0.667026
9	0.0170947	0.153852
10	0.0614793	0.614793

k	$\mathrm{cs}\left(T_{-12}(\frac{2\pi}{k})\right)$	$\mathrm{cs}\left(M_k(T_{-12})\right)$
3	0.119699	0.359097
4	0.163139	0.652556
5	0.0170874	0.0854371
6	0.0558073	0.334844
7	0.0683200	0.478240
8	0.0693583	0.554866
9	0.00963738	0.0867365
10	0.0587154	0.587154

k	$\mathrm{cs}\left(T_{-14}(\frac{2\pi}{k})\right)$	$\mathrm{cs}\left(M_k(T_{-14})\right)$
3	0.0758416	0.227525
4	0.0503095	0.201238
5	0.0493320	0.246660
6	0.0125167	0.0751005
7	0.0397753	0.278427
8	0.0503822	0.403058
9	0.0527761	0.474985
10	0.0509898	0.509898

k	$\mathrm{cs}\left(T_{-16}(\frac{2\pi}{k})\right)$	$\mathrm{cs}\left(M_k(T_{-16})\right)$
3	0.0291847	0.0875541
4	0.184443	0.737773
5	0.0785018	0.392509
6	0.132806	0.796838
7	0.00813976	0.0569783
8	0.0283132	0.226505
9	0.0372653	0.335388
10	0.0401699	0.401699

k	$\mathrm{cs}\left(T_{-18}(\frac{2\pi}{k})\right)$	$\mathrm{cs}\left(M_k(T_{-18})\right)$
3	0.147267	0.441800
4	0.0665438	0.266175
5	0.00562151	0.0281075
6	0.0843746	0.506248
7	0.0458762	0.321134
8	0.00418700	0.0334960
9	0.0196972	0.177275
10	0.0272925	0.272925

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Table 3: Chern-Simons invariant of X_{2n} for n between 1 and 9 and for n between -9 and -1).

2n	α_0	cs (X_{2n})
2	2.40717	0.346796
4	2.75511	0.187220
6	2.87826	0.116482
8	2.94175	0.0787607
10	2.98054	0.0554891
12	3.00671	0.0397296
14	3.02556	0.0283589
16	3.03978	0.0197708
18	3.05090	0.0130565

2n	α_0	cs (T_{2n})
-2	2.09440	0
-4	2.68404	0.202492
-6	2.84713	0.287081
-8	2.92433	0.330333
-10	2.96942	0.356274
-12	2.99899	0.373511
-14	3.01989	0.385781
-16	3.03545	0.394957
-18	3.04747	0.402076

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