A Proposal On Culling & Filtering A Coxeter Group
For 4D, $\mathcal{N} = 1$ Spacetime SUSY Representations

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ABSTRACT

We review the mathematical tools required to cull and filter representations of the Coxeter Group BC_4 into providing bases for the construction of minimal off-shell representations of the 4D, $\mathcal{N} = 1$ spacetime supersymmetry algebra. Of necessity this includes a description of the mathematical mechanism by which four dimensional Lorentz symmetry appears as an emergent symmetry in the context of one dimensional adinkras with four colors described by the Coxeter Group BC_4.

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1 Introduction

Recently, one of us (D.E.A.G.) gave a presentation at the 2015 Miami Topical Physics Conference and during a question-and-answer session afterward there arose a query from Prof. J. Lukierski about the non-obvious relationship between the Euclidean SO(4) symmetry manifest in the use of adinkras [1] and their adjacency matrices [2,3] versus the Lorentzian structure required to describe the usual theories of interest realizing the 4D, $\mathcal{N} = 1$ spacetime supersymmetry algebra.

This exchange motivates us to review tools developed in previous works to show evidence that although adinkras with four colors, four open nodes, and four closed nodes manifestly realize a Euclidean SO(4) symmetry there is a "hidden path" that also relates each of them to a set of SO(1,3) Dirac $\boldsymbol{\gamma}$-matrices.

First, we review results that are standard to the usual Dirac matrices appropriate for a four dimensional Minkowski space. We also show how the generators of SO(4) possess an apparently often overlooked relationship to the Dirac matrices appropriate for a four dimensional Minkowski space. There follows a discussion of the L-matrices and R-matrices [2] associated with every adinkra graph [1] obtainable by a modification of the standard notion of an adjacency graph [3]. The commutator algebra of the L-matrices and R-matrices define the 'holoraumy' [4,5,6] associated with the graph.

In this presentation, we restrict ourselves to the case of adinkras with four colors, four open nodes, and four closed nodes because we constructed all possible representations of this kind in a previous analysis [8]. Using a computer software program, it was found one can start with the Coxeter Group BC_4 and associate with every element of this group to some adinkra. In particular, quartets of elements of BC_4 form representations of the adjacency matrices associated with this class of adinkras. This previous work showed there are 1,536 such quartets. By taking the elements of the BC_4 Coxeter Group as our starting point, we have a rigorous mathematically well-defined beginning for our analysis. The fifth section presents the criteria by which a subset of the elements of Coxeter Group BC_4 can be consistently interpreted as a projection of the 4D, $\mathcal{N} = 1$ fundamental irreducible supermultiplet representations. Thus we also identify obstructions that prevent such identifications for all the elements of BC_4.

2 Connecting Dirac SO(1,3) γ-Matrices To SO(4) Rotation Matrices

A set of Dirac γ-matrices is provided by $\boldsymbol{\gamma}^\mu = (\boldsymbol{\gamma}^0, \boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2, \boldsymbol{\gamma}^3)$ and must satisfy the usual condition

$$\boldsymbol{\gamma}^\mu \boldsymbol{\gamma}^\nu + \boldsymbol{\gamma}^\nu \boldsymbol{\gamma}^\mu = 2\eta^{\mu\nu} \mathbf{I}_4 \quad , \tag{2.1}$$

where the 4×4 identity matrix is denoted by $\mathbf{I}_4$ and the Minkowski metric $\eta^{\mu\nu}$ in (2.1) has non-vanishing diagonal entries $(-1, +1, +1, +1)$.

Given a set of Dirac gamma matrices $\boldsymbol{\gamma}^\mu$, we define $\boldsymbol{\gamma}^5$ via the usual definition

$$\boldsymbol{\gamma}^5 = i\boldsymbol{\gamma}^0\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 \quad , \tag{2.2}$$

and define a representation of the generators of spatial rotations provided by the set containing the

three elements $\{\,\boldsymbol{\sigma}^{12},\,\boldsymbol{\sigma}^{23},\,\boldsymbol{\sigma}^{31}\,\}$ where

$$\boldsymbol{\sigma}^{12}\;=\;-\,i\,\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\;\;,\;\;\;\boldsymbol{\sigma}^{23}\;=\;-\,i\,\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3\;\;,\;\;\;\boldsymbol{\sigma}^{31}\;=\;-\,i\,\boldsymbol{\gamma}^3\boldsymbol{\gamma}^1\;\;.\tag{2.3}$$

The commutator algebra of these takes the usual form

$$[\,\boldsymbol{\sigma}^{12}\,,\,\boldsymbol{\sigma}^{23}\,]\;=\;i\,2\,\boldsymbol{\sigma}^{31}\;\;,\;\;\;[\,\boldsymbol{\sigma}^{23}\,,\,\boldsymbol{\sigma}^{31}\,]\;=\;i\,2\,\boldsymbol{\sigma}^{12}\;\;,\;\;\;[\,\boldsymbol{\sigma}^{31}\,,\,\boldsymbol{\sigma}^{12}\,]\;=\;i\,2\,\boldsymbol{\sigma}^{23}\;\;,\tag{2.4}$$

with all other commutators vanishing.

We now introduce another set of matrices containing three elements $\{\,i\,\boldsymbol{\gamma}^0,\,\boldsymbol{\gamma}^5,\,\boldsymbol{\gamma}^0\boldsymbol{\gamma}^5\,\}$. The commutator algebra of these elements is

$$[\,i\,\boldsymbol{\gamma}^0\,,\,\boldsymbol{\gamma}^5\,]\;=\;i\,2\,\boldsymbol{\gamma}^0\boldsymbol{\gamma}^5\;\;,\;\;\;[\,\boldsymbol{\gamma}^5\,,\,\boldsymbol{\gamma}^0\boldsymbol{\gamma}^5\,]\;=\;i\,2\,(i\,\boldsymbol{\gamma}^0)\;\;,\;\;\;[\,\boldsymbol{\gamma}^0\boldsymbol{\gamma}^5\,,\,i\,\boldsymbol{\gamma}^0\,]\;=\;i\,2\,(\boldsymbol{\gamma}^5)\;\;,\tag{2.5}$$

with all other commutators vanishing. The form of this commutator algebra (2.5) is identical to the one in (2.4) and both are recognizable as SU(2) algebras. Furthermore, it is easy to show

$$[\,i\,\boldsymbol{\gamma}^0\,,\,\boldsymbol{\sigma}^{ij}\,]\;=\;0\;\;,\;\;\;[\,\boldsymbol{\gamma}^5\,,\,\boldsymbol{\sigma}^{ij}\,]\;=\;0\;\;,\;\;\;[\,\boldsymbol{\gamma}^0\boldsymbol{\gamma}^5\,,\,\boldsymbol{\sigma}^{ij}\,]\;=\;0\;\;.\tag{2.6}$$

This all implies given a set of Dirac $\boldsymbol{\gamma}$-matrices together with a $\boldsymbol{\gamma}^5$-matrix it is possible to construct the six matrices above (2.3) and (2.5), each forming a representation of an SU(2) algebra, and each SU(2) algebra commutes with the other.

A standard result for $\boldsymbol{\gamma}$-matrices implies

$$\boldsymbol{\gamma}^1\;=\;\boldsymbol{\gamma}^0\boldsymbol{\gamma}^5\,\boldsymbol{\sigma}^{23}\;\;,\;\;\;\boldsymbol{\gamma}^2\;=\;\boldsymbol{\gamma}^0\boldsymbol{\gamma}^5\,\boldsymbol{\sigma}^{31}\;\;,\;\;\;\boldsymbol{\gamma}^3\;=\;\boldsymbol{\gamma}^0\boldsymbol{\gamma}^5\,\boldsymbol{\sigma}^{12}\;\;,\tag{2.7}$$

which demonstrates that given the data of the two distinct SU(2) matrices, the three spatial $\boldsymbol{\gamma}$-matrices can be reconstructed. Actually, the information in both commuting SU(2)'s is over-complete as it is only the "third" component of the "non-orbital" SU(2) along the three components of the "orbital" SU(2) that are required.

At this point, a different set of 4×4 matrices can be introduced via the definitions

$$\begin{aligned}
\boldsymbol{\alpha}^1\;&=\;\boldsymbol{\sigma}^2\otimes\boldsymbol{\sigma}^1\;\;,\;\;\;&\boldsymbol{\beta}^1\;&=\;\boldsymbol{\sigma}^1\otimes\boldsymbol{\sigma}^2\;\;,\\
\boldsymbol{\alpha}^2\;&=\;\mathbf{I}\otimes\boldsymbol{\sigma}^2\;\;,\;\;\;&\boldsymbol{\beta}^2\;&=\;\boldsymbol{\sigma}^2\otimes\mathbf{I}\;\;,\\
\boldsymbol{\alpha}^3\;&=\;\boldsymbol{\sigma}^2\otimes\boldsymbol{\sigma}^3\;\;,\;\;\;&\boldsymbol{\beta}^3\;&=\;\boldsymbol{\sigma}^3\otimes\boldsymbol{\sigma}^2\;\;,
\end{aligned}\tag{2.8}$$

where these matrices satisfy the identities

$$\begin{aligned}
\boldsymbol{\alpha}^{\widehat{I}}\boldsymbol{\alpha}^{\widehat{K}}\;&=\;\delta^{\widehat{I}\widehat{K}}\,\mathbf{I}_4\;+\;i\,\epsilon^{\widehat{I}\widehat{K}\widehat{L}}\,\boldsymbol{\alpha}^{\widehat{K}}\;\;,\\
\boldsymbol{\beta}^{\widehat{I}}\boldsymbol{\beta}^{\widehat{K}}\;&=\;\delta^{\widehat{I}\widehat{K}}\,\mathbf{I}_4\;+\;i\,\epsilon^{\widehat{I}\widehat{K}\widehat{L}}\,\boldsymbol{\beta}^{\widehat{K}}\;\;,\\
[\,\boldsymbol{\alpha}^{\widehat{I}}\,,\,\boldsymbol{\beta}^{\widehat{J}}\,]\;&=\;0\;\;.
\end{aligned}\tag{2.9}$$

The commutator algebra derivable from (2.9) allows us to identify the six matrices (2.8) as the hermitian 4×4 matrix generators of SO(4). We also have

$$\mathrm{Tr}\big(\boldsymbol{\alpha}^{\widehat{I}}\boldsymbol{\alpha}^{\widehat{J}}\big)\;=\;\mathrm{Tr}\big(\boldsymbol{\beta}^{\widehat{I}}\boldsymbol{\beta}^{\widehat{J}}\big)\;=\;4\,\delta^{\widehat{I}\widehat{J}}\;\;,\;\;\;\mathrm{Tr}\big(\boldsymbol{\alpha}^{\widehat{I}}\boldsymbol{\beta}^{\widehat{J}}\big)\;=\;0\;\;,$$
$$\mathrm{Tr}\big(\boldsymbol{\alpha}^{\widehat{I}}\big)\;=\;\mathrm{Tr}\big(\boldsymbol{\beta}^{\widehat{I}}\big)\;=\;0\;\;.\tag{2.10}$$

3

However, the commutator algebra defined by (2.4), (2.5), and (2.6) is isomorphic to that which is derivable from (2.9). Hence both are representations of SO(4). Therefore, the Dirac gamma matrices can be expressed using the '$\boldsymbol{\alpha}$-set' and '$\boldsymbol{\beta}$-set'. One such set of definitions are

$$\boldsymbol{\gamma}^0 = i\,\boldsymbol{\beta}^3 \quad , \quad \boldsymbol{\gamma}^1 = \boldsymbol{\alpha}^1\boldsymbol{\beta}^2 \;\; , \quad \boldsymbol{\gamma}^2 = \boldsymbol{\alpha}^2\boldsymbol{\beta}^2 \quad , \quad \boldsymbol{\gamma}^3 = \boldsymbol{\alpha}^3\boldsymbol{\beta}^2 \;\; , \tag{2.11}$$

which imply

$$\boldsymbol{\sigma}^{1\,2} \;=\; \boldsymbol{\alpha}^3 \;\; , \quad \boldsymbol{\sigma}^{2\,3} \;=\; \boldsymbol{\alpha}^1 \;\; , \quad \boldsymbol{\sigma}^{3\,1} \;=\; \boldsymbol{\alpha}^2 \;\; . \tag{2.12}$$

These equations make manifest that the $\boldsymbol{\sigma}^{ij}$-matrices are the generators of SU(2). A definition of the $\boldsymbol{\gamma}^5$ matrix chosen as

$$\boldsymbol{\gamma}^5 \;=\; -\,\boldsymbol{\beta}^1 \quad , \tag{2.13}$$

together with the definition of $\boldsymbol{\gamma}^0$ shown above implies

$$\boldsymbol{\gamma}^0\boldsymbol{\gamma}^5 \;=\; \boldsymbol{\beta}^2 \quad . \tag{2.14}$$

It is not commonly noted that given a set of four dimensional gamma matrices, it is possible to use them to define *two mutually commuting* SU(2)'s and this is valid independent of the representation chosen for the gamma matrices.

Let us also note the identifications between the $\boldsymbol{\gamma}$-matrices on the one side of the equations (2.11) and (2.12) and the $\boldsymbol{\alpha}$-matrices and the $\boldsymbol{\beta}$-matrices on the other are not unique. One can cyclically permute, independently, the $\boldsymbol{\alpha}$-matrices and the $\boldsymbol{\beta}$-matrices in these equation and this still leads to a properly defined set of $\boldsymbol{\gamma}$-matrices. Similarly, one can perform the exchanges of the form $\boldsymbol{\alpha}^{\widehat{I}} \leftrightarrow \boldsymbol{\beta}^{\widehat{I}}$ simultaneously in the equations (2.11) and (2.12) and this also leads to a properly defined set of $\boldsymbol{\gamma}$-matrices. It is an open question to ask, "What is the group of transformations acting on the $\boldsymbol{\alpha}$-matrices and the $\boldsymbol{\beta}$-matrices in (2.11) such that the so defined $\boldsymbol{\gamma}$-matrices satisfy (2.1)?" The bottom line is there is *not* a unique way for a set of Dirac $\boldsymbol{\gamma}$-matrices for a four dimensional Minkowski space with Lorentz symmetry to emerge from the two commuting SU(2) groups constructed from SO(4).

However, the fact that SO(4) can 'secretly' carry information about SO(1,3) spinors and $\boldsymbol{\gamma}$-matrices is one of the important mechanisms for use of adinkras with four colors to describe 4D, $\mathcal{N}$ = 1 SUSY theories that utilize Minkowski space spinors.

3   From L-matrices, R-matrices to Dirac γ-matrices

In this section, we will show how the L-matrices and R-matrices [2] that occur in the description of any adinkra graph [1], [3] with four colors (I $= 1, \ldots, 4$) four open nodes ($i = 1, \ldots, 4$), and four closed nodes ($\hat{k} = 1, \ldots, 4$) are related to a set of SO(4) rotation matrices. From the last section, we showed there exist a possibility of linking the $\boldsymbol{\gamma}$-matrices of SO(1,3) to the SO(4) rotation matrices. Combining these two results, we thus find a pathway that connects all adinkras with four colors, four open nodes, and four closed nodes to the representations of SO(1,3) $\boldsymbol{\gamma}$-matrices.

Every adinkra representation ($\mathcal{R}$) leads to a set of four adjacency matrices denoted by $\mathrm{L}_{\mathrm{I}}^{(\mathcal{R})}$ and

$R_I^{(\mathcal{R})}$ which satisfy the conditions

$$(L_I^{(\mathcal{R})})_i{}^{\hat{j}} \, (R_J^{(\mathcal{R})})_{\hat{j}}{}^{k} \; + \; (L_J^{(\mathcal{R})})_i{}^{\hat{j}} \, (R_I^{(\mathcal{R})})_{\hat{j}}{}^{k} = 2 \, \delta_{IJ} \, \delta_i{}^k \ ,$$
$$(R_J^{(\mathcal{R})})_{\hat{i}}{}^{j} \, (L_I^{(\mathcal{R})})_j{}^{\hat{k}} \; + \; (R_I^{(\mathcal{R})})_{\hat{i}}{}^{j} \, (L_J^{(\mathcal{R})})_j{}^{\hat{k}} = 2 \, \delta_{IJ} \, \delta_{\hat{i}}{}^{\hat{k}} \ ,$$
$$(R_I^{(\mathcal{R})})_{\hat{j}}{}^{k} \, \delta_{ik} = (L_I^{(\mathcal{R})})_i{}^{\hat{k}} \, \delta_{\hat{j}\hat{k}} \ . \tag{3.1}$$

This we call the "Garden Algebra." Given a set of L-matrices and R-matrices for a specified adinkra representation ($\mathcal{R}$), we can define two additional matrix sets denoted by $\boldsymbol{V_{IJ}^{(\mathcal{R})}}$ and $\boldsymbol{\widetilde{V}_{IJ}^{(\mathcal{R})}}$ [4,5,6] via the equations

$$(L_I^{(\mathcal{R})})_i{}^{\hat{j}} \, (R_J^{(\mathcal{R})})_{\hat{j}}{}^{k} \; - \; (L_J^{(\mathcal{R})})_i{}^{\hat{j}} \, (R_I^{(\mathcal{R})})_{\hat{j}}{}^{k} = i \, 2 \, (V_{IJ}^{(\mathcal{R})})_i{}^{k} \ ,$$
$$(R_J^{(\mathcal{R})})_{\hat{i}}{}^{j} \, (L_I^{(\mathcal{R})})_j{}^{\hat{k}} \; - \; (R_I^{(\mathcal{R})})_{\hat{i}}{}^{j} \, (L_J^{(\mathcal{R})})_j{}^{\hat{k}} = i \, 2 \, (\widetilde{V}_{IJ}^{(\mathcal{R})})_{\hat{i}}{}^{\hat{k}} \ . \tag{3.2}$$

We have given the name of "bosonic holoraumy matrices" to the quantities $\boldsymbol{V_{IJ}^{(\mathcal{R})}}$ and "fermionic holoraumy matrices" to the quantities $\boldsymbol{\widetilde{V}_{IJ}^{(\mathcal{R})}}$ defined here. Due to the definitions in (3.1), it follows that both sets of holoraumy matrices satisfy the commutator algebra that describes SO(4). Since the $\boldsymbol{\widetilde{V}_{IJ}^{(\mathcal{R})}}$ matrices act in the spinor space of the adinkras, we concentrate upon it. This means we can write an equation of the form

$$\boldsymbol{\widetilde{V}}_{IJ}^{(\mathcal{R})} \; = \; \left[\, \ell_{IJ}^{(\mathcal{R})1} \, \boldsymbol{\alpha^1} \; + \; \ell_{IJ}^{(\mathcal{R})2} \, \boldsymbol{\alpha^2} \; + \; \ell_{IJ}^{(\mathcal{R})3} \, \boldsymbol{\alpha^3} \, \right]$$
$$+ \; \left[\, \widetilde{\ell}_{IJ}^{(\mathcal{R})1} \, \boldsymbol{\beta^1} \; + \; \widetilde{\ell}_{IJ}^{(\mathcal{R})2} \, \boldsymbol{\beta^2} \; + \; \widetilde{\ell}_{IJ}^{(\mathcal{R})3} \, \boldsymbol{\beta^3} \, \right] \ , \tag{3.3}$$

for some set of coefficients $\ell_{IJ}^{(\mathcal{R})1}$, $\ell_{IJ}^{(\mathcal{R})2}$, $\ell_{IJ}^{(\mathcal{R})3}$, $\widetilde{\ell}_{IJ}^{(\mathcal{R})1}$, $\widetilde{\ell}_{IJ}^{(\mathcal{R})2}$, and $\widetilde{\ell}_{IJ}^{(\mathcal{R})3}$. Using the results of the last chapter, this becomes

$$\boldsymbol{\widetilde{V}}_{IJ}^{(\mathcal{R})} \; = \; \left[\, \ell_{IJ}^{(\mathcal{R})1} \, \boldsymbol{\Sigma^{23}} \; + \; \ell_{IJ}^{(\mathcal{R})2} \, \boldsymbol{\Sigma^{31}} \; + \; \ell_{IJ}^{(\mathcal{R})3} \, \boldsymbol{\Sigma^{12}} \, \right]$$
$$+ \; \left[\, - \, \widetilde{\ell}_{IJ}^{(\mathcal{R})1} \, \boldsymbol{\gamma^5} \; + \; \widetilde{\ell}_{IJ}^{(\mathcal{R})2} \, \boldsymbol{\gamma^0 \gamma^5} \; + \; i \, \widetilde{\ell}_{IJ}^{(\mathcal{R})3} \, \boldsymbol{\gamma^0} \, \right] \ . \tag{3.4}$$

We have referred to (3.4) in the past [4] as the "Adinkra/$\boldsymbol{\gamma}$-matrix Holography Equation."

The importance of (3.4) when combined with (2.7) is that it implies that for any four color, four open-node, four-closed node adinkra along with the introduction of the complete specification of two distinct commuting SU(2) algebras, $\{\Sigma^{ij}\}$ and $\{i\,\gamma^0, \gamma^5, \gamma^0\gamma^5\}$, derivable from adinkras, it is possible to find a set of three spatial γ-matrices and connect to the Lorentz symmetries. The link between any specific adinkra, of the type under consideration, to the representations of the Minkowski space SU(2) algebras, $\{\Sigma^{ij}\}$ and $\{i\,\gamma^0, \gamma^5, \gamma^0\gamma^5\}$, is specified by the constants $\ell_{IJ}^{(\mathcal{R})1}$, $\ell_{IJ}^{(\mathcal{R})2}$, $\ell_{IJ}^{(\mathcal{R})3}$, $\widetilde{\ell}_{IJ}^{(\mathcal{R})1}$, $\widetilde{\ell}_{IJ}^{(\mathcal{R})2}$, and $\widetilde{\ell}_{IJ}^{(\mathcal{R})3}$.

4 The Coxeter Group BC_4 Embedding Starting Point

For our purposes, we can define the elements of BC_4 [7] in the following manner. Consider the set of all real 4×4 matrices of the form [8]

$$\mathbf{L} \; = \; \boldsymbol{\mathcal{S} \, \mathcal{P}} \tag{4.1}$$

We call the matrix $\mathcal{S}$ the "Boolean Factor" [8] as it is a real diagonal 4×4 matrix that squares to the identity. The matrix $\mathcal{P}$ is a matrix representation of a permutation of 4 objects. There are $2^d d! = 4! \times 2^4 = 384$ ways to choose the Boolean Factor *and* the Permutation matrix. This is the dimension of the Coxeter group BC_4.

By embedding the L-matrices as the elements in the entirety of BC_4 we know that for each one we can write the equation

$$(\mathrm{L}_\mathrm{I}{}^{(\mathcal{R})})_i{}^{\hat{k}} = [\mathcal{S}^{(\mathrm{I})(\mathcal{R})}]_i{}^{\hat{\ell}} [\mathcal{P}_{(\mathrm{I})}{}^{(\mathcal{R})}]_{\hat{\ell}}{}^{\hat{k}}, \qquad \text{for each fixed I} = 1, 2, \ldots, N. \tag{4.2}$$

This notation anticipates that there are distinct adinkra representations denoted by the label $(\mathcal{R})$ and each adinkra leads to four matrices labeled by the index I. In other words, the L-matrix for a single fixed value of I can be chosen to be any element in the Coxeter group BC_4.

So if we were simply picking quartets of distinct elements of the Coxeter group BC_4 in an arbitrary manner there would be n_4 where

$$n_4 = \frac{384 \cdot 383 \cdot 382 \cdot 381}{4!} = 891,881,376 \tag{4.3}$$

ways to pick the elements. However, we wish to pick the distinct four elements of the BC_4 Coexeter Group so that they satisfy the "Garden Algebra." This requirement is so severe there are only 1,536 ways in which the four elements of the BC_4 Coexeter Group can be chosen to form a supersymmetry quartet. This was discovered by utilizing a code [8] to exhaustively construct all possible quartets starting from the BC_4 Coexeter Group elements. The label $(\mathcal{R})$ written in (4.2) takes its values over these representations.

This startlingly smaller number is mostly determined by the permutation elements from which any L-matrix is constructed. It turns out only particular choices of the permutation elements can appear within any given quartet. This is shown in the following collections of sets

$$\begin{aligned}
\{\mathcal{P}_1\} &= \{(123), (134), (142), (243)\} = (123)\{\mathcal{V}\} \quad, \\
\{\mathcal{P}_2\} &= \{(124), (132), (143), (234)\} = (124)\{\mathcal{V}\} \quad, \\
\{\mathcal{P}_3\} &= \{(14), (23), (1243), (1342)\} = (14)\{\mathcal{V}\} \quad, \\
\{\mathcal{P}_4\} &= \{(13), (24), (1234), (1432)\} = (13)\{\mathcal{V}\} \quad, \\
\{\mathcal{P}_5\} &= \{(12), (34), (1243), (1324)\} = (12)\{\mathcal{V}\} \quad, \\
\{\mathcal{P}_6\} &= \{(), (12)(34), (13)(24), (14)(23)\} = \{\mathcal{V}\} \quad,
\end{aligned} \tag{4.4}$$

where we have used cycle notation to indicate the distinct permutations and relate all the permutations to the Vierergruppe[3] denoted by $\{\mathcal{V}\}$ [9] thus making manifest its critical role.

The action of transposition (denoted by the symbol *) on these sets is straightforward to calculate and we find

$$\begin{aligned}
&{}^*\{\mathcal{P}_1\} = \{\mathcal{P}_2\}, & &{}^*\{\mathcal{P}_3\} = \{\mathcal{P}_3\}, & &{}^*\{\mathcal{P}_5\} = \{\mathcal{P}_5\}, \\
&{}^*\{\mathcal{P}_2\} = \{\mathcal{P}_1\}, & &{}^*\{\mathcal{P}_4\} = \{\mathcal{P}_4\}, & &{}^*\{\mathcal{P}_6\} = \{\mathcal{P}_6\},
\end{aligned} \tag{4.5}$$

and in writing this, we define two sets to be the same if they contain the same elements irrespective of the order in which they appear. In Figure (1) these subsets of permutations together with the action of the * map are shown.

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[3] We thank our colleague K. Iga for this observation.
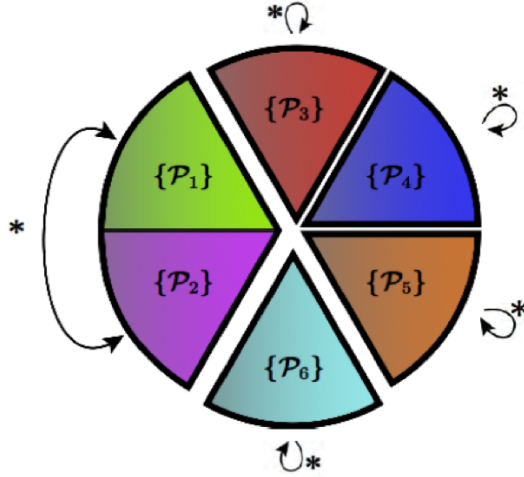
Figure 1: Orbit space of $\mathcal{P}$ permutation matrices under the $*$ map

5   Culling & Filtering

In a previous work [10], there was presented an obstruction that indicated when an adinkra with two colors was compatible with being the projection of a two dimensional supermultiplet. It was shown there exist what we may call the "no two-color ambidextrous bow tie" theorem which asserted if an adinkra graph contained a certain structure, then it was not possible to consistently "lift" the adinkra graph in such a way that it could be associated with a supermultiplet defined on a Minkowski space with a Lorentzian metric with diagonal entries of the (-1,1) variety.

Up until now we have made no similar comments about when an arbitrary four-color adinkra can be regarded as being associated with the projection of a supermultiplet defined on a Minkowski space with a Lorentzian metric with diagonal entries of the (-1, 1, 1, 1) variety. Due to our analysis of BC_4, we now have enough hints so as be comfortable laying out a set of analogous requirements for all adinkras based on BC_4 elements. We are now ready to describe how a subset of the elements of BC_4 can be used to construct the off-shell minimal representations of 4D, $\mathcal{N} = 1$ SUSY.

This chapter will depend crucially on some conjectures for which we do not have closed-form explicit mathematical proofs.

> *Conjecture # 1:*
> *Given an arbitrary element in BC_4 it is always possible to find three additional distinct elements so that this quartet of distinct elements satisfies the "Garden Algebra" with four colors.*

> *Conjecture # 2:*
> *Given an a quartet of elements in BC_4 that satisfies the "Garden Algebra" with four colors, their holoramy tensors always takes the form given in (3.3) and (3.4) with either all of the ℓ-coefficients equal to zero or all of the $\widetilde{\ell}$-coefficients equal to zero.*

Although we do not have a closed-form explicit mathematical proof of the first conjecture, the

explicit construction in the work of [8] gives us confidence in its validity. The code described therein constitutes a proof by exhaustive examination. For the second conjecture, we can make no such claim. We have examined some number of specific cases and these all indicated such a result holds. Under the assumption of the correctness of the two conjectures, the process of culling and filtering of the elements of BC_4 to consistently describe 4D, $\mathcal{N} = 1$ spacetime supermultiplets only requires the application of the tools of the $*$-map and the holoraumy tensor $\widetilde{\boldsymbol{V}}_{\text{IJ}}^{(\mathcal{R})}$.

One can pick an arbitrary element of BC_4 and examine how it behaves with respect to the $*$-map acting on the permutation upon which the element is constructed. The permutation associated with the element will be in one of the "even" sets ($\{\mathcal{P}_3\}$ thru $\{\mathcal{P}_6\}$) or one of the "odd" sets ($\{\mathcal{P}_1\}$ or $\{\mathcal{P}_2\}$). If the permutation associated with the element is in one of the "even" sets, we next calculate the holoraumy associated with it. Let us call the element our base element. By conjecture (# 2) this must take to the form of (3.3) with half of the coefficients vanishing.

Now there comes a subtlety. In going from (3.3) to (3.4) there is an ambiguity. To go from the former to the latter required the identifications made in (2.11) and (2.12). However, as we discussed below the latter equations, there is always an inherent ambiguity as identified in the discussion above (2.7). So using this ambiguity we can simply declare that whatever explicit matrices emerge from the holoraumy associated with this base element are associated with the orbital SU(2).

To the skeptical reader on this point, we should also note this also emphasizes that the 4D Lorentz symmetry is an emergent symmetry. Before the choice of which adinkra based SU(2) symmetry corresponds to the orbital SU(2) of Minkowski space, both adinkra based SU(2) symmetry groups are equivalent.

This choice immediately culls and filters the rest of the BC_4 elements dependent on even permutations. Given a second element dependent upon an even permutation, if its holoraumy tensor commutes with that of the base element, this second element does not provide an example that can be reached by projection of any 4D, $\mathcal{N} = 1$ spacetime supermultiplet. On the other hand, given a second element dependent upon an even permutation, if its holoraumy tensor does not commute with that of the base element, this second element does provide an example that can be reached by projection of a 4D, $\mathcal{N} = 1$ spacetime supermultiplet.

The process we have described above provides a set theoretic definition of a 4D, $\mathcal{N} = 1$ spacetime vector supermultiplet based solely on the properties of elements of BC_4. We must still consider the BC_4 elements that depend on odd permutations.

For the BC_4 elements dependent on odd permutations, we now imagine calculating the holoraumy tensors. Given conjecture (# 2), some of these will have holoraumy tensors that commute with the vector supermultiplet holoraumy tensor as defined above. Others will have holoraumy tensors that do not commute with the vector supermultiplet holoraumy tensor as defined above.

If the BC_4 elements dependent on odd permutations possess holoraumy tensors that commute with the vector supermultiplet holoraumy tensor as defined above, then such elements describe the projections of 4D, $\mathcal{N} = 1$ spacetime chiral supermultiplets.

If the BC_4 elements dependent on odd permutations possess holoraumy tensors that do not commute with the vector supermultiplet holoraumy tensor as defined above, then such elements describe the projections of 4D, $\mathcal{N} = 1$ spacetime tensor supermultiplets.

8

Notice that the definitions given above depend only on structures that are intrinsic to BC_4. So these are "BC_4-centric" definitions of the chiral, vector, and tensor multiplet adinkras that do not require any information from the higher dimensional supermultiplets. Only the behavior of the BC_4 elements under the $*$-map and the holoraumy tensors associated with each BC_4 element have been used to define the respective adinkras to be associated with each off-shell supermultiplet. Although there is nothing in these definition that depend on structures outside BC_4, the requirement on the commutativity or non-commutativity of the various holoraumies is motivated by the study [6] where these conditions were found to hold in four dimensional description of these supermultiplets.

The arguments are a little bit more involved if one begins the analysis from a starting point of picking a element of BC_4 that depends on odd permutations. But with appropriate modifications, the same final result occurs.

6 Conclusion

Before leaving entirely the realm of conjectures, there is one more that we would like to present. This one is not confined to adinkras related to BC_4.

In a recent fascinating development [14] in this general line of research on Garden Algebras, adinkras, and codes (GAAC), there has appeared indications that adinkras can be interpreted as objects possessing algebraic geometrical descriptions as punctures of Riemann surfaces. With this occurrence, we conjecture the holoraumy matrices defined by $\widetilde{\boldsymbol{V}}_{\mathrm{IJ}}^{(\mathcal{R})}$ will also likely be related to a construction with a basis in algebraic geometry.

> *Conjecture # 3*:
> *The holoraumy matrices $\widetilde{\boldsymbol{V}}_{\mathrm{IJ}}^{(\mathcal{R})}$ that can be constructed from the L-matrices*
> *and R-matrices of the "Garden Algebra" may be obtained from an algebraic*
> *geometrical construction based on monodromy matrices.*

In this paper, we have attempted to repeat the path pioneered by the work in [10] that showed how adinkras in one dimension can be extended to understand when such adinkras also allow the interpretation of being the reductions of 2D, $\mathcal{N} = 1$ supermultiplets. The work in [10] can be interpreted as the analog of the integration of a 1-cycle along a closed path. More recently [11], however, there has been introduced another methodology only based on the codes. Older works, [12] had made note of the role of codes in defining irreducible representations of adinkras that descend from four dimensions. But the work in [11] emphasizes that codes also play a role in understanding dimensional enhancement from 1D to 2D. In the light of the result in [13] on fermionic dimensional enhancement, it would be an interesting investigation to see how codes play a role in that result.

If the conjectures made in this paper are valid, the path now seems open for deriving how adrinkras and restriction places there upon give rise to supersymmetrical representations in *all* higher dimensions.

"A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas."

- G. H. Hardy

References

[1] M. Faux, S. J. Gates Jr. "Adinkras: A Graphical Technology for Supersymmetric Representation Theory," Phys. Rev. **D71** (2005) 065002, [hep-th/0408004].

[2] S. J. Gates and L. Rana, "A Theory of Spinning Particles for Large N-extended Supersymmetry (I)," Phys. Lett. **B352** (1995) 50, arXiv [hep-th:9504025]; S. J. Gates, Jr., and L. Rana, "A Theory of Spinning Particles for Large N-extended Supersymmetry (II)," ibid. Phys. Lett. **B369** (1996) 262, arXiv [hep-th:9510151].

[3] B. Bollobás, *Modern Graph Theory*, Springer (1998), p. 52.

[4] S. J. Gates, Jr., "The Search for Elementarity Among Off-Shell SUSY Representations," Korean Institute for Advanced Study (KIAS) Newsletter, Vol. 5 1012, p. 19; S. J. Gates, Jr., T. Hübsch, and K. Stiffler, "Adinkras and SUSY Holography: Some Explicit Examples," Int. J. Mod. Phys. **A29** (2014) 07, 1450041, e-Print: arXiv:1208.5999 [hep-th].

[5] S. J. Gates, Jr., T. Hübsch, and K. Stiffler, "On Clifford-algebraic Dimensional Extension and SUSY Holography," Int. J. Mod. Phys. **A30** (2015) 09, 1550042, e-Print: arXiv:1409.4445 [hep-th]; M. Calkins, D. E. A. Gates, S. J. Gates, Jr., and K. Stiffler, "Adinkras, 0-branes, Holoraumy and the SUSY QFT/QM Correspondence," Int. J. Mod. Phys. **A30** (2015) 11, 1550050, e-Print: arXiv:1501.00101 [hep-th].

[6] S. J. Gates, Jr., , T. Grover, M. David Miller-Dickson, B. A. Mondal, A. Oskoui, S. Regmi, E. Ross, and R. Shetty "A Lorentz Covariant Holoraumy-Induced Gadget From Minimal Off-shell 4D, N=1 Supermultiplets," JHEP **1511** (2015) 113, arXiv:1508.07546 [hep-th].

[7] H. S. M. Coxeter, "Discrete groups generated by reflections," Ann. Of Math. **35** (3): 588621, (1934) JSTOR 1968753; N. Bourbaki, *Elements of Mathematics*, Lie Groups and Lie Algebras: Ch. 4-6, (2002) Springer, ISBN 978-3-540-42650-9, Zbl 0983.1700.1

[8] I. Chappell, II, S. J. Gates, Jr, and T. Hübsch, "Adinkra (In)Equivalence From Coxeter Group Representations: A Case Study," Int. J. Mod. Phys. **A29** (2014) 06, 1450029 e-Print: arXiv:1210.0478 [hep-th].

[9] Klein, F. "Vorlesungen ueber das Ikosaeder und die Aufloesung der Gleichungen vom fuenften," Grade. 1884. Reprinted as Klein, F. Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree, 2nd rev. ed. New York: Dover, 1956; Arfken, G. Mathematical Methods for Physicists, 3rd ed. Orlando, FL: Academic Press, pp. 184-185 and 239-240, 1985.

[10] S. J. Gates, Jr., and T. Hübsch, "On Dimensional Extension of Supersymmetry: From Worldlines to Worldsheets," J. Phys. **A42** (2009) 415206, e-Print: arXiv:0904.4719 [hep-th].

[11] K. Iga, and Y. X. Zhang "Structural Theory of 2-d Adinkras," e-Print: arXiv:1508.00491 [hep-th]; K. Iga, and Y. X. Zhang, "Structural Theory and Classification of 2-D Adinkras," private communication. .

[12] C. F. Doran, M. G. Faux, S. J. Gates, Jr., T. Hübsch, K. M. Iga, G. D. Landweber, and R. L. Miller, "Codes and Supersymmetry in One Dimension," Adv. Theor. Math. Phys. **15** (2011) 6, 1909-1970; e-Print: arXiv:1108.4124 [hep-th]; C. F. Doran, M. G. Faux, S. J. Gates, Jr., T. Hübsch, K. M. Iga, and G. D. Landweber, "Relating Doubly-Even Error-Correcting Codes, Graphs, and Irreducible Representations of N-Extended Supersymmetry," UMDEPP-07-012 SUNY-O-663, e-Print: arXiv:0806.0051 [hep-th]; C. F. Doran, M. G. Faux, S. J. Gates, Jr., T. Hübsch, K. M. Iga, G. D. Landweber, and R. L. Miller, "Topology Types of Adinkras and the Corresponding Representations of N-Extended Supersymmetry," UMDEPP-08-010, SUNY-O-667, e-Print: arXiv:0806.0050 [hep-th]; I. Chappell, S. J. Gates, Jr., W. D. Linch, J, Parker, S. Randall, A. Ridgway, and K. Stiffler, "4D, N=1 Supergravity Genomics," JHEP **1310** (2013) 004, DOI: 10.1007/JHEP10(2013)004, e-Print: arXiv:1212.3318 [hep-th]; S. J. Gates, Jr., J. Hallett, T. Hübsch, and K. Stiffler, "The Real Anatomy of Complex Linear Superfields," Int. J. Mod. Phys. **A27** (2012) 1250143, DOI: 10.1142/S0217751X12501436, e-Print: arXiv:1202.4418 [hep-th]; S. J. Gates, Jr., J. Hallett, J. Parker, V. G. J. Rodgers, and K. Stiffler "4D, N = 1 Supersymmetry Genomics (II)," JHEP **1206** (2012) 071, DOI: 10.1007/JHEP06(2012)071, e-Print: arXiv:1112.2147 [hep-th].

[13] S. J. Gates, Jr., and K. Stiffler, "Adinkra Color Confinement In Exemplary Off-Shell Constructions Of 4D, N = 2 Supersymmetry Representations," JHEP **1407** (2014) 051, e-Print: arXiv:1405.0048 [hep-th].

[14] C. Doran, J. Kostiuk, K. Iga, G. Landweber, and S. Méndez-Diez, "Geometrization of N-Extended 1-Dimensional Supersymmetry Algebras, I," private communication, C. Doran, K. Iga, G. Landweber, and S. Méndez-Diez, "Geometrization of N-Extended 1-Dimensional Supersymmetry Algebras," e-Print: arXiv:1311.3736 [hep-th].