

Lie transformation method of quantum state evolution controlled by classical dynamics on a general time-dependent driven and damped parametric oscillator

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A variety of dynamics in this world can be approximately treated by a model: a driven and damped parametric oscillator. An intensive investigate of this time-dependent model from both classical and quantum mechanical points of view provides a dynamical insight to resolve the classical and quantum dynamics in order to understand the more complicated time-dependent evolutions not only happens in the macroscopic classical scale for the synchronized behaviors but also in the coherent microscopic scale for a quantum state control. By using a Lie transformation on a general time-dependent quadratic Hamiltonian, we exactly solve the dynamical behaviors of the driven and damped parametric oscillator to give a universal physical mechanism for the classical parametric control on the quantum state evolution in a dissipation system.

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I. INTRODUCTION

The dynamics of a classical driven and damped harmonic oscillator is now a closed fundamental problem discussed in many textbooks and all its behaviors are well known [1, 2]. However, the driven and damped harmonic oscillator can be used to understand the dynamics of a more general system which has an internal (intrinsic) frequency along with an external environment inducing dissipation. A more complicated system just owns a richer frequency structure but only one frequency window plays a dominant role in a certain parametric region. The external driving force can be used to simulate an input energy in order to maintain or control the internal freedom. A more complex driving signal often tells a similar story as that does by a simple periodic driving. Up to now, some behaviors of the damped harmonic oscillator driven by a simple periodic force still exhibit interesting properties, especially due to a very quickly developing field for the optomechanical systems [3] and for the quantum shortcut control problems [4]. In this paper, we want to reconsider it in a different way to focus on the transient dynamics for a purpose of dynamical control. Among many control models, the driven and damped harmonic oscillator is the most simple and fundamental one to reveal the main properties of the control dynamics. Although a classical forced damping harmonic oscillator will finally follow the external driving force and its behavior is totally deterministic, its transient way to a final locking state or its dynamical response to an external force is still an important problem to be explored and generalized to the quantum region in a point of view of dynamical controlling. For any classical or quantum control problems, the controlled systems are nonconservative and the corresponding theory for a time-dependent Hamiltonian is still a developing field exhibiting rich and novel behaviors beyond the adiabatic theory [5]. In this paper, we will extensively investigate this simple but fundamental model, a forced damping parametric oscillator, in both classical and quantum mechanical points of view, in order to find a dynamical connection between the classical and quantum evolutions. We first give a brief but complete classical review on the classical behavior of a pure forced damping harmonic oscillator to lay down a basic knowledge for a classical control and then generalize it to a driven and damped parametric oscillator in quantum regime by using a time-dependent transformation method of Lie algebra.

II. CLASSICAL DYNAMICS CONTROLLED BY AN EXTERNAL DRIVING

A. The classical dynamics

The equation of motion to govern a classical driven and damped harmonic oscillator with constant parameters is ($m = 1$)

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_0^2x(t) = F(t), \quad (1)$$

where ω_0 is the intrinsic frequency of the harmonic oscillator, γ is its damping rate related to its Q factor by $Q = \omega_0/2\gamma$, and $F(t)$ is the driving force. Above equation is a very fundamental equation to understand a general driven and damped parametric oscillator emerging in many physical systems. Besides the traditional weak vibration in a mechanical system, one important model emerging in the plasma is the motion of a charged particle driven by the

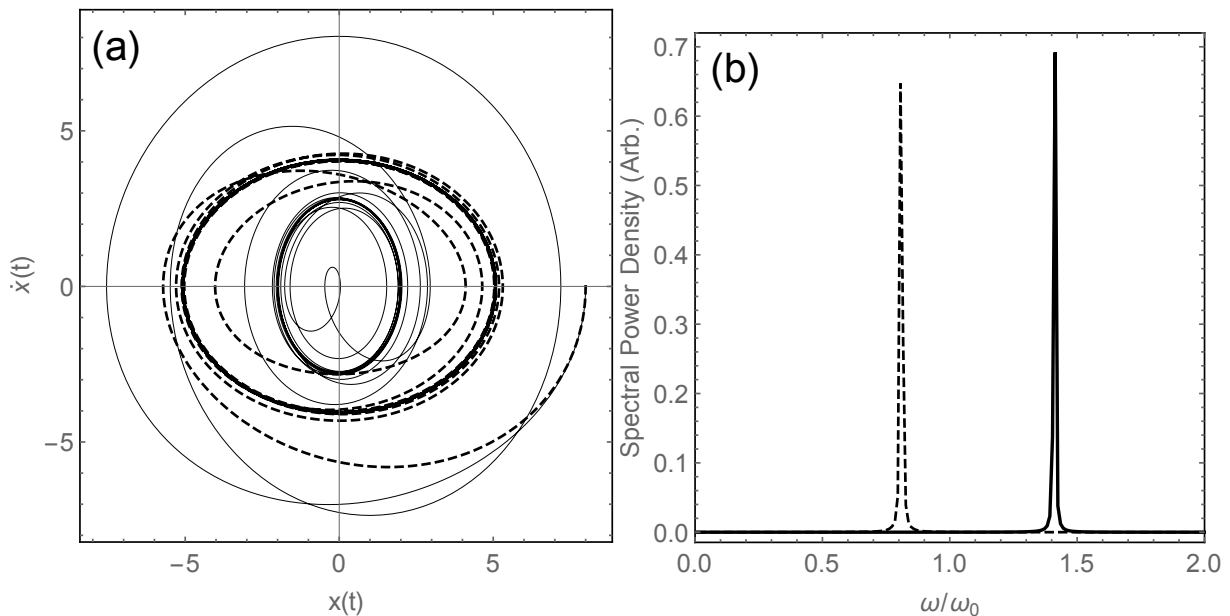


FIG. 1: Two typical trajectories and their corresponding frequency power spectra of the damped harmonic oscillator under a red-detuned driving ($\Omega = 0.8$, the dashed line) and a blue-detuned driving ($\Omega = 1.4$, the solid line) with respect to the intrinsic frequency of ω_0 . The other parameters are $\omega_0 = 1, \gamma = 0.1, F_0 = 2$ and the initial conditions are $x(0) = 8, \dot{x}(0) = 0$.

electromagnetic fields, which is called Lorentz oscillator there. The other typical motion is about the electromagnetic field in the RLC circuits or in a closed cavity driving by an input field. Surely, many other phenomena in chemical, biological or economical field can also be explained or understood by this simple model.

The response of a damped oscillator to a driving force exhibits a very important phenomenon: resonance, and many phenomena in this world can be explained by this resonant effect. As most of the controlling or driving signals can be expanded by the simple harmonic components, most studies focus on a monochromatic driving which has a constant amplitude and a fixed frequency, then Eq.(1) can be simplified by

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F_0 \cos(\Omega t + \phi), \quad (2)$$

where F_0, Ω and ϕ are the driving strength, frequency and initial phase, respectively. This equation has been sufficiently studied and Fig.1(a) demonstrates two typical motions of the oscillator under two robust harmonic driving signals with the same initial state. After a complex transient motion (see the historical trajectories in the phase space), the oscillator will finally lock itself to the driving force as shown by two closed orbits in the phase space and the corresponding frequency spectra are shown in Fig.1(b). That the final motion of the oscillator asymptotically follows the driving force is actually a universal “synchronized” dynamics for a damped system induced by a robust external driving. Although the force-induced oscillation will disappear and the forced oscillation will immediately return back to its damping intrinsic oscillation when the force is removed, this kind of synchronized motion is still a universal phenomena induced by an external driving on a (nearly) harmonic oscillator. Strictly, the force-reduced oscillation is not a self-oscillation and the final circle in the phase space is not a limit circle but just a passive orbit due to the driving force. However this synchronized behavior to an robust external signals happens in many classical low energy synchronization processes, especially in a classically driven quantum system.

A well-known solution to above inhomogeneous differential equation is a sum of a general solution of its homogeneous equation and a particular solution appropriate to the inhomogeneous one. It has the form of

$$x(t) = x_H(t) + x_I(t), \quad (3)$$

where the first free part $x_H(t)$ is

$$x_H(t) = A_1 e^{-\gamma t} \cos(\omega t + \phi_1),$$

and the damping shifted frequency ω is defined by $\omega = \sqrt{\omega_0^2 - \gamma^2}$. Because the free part will finally damped out, the question focuses on the second part of $x_I(t)$.

a. *A normal form:*

A normal form of $x_I(t)$ is chosen as

$$x_I(t) = A_2 \cos(\Omega t + \phi_2). \quad (4)$$

Substituting it into Eq.(2), we have

$$A_2 = \frac{F_0}{\sqrt{(2\gamma\Omega)^2 + (\omega_0^2 - \Omega^2)^2}}, \quad \tan(\phi_2 - \phi) = \frac{-2\gamma\Omega}{\omega_0^2 - \Omega^2}.$$

This solution clearly shows that the frequency of a forced damping harmonic oscillator is only determined by the driving force because its intrinsic oscillation will finally damp out, as shown in Fig.1(b). While the final amplitude of the driven oscillation depends on all the parameters such as the intrinsic frequency, the damping rate, the driving frequency and the driving strength. Surely, the final oscillation doesn't depend on the initial conditions.

b. *Quadrature form:*

In some cases, another form of x_I sets

$$x_I(t) = X_1 \cos(\Omega t) + X_2 \sin(\Omega t), \quad (5)$$

where X_1 and X_2 are two quadrature amplitudes which can be called in-phase and out-of-phase part with respect to the force $F_1 = F_0 \cos \phi$ and $F_2 = F_0 \sin \phi$, respectively. Substitute it into Eq.(2), we obtain

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{bmatrix} \frac{\omega_0^2 - \Omega^2}{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2} & \frac{2\gamma\Omega}{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2} \\ \frac{2\gamma\Omega}{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2} & -\frac{\omega_0^2 - \Omega^2}{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2} \end{bmatrix} \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix} F_0.$$

The solution of Eq.(5) is surely equivalent to Eq.(4) if we define a rotation angle θ as

$$\cos \theta = \frac{\omega_0^2 - \Omega^2}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2}}, \quad \sin \theta = \frac{2\gamma\Omega}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2}}, \quad (6)$$

then x_I becomes

$$x_I(t) = \frac{F_0}{\sqrt{(\Omega^2 - \omega_0^2)^2 + (2\gamma\Omega)^2}} \cos(\Omega t + \phi - \theta), \quad (7)$$

where the phase shifted with respect to the initial force phase is

$$\theta = \arctan\left(\frac{2\gamma\Omega}{\omega_0^2 - \Omega^2}\right), \quad (8)$$

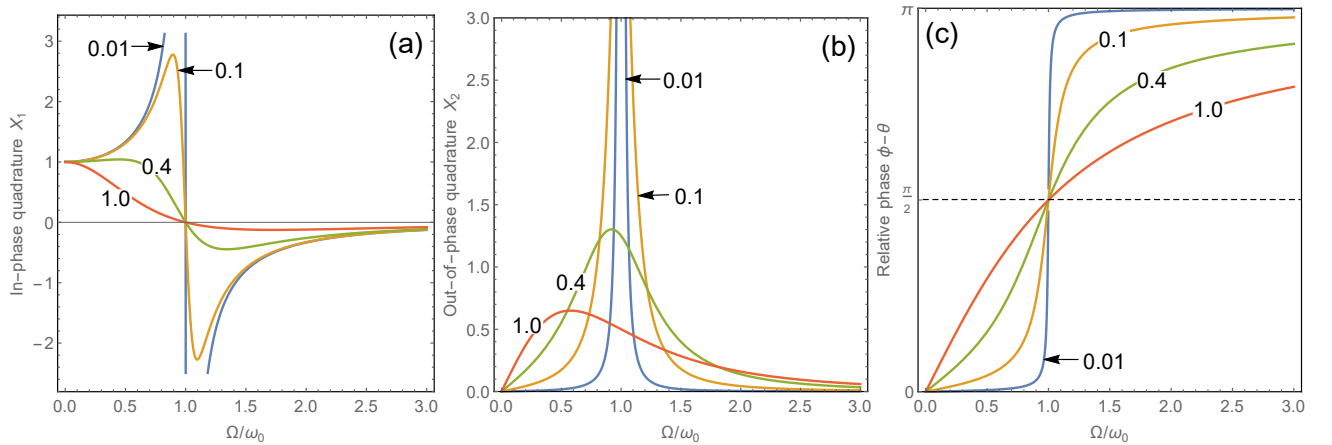


FIG. 2: The amplitudes of (a) in-phase quadrature X_1 , (b) out-of-phase quadrature X_2 with $\phi = 0$ and (c) the phase shift of the final oscillation under different damping rates labeled by the numbers in the figures. The force strength is set to $F_0 = 1$.

and the final phase of the forced oscillation can be easily find by

$$\phi - \theta = \arctan\left(\frac{X_2}{X_1}\right).$$

Above two different forms of solution Eq.(4) and Eq.(5) can be used in different control situations. Based on above solutions, Fig.2 shows the quadrature amplitudes of X_1 , X_2 and the relative phase of the final oscillation with respect to the driving under different damping rates. If the frequency of the force equals to the intrinsic frequency of the harmonic oscillator, the system exhibits a very universal phenomenon: resonance, which we should closely discussed in the next part. Fig.2 demonstrates that the two quadratures X_1 and X_2 owns different responsive properties at the resonant frequency window. The in-phase quadrature X_1 (Fig.2(a)) conducts an abrupt transition from positive response to a negative response and the out-of-phase quadrature of X_2 (Fig.2(a)) exhibits a resonant peak profile. Under a resonant driving, the phase of a high-Q oscillator will be very sensitive to the detuning fluctuation according to Eq.(8). If the frequency of the driving is larger than the natural frequency ($\Omega > \omega_0$, blue detuned), the motion of the oscillator is lagged and tend to be in phase with the driving force because the oscillator tries to keep with the pace of the force, and if $\Omega < \omega_0$ (red detuned), the motion of the harmonic oscillator will be pulled back by the driving force and tend to be out of the driving pace.

B. Input-output balance: amplification versus damping

Above solutions of a damped harmonic oscillator driven by an external force can be understood by the energy balance within the harmonic oscillators. Eq.(2) actually describes two coupled harmonic oscillations with different parameters, one is a passive oscillator with a damping amplitude and the other is an active oscillator (driving) with an undepleted amplitude. The energy flow between two oscillators is determined by the parametric matching between their dynamics. When their parameters are well matched, the energy stored in the harmonic oscillator will reach a maximum value. Now let's look at the most important phenomenon happens on a forced system: the resonant response of the system. For a simple oscillator, the response of an oscillator can be measured by the final amplitude or energy of the oscillator. Conveniently, a magnification factor M of its amplitude under an unit force driving is defined by

$$M(\gamma, \Omega) = |\chi(\Omega)| = \frac{1}{\sqrt{(\Omega^2 - \omega_0^2)^2 + (2\gamma\Omega)^2}}.$$

Then for $\Omega > 0$, the maximum amplification of M will happen at a driving frequency of

$$\Omega_R = \sqrt{\omega_0^2 - 2\gamma^2},$$

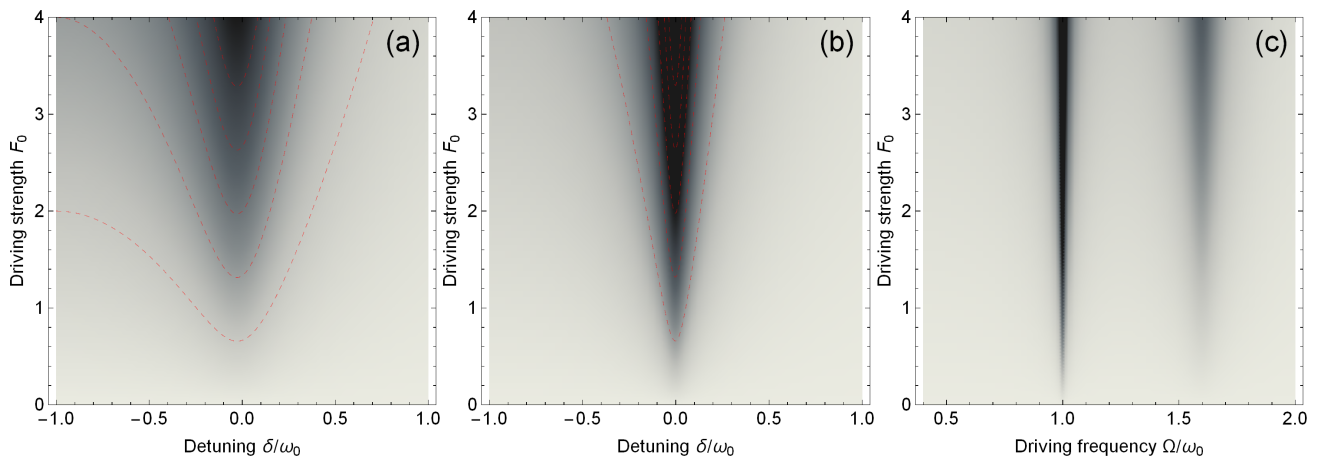


FIG. 3: The amplitude responses of the damped harmonic oscillator driving by the external forces with (a) $Q = 3$, (b) $Q = 10$ and (c) with two different resonant windows at $\Omega = \omega_0$ and $1.6\omega_0$. The dashed contour lines are samples for references.

which we call the resonant frequency. Then the final amplitude A and phase shift Θ are

$$A = \frac{F_0}{\omega_0^2} \frac{4Q^2}{\sqrt{\delta^2 (4Q + \delta)^2 + (4Q + 2\delta)^2}},$$

$$\Theta = \theta - \phi = \tan^{-1} \left[\frac{-4Q - 2\delta}{-4Q\delta - \delta^2} \right],$$

where $\delta = (\Omega - \omega_0)/\gamma$ is the scaled detuning. For a harmonic oscillator with a very high quality factor $Q = \omega_0/2\gamma \gg 1$, the amplitude and phase reduce to

$$A \approx \frac{F_0}{\sqrt{1 + \delta^2}} Q, \quad \Theta \approx \tan^{-1} \left(\frac{-1}{-\delta} \right).$$

Fig.3 shows the typical resonant density profiles of a damped harmonic oscillator under the periodic driving with respect to the driving strength (power) and the frequency detuning of the force. These typical response pictures of Fig.3(a) and Fig.3(b) display the main properties the responsive structures detected in many complicated driving systems. Fig.3(c) shows a resonant profile of an oscillator with two well-separated resonant modes (without coupling), which can be safely treated by two independent driven and damped harmonic oscillators giving two resonant windows of different window width (the left Q factor is 100 and the right one is 10).

Here we'd like to discuss the resonant effect through the energy stored in the harmonic oscillator under the driving force. Because the final amplitude of the oscillation is determined by an energy power balance between the input and output processes. The power of the force exerted on the oscillator at time t can be calculated by

$$P(t) = F(t) \dot{x}(t) = \dot{E}(t) + 2\gamma \dot{x}^2(t)$$

where the total energy stored in the the harmonic oscillator can be defined by

$$E(t) = E_k(t) + E_p(t) = \frac{1}{2} \dot{x}^2(t) + \frac{1}{2} \omega_0^2 x^2(t).$$

Then the energy changing rate of $E(t)$ will be determined by

$$\dot{E}(t) = F(t) \dot{x}(t) - 2\gamma \dot{x}^2(t) = [F(t) - 2\gamma \dot{x}(t)] \dot{x}(t),$$

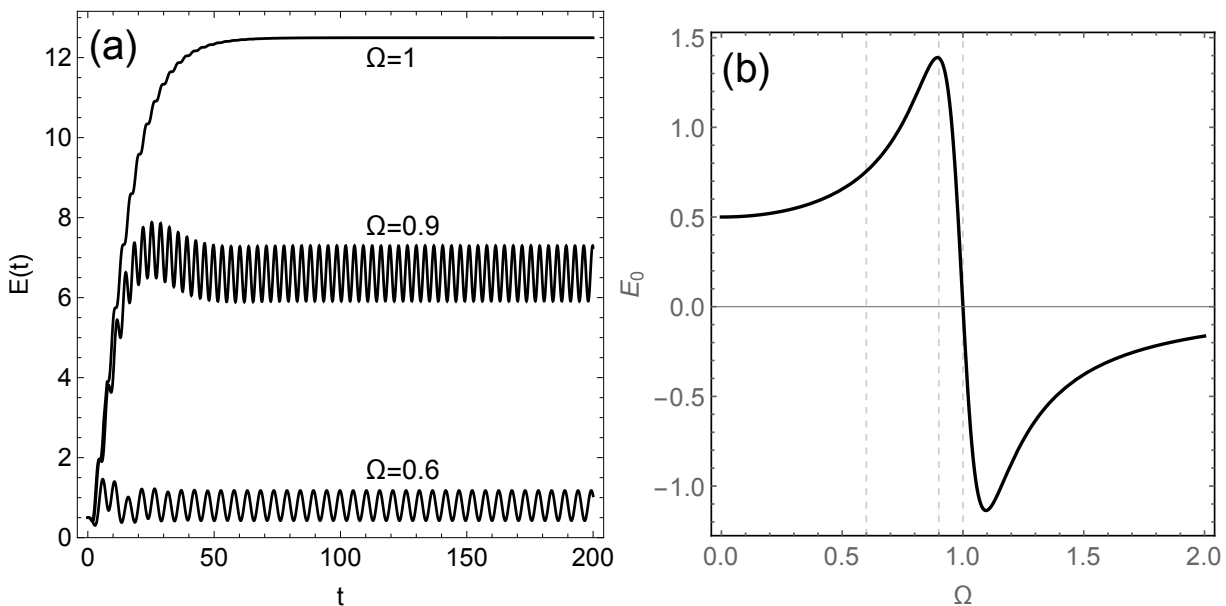


FIG. 4: (a) The variation of the energy stored in the harmonic oscillator under different driving frequencies. (b) The oscillating amplitudes of the varying energy for the harmonic oscillator. The parameters are $\omega_0 = 1, \gamma = 0.1, F_0 = 1$.

which means the energy stored in the harmonic oscillator is determined by the power balance between the driving and the damping. We notice that if

$$\dot{x}(t) = \frac{F(t)}{2\gamma},$$

i.e.

$$x(t) = \frac{1}{2\gamma} \int_0^t F(t') dt',$$

the stored energy will go to an extreme constant value and the motion will lock to a specific closed orbit in the phase space. Then this indicates that a final motion or the energy stored in the harmonic oscillation can be controlled by the driving force. For a harmonic driving force, we can see that the final motion will be

$$x(t) = \frac{1}{2\gamma} \int_0^t F(t') dt' = \frac{F_0}{2\gamma} \int_0^t \cos(\Omega t' + \phi) dt' = \frac{F_0}{2\Omega\gamma} \sin(\Omega t + \phi) + C_0,$$

where the integral constant C_0 is trivial here but only giving a phase shift of the solution. Compared with the rigid solution Eq.(7), we find that above solution is just the resonant solution when the energy stored in the driven oscillator reaches a maximum value. In a general case, the energy stored in the forced harmonic oscillator will exhibit a periodic behavior

$$E(t) = E_0 \cos(\Omega t + \varphi), \quad (9)$$

where the oscillating amplitude E_0 is determined by

$$E_0 = \frac{1}{2} \frac{\omega_0^2 - \Omega^2}{(\omega_0^2 - \Omega^2)^2 + (2\gamma\Omega)^2} F_0^2.$$

Fig.4(a) display the energy $E(t)$ stored in the harmonic oscillator and its varying amplitude E_0 is shown in Fig.4(b). The energy variation indicates a passive response of the harmonic oscillator and the final varying frequency of the energy is locked to the driving force. We can see that the amplitude E_0 is closely related to the in-phase quadrature of X_1 and the energy $E(t)$ will cease to change and finally reach to a maximum constant when the driving is at resonant. This interesting property can be used to develop a feed back control to keep the driving to the resonant point by detecting the energy output of the driven harmonic oscillator.

In order to reveal the critical role of damping for a final state of a driven oscillator, we introduce a complex variable in the phase space as

$$a(t) = x(t) + i \frac{\dot{x}(t)}{\omega_1}, \quad a^*(t) = x(t) - i \frac{\dot{x}(t)}{\omega_1^*}, \quad (10)$$

then solution Eq.(1) can be written in a form of

$$a(t) = a(0) e^{i\omega_1 t} + \frac{e^{i\omega_1 t}}{i\omega_1} \int_0^t F(\tau) e^{-i\omega_1 \tau} d\tau,$$

where

$$\omega_1 = i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}.$$

The first term of above solution describes an intrinsic rotation with a damping modified frequency ω_1 . The rotation is only determined by the parameters of the driven system, such as its intrinsic frequency, the damping rate and the initial state, and, finally, this rotation disappears because of the damping. If there is no damping or damping is very small for $\gamma \sim 0$, the forced motion, $x(t)$, will definitely maintain its intrinsic rotation for a long time and can not completely lock on the driving pace. That means the final motion of a driven harmonic oscillator will depend on its intrinsic frequency and its initial conditions, and never can exactly follows the pace of driving if damping is zero. In this case, the superposition of non-damping intrinsic oscillation and the driving oscillation will be mixed by (see Eq.(3))

$$x(t) = A_1 \cos(\omega t + \phi_1) + A_2 \cos(\Omega t + \phi_2),$$

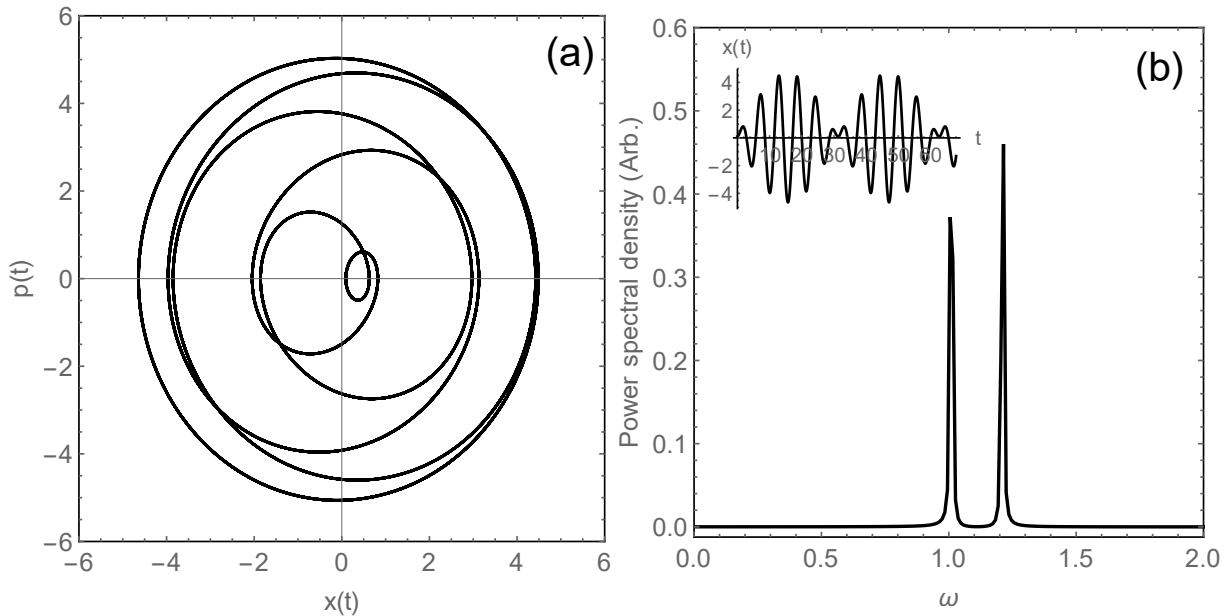


FIG. 5: (a) The phase trajectory of the driven harmonic oscillator with an extremely small damping. (b) The corresponding power spectral density of the motion (Inset: the beating dynamics of $x(t)$). The parameters are $\omega_0 = 1, F_0 = 1, \Omega = 1.2$ and the initial conditions are $x(0) = 0.1, p(0) = 0.1$.

which can lead to a beating behavior of $x(t)$ at the resonant driving window, $\Omega \sim \omega$ (see the inset of Fig.5(b)). Surely, this beating behavior can be used to detect the intrinsic frequency by an external driving signal. In order to show this motion, Fig.5 calculate the dynamic of the harmonic oscillator when the damping rate $\gamma = 10^{-5}$. Fig.5(a) shows a closed phase trajectory whose shape depends on the initial conditions and never seems to damp to a closed orbits as that shown in Fig.1(a) at a very large time scale. The longtime power spectral density of $x(t)$ (see Fig.5(b)) gives a two-peak profile (compared with Fig.1(b)), one peak is at the intrinsic frequency of $\omega_0 = 1$ and the other is at the driving frequency of $1.2\omega_0$. Therefore, above discussions indicate that the damping is a very important reason for the forced harmonic oscillator finally locking its motion to a driving force.

C. The linear response theory: susceptibility function

We can understand further about above behaviors through a response theory of a harmonic oscillator to the driving force. Taking a Fourier transformation on Eq.(1), we will get

$$x(\omega) = \chi(\omega) F(\omega), \quad (11)$$

where

$$\chi(\omega) = \frac{1}{(\omega_0^2 - \omega^2) - i(2\gamma\omega)} \quad (12)$$

is called displacement susceptibility and

$$F(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} F(t) dt$$

is the spectrum of the force. Therefore, if $\chi(\omega)$ are known, a reverse Fourier transformation gives the oscillator's position at any time by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \chi(\omega) F(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(\omega) e^{-i\omega t}}{(\omega_0^2 - \omega^2) - i2\gamma\omega} d\omega. \quad (13)$$

Therefore the susceptibility function $\chi(\omega)$ reveals the responsive properties of an oscillator to an external force at a certain frequency. Usually, $\chi(\omega)$ is a complex function because the displacement response, naturally, should include the magnitude and the phase, which can be directly written as

$$\chi(\omega) = |\chi(\omega)| e^{i \text{Arg}[\chi(\omega)]},$$

where, in the present model,

$$|\chi(\omega)| = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}}, \quad \text{Arg}[\chi(\omega)] = \arctan\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right).$$

If the driving force is $F_0 \cos(\Omega t + \phi)$, then this gives the solution of

$$x(t) = |\chi(\Omega)| F_0 \cos(\Omega t + \phi - \text{Arg}[\chi(\Omega)]),$$

which is surely the solution of Eq.(7). Or, directly, it is

$$\chi(\omega) = \chi_R(\omega) + i\chi_I(\omega), \quad (14)$$

where

$$\chi_R(\omega) = \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}, \quad \chi_I(\omega) = \frac{2\gamma\omega}{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2},$$

and then the displacement response can be separated into in phase and out of phase parts if the force spectrum $F(\omega)$ is real, says

$$x(\omega) = \chi_R(\omega) F(\omega) + i\chi_I(\omega) F(\omega),$$

which, clearly, corresponds the quadrature form of Eq.(5). The real part is often called reactive part which describes a direct respond to the driving, and the imaginary part is called dissipative or absorptive part which is also know as the spectral function. A reverse Fourier transformation on Eq.(11) is conducted by using the convolution formula, we have

$$x(t) = \int_{-\infty}^t \chi(t-t') F(t') dt', \quad (15)$$

where $\chi(t)$ is the response function with its Fourier transformation being $\chi(\omega)$. For the driven damping harmonic oscillator,

$$\chi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \chi(\omega) d\omega = \frac{1}{2\pi} \frac{\sin(\sqrt{\omega_0^2 + \gamma^2}t)}{\sqrt{\omega_0^2 + \gamma^2}} [\cos(\gamma t) - \sin(\gamma t)].$$

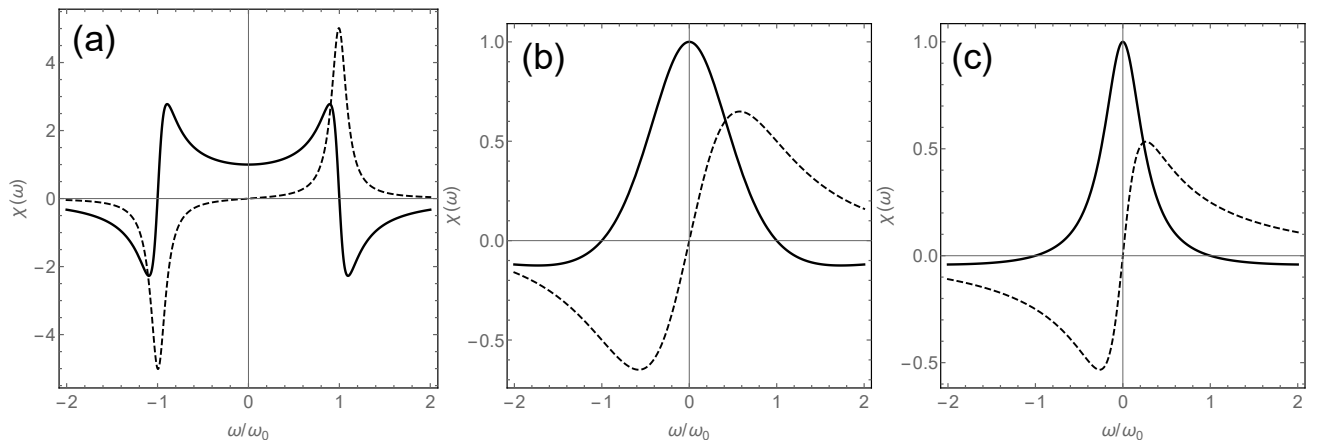


FIG. 6: The real part (solid lines) and imaginary part (dashed lines) of the displacement susceptibility of the damped harmonic oscillator in a case of (a) under-damped $\gamma = 0.1\omega_0$, (b) critical-damped $\gamma = \omega_0$ and (c) over-damped $\gamma = 2\omega_0$.

Eq.(15) reveals that the response of $x(t)$ is related to an input-output relation between the input driving signal (such as $F(t)$) and the output state of the system (such as $x(t)$). The present position of $x(t)$ depends not only on the present value of $F(t)$, but also on all its past values ($t' \leq t$) weighted by the response function of $\chi(t)$. However if the system is nonlinear, Eq.(15) should be an expanding series by including higher order responsive functions.

Based on the physical properties of the linear response theory, we can obtain some general properties of the susceptibility function. First, as $x(t)$ is real, then we have

$$\chi(\omega) F(\omega) = \chi^*(-\omega) F^*(-\omega),$$

which implies a symmetric properties of $\chi(\omega)$. When the force spectrum $F(\omega)$ is real and symmetric, $F(\omega) = F^*(-\omega)$, then

$$\chi(\omega) = \chi^*(-\omega) \Rightarrow \chi_R(\omega) = \chi_R(-\omega); \chi_I(\omega) = -\chi_I(-\omega).$$

That means the real part of susceptibility function is symmetric and its imaginary part is an odd function of ω under a real driving force, as shown in Fig.6. Second, following the causality principle that an output signal is always slower than an input one, the susceptibility function (analytic and local) must have poles only in the lower half-plane of ω , which leads to the famous Kramershas-Kronig relations giving a relation between its real part and imaginary part as

$$\chi_R(\omega) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{+\infty} \frac{\chi_I(\omega')}{\omega' - \omega} d\omega', \quad \chi_I(\omega) = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{+\infty} \frac{\chi_R(\omega')}{\omega' - \omega} d\omega',$$

where \mathcal{P} denotes the Cauchy principal value. That means the real part and imaginary part of susceptibility function are not independent functions (see Fig.6) and any part can be used to determine the total susceptibility function, for example,

$$\chi(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\chi_I(\omega')}{\omega' - \omega - i\epsilon} d\omega',$$

where ϵ is a real positive number $\epsilon \rightarrow 0^+$. As for a complex function of $\chi(\omega)$, its poles play a very important role in determining its responsive properties in the complex plane ω . The poles of $\chi(\omega)$ can be readily determined by its denominator of

$$(\omega_0^2 - \omega^2) - i(2\gamma\omega) = 0 \Rightarrow \omega = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}.$$

If $\gamma < \omega_0$ (under-damped case), there are two poles in the lower half plane with a symmetric locations with respect to imaginary axis (see the two symmetric peaks in Fig.6(a)). When γ increases, the two poles come near to the imaginary axis and when $\gamma \geq \omega_0$ (over-damped case), both poles sit on the imaginary axis (only one peak left in Fig.6(b)and(c)). Fig.6 demonstrates the real and imaginary part of $\chi(\omega)$ for different damping cases and we can see the different responses of a forced harmonic oscillator. If the damping is very small when $\gamma \ll \omega_0$, the susceptibility will be

$$\chi(\Delta) \approx -\frac{1}{2\omega_0^2} \frac{\omega_0}{\Delta + i\gamma},$$

where the detuning $\Delta = \omega - \omega_0$. As $\gamma \rightarrow 0$, the real part and imaginary part of Eq.(14) become $\chi_R(\omega) \rightarrow 1/(\omega_0^2 - \omega^2)$, $\chi_I(\omega) \rightarrow \pi\delta(\omega \pm \omega_0)$, respectively. Resorting to the linear response theory, we can obtain the linear response function of the forced harmonic oscillator as

$$\chi(t) = \frac{1}{2\pi} \frac{\cos(\gamma t) - \sin(\gamma t)}{\sqrt{\omega_0^2 + \gamma^2}} \sin\left(\sqrt{\omega_0^2 + \gamma^2} t\right).$$

Particularly for $\gamma = 0$, the solution for the forced harmonic oscillator will be

$$x(t) = \frac{1}{\omega_0} \int_{-\infty}^t \sin[\omega_0(t-t')] F(t') dt'. \quad (16)$$

Given a harmonic driving force $F(t) = F_0 \cos(\Omega t + \phi)$, we have

$$F(\omega) = F_0 [\pi\delta(\omega + \Omega) e^{i\phi} + \pi\delta(\omega - \Omega) e^{-i\phi}],$$

and we definitely get the same solution of Eq.(4) from Eq.(13). We can see that the final oscillation of the damping harmonic oscillator taking the same frequency as that of the force is just due to the delta responsive function of the force spectrum.

From above analysis, we know that the classical response of the driven and damped system can be understood by the susceptibility function which always has a finite responsive window for the classical resonant phenomenon. We should notice that all the classical dynamics we discussed above are deterministic and includes no internal fluctuations on the dynamics such as introducing an effective damping rate. Including external noises to the system will leads no new features to above classical dynamics because any external noise can safely be treated as external driving. However, for a quantum system, the internal fluctuations will definitely bring new features to a system above its classical dynamics.

III. THE QUANTUM DYNAMICS FOR A FORCED AND DAMPED SYSTEM

A. A brief note on dissipative quantum system

Similar to a classical system, any quantum system is also influenced by an environment which definitely leads to energy dissipations of the system. In classical scale, this influence can be safely described by phenomenologically introducing an effective damping rate. However, a damped harmonic oscillator in quantum scale becomes very different from a relevant classical counterpart, which have been extensively discussed in many papers (see the review paper [6, 7]). One reason of the difficulties to study a damped system in quantum regime is that the system doesn't admit a standard Hamiltonian for the quantization [8, 9]. The traditional theory on this problem is somehow to construct a non-standard time-dependent Hamiltonian and the quantization of the damping harmonic oscillator leads to the dual Bateman Hamiltonian by intruding an auxiliary oscillator (time-reversed counterpart) based on non-standard Hamiltonians [7, 9]. As a brief example, you can easily verify that the following simple equation of motion

$$\ddot{x} + 2\gamma\dot{x} = 0$$

does not have any standard Lagrange function to be derived from a Euler-Lagrange equation. This is easy to be understood because a constant total energy (Hamiltonian) of a dissipative system does not exist due to the energy loss. But a non-standard Hamiltonian description can still be developed in the framework of Euler-Lagrange equation [10], such as by using a time-dependent quantity of

$$\mathcal{L} = e^{2\gamma t} \frac{1}{2} m \dot{x}^2,$$

as its Lagrange [11]. For the driven damped harmonic oscillator, its classical equation of motion is ($m=1$)

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_0^2 x(t) = F(t),$$

and, clearly, it does not have a standard Hamiltonian. However, we can still find many formal Hamiltonians from which above equations of motion can be derived by using its Hamiltonian equations [12]. The time-dependent damped driven harmonic oscillator based on the well-known Caldirola-Kanai Hamiltonian is [9]

$$H(t) = e^{-2\gamma t} \frac{p^2}{2m} + e^{2\gamma t} \left[\frac{1}{2} m \omega_0^2 x^2 - x F(t) \right], \quad (17)$$

and the Hamilton's equations of motion leads to

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F(t)}{m}.$$

However, this treatment seems to be formally and a complete consideration on quantum damped oscillator should use an open quantum theory by including a continuum reservoir which can be treated as an ensemble of harmonic oscillators with infinite degree of freedom [13]. Surely, this complete theory is somehow cumbersome and heavily depends on the statistical properties of the reservoirs that we should known in advance. Alternatively, a simple but equivalent method to deal with a damped quantum system is to use a non-Hermitian effective Hamiltonian as [14]

$$\hat{H} = \hat{H}_0 - i\hat{\Gamma}, \quad (18)$$

which is decomposed into a Hermitian part with $\hat{H}_0^\dagger = \hat{H}_0$, and an anti-Hermitian part with $\hat{\Gamma}^\dagger = \hat{\Gamma}$. Then the dynamics of a pure state $|\Psi\rangle$ is still governed by the conventional Schrödinger equation of

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle.$$

However, above non-Hermitian Hamiltonian leads to a non-unitary state evolution because the norm of the state is not conserved to 1 any more. One well-known Hamiltonian for a damped oscillator in units of Eq.(33) is

$$\hat{H} = \frac{1}{2} \left(\hat{P}^2 + \hat{X}^2 \right) - i\frac{\Gamma}{2} \left(\hat{X}\hat{P} + \hat{P}\hat{X} \right) = \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) - \frac{\Gamma}{2} (\hat{a}^{\dagger 2} - \hat{a}^2), \quad (19)$$

and which can restore the classical equation of motion for a damped harmonic oscillator by

$$\begin{aligned} \dot{x} &= p - \Gamma x, \\ \dot{p} &= -x - \Gamma p, \end{aligned}$$

or

$$\ddot{x} + 2\Gamma\dot{x} + (1 + \Gamma^2)x = 0,$$

where x and p are defined by $\alpha = (x + ip)/\sqrt{2}$ for the coherent state $|\alpha\rangle$. Here, the classical Hamiltonian reads

$$H_c = \langle \alpha | \hat{H} | \alpha \rangle = \left(\alpha^* \alpha + \frac{1}{2} \right) - \frac{\Gamma}{2} (\alpha^{*2} - \alpha^2) = \frac{1}{2} (x^2 + p^2) - i\Gamma xp \equiv H_{c0} - i\Gamma_c,$$

and the classical equation of motion in the phase-space follows [14]

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \Omega^{-1} \nabla H_{c0} - G^{-1} \nabla \Gamma_c,$$

where ∇ is the phase space gradient operator, the symplectic unit matrix Ω is

$$\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Omega^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the phase space metric G is

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The spectrum of Hamiltonian Eq.(19) can be obtained by a similarity transformation of

$$\hat{H}' = \hat{R} \hat{H} \hat{R}^{-1} = \frac{\omega}{2} \left(\hat{P}^2 + \hat{X}^2 \right), \quad \hat{R} = e^{-\frac{\theta}{2} (\hat{P}^2 - \hat{X}^2)},$$

where

$$\theta = -\Gamma, \quad \omega = \sqrt{1 + \Gamma^2}.$$

Above transformation can be calculated under $su(1,1)$ algebra by defining operators as

$$\hat{J}_0 = \frac{1}{2} \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad \hat{J}_+ = \frac{1}{2} \hat{a}^{\dagger 2}, \quad \hat{J}_- = \frac{1}{2} \hat{a}^2,$$

and the Hamiltonian of Eq.(19) will be written as

$$\hat{H} = 2\hat{J}_0 - i\Gamma \left(\hat{J}_+ - \hat{J}_- \right).$$

However, there are many Hamiltonians which can be used to describe the damped oscillators with different damping mechanisms. For example, if $\hat{\Gamma} = \gamma \hat{X}^2/2$, then another Hamiltonian for a damped oscillator is

$$\hat{H} = \frac{1}{2} \left(\hat{P}^2 + \hat{X}^2 \right) - i\frac{\gamma}{2} \hat{X}^2 = \frac{1}{2} \hat{P}^2 + \frac{1}{2} g \hat{X}^2, \quad (20)$$

where $g = 1 - i\gamma$ is a complex number. Actually, the non-Hermitian Hamiltonian (20) is related to a general class of \mathcal{PT} -symmetric Hamiltonian of [15]

$$\hat{H} = \hat{P}^2 - (i\hat{X})^\epsilon \hat{X}^2,$$

where the parameter ϵ is real. The study on above non-Hermitian Hamiltonian shows that, when $\epsilon \geq 0$, all of the eigenvalues of the Hamiltonians are entirely real and positive (unbroken \mathcal{PT} -symmetric parametric region), but when $\epsilon < 0$, there are complex eigenvalues (broken \mathcal{PT} -symmetric parametric region) leading to the damping process [16]. The non-Hermitian Hamiltonians are called \mathcal{PT} -symmetric under the transformation of $\hat{\mathcal{P}}\hat{\mathcal{T}}\hat{H}\hat{\mathcal{P}}\hat{\mathcal{T}} = \hat{H}$, that is,

$$[\hat{H}, \hat{\mathcal{P}}\hat{\mathcal{T}}] = 0,$$

where $\hat{\mathcal{P}}\hat{X}\hat{\mathcal{P}} = -\hat{X}$, $\hat{\mathcal{P}}\hat{P}\hat{\mathcal{P}} = -\hat{P}$, $\hat{\mathcal{T}}\hat{X}\hat{\mathcal{T}} = \hat{X}$, $\hat{\mathcal{T}}\hat{P}\hat{\mathcal{T}} = -\hat{P}$, $\hat{\mathcal{T}}i\hat{\mathcal{T}} = -i$, and $\hat{\mathcal{P}}^2 = 1$, $\hat{\mathcal{T}}^2 = 1$, $\hat{\mathcal{P}}\hat{\mathcal{T}} = \hat{\mathcal{T}}\hat{\mathcal{P}}$. Because $\hat{\mathcal{P}}\hat{\mathcal{T}}$ is not linear, the eigenstates of \hat{H} may or may not be the eigenstates of $\hat{\mathcal{P}}\hat{\mathcal{T}}$. In this sense we can say that if every eigenfunction of a \mathcal{PT} -symmetric Hamiltonian is also an eigenfunction of the $\hat{\mathcal{P}}\hat{\mathcal{T}}$ operator, then \mathcal{PT} -symmetry of \hat{H} is unbroken. Conversely, if some of the eigenfunctions of a \mathcal{PT} -symmetric Hamiltonian are not simultaneously eigenfunctions of the $\hat{\mathcal{P}}\hat{\mathcal{T}}$ operator, we say that the \mathcal{PT} -symmetry of \hat{H} is broken.

B. Driven quantum Harmonic oscillator beyond damping

In order to manifest the role of damping and reveal the main properties of the method we use later, we first consider a driven quantum harmonic oscillator without damping described by a Hermite Hamiltonian in a form of [17]

$$\hat{H} = \hat{H}_0 - \hat{x}F(t),$$

where \hat{H}_0 is the Hamiltonian of a free harmonic oscillator. Usually, \hat{H}_0 is time-independent, but many works have been done on a general time-dependent harmonic oscillator with time varying mass and frequency [18]. Here we start with a simple case for a time-independent \hat{H}_0 and then generalize it to a general time-dependent model.

In order to compare with the former results on the classical harmonic oscillator, we use the following specific Hamiltonian as [19]

$$\hat{H}(t) = \frac{\hat{p}^2}{2m_0} + \frac{1}{2}m_0\omega_0^2\hat{x}^2 - m_0\omega_0^2f(t)\hat{x}, \quad (21)$$

where $f(t)$ is some specific time-dependent function with a dimension of length. As $f(t)$ is a classical parameter for a quantum system, we can define a position shift operator as [20]

$$\hat{x}_f(t) = \hat{x} - f(t),$$

then the quantized Hamiltonian will be

$$\hat{H}_f(t) = \hbar\omega_0 \left(\hat{a}_f^\dagger \hat{a}_f + \frac{1}{2} \right) - \frac{1}{2}m_0\omega_0^2f^2(t), \quad (22)$$

where

$$\hat{a}_f = \sqrt{\frac{\hbar m_0 \omega_0}{2}} \left[(\hat{x} - f) + i \frac{\hat{p}}{\hbar m_0 \omega_0} \right],$$

and the instantaneous eigenfunctions and eigenvalues of $\hat{H}_f(t)$ take the wave function of harmonic oscillator (denoted by $\varphi_n(x, t)$) as

$$\begin{aligned} \psi_n(x, t) &= \varphi_n(x - f, t), \\ E_n(t) &= \left(n + \frac{1}{2} \right) \hbar\omega_0 - \frac{1}{2}m_0\omega_0^2f^2(t), \end{aligned}$$

satisfying

$$\hat{H}_f(t) \psi_n(x, t) = E_n(t) \psi_n(x, t).$$

Clearly above instantaneous eigenfunctions are not solutions of the Schrödinger equation of $\hat{H}(t)$. If the initial state is in the n th state of a free harmonic oscillator,

$$\Psi(x, 0) = \varphi_n(x),$$

and the force exerts on the oscillator at time $t = 0$, then the solution for the $\hat{H}(t)$ at time t will be [21]

$$\Psi(x, t) = e^{\frac{i}{\hbar}[-(n+\frac{1}{2})\hbar\omega_0 t + m_0\dot{x}_c(x-\frac{x_c}{2}) + \frac{1}{2}m_0\omega_0^2 \int_0^t f(t')x_c(t')dt']} \varphi_n(x - x_c), \quad (23)$$

where $x_c(t)$ is the classical solution of the driven harmonic oscillator (see Eq.(16) with $m = 1$)

$$x_c(t) = \omega_0 \int_0^t \sin[\omega_0(t-t')] f(t') dt'$$

with an initial condition of $x_c(0) = \dot{x}_c(0) = 0$. It can be seen that the quantum state of Eq.(23) is obtained by conducting a spatial shift and a unitary transformation on the initial state as shown in Ref.[22].

Now we begin to use an algebraic transformation method to solve this problem as an example on the following time-dependent schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \hat{H}(t) \Psi(x, t) = \left[-\frac{\hbar^2}{2m_0} \frac{\partial^2}{\partial x^2} + \frac{1}{2}m_0\omega_0^2 x^2 - xF(t) \right] \Psi(x, t), \quad (24)$$

where $F(t) = m_0\omega_0^2 f(t)$. Usually, a solution of the time-dependent system is resorted to a time-dependent transformation as

$$\Psi(x, t) = \hat{U}(t) \psi(x, t),$$

then an initial schrödinger equation will be changed into

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = \left(\hat{U}^{-1} \hat{H} \hat{U} - i\hbar \hat{U}^{-1} \frac{\partial \hat{U}}{\partial t} \right) \psi(x, t) \equiv \hat{H}_U \psi(x, t).$$

In above transformation, the time-dependent operators $\hat{U}(t)$ should have its inverse operator and it can be properly chosen in order to turn the new Hamiltonian \hat{H}_U into a solvable form, which is, also, a method to be used to control a system's evolution by \hat{H}_U with a particularly designed transformation operator $\hat{U}(t)$. Now we use this method on Eq.(24) and verify the solution given by Eq.(23). As the driving term in Eq.(24) is related to the Heisenberg algebra $h(4)$ [23], which having at least three independent transformation parameters (3-dimensional parametric space), we can choose $\hat{U}(t)$ as

$$\hat{U}(t) = e^{-is(t)/\hbar} e^{-i\alpha(t)\hat{p}/\hbar} e^{-i\beta(t)\hat{x}/\hbar},$$

and its reverse operator is

$$\hat{U}^{-1}(t) = e^{i\beta(t)\hat{x}/\hbar} e^{i\alpha(t)\hat{p}/\hbar} e^{is(t)/\hbar}.$$

The three transformation parameters (or three control parameters) are often real in order to keep the transformation unitary. However, in some cases, especially, in a dissipative environment, the control parameters can be generalized to the complex functions. Conduct it on the initial Hamiltonian, we have

$$\hat{H}_U(t) = \frac{\hat{p}^2}{2m_0} + \frac{1}{2}m_0\omega_0^2 \hat{x}^2 + (m_0\omega_0^2\alpha - F - \dot{\beta}) \hat{x} - \left(\frac{\beta}{m_0} + \dot{\alpha} \right) \hat{p} + \left(\frac{\beta^2}{2m_0} + \frac{1}{2}m_0\omega_0^2\alpha^2 - \alpha F + \dot{\alpha}\beta - \dot{s} \right). \quad (25)$$

During the derivation, the following formulas are used

$$\begin{aligned} e^{i\alpha(t)\hat{p}/\hbar} \hat{x} e^{-i\alpha(t)\hat{p}/\hbar} &= \hat{x} + \alpha(t), \\ e^{i\beta(t)\hat{x}/\hbar} \hat{p} e^{-i\beta(t)\hat{x}/\hbar} &= \hat{p} - \beta(t). \end{aligned}$$

If we set $\alpha(t)$, $\beta(t)$ and $s(t)$ in the Hamiltonian (25) to be zero, then we have dynamical equations of

$$m_0\omega_0^2\alpha - \dot{\beta} - F = 0, \quad (26)$$

$$\frac{\beta}{m_0} + \dot{\alpha} = 0, \quad (27)$$

$$\frac{\beta^2}{2m_0} + \frac{1}{2}m_0\omega_0^2\alpha^2 - \alpha F + \dot{\alpha}\beta - \dot{s} = 0, \quad (28)$$

the \hat{H}_U will return back to a free harmonic form of

$$\hat{H}_U(t) = \frac{1}{2m_0}\hat{p}^2 + \frac{1}{2}m_0\omega_0^2\hat{x}^2,$$

and which can be strictly solved by

$$\hat{H}_U(t)\varphi_n(x,t) = E_n\varphi_n(x,t),$$

where $\varphi_n(x,t)$ is the eigenfunction of the harmonic oscillator with the normal energy of $E_n = (n + 1/2)\hbar\omega_0$.

Now we will look at how the transformation parameters in the operator of $\hat{U}(t)$ to control a driven harmonic oscillator back to an undriven one. From Eq.(26) and Eq.(27), we find parameter $\alpha(t)$ satisfies

$$\ddot{\alpha} + \omega_0^2\alpha = \frac{F}{m_0}, \quad (29)$$

which clearly obeys the classical dynamical equation of a driven harmonic oscillator. In this case, we set $\alpha(t) = x_c(t) = \langle \hat{x} \rangle$ and

$$\beta(t) = -m_0\dot{\alpha} = -m_0\dot{x}_c(t),$$

which corresponds the classical momentum but in a reverse direction. Therefore the parameter $s(t)$ determined by Eq.(28) is

$$s(t) = \int \left(\frac{1}{2}m_0\dot{x}_c^2 + \frac{1}{2}m_0\omega_0^2x_c^2 - x_cF - m_0\dot{x}_c^2 \right) dt \equiv \int_0^t \mathcal{L}(x_c, \dot{x}_c, t') dt'.$$

We can easily check that the function of

$$\mathcal{L}(x_c, \dot{x}_c, t) = - \left(\frac{1}{2}m_0\dot{x}_c^2 - \frac{1}{2}m\omega_0^2x_c^2 + x_cF \right) \quad (30)$$

is just the classical Lagrangian for a driven harmonic oscillator with a negative sign and the parameter s is a negative classical action. For this reason, we can call above method an inverse transformation method. Here we should notice that different forms of $\hat{U}(t)$ (such as different product orders for the tree operators) will definitely lead to different classical Lagrangian because the Lagrangian leading to a same dynamical equation is not unique [24] and many types of linear canonical transformations on Eq.(30) can generate different forms of $\mathcal{L}(x_c, \dot{x}_c, t)$ [6].

Therefore, if the initial quantum state (before the force is exerted on) is in the harmonic state $\varphi_n(x)$, the solution of Eq.(24) at time t will be

$$\Psi_n(x,t) = e^{-is_n(t)/\hbar} e^{-i\alpha_n(t)\hat{p}/\hbar} e^{-i\beta_n(t)\hat{x}/\hbar} \varphi_n(x,t) = e^{-\frac{i}{\hbar} \int_0^t \mathcal{L}(x_c, \dot{x}_c, t') dt'} e^{im_0\dot{x}_c(x-x_c)/\hbar} \varphi_n(x-x_c), \quad (31)$$

where the displace operator gives

$$e^{-ix_c\hat{p}/\hbar}\varphi(x,t) = \varphi(x-x_c,t).$$

Here the subscript n in the parameters of s_n, α_n and β_n means the different initial conditions of s, α and β for the different initial wave function of φ_n . Clearly, we can easily check that above solution of Eq.(31) is dynamically equivalent to that given by Eq.(23).

In Fig.7, we calculate the quantum evolutions of the driven harmonic oscillators starting with three different quantum states. We can see a clear connection between the classical dynamics and the quantum dynamics by this transformation method. The transformation reveals that all the physical quantities related to the classical dynamics now becomes the parameters of the quantum wave function. Fig.7 shows that the driven harmonic oscillator maintain its density distributions (quantum states) around the classical orbits (dashed lines) and the driving force only classically shift the mean position of the initial distribution following the classical motion. This displacement behavior of a driven harmonic oscillator can be revealed by the so called displaced number state [25]. With Dirac notation, Eq.(31) becomes

$$|n, \alpha_c, t\rangle = e^{-is(t)/\hbar} e^{i\frac{\alpha\beta}{2\hbar}} \hat{D}(\alpha_c) |n\rangle = e^{-is(t)/\hbar} e^{i\frac{\alpha\beta}{2\hbar}} |n, \alpha_c\rangle \quad (32)$$

where $\hat{D}(\alpha_c)$ is the displacement operator with the parameter of

$$\alpha_c = \frac{1}{\sqrt{2}} \left(\frac{x_c}{x_{zpf}} + i \frac{p_c}{p_{zpf}} \right).$$

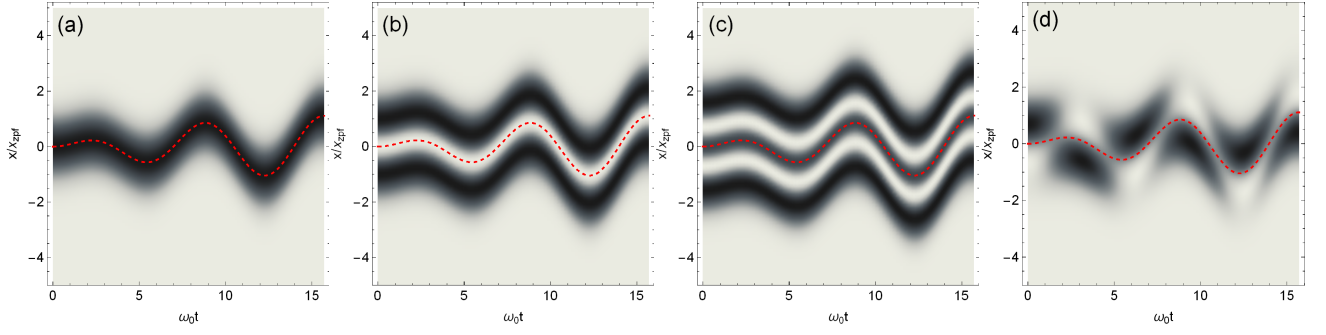


FIG. 7: The evolution of the quantum distribution and the classical orbit (red dashed lines) of a driven harmonic oscillator starting from initial number state of (a) $|0\rangle$, (b) $|1\rangle$, (c) $|2\rangle$ and (d) $(|0\rangle + |1\rangle)/\sqrt{2}$. The driving parameters are $f = 0.2$, $\Omega = 0.8$ and the classical initial conditions are set to $x_c(0) = p_c(0) = 0$ in order to highlight the quantum distribution.

The typical units x_{zpf} and p_{zpf} used above are zero-point fluctuations of

$$x_{zpf} = \sqrt{\frac{\hbar}{m_0\omega_0}}, \quad p_{zpf} = \sqrt{\hbar m_0\omega_0}. \quad (33)$$

Usually, the displaced number state $|n, \alpha\rangle$ is defined by

$$|n, \alpha\rangle = \hat{D}(\alpha)|n\rangle = e^{(\alpha\hat{a}^\dagger - \alpha^*\hat{a})}|n\rangle,$$

and it is also called generalized coherent state for a Heisenberg algebra [25]. The displaced number state can be expanded by [25, 26]

$$|n, \alpha\rangle = \sum_m |m\rangle \langle m|n, \alpha\rangle = \sum_m c_{m,\alpha} e^{i\varphi_m} |m\rangle,$$

where $\alpha = |\alpha| e^{i\varphi}$ and

$$c_{m,\alpha} = \sqrt{\frac{n!}{m!}} (-1)^{n-m} |\alpha|^{n-m} e^{-|\alpha|^2/2} L_m^{n-m}(|\alpha|^2), \quad \varphi_m = (n-m)\varphi.$$

Then the projective probability of a driven harmonic starting on its initial state of $|n\rangle$ will be

$$P_n = \left| \langle n | \hat{U}(t) | n \rangle \right|^2 = \left| \langle n | \hat{D}(\alpha_c) | n \rangle \right|^2$$

where

$$\alpha_c = \frac{1}{\sqrt{2}} (x_c + ip_c)$$

is in the units of Eq.(33). Specifically, the probability on $|n\rangle$ will be

$$P_n(t) = |c_{n,\alpha_c}|^2 = e^{-|\alpha_c|^2} \left[L_n(|\alpha_c|^2) \right]^2 = e^{-\frac{x_c^2 + p_c^2}{2}} \left[L_n\left(\frac{x_c^2 + p_c^2}{2}\right) \right]^2 = e^{-E_c(t)} [L_n(E_c(t))]^2,$$

where $L_n(x)$ is Laguerre polynomial functions. The probability remains on the initial state depends on the classical energy $E_c(t)$ of the driven harmonic oscillator for $E_c(0) = E_n$. We know that the classical motion of a driven harmonic oscillator is determined by the response of force as

$$x_c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \chi_c(\omega) F(\omega) d\omega.$$

If a simple harmonic driving force is exerted, the final classical motion will be (see Eq.(3))

$$x_c(t) = A_2 \cos(\Omega t + \phi_2)$$

and the final probability in the initial number state will be

$$P_n(t) = e^{-E_f^c(t)} [L_n(E_f^c)]^2,$$

where $E_f^c(t)$ is the final classical energy stored in the harmonic oscillator. Above equation means that, at a long time, the probability remaining on the initial state will be influenced only by the classical energy of Eq.(9) due to the spacial shift of the wave function relative to its initial wave function. However, the oscillator will not be excited by the classical resonant driving force because the oscillator remains on its initial density distribution (see Fig.7) without any quantum transitions between different number states due to a unitary evolution of the wave function. As there is no damping, the classical shift of the wave packet will continuously increased with time as a result of the classical resonant driving and the quantum behavior will be masked by an increasing amplitude of the classical displacement.

C. Lie transformation method on the driven and damped parametric oscillator

1. The general quadratic Hamiltonian and its Lie algebra

Now we consider a general time-dependent system including both driven and damping in a unified algebraic framework. A general driven and damped parametric oscillator can be described by the following time-dependent quadratic Hamiltonian as [27-29]

$$\hat{H} = \frac{1}{2} [A(t)\hat{p}^2 + B(t)\hat{x}^2 + C(t)(\hat{x}\hat{p} + \hat{p}\hat{x})] + D(t)\hat{p} + E(t)\hat{x} + F(t), \quad (34)$$

where all the time-dependent functions $A(t) \cdots F(t)$ are continuous functions of time which can be used to describe the classical driving or parametric controls on a quantum system. We consider for simplicity the one-dimensional model and a generalization to d -dimensional is straightforward [30]. Totally, this Hamiltonian owns a well known real symplectic group $Sp(2d, \mathbb{R})$ [31] and we will consider it in a decomposed space with a Lie algebraic structure of $su(2) \oplus h(4)$ [33]. Therefore six generators can be separately defined by

$$\hat{J}_+ = \frac{1}{2\hbar} \hat{x}^2, \quad \hat{J}_- = \frac{1}{2\hbar} \hat{p}^2, \quad \hat{J}_0 = \frac{i}{4\hbar} (\hat{x}\hat{p} + \hat{p}\hat{x}),$$

which satisfy the commutation relations of $su(2)$ algebra [34]

$$[\hat{J}_+, \hat{J}_-] = 2\hat{J}_0, \quad [\hat{J}_0, \hat{J}_\pm] = \pm\hat{J}_\pm,$$

and

$$\hat{T}_1 = \frac{\hat{x}}{\hbar}, \hat{T}_2 = \frac{\hat{p}}{\hbar}, \hat{T}_0 = \frac{i}{\hbar}$$

which satisfy the Heisenberg-Weyl algebra of

$$[\hat{T}_1, \hat{T}_2] = \hat{T}_0, \quad [\hat{T}_1, \hat{T}_0] = 0, \quad [\hat{T}_2, \hat{T}_0] = 0.$$

Then the Hamiltonian can be written as

$$\hat{H} = \hbar \left(A\hat{J}_- + B\hat{J}_+ - i2C\hat{J}_0 \right) + \hbar \left(D\hat{T}_2 + E\hat{T}_1 - iF\hat{T}_0 \right).$$

Above Hamiltonian has been extensively studied by many authors [27, 35] and we will follow the algebraic method firstly proposed by Wei and Norman [33, 36] and, recently, summarized in Ref.[47]. Actually, the method we use here is a reverse procedure proposed by Lohe in Ref.[30]. For simplify, we use the typical units of zero-point fluctuations $x_{zpf} = \sqrt{\hbar/(m_0\omega_0)}$ and $p_{zpf} = \sqrt{\hbar m_0\omega_0}$ to scale the Hamiltonian by

$$\hat{\mathcal{H}}(t) = \frac{1}{2}a(t)\hat{P}^2 + \frac{1}{2}b(t)\hat{X}^2 + \frac{1}{2}c(t)\left(\hat{X}\hat{P} + \hat{P}\hat{X}\right) + d(t)\hat{P} + e(t)\hat{X} + f(t), \quad (35)$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$, $e(t)$ and $f(t)$ are scaled time-dependent functions and the commutation relation between the scaled position and momentum operators becomes $[\hat{X}, \hat{P}] = i$. The mass m_0 and the frequency ω_0 used in above

scaled parameters are the intrinsic or characteristic parameters which can be properly chosen according to the initial conditions for a control problem. Then the six generators of $su(2)$ and $h(4)$ algebras are

$$\hat{J}_+ = \frac{1}{2}\hat{X}^2, \quad \hat{J}_- = \frac{1}{2}\hat{P}^2, \quad \hat{J}_0 = \frac{i}{4}(\hat{X}\hat{P} + \hat{P}\hat{X}), \quad \hat{T}_1 = \hat{X}, \quad \hat{T}_2 = \hat{P}, \quad \hat{T}_0 = i.$$

The merit of an algebraic method is that above generators can be easily generalized to other realization forms to study a time-dependent Hamiltonian beyond the quadratic style [32].

If all the operators in Eq.(35) are treated with their corresponding mean-value c numbers such as $\hat{X} \rightarrow X$, the classical dynamical equation of motion reads

$$\ddot{X} - \frac{\dot{a}}{a}\dot{X} + \left(ab + \frac{\dot{a}}{a}c - c^2 - \dot{c}\right)X = cd - ae + \dot{d} - \frac{\dot{a}}{a}d. \quad (36)$$

Based on above closed Lie algebra, the time-dependent quantum system can be parameterized by six time-dependent functions $a(t) \cdots f(t)$ and its state evolution can be decomposed into different time-dependent transformations within the corresponding 6-dimensional parametric space.

2. Time-dependent transformations on Lie algebra

Now we give the main idea of the Lie transformation method to solve or control the general quadratic Hamiltonian of Eq.(35) [33]. If an original Schrödinger equation is

$$i\hbar \frac{\partial}{\partial t} \Psi(t) = \hat{\mathcal{H}}(t) \Psi(t), \quad (37)$$

and often a time-dependent transformation on $\Psi(t)$ is introduced by

$$\Psi(t) = \hat{U}(t) \psi(t), \quad (38)$$

then the Schrödinger equation for the new state $\psi(t)$ will be

$$i\hbar \frac{\partial \psi(t)}{\partial t} = \hat{\mathcal{H}}_U(t) \psi(t), \quad (39)$$

where

$$\hat{\mathcal{H}}_U(t) = \hat{U}^{-1}(t) \hat{\mathcal{H}}(t) \hat{U}(t) - i\hbar \hat{U}^{-1}(t) \frac{\partial \hat{U}(t)}{\partial t}. \quad (40)$$

Following Ref.[47], we can call above transformation of Eq.(40) as a Floquet U -transformation (FUT), and the transformed Hamiltonian $\hat{\mathcal{H}}_U$, generally, becomes non-Hermitian under FUT for that $\hat{\mathcal{H}}_U \neq \hat{\mathcal{H}}_U^\dagger$. Properly, if we select a successive transformations of $\hat{U}(t)$ based on a closed Lie algebra, we can finally simplify the original Hamiltonian $\hat{H}(t)$ into a solvable $\hat{\mathcal{H}}_U(t)$ whose eigenstates $\psi(t)$ can easily be obtained, then the original wave function for system $\hat{H}(t)$ will be determined by Eq.(38) if the initial conditions are given.

If a control driving is exerted at $t = 0$ on a quantum system, then we can set all the transformation parameters of $\hat{U}(t)$ to be zero at $t = 0$ to give $\hat{U}(0) = 1$. Therefore the initial state of a time-dependent controlled system can be determined by the initial Hamiltonian of $\hat{\mathcal{H}}(0)$ with its time-independent eigenequation of

$$\hat{\mathcal{H}}(0)\varphi_n = E_n\varphi_n.$$

Therefore any initial state of the system can then be expanded by above basis as

$$\Psi(0) = \sum_n C_n \varphi_n,$$

and

$$\hat{\mathcal{H}}_U(0) = \hat{\mathcal{H}}(0), \quad \psi(0) = \sum_n C_n \varphi_n. \quad (41)$$

Therefore, for the quadratic Hamiltonian of Eq.(35), we can define six independent transformations on $su(2) \oplus h(4)$ algebra by

$$e^{-is}, e^{-i\beta\hat{X}}, e^{-i\alpha\hat{P}}, e^{-i\theta_-\hat{J}_-}, e^{-2\theta_0\hat{J}_0}, e^{-i\theta_+\hat{J}_+},$$

where $s, \alpha, \beta, \theta_+, \theta_0, \theta_-$ are six transformation parameters. We can see that the total transformation $\hat{U}(t)$ can be any combination of above operators in different orders. For simplicity, we choose the following successive groups of transformations separated by $h(4)$ algebra and $su(2)$ algebra as [33]

$$\hat{U}(t) = \hat{U}_1(t) \hat{U}_2(t), \quad (42)$$

where

$$\hat{U}_1(t) = e^{-is(t)} e^{-i\alpha(t)\hat{P}} e^{-i\beta(t)\hat{X}}$$

is defined on $h(4)$ algebra which can generate a general coherent state of $h(4)$, and

$$\hat{U}_2(t) = e^{-i\theta_+\hat{J}_+} e^{-2\theta_0\hat{J}_0} e^{-i\theta_-\hat{J}_-} = e^{-i\theta_+\hat{X}^2/2} e^{-i\theta_0(\hat{X}\hat{P}+\hat{P}\hat{X})/2} e^{-i\theta_-\hat{P}^2/2},$$

is defined on $su(2)$ algebra which can provide a general $su(2)$ coherent state. Surely, above $\hat{U}(t)$ is a specific choice for our presentation and the relations between the transformations in different order can be obtained by using the Baker-Campbell-Hausdorff formula on the relevant algebra [38]. Generally, the six parameters $\alpha(t)$, $\beta(t)$, $s(t)$ and $\theta_+(t)$, $\theta_0(t)$, $\theta_-(t)$ are continuous functions of time defined within the controlling time interval. The real parametric functions will keep the system under a unitary evolution while the complex functions will break it in order to effectively describe a damping process. Anyway, according to Eq.(38), above Lie transformations not only induce transitionless evolution of the wave function (adiabatic process), but also produce non-adiabatic transitions controlled by the parametric functions. Substituting above successive transformations into Eq.(40), we have

$$\hat{\mathcal{H}}_U = \hat{U}_2^{-1} \left[\hat{U}_1^{-1} \hat{\mathcal{H}} \hat{U}_1 - i \hat{U}_1^{-1} \frac{\partial \hat{U}_1}{\partial t} \right] \hat{U}_2 - i \hat{U}_2^{-1} \frac{\partial \hat{U}_2}{\partial t},$$

where

$$\begin{aligned} \hat{\mathcal{H}}_1 &\equiv \hat{U}_1^{-1} \hat{\mathcal{H}} \hat{U}_1 - i \hat{U}_1^{-1} \frac{\partial \hat{U}_1}{\partial t} \\ &= \frac{1}{2} a \hat{P}^2 + \frac{1}{2} b \hat{X}^2 + \frac{1}{2} c (\hat{X} \hat{P} + \hat{P} \hat{X}) \\ &\quad + (c\alpha - a\beta + d - \dot{\alpha}) \hat{P} + (b\alpha - c\beta + e - \dot{\beta}) \hat{X} \\ &\quad + \frac{1}{2} a \beta^2 + \frac{1}{2} b \alpha^2 - c\alpha\beta - d\beta + e\alpha + f + \dot{\alpha}\beta - \dot{s}, \end{aligned}$$

and \hat{H} is in the unit of $\hbar\omega_0$. In order to simplify the Hamiltonian, we set the coefficients before the operators of $h(4)$ algebra to be zero. Then the dynamical equation of motion for the transformation parameters are

$$\begin{aligned} \dot{\alpha} &= c\alpha - a\beta + d, \\ \dot{\beta} &= b\alpha - c\beta + e, \\ \dot{s} &= \frac{1}{2} a \beta^2 + \frac{1}{2} b \alpha^2 - c\alpha\beta - d\beta + e\alpha + f + \dot{\alpha}\beta, \end{aligned} \quad (43)$$

which can reduce the original Hamiltonian into a standard quadratic form on $su(2)$ algebra as

$$\hat{\mathcal{H}}_1 = \frac{1}{2} a \hat{P}^2 + \frac{1}{2} b \hat{X}^2 + \frac{1}{2} c (\hat{X} \hat{P} + \hat{P} \hat{X}) = a \hat{J}_- + b \hat{J}_+ - 2ic \hat{J}_0. \quad (44)$$

The first two parametric equations of Eq.(43) lead to

$$\ddot{\alpha} - \frac{\dot{a}}{a} \dot{\alpha} + \left(ab + \frac{\dot{a}c}{a} - c^2 - \dot{c} \right) \alpha = cd - ae + \dot{d} - \frac{\dot{a}}{a} d \quad (45)$$

which exactly recovers the classical mean-value dynamics of Eq.(36). The last equation of Eq.(43) leads to a classical dynamical action of

$$s(t) = \int_0^t \mathcal{L}(\alpha, \beta, t') dt',$$

where the classical Lagrangian can be similarly defined by

$$\mathcal{L}(\alpha, \beta, t) = \frac{1}{2}b\alpha^2 - \frac{1}{2}a\beta^2 + e\alpha + f. \quad (46)$$

It should be noted again that this Lagrangian is not a unique form corresponding to the dynamical equation of Eq.(45) due to the existence of many linear canonical transformations in this system [6, 31]. As the parameter α is a solution of the classical dynamical equation of Eq.(36), we can similarly denote it as a classical position by $\alpha \rightarrow X_c = \langle \hat{X} \rangle$ and Eq.(45) can be written in a general form of

$$\ddot{X}_c + \chi(t) \dot{X}_c + \xi(t) X_c = \eta(t), \quad (47)$$

where

$$\begin{aligned} \chi(t) &= -\frac{\dot{a}}{a}, \\ \xi(t) &= ab + \frac{\dot{a}c}{a} - c^2 - \dot{c}, \\ \eta(t) &= cd - ae + \dot{d} - \frac{\dot{a}}{a}d. \end{aligned}$$

After the first series of transformations of \hat{U}_1 on Heisenberg-Weyl algebra $h(4)$, the original Hamiltonian (35) becomes a standard quadratic time-dependent Hamiltonian which has been exactly solved through Lewis-Riesenfeld invariant method [18, 35]. Although the dynamical invariant method is powerful, there is no general procedure to find an explicit invariant for a certain time-dependent Hamiltonian (the invariant for Eq.(44) is not unique). Subsequently, in the present method, a second series of independent FUTs on $su(2)$ algebra gives

$$\hat{\mathcal{H}}_U = \hat{U}_2^{-1} \hat{\mathcal{H}}_1 \hat{U}_2 - i \hat{U}_2^{-1} \frac{\partial \hat{U}_2}{\partial t}, \quad (48)$$

where (see Appendix A)

$$\begin{aligned} \hat{\mathcal{H}}_U &= \frac{1}{2} e^{2\theta_0} \left(a\theta_+^2 - 2c\theta_+ - \dot{\theta}_+ + b \right) \hat{X}^2 \\ &+ \frac{1}{2} \left[a e^{-2\theta_0} + e^{2\theta_0} \left(a\theta_+^2 - 2c\theta_+ - \dot{\theta}_+ + b \right) \theta_-^2 + 2\theta_- \left(c - a\theta_+ - \dot{\theta}_0 \right) - \dot{\theta}_- \right] \hat{P}^2 \\ &+ \frac{1}{2} \left[e^{2\theta_0} \left(a\theta_+^2 - 2c\theta_+ - \dot{\theta}_+ + b \right) \theta_- + \left(c - a\theta_+ - \dot{\theta}_0 \right) \right] \left(\hat{X}\hat{P} + \hat{P}\hat{X} \right). \end{aligned}$$

Similarly, if we set all the coefficients of $su(2)$ operators (\hat{X}^2 , \hat{P}^2 and $\hat{X}\hat{P} + \hat{P}\hat{X}$) in above transformed Hamiltonian to be zero, then we get the dynamical equations for θ s as

$$\begin{aligned} \dot{\theta}_+ &= a\theta_+^2 - 2c\theta_+ + b, \\ \dot{\theta}_- &= a e^{-2\theta_0}, \\ \dot{\theta}_0 &= c - a\theta_+. \end{aligned} \quad (49)$$

That means if all the parameters θ s are controlled by Eq.(49), the transformed Hamiltonian $\hat{\mathcal{H}}_U$ will shift to a zero point of energy and the system will stay on any initial state $|\varphi(t_0)\rangle$ beyond any quantum evolution. According to Eq.(48), we can see, in this case,

$$i \frac{\partial \hat{U}_2}{\partial t} = \hat{\mathcal{H}}_1 \hat{U}_2, \quad (50)$$

and the solution of $\hat{\mathcal{H}}_1$ can be exactly solved by the transformation \hat{U}_2 . Combined with transformation \hat{U}_1 , the complete solution of the initial Hamiltonian of Eq.(35) can be obtained as

$$\Psi(X, t) = e^{-is(t)} e^{-i\alpha(t)\hat{P}} e^{-i\beta(t)\hat{X}} e^{-i\theta_+\hat{X}^2/2} e^{-i\theta_0(\hat{X}\hat{P}+\hat{P}\hat{X})/2} e^{-i\theta_-\hat{P}^2/2} \varphi(X, 0), \quad (51)$$

where $\varphi(X, 0)$ is the initial state of the system. Therefore, if all the six transformation parameters α, β, s and $\theta_+, \theta_0, \theta_-$ are determined or controlled by the corresponding classical dynamical equations of Eq.(47) and Eq.(49), the solution of the system described by Eq.(35) can be exactly solved by this method. Above solution accompanied with two classical parametric equations can perfectly resolve the quantum evolution of an initial state into the classical dynamics and the quantum dynamics and it can naturally discriminate the dynamical phase and the geometric phase in the parametric space during an quantum evolution. However, by considering the physical meanings of the six parameters in Eq.(51), we can see some problems of solving the parametric equation of Eq.(49) for a dissipative system.

According to the mathematical properties of these transformations, we notice that the transformations $e^{-i\beta(t)\hat{X}}$ and $e^{-i\alpha(t)\hat{P}}$ are the momentum and spatial displacement operators, which will induce overall shifts to an initial wave packet in the momentum space and in the real space, respectively. The transformations $e^{-i\theta_+\hat{X}^2/2}$ and $e^{i\theta_-\hat{P}^2/2}$ are dispersion operators which leads to width modifications on a wave packet in momentum or in real space, and the transformation $e^{-i\theta_0(\hat{X}\hat{P}+\hat{P}\hat{X})/2}$ will result in a squeezing (or dilation) on the wave packet. These properties of the transformations can be clearly seen in a Heisenberg picture, in which the position and momentum operators evolve as [47]

$$\begin{aligned} \hat{X}_h(t) &= \hat{U}^{-1}\hat{X}\hat{U} = e^{\theta_0}\hat{X} + \theta_-e^{\theta_0}\hat{P} + \alpha, \\ \hat{P}_h(t) &= \hat{U}^{-1}\hat{P}\hat{U} = (e^{-\theta_0} - e^{\theta_0}\theta_+\theta_-)\hat{P} - e^{\theta_0}\theta_+\hat{X} - \beta. \end{aligned}$$

In a matrix form, above transformation $\hat{U}(t)$ reads

$$\begin{pmatrix} \hat{X}_h \\ \hat{P}_h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\theta_+ & 1 \end{pmatrix} \begin{pmatrix} e^{\theta_0} & 0 \\ 0 & e^{-\theta_0} \end{pmatrix} \begin{pmatrix} 1 & \theta_- \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{X} \\ \hat{P} \end{pmatrix} + \begin{pmatrix} \alpha \\ -\beta \end{pmatrix},$$

which clearly indicates a successive operation of translation, rotation and dilation on the operators \hat{X} and \hat{P} and they are actually equivalent to a canonical transformation on \hat{X} and \hat{P} . Specifically, if the initial wave function is in a number state of $|n\rangle$, then we can see the average values of

$$\begin{aligned} X_h(t) &= \langle n | \hat{X}_h | n \rangle = \alpha(t), \\ P_h(t) &= \langle n | \hat{P}_h | n \rangle = -\beta(t), \end{aligned}$$

which shows a total displacement of the wave function in real space or in the momentum space. The standard deviations of the wave function will be

$$\Delta X_h = e^{\theta_0} \sqrt{1 + \theta_-^2} \sqrt{n + \frac{1}{2}}, \quad (52)$$

$$\Delta P_h = e^{\theta_0} \sqrt{\theta_+^2 + (e^{-2\theta_0} - \theta_+\theta_-)^2} \sqrt{n + \frac{1}{2}}, \quad (53)$$

which leads to a profile modification of a wave packet in real space or in momentum space. Therefore, in the corresponding Schrödinger picture, above transformations indicate that the average position and the width of the wave packet will be modified during the quantum evolution controlled by the classical dynamic equations of Eq.(47) and Eq.(49). Furthermore, by using the transformation operator $\hat{U}(t)$, the propagator of the wave function will be easily obtained as shown in Ref.[47].

Now we try to find a solution of Eq.(49) in order to finally determined the transformation parameters of θ_s . We can easily find that the first equation of Eq.(49) is decoupled from the other two equations as

$$\dot{\theta}_+ = a\theta_+^2 - 2c\theta_+ + b. \quad (54)$$

Above equation is the Riccati differential equation [39] and its real solution can be solved by a Riccati transformation of

$$\theta_+ = -\frac{\dot{u}}{au}, \quad (55)$$

where u should satisfy the following linear second order ordinary differential equation

$$\ddot{u} + \left(2c - \frac{\dot{a}}{a}\right) \dot{u} + abu = 0. \quad (56)$$

Therefore, if θ_+ is determined, the other two parameters are

$$\begin{aligned} \theta_0(t) &= \int_0^t c(\tau) d\tau + \ln \frac{u(t)}{u(0)}, \\ \theta_-(t) &= a(t) \left[\frac{u(0)}{u(t)} \right]^2 e^{-2 \int_0^t c(\tau) d\tau}, \end{aligned}$$

where we can set the initial condition of $\theta_+(0) = \theta_-(0) = \theta_0(0) = 0$ for a control problem. Finally, as all the six parameters are solved according to Eq.(47) and Eq.(49), the quantum wave function of Eq.(51) will be easily obtained by this method.

3. Problems for Lie transformation method

Although above method seems perfect, the solution given by above Lie transformations can not always exhibit well behaviors, especially for an open quantum system with dissipations, because the real parameters θ s often become divergent and no physical solution can exist (see the example in the next section). Therefore, more control parameters should be introduced by conducting further transformations in order to avoid the singular dynamics of Eq.(49). Another way to smooth these divergent dynamics for a damping case without increasing the parametric dimension is to use a non-Hermitian effective Hamiltonian (the parameters $a(t), b(t)$ or $c(t)$ may be complex functions) to find a complex solution for the Riccati Eq.(54) by taking a form of [39]

$$\theta_+ = -\frac{\dot{\rho}}{a\rho} - i\frac{1}{\rho^2}, \quad (57)$$

and ρ will satisfy the following differential equation instead

$$\ddot{\rho} + \left(2c - \frac{\dot{a}}{a}\right) \dot{\rho} + ab\rho = \frac{a^2}{\rho^3}, \quad (58)$$

where we suppose $a(t)$ is still a real function in this case. Above complex solution of Eq.(57) has a better dynamical behavior and Eq.(58) is the fatuous Ermakov equation which always appears in the problems of a time-dependent parametric harmonic oscillator. In this case, the other two parameters become

$$\begin{aligned} \theta_0(t) &= \int_0^t c(\tau) d\tau + \ln \frac{\rho(t)}{\rho(0)} + i \int_0^t \frac{a}{\rho^2} d\tau, \\ \theta_-(t) &= a \left[\frac{\rho(0)}{\rho(t)} \right]^2 e^{-2 \int_0^t c(\tau) d\tau} e^{-2i \int_0^t \frac{a}{\rho^2} d\tau}. \end{aligned}$$

However, we should notice that the parameters θ s are complex numbers, and the unitary evolution of the wave function will be subsequently broken as expected for a dissipative system.

Because of the pathological behavior of the dynamical equation of Eq.(49) for a dissipative system, we can not always obtain ideal solutions for θ s through Eq.(49) in order to determined a perfect quantum solution of Eq.(51). Anyway, we can always choose a transformation of \hat{U} to reduce the original Hamiltonian (35) into a solvable form of $\hat{\mathcal{H}}_U$ by generally setting the corresponding coefficients in front of $su(2)$ operators of Eq.(48) to be arbitrary designed functions, say $K_1(t), K_2(t), K_3(t)$ (K s). Then, for the Lie transformations of Eq.(42), a general parametric equation will be

$$\begin{aligned} \dot{\theta}_+ &= a\theta_+^2 - 2c\theta_+ - K_2 e^{-2\theta_0} + b, \\ \dot{\theta}_- &= a e^{-2\theta_0} - K_2 \theta_-^2 + 2K_3 \theta_- - K_1, \\ \dot{\theta}_0 &= K_2 \theta_- - a\theta_+ + c - K_3, \end{aligned} \quad (59)$$

and the final transformed Hamiltonian becomes

$$\hat{\mathcal{H}}_U(t) = \frac{1}{2}K_1(t)\hat{P}^2 + \frac{1}{2}K_2(t)\hat{X}^2 + \frac{1}{2}K_3(t)\left(\hat{X}\hat{P} + \hat{P}\hat{X}\right), \quad (60)$$

which again takes a same form as Eq.(44). Therefore, in principle, the repeated FUT transformations of \hat{U}_2 will connect all the relevant quadratic Hamiltonians in a form of (44) and finally gives a closed group. As different functions assigned to K s will leads to different parametric equations generating different wave functions, now, the key problem here is how to determine or design the time-dependent functions of K s. The freedom of choice on K s indicates a trick that the FUTs should simplify or construct the original Hamiltonian into a solvable form according to different problems with specific initial conditions, which we will show in the next section by the specific driven and damped parametric oscillators as examples.

Since the Lewis-Riesenfeld invariants for Hamiltonian $\hat{\mathcal{H}}_1$ have been obtained and extensively discussed in many literatures [35, 40, 41], we can connect this Lie transformation with the invariant method to determined the proper functions of K s. As, in principle, there exist many dynamical invariants \hat{I} for a given Hamiltonian [30, 40], so, generally, we suppose that the transformed Hamiltonian $\hat{\mathcal{H}}_U$ can be reduced to a real function of the dynamical invariant of $\hat{I}_1(t)$ for $\hat{\mathcal{H}}_1$ [42], i.e.,

$$\hat{\mathcal{H}}_U(t) = F(\hat{I}_1) \quad (61)$$

with

$$\frac{d}{dt}\hat{I}_1(t) = \frac{\partial \hat{I}_1}{\partial t} + \frac{1}{i}\left[\hat{I}_1, \hat{\mathcal{H}}_1\right] = 0. \quad (62)$$

Surely, if above real function $F(x)$ is properly designed for a specific control problem, all the time-dependent quadratic Hamiltonians connected by FUT with $su(2)$ operators can be exactly solved only by considering the invariants of the system. Among different invariants of $\hat{I}_1(t)$, we only focus on the simplest one, such as the basic invariant [43], or on a suitable one for a specific controlled system. Incidentally, if $K_1(t) = a(t)$, $K_2(t) = b(t)$, $K_3(t) = c(t)$, the Hamiltonian is invariant under the FUT and the corresponding transformation operator \hat{U}_2 itself will be an invariant operator following Eq.(62). This indicates a convergent case for the FUT transformations.

Simply, if Eq.(60) can be reduced to *one* of invariants of $\hat{\mathcal{H}}_1$, the invariant requirement of Eq.(62) will lead to a dynamical equation for the parameters K s as

$$\begin{aligned} \dot{K}_1 &= 2cK_1 - 2aK_3, \\ \dot{K}_2 &= 2bK_3 - 2cK_2, \\ \dot{K}_3 &= bK_1 - aK_2. \end{aligned} \quad (63)$$

As shown in Ref.[35], a proper solution of Eq.(63) can be constructed by setting

$$\begin{aligned} K_1(t) &= \rho^2, \\ K_2(t) &= \frac{\Omega^2}{\rho^2} + \frac{1}{a^2}(c\rho - \dot{\rho})^2, \\ K_3(t) &= \frac{\rho}{a}(c\rho - \dot{\rho}), \end{aligned} \quad (64)$$

where an auxiliary time-dependent function $\rho(t)$ is introduced to solve Eq.(63) and it should obey a further dynamical equation of

$$\ddot{\rho} + \chi(t)\dot{\rho} + \xi(t)\rho = \Omega^2 \frac{a^2}{\rho^3}, \quad (65)$$

where Ω is an arbitrary real constant, $\chi(t)$ and $\xi(t)$ are defined in Eq.(47). We can see that Eq.(65) is different from the classical dynamical equation of Eq.(47) but is similar to the Ermakov equation of Eq.(58) obtained in the construction of a complex solution for Riccati equation. The new dynamical equation of Eq.(65) coming up here is due to the requirement of keeping the transformed $\hat{\mathcal{H}}_U$ to be an invariant for the standard quadratic Hamiltonian of Eq.(44), for example, to the solution of Eq.(64),

$$\hat{\mathcal{H}}_U = \frac{1}{2}\left(\frac{\Omega}{\rho}\right)^2 \hat{X}^2 + \frac{1}{2}\left[\rho\hat{P} + \frac{1}{a}(c\rho - \dot{\rho})\hat{X}\right]^2. \quad (66)$$

As the functions of K s are determined by Eq.(64), the transformation parameters of θ s in Eq.(59) can provide a general Lie transformation to exactly solve the wave function of Eq.(51). Surely, for a control problem to obtain a target state, we can “design” the functions of K s for a controlled system from its initial Hamiltonian $\hat{H}(0)$ to a final $\hat{H}(T)$ by an inverse Lie transformation during a finite control time interval of T [30, 44].

Certainly, the simplest case to determined the Lie transformations is to set all the three parameters K s to be constant (the constant K s are the specific solutions of Eq.(63)). In this case, $\hat{\mathcal{H}}_U$ will naturally become an invariant and the initial time-dependent system $\hat{\mathcal{H}}(t)$ can be turned into a time-independent system (conservative system) by the time-dependent FUT transformations of $\hat{U}(t)$. Theoretically, the constant K s can always be permitted by this method but, unfortunately, sometimes the constant K s will lead to pathological solutions of θ s which failed to determine the wave functions. In this case, a constructed invariant Hamiltonian from $\hat{\mathcal{H}}_U$ by an inverse Lie transformation of $\hat{\mathcal{H}}(t) = \hat{U}(t)\hat{\mathcal{H}}_U\hat{U}^{-1}(t)$ is needed. In the following part, we will use this Lie transformation method on the driven and damped parametric oscillators to show how to use this method to solve or construct a quadratic time-dependent system under different constant of K s. A general case of K s are simply considered in the end.

D. Specific models on driven and damped parametric oscillator

Now we will apply above method to consider the quantum dynamics for the specific driven and damped parametric oscillators under a general Hamiltonian of (35) by using different driving functions of $a(t), \dots, f(t)$. A very typical Hamiltonian widely used in the literature to described a driven and damped harmonic oscillator is [9]

$$\hat{H}(t) = e^{-2\gamma t} \frac{\hat{p}^2}{2m_0} + e^{2\gamma t} \left[\frac{1}{2} m_0 \omega_0^2 \hat{x}^2 - \hat{x} F(t) \right]. \quad (67)$$

Clearly, this Hamiltonian is Hermitian with real parameters and its scaled form is

$$\hat{\mathcal{H}}(t) = \frac{1}{2} e^{-2\gamma t} \hat{P}^2 + \frac{1}{2} e^{2\gamma t} \hat{X}^2 - e^{2\gamma t} f_c(t) \hat{X}, \quad (68)$$

and the parameters for the system are $a(t) = e^{-2\gamma t}$, $b(t) = e^{2\gamma t}$ and $e(t) = -e^{2\gamma t} f_c(t)$. We can see that the initial Hamiltonian of this model is

$$\hat{\mathcal{H}}(0) = \frac{1}{2} \hat{P}^2 + \frac{1}{2} \hat{X}^2 - f_c(0) \hat{X},$$

which means initially the system is a normal harmonic oscillator with a driving force $f_c(t)$ gradually exerting on it and we can set $f_c(0) = 0$ for simplicity. By using the time-dependent Lie transformation method, we can directly solve Hamiltonian (68). Then Eq.(47) gives

$$\ddot{X}_c + 2\gamma \dot{X}_c + X_c = f_c(t), \quad (69)$$

which recovers the classical dynamical equation of Eq.(1). Here, for simplicity, we consider again the simple driving force as

$$f_c(t) = f_0 \cos(\Omega t + \phi),$$

and the classical Lagrangian given by Eq.(46) is

$$\mathcal{L}(X_c, \dot{X}_c, t) = \frac{1}{2} e^{2\gamma t} \dot{X}_c^2 - \frac{1}{2} e^{2\gamma t} X_c^2 - f_0 e^{2\gamma t} \cos(\Omega t + \phi) X_c.$$

After the transformation of \hat{U}_1 on $h(4)$ algebra, the Hamiltonian reduces to a quadratic form of

$$\hat{\mathcal{H}}_1 = \frac{1}{2} e^{2\gamma t} \hat{P}^2 + \frac{1}{2} e^{-2\gamma t} \hat{X}^2, \quad (70)$$

and it reduces to the well-known Hamiltonian of free harmonic oscillator with exponentially decreasing mass. Then the followed transformations on $su(2)$ algebra gives

$$\hat{\mathcal{H}}_U(t) = \frac{1}{2} K_1 \hat{P}^2 + \frac{1}{2} K_2 \hat{X}^2 + \frac{1}{2} K_3 (\hat{X} \hat{P} + \hat{P} \hat{X}), \quad (71)$$

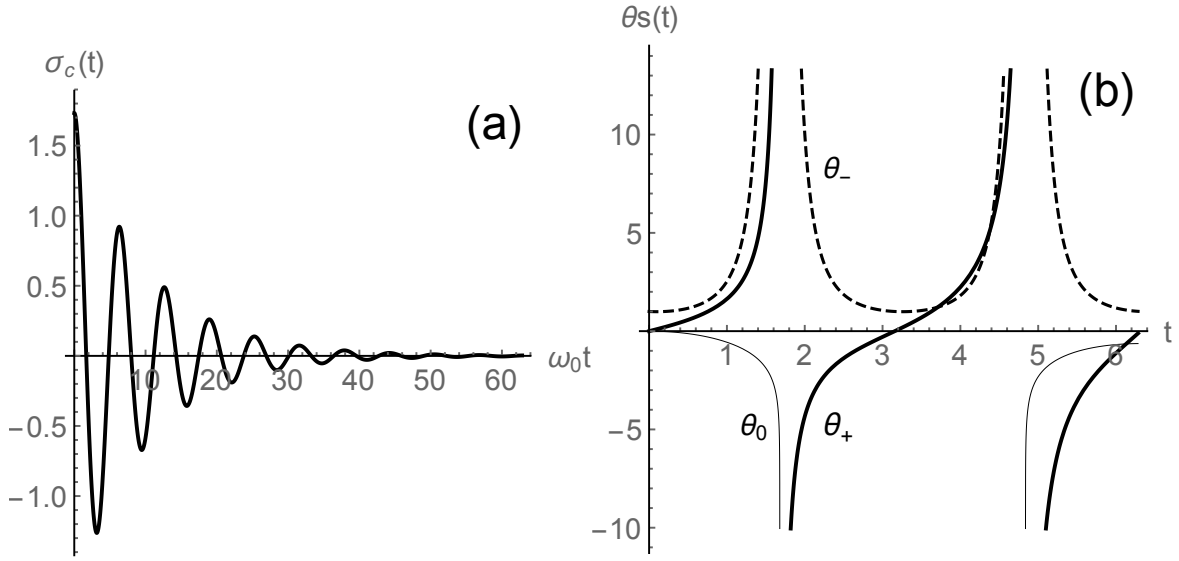


FIG. 8: (a) The dynamics of parameter $\sigma(t)$ and (b) the dynamical behaviors of the parameters θ_+ , θ_0 , θ_- . The other parameters are $\gamma = 0.1$ and $\sigma_0 = \sqrt{3}$.

with the parametric equation of

$$\begin{aligned}\dot{\theta}_+ &= e^{-2\gamma t}\theta_+^2 - K_2e^{-2\theta_0} + e^{2\gamma t}, \\ \dot{\theta}_- &= e^{-2\gamma t}e^{-2\theta_0} - K_2\theta_-^2 + 2K_3\theta_- - K_1, \\ \dot{\theta}_0 &= K_2\theta_- - e^{-2\gamma t}\theta_+ - K_3.\end{aligned}\quad (72)$$

Surely, the solutions of above parameters depend on the choice of K s. Now we will consider different K s to determine $su(2)$ transformation parameters of θ s by reducing the final Hamiltonian $\hat{\mathcal{H}}_U$ into different solvable forms.

1. Exact solvable case of $K_s=0$

The simplest case is to set $K_1 = K_2 = K_3 = 0$ in Eq.(72) and the transformed Hamiltonian Eq.(60) will shift to a zero point of energy. In principle, no constraints exclude this simplest case and the real solution for the transformation parameter θ_+ is

$$\theta_+(t) = -e^{2\gamma t} \frac{\dot{\sigma}_c}{\sigma_c}, \quad (73)$$

where σ_c satisfies a homogenous damped harmonic equation as

$$\ddot{\sigma}_c + 2\gamma\dot{\sigma}_c + \sigma_c = 0. \quad (74)$$

The solution of Eq.(74) is well known as

$$\sigma_c(t) = \sigma_0 e^{-\gamma t} \left[\cos(\sqrt{1-\gamma^2}t) + \frac{\gamma}{\sqrt{1-\gamma^2}} \sin(\sqrt{1-\gamma^2}t) \right],$$

where σ_0 is the initial value of σ_c . Then the other two transformation parameters read

$$\begin{aligned}\theta_0(t) &= \ln\left(\frac{\sigma_c}{\sigma_0}\right), \\ \theta_-(t) &= e^{-2\gamma t} \left(\frac{\sigma_0}{\sigma_c}\right)^2.\end{aligned}$$

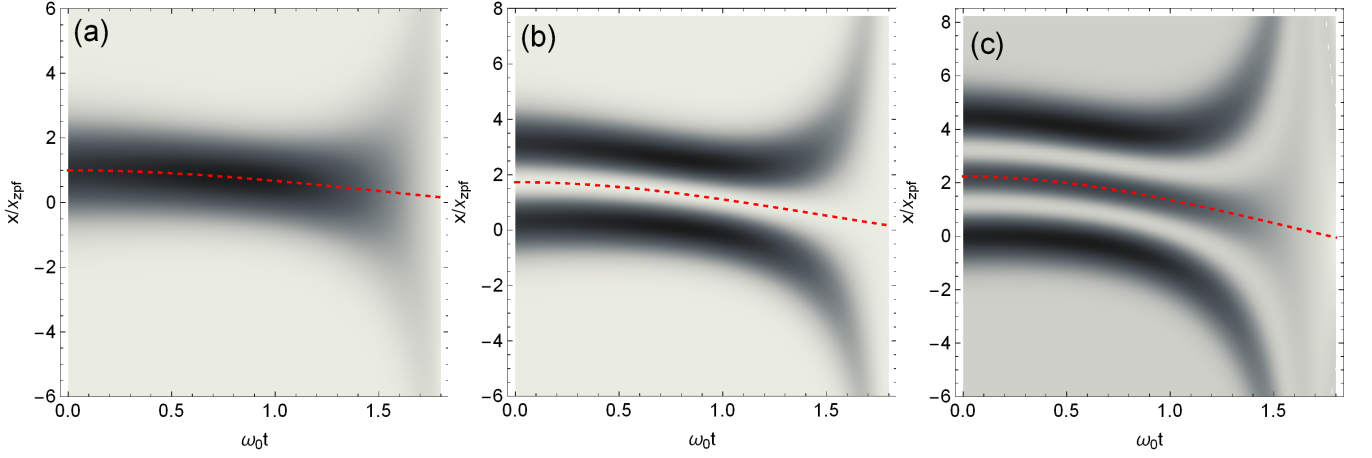


FIG. 9: The evolution of probability distributions around the classical orbits (red dashed lines) of a driven and damped harmonic oscillator starting from different initial states of (a) $|0\rangle$, (b) $|1\rangle$ and (c) $|2\rangle$ under a harmonic driving force of $f_c = 0.2 \cos(0.8\omega_0 t)$. The damping rate of the harmonic oscillator is $\gamma = 0.2$ and the initial conditions for (a) $X_c(0) = 1, P_c(0) = 0$, (b) $X_c(0) = \sqrt{3}, P_c(0) = 0$ and (c) $X_c(0) = \sqrt{5}, P_c(0) = 0$ according to the corresponding energies of the number state.

Fig.8 shows the dynamics of the parameters of $\sigma_c(t)$ and θ_s in this simplest case. As the parameter $\sigma_c(t)$ follows Eq.(74), we can see clearly that θ_s becomes divergent when

$$\cot\left(\sqrt{1-\gamma^2}t\right) = -\frac{\gamma}{\sqrt{1-\gamma^2}}. \quad (75)$$

At the time points determined by Eq.(75), θ_s are singular and the solution obtained in this case will lose its physical meaning. If the initial wave function is in a pure wave function of harmonic oscillator,

$$\varphi_n(X, 0) = A_n e^{-\frac{1}{2}X^2} H_n(X), \quad |A_n|^2 = \frac{1}{\sqrt{\pi}2^n n!},$$

then the wave function of Eq.(51) at time t will be

$$\Psi(X, t) = e^{-i\int_0^t \mathcal{L}(X_c, \dot{X}_c, t) dt} e^{-\theta_0/2} e^{-iX_c \hat{P}} e^{i(e^{-2\gamma t} \dot{X}_c) \hat{X}} e^{-i\theta_+ \hat{X}^2/2} e^{-i\theta_0 \hat{X} \hat{P}} e^{-i\theta_- \hat{P}^2/2} \varphi_n(X). \quad (76)$$

The density distributions of the wave function at different time starting from different initial states are displayed in Fig.9. By using the formula such as

$$e^{\theta \frac{\partial^2}{\partial x^2}} f(x) = \frac{1}{\sqrt{4\pi\theta}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\theta}} f(y) dy,$$

$$e^{\theta x \frac{\partial}{\partial x}} f(x) = f(xe^\theta),$$

an explicit solution of the wave function can be obtained. In Fig.9, we calculate the probability distribution of $\Psi(X, t)$ by a Fourier transformation method and find a singular behavior compared with the case without damping in Fig.7. When the parameters θ_s becomes divergent, the wave function deviated from the classical orbit (dashed lines) and is totally dilated by θ_0 (see Eq.(52) and Eq.(53)) at a time determined by Eq.(75) due to the damping solution of Eq.(74). We can see that this simplest case for $K_s = 0$ can not give a proper longtime solution for a driven and damped harmonic oscillator described by Caldirola-Kanai Hamiltonian of (67).

Since the divergent dynamics for a damping case, we can formally find the complex parameters for a system described by a non-Hermitian Hamiltonian, such as Eq.(19) where $c(t) = -i\Gamma$. But we still want to use the Hermitian Hamiltonian of Eq.(67) to construct a complex solution for θ_+ as

$$\theta_+(t) = -e^{2\gamma t} \frac{\dot{\sigma}_c}{\sigma_c} - i \frac{1}{\sigma_c^2},$$

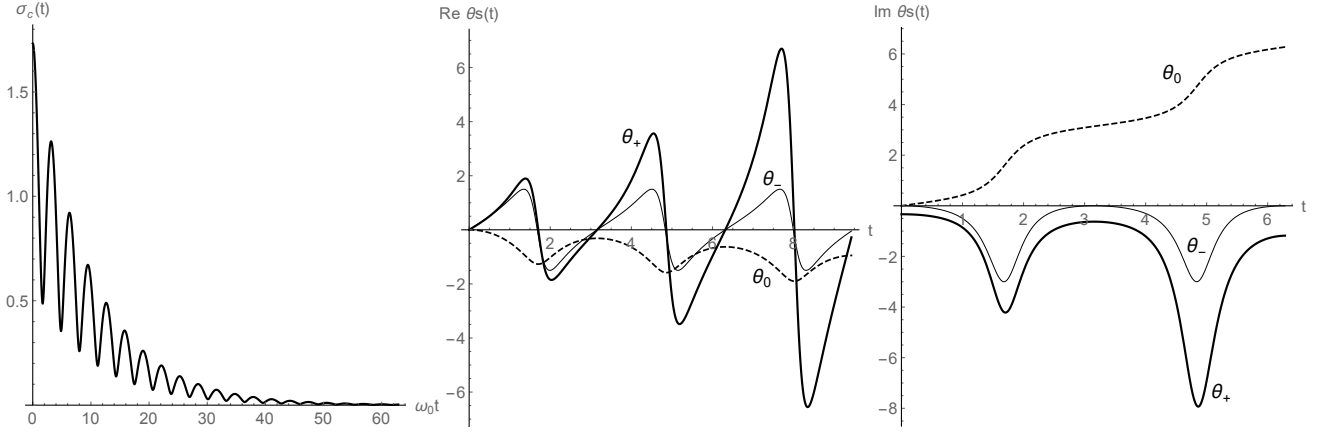


FIG. 10: The dynamics of parameter $\sigma_c(t)$ and the dynamical behaviors of the real and the imaginary part of θ_+ , θ_0 , θ_- . The other parameters are the same as that of Fig.8.

and the auxiliary classical equation changes to

$$\ddot{\sigma}_c + 2\gamma\dot{\sigma}_c + \sigma_c = \frac{e^{-4\gamma t}}{\sigma_c^3}.$$

The other two parameters are

$$\begin{aligned}\theta_0(t) &= \ln\left(\frac{\sigma_c}{\sigma_0}\right) + i\theta(t), \\ \theta_-(t) &= e^{-2\gamma t} \left(\frac{\sigma_0}{\sigma_c}\right)^2 e^{-2i\theta(t)},\end{aligned}$$

with

$$\theta(t) = \int_0^t \frac{e^{-2\gamma\tau}}{\sigma_c^2} d\tau.$$

In this case the dynamics of the transformation parameters θ s are improved at the divergent times, as shown in Fig.10, but, still, the parameters will go to infinity at a long time. As the complex θ s breaks the unitary evolution of

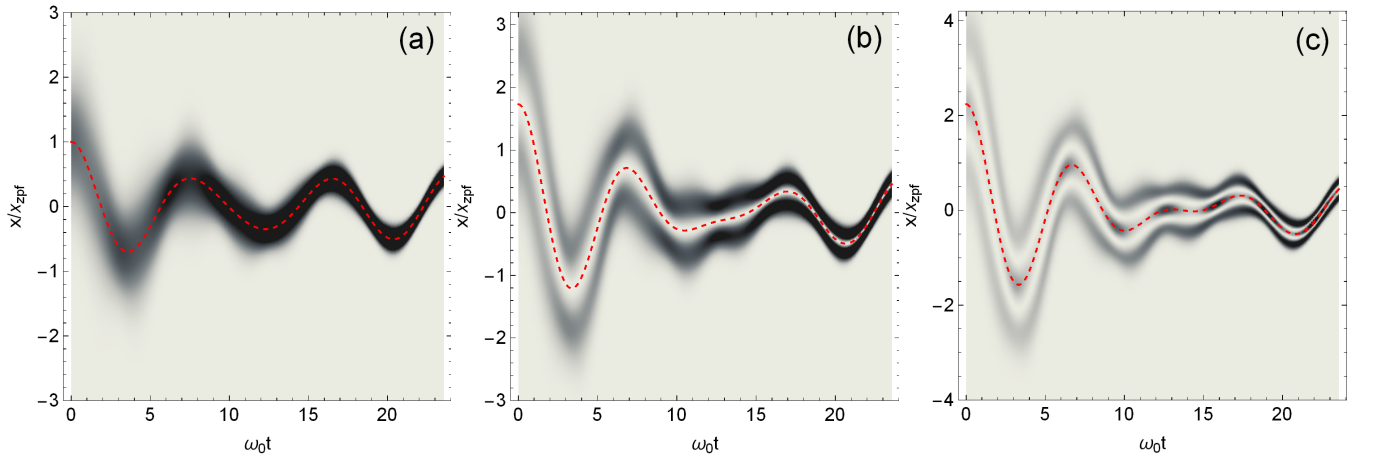


FIG. 11: The evolution of the wave packet starting from (a) $|0\rangle$, (b) $|1\rangle$ and (c) $|2\rangle$ number states by using the complex parameter method. The other parameters are the same as that of Fig.9.

the wave function, the probability density of the wave function will be distorted by the imaginary parts of θ s compared with that of Eq.(76) in Fig.9. In this case, the wave function reads

$$\Psi(X, t) = e^{-i \int_0^t \mathcal{L}(X_e, \dot{X}_e, t) dt} e^{-\theta_0/2} e^{-i\alpha \hat{P}} e^{-i\beta \hat{X}} e^{-i\theta_+ \hat{X}^2/2} e^{-i\theta_0 \hat{X} \hat{P}} e^{-i\theta_- \hat{P}^2/2} \varphi_n(X)$$

and its evolutions are displayed in the lower frames of Fig.11. We can see that the wave packets are dramatically distorted and localized by the imaginary part of θ s preventing divergence as that shown in Fig.9. As the parameters increase with time, the local probability density will finally exceed a physical value when the solution becomes invalid. Above results demonstrate that although the Lie transformation method seems neat if we set all the parameters in Eq.(48) to be zero, we can not always find a good solution for a quantum system, especially, for a dissipative one. Certainly, a better solution can be obtained if we don't set all K 's to be zero.

2. The driven parametric harmonic oscillator

If, initially, the system $\hat{\mathcal{H}}(0)$ is a free harmonic oscillator, then we can naturally set $K_1 = 1$, $K_2 = \Omega^2$ and $K_3 = 0$ for a better choice to improve the dynamical behavior of the transformation parameters. In this case, the time-dependent Hamiltonian of Eq.(44) or Eq.(70) can be transformed into a time-independent harmonic oscillator as

$$\hat{\mathcal{H}}_U = \frac{1}{2} \hat{P}^2 + \frac{1}{2} \Omega^2 \hat{X}^2 \quad (77)$$

provided that the parameters of $su(2)$ transformation in $\hat{U}_2(t)$ obey the following equations of motion

$$\begin{aligned} \dot{\theta}_+ &= e^{-2\gamma t} \theta_+^2 - \Omega^2 e^{-2\theta_0} + e^{2\gamma t}, \\ \dot{\theta}_- &= e^{-2\gamma t} e^{-2\theta_0} - \Omega^2 \theta_-^2 - 1, \\ \dot{\theta}_0 &= \Omega^2 \theta_- - e^{-2\gamma t} \theta_+. \end{aligned} \quad (78)$$

If the parameters θ s are perfectly solved, the wave function will be

$$\Psi(X, t) = \hat{U}(t) \Psi(X, 0) = \hat{U}_1(t) \hat{U}_2(t) \Psi(X, 0),$$

and the general initial wave function $\Psi(X, 0)$ can be expressed by

$$\Psi(X, 0) = \sum_n C_n \varphi_n(X, 0), \quad (79)$$

where C_n are complex constants determined by the initial population distribution and

$$\begin{aligned} \varphi_n(X, t) &= \frac{1}{\sqrt{\sqrt{\pi} 2^n n!}} e^{-\frac{1}{2} X^2} H_n(X) e^{-i E_n t}, \\ E_n &= \Omega \left(n + \frac{1}{2} \right). \end{aligned}$$

Clearly, above wave function $\Psi(X, t)$ is solved based on a basis of Eq.(77). The upper frames of Fig.12 show the dynamics of θ s (left), the variances ΔX_h and ΔP_h and the position-momentum uncertainty $\Delta X_h \Delta P_h$ (right) of wave packets starting from the ground state $|0\rangle$. The lower frames Fig.12 demonstrate the probability density of the wave functions starting from different pure number states. Fig.12 reveals that a squeezing of the fluctuations in position is in the expense of a dilation in the fluctuations of momentum, and the wave function will gradually go to a localized displaced number state in real space, then, finally, it explodes due to $\Delta X_h \rightarrow 0$ or $\Delta P_h \rightarrow \infty$. Unfortunately, a further parameter Ω still can not stabilize the longtime evolution of the wave packet for that only $\Omega = 1$ is a better choice for the damping case because Eq.(78) becomes unstable when Ω deviates a little from 1. During the state evolution, the damping rate γ will squeeze the width of the wave packet in real space through θ s and its dynamics is the same as that shown in Fig.11 by the complex transformation variables of θ s. The longtime localized wave function is clearly unreasonable, but for a control problem, the solution for a short time interval is physically acceptable.

In order to give a more physical wave function for a long time evolution, we can resort to the invariant operator to solve the problem based on the Hamiltonian of Eq.(77). If a controlled time-dependent system (35) starts with a

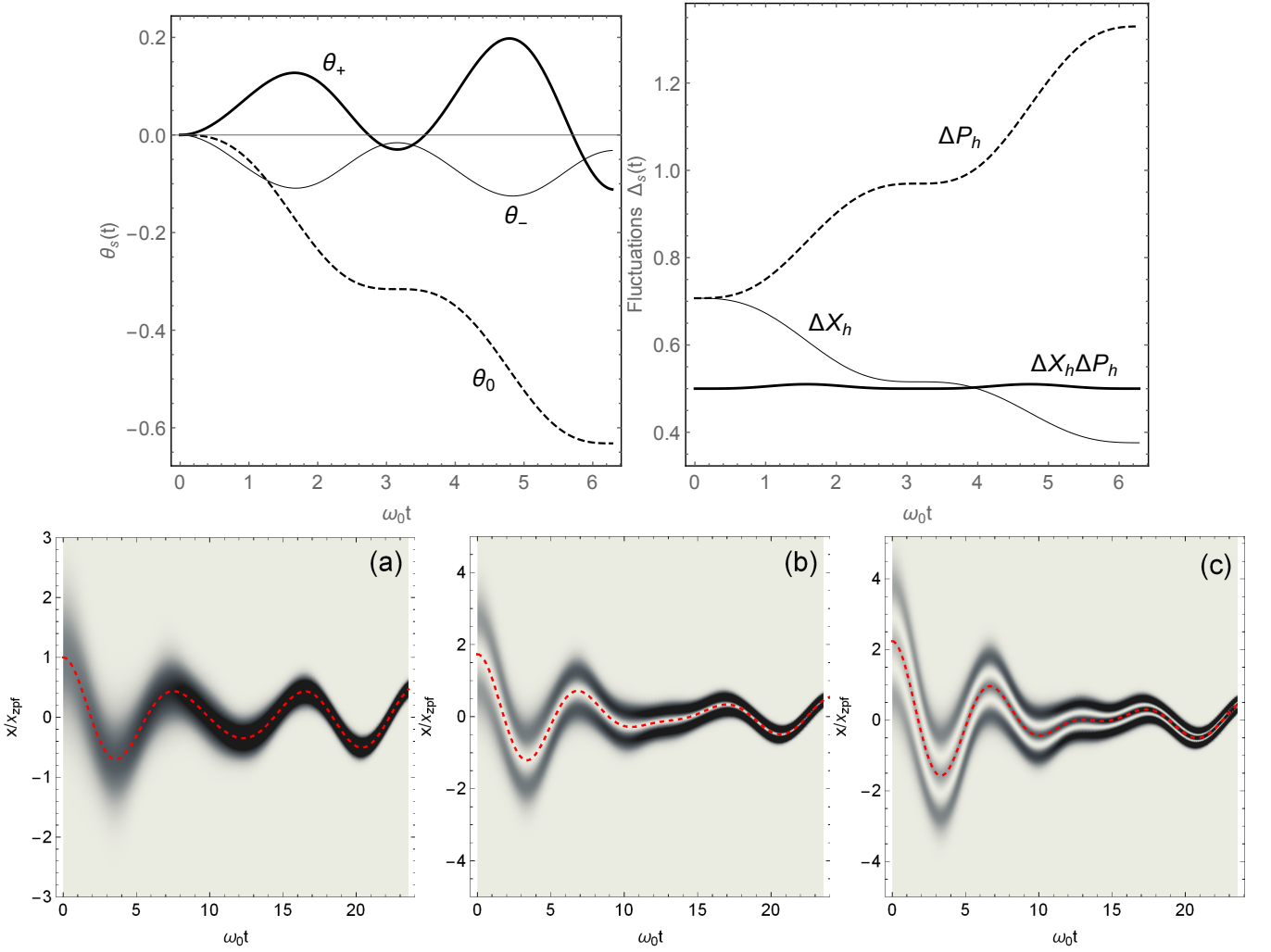


FIG. 12: (Upper frames) The dynamical behaviors of the parameters of θ_+ , θ_0 , θ_- and the fluctuations of position ΔX_h and momentum ΔP_h as well as the uncertainty of $\Delta X_h \cdot \Delta P_h$ for $\Omega = 1$. (Lower frames) The evolution of the wave packet starting from the initial states of (a) $|0\rangle$, (b) $|1\rangle$ and (c) $|2\rangle$. The other parameters are the same as that of Fig.9.

Hamiltonian of $\hat{\mathcal{H}}(0) = \hat{\mathcal{H}}_U$ as shown by Eq.(41), a general invariant operator for system (67) can be constructed by an inverse transformation of $\hat{U}(t)$,

$$\hat{I}(t) = \hat{U} \hat{\mathcal{H}}_U \hat{U}^{-1} = \frac{1}{2} e^{2\theta_0} \left[(\hat{P} + \beta) + \theta_+ (\hat{X} - \alpha) \right]^2 + \frac{\Omega^2}{2} \left[(e^{-\theta_0} - e^{\theta_0} \theta_- \theta_+) (\hat{X} - \alpha) - e^{\theta_0} \theta_- (\hat{P} + \beta) \right]^2.$$

Clearly, if we set $\theta_- = 0$ in Eq.(78), the invariant operator reduces to

$$\hat{I}(t) = \frac{1}{2} e^{2\theta_0} \left[(\hat{P} + \beta) + \theta_+ (\hat{X} - \alpha) \right]^2 + \frac{1}{2} \Omega^2 e^{-2\theta_0} (\hat{X} - \alpha)^2, \quad (80)$$

which is equivalent to the invariant operator given in Ref.[27, 28] when we use a different function of $\theta_0 = \ln \rho$. The invariant obtained here is also equivalent to that derived from Eq.(66), which we will investigate in detail for a general case of K s by using Eq.(59) and Eq.(63).

In order to give more insight into the state evolution based on a reduced harmonic oscillator, we consider another model of harmonic oscillator with time-dependent mass and frequency as

$$\hat{H}(t) = \frac{1}{2m(t)} \hat{p}^2 + \frac{1}{2} m(t) \omega^2(t) \hat{x}^2. \quad (81)$$

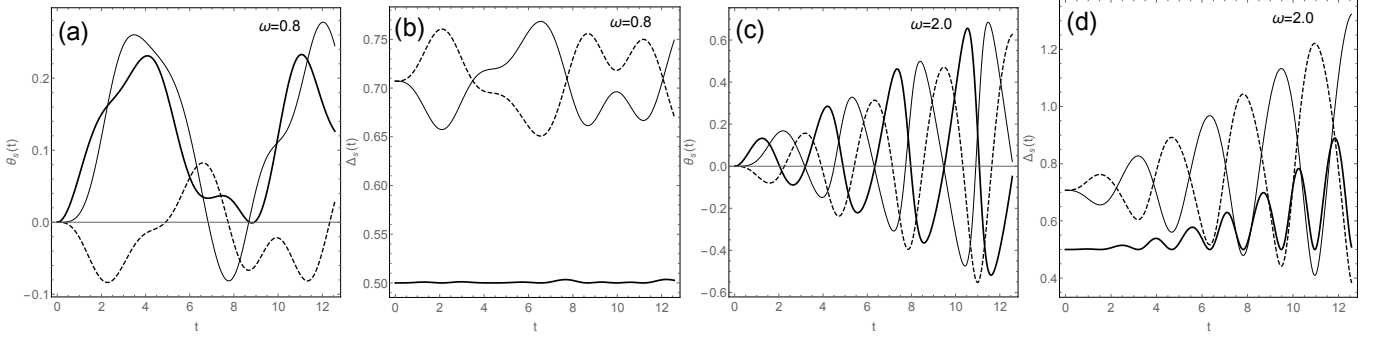


FIG. 13: (a)(c)The dynamical behaviors of parameters θ_+ (thick lines), θ_0 (dashed lines), θ_- (thin lines) and (b)(d)their fluctuations of position ΔX_h (thin lines) and momentum ΔP_h (dashed lines) as well as the uncertainty of $\Delta X_h \cdot \Delta P_h$ (thick lines) for the parametric harmonic oscillator starting from ground state $|0\rangle$ under different modulating frequencies ω . The modulating amplitude of the frequency is $\lambda = 0.2$.

If $m(0) = m_0$ and $\omega(0) = \omega_0$, and the scaled Hamiltonian in the units of Eq.(33) is

$$\hat{\mathcal{H}} = \frac{1}{2M(t)} \hat{P}^2 + \frac{1}{2} M(t) \Omega^2(t) \hat{X}^2,$$

where $M(t) = m(t)/m_0$, $\Omega(t) = \omega(t)/\omega_0$ and $\hat{\mathcal{H}}$ is in unit of $\hbar\omega_0$. Above Hamiltonian has a $su(2)$ algebraic structure and we can use Lie transformations of

$$\hat{U}_2(t) = e^{-i\frac{\theta_+}{2}\hat{X}^2} e^{-i\frac{\theta_0}{2}(\hat{X}\hat{P}+\hat{P}\hat{X})} e^{-i\frac{\theta_-}{2}\hat{P}^2} \quad (82)$$

to reverse it into an initial Hamiltonian of $\hat{\mathcal{H}}(0)$ as

$$\hat{\mathcal{H}}_U = \hat{\mathcal{H}}(0) = \frac{1}{2}\hat{P}^2 + \frac{1}{2}\hat{X}^2,$$

where $M(0) = \Omega(0) = 1$. Therefore, in this case, the transformation parameters must follow ($K_1 = K_2 = 1$)

$$\begin{aligned} \dot{\theta}_+ &= \frac{1}{M(t)}\theta_+^2 - e^{-2\theta_0} + M(t)\Omega^2(t), \\ \dot{\theta}_- &= \frac{1}{M(t)}e^{-2\theta_0} - \theta_-^2 - 1, \\ \dot{\theta}_0 &= \theta_- - \frac{1}{M(t)}\theta_+, \end{aligned} \quad (83)$$

which, clearly, is a specific case of Eq.(77) for $\Omega = 1$. Actually, the condition of above equation having a steady state is only for $\Omega^2(t) = 1$. In this case, the classical dynamic equation for a parametric oscillator determined by Eq.(47) is

$$\ddot{X}_c + \frac{\dot{M}(t)}{M(t)}\dot{X}_c + \Omega^2(t)X_c = 0. \quad (84)$$

Certainly, as the original Hamiltonian (81) only has an algebraic structure of $su(2)$, the Lie transformation \hat{U}_1 of $h(4)$ is not necessarily needed (no driving case) and the classical dynamical equation of Eq.(84) should not be included. However, above model can be easily generalized to a driven parametric oscillator with classical dynamic equation as Eq.(69)

$$\ddot{X}_c + \frac{\dot{M}(t)}{M(t)}\dot{X}_c + \Omega^2(t)X_c = \frac{f_0}{M(t)}\cos(\Omega t + \phi), \quad (85)$$

which will lead to a wave function of

$$\Psi(X, t) = \hat{U}_1(t)\hat{U}_2(t)\Psi(X, 0)$$

with $\Psi(X, 0)$ being defined by Eq.(79). Now we consider a specific model with the controlled parameters as

$$M(t) = 1, \quad \Omega^2(t) = 1 + \lambda \sin(\omega t),$$

to demonstrate the quantum evolution of a driven parametric harmonic oscillator starting from different number states. In Fig.13 we first show the temporal behavior of the transformation parameters of θ s and the corresponding fluctuations of position and momentum starting from the ground state $|0\rangle$. We can clearly see an alternative squeezing of position or momentum below $1/2$ for a modulating frequency of $\omega = 2\omega_0$ and the probability distribution of the wave functions display a manifest distortion for this double parametric resonance as shown in Fig.14.

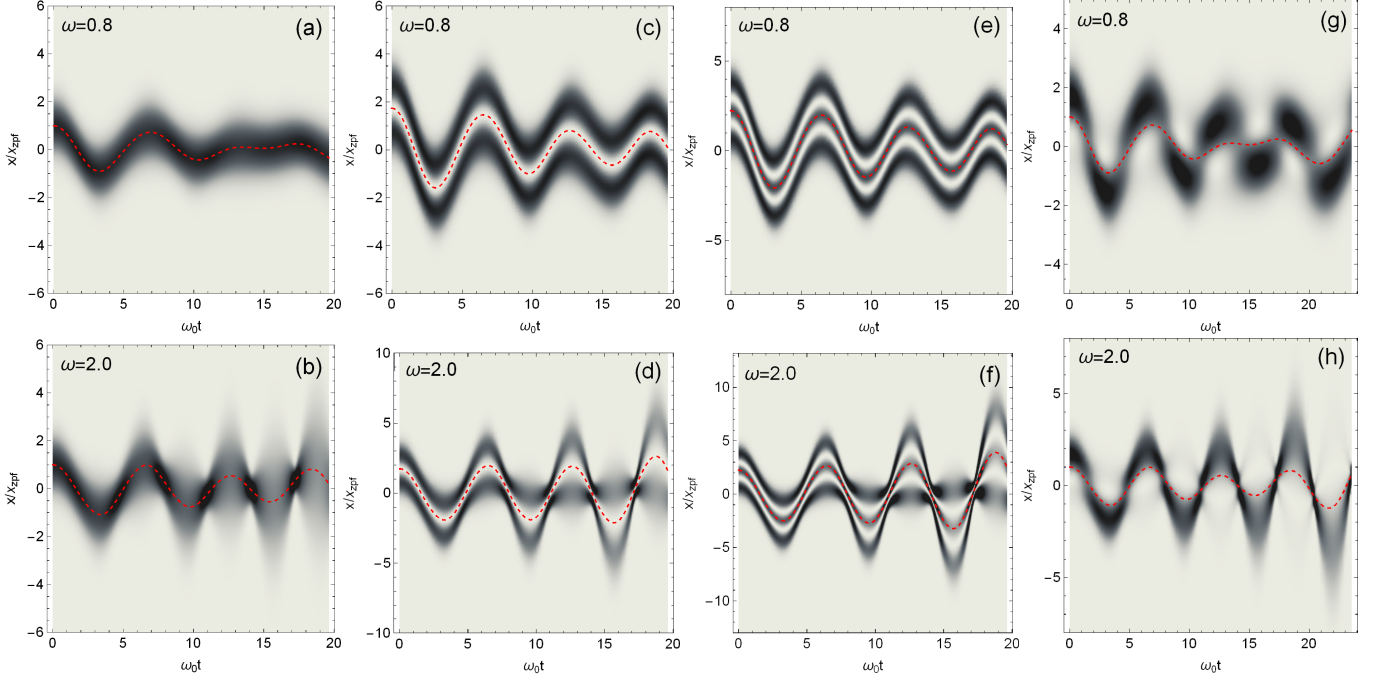


FIG. 14: The evolution of the wave packet of driven parametric oscillator starting from initial states of (a)(b) $|0\rangle$, (c)(d) $|1\rangle$, (e)(f) $|2\rangle$ and (g)(h) $(|0\rangle + |1\rangle)/\sqrt{2}$. The driving amplitude $f_0 = 0.2$ and the driving frequency $\Omega = 0.8$. The other parameters are the same as that of Fig.13.

For a harmonic oscillator with a time-dependent mass of $M(t)$, we can further simplify the transformation of \hat{U}_2 in Eq.(82) by setting $\theta_- = 0$, and then we can get solution of

$$\begin{aligned} \theta_+(t) &= \frac{1}{2} \dot{M}(t), \\ \theta_0(t) &= -\frac{1}{2} \ln M(t), \end{aligned}$$

and the mass should be controlled by an equation of

$$\ddot{M} - \frac{\dot{M}^2}{2M} + 2M(1 - \Omega^2) = 0$$

in order to meet Eq.(83). Then the transformation in this case will be

$$\hat{U}_2(t) = e^{-i\frac{1}{4}\dot{M}(t)\hat{X}^2} e^{i\frac{1}{4}[\ln M(t)](\hat{X}\hat{P} + \hat{P}\hat{X})},$$

and the solution without driving will be

$$\Psi(X, t) = M^{1/4} e^{-i\frac{1}{4}\dot{M}(t)X^2} \Psi(\sqrt{M(t)}X, 0).$$

The wave function indicates a dilation of the position coordinate with an increase of the mass of the parametric oscillator, which implying a localization of the wave packet as shown in Fig.12. Surely, another special transformation of \hat{U}_2 to control the evolution of the wave function can set $\theta_+ = 0$.

3. The driven free particle in a well with moving boundary

If the original Hamiltonian (35) with an initial form of $\hat{\mathcal{H}}(0)$ being a free particle ($a(0) = 1, b(0) = 1, c(0) = 0$) and the system is initially in the eigenstate of $\hat{\mathcal{H}}(0)$, then we can use the parameters $K_1 = K(t)$ and $K_2 = K_3 = 0$ to reduce Hamiltonian (35) into a free-particle like Hamiltonian of

$$\hat{\mathcal{H}}_U = \frac{1}{2}K(t)\hat{P}^2. \quad (86)$$

Here the controlled parametric equations are

$$\begin{aligned} \dot{\theta}_+ &= a\theta_+^2 - 2c\theta_+ + b, \\ \dot{\theta}_- &= ae^{-2\theta_0} - K(t), \\ \dot{\theta}_0 &= c - a\theta_+. \end{aligned} \quad (87)$$

In this case, we should solve the eigenstate of $\hat{\mathcal{H}}_U$ by

$$i\frac{\partial}{\partial t}\varphi(X, t) = \frac{1}{2}K(t)\hat{P}^2\varphi(X, t), \quad (88)$$

and the separation variables of $\varphi(X, t) = f(X)h(t)$ gives

$$\varphi(X, t) = \frac{1}{\sqrt{2\pi\hbar}}e^{iK_0X}e^{-iE_0\int_0^t K(\tau)d\tau},$$

where E_0 is the separation constant (the kinetic energy) and the scaled momentum K_0 is determined by the eigenequation of the momentum operator

$$\hat{P}f_{K_0}(X) = K_0f_{K_0}(X), \quad f_{K_0}(X) = \frac{1}{\sqrt{2\pi\hbar}}e^{iK_0X}, \quad (89)$$

where $K_0 = \sqrt{2E_0}$. Therefore, if the parameters of θ s can be properly controlled by Eq.(87), the final wave function for the driven and damped harmonic oscillator will be got by the transformations of \hat{U}_1 and \hat{U}_2 as

$$\Psi(X, t) = \hat{U}_1(t)\hat{U}_2(t)\varphi(X, t).$$

By using the parameters of driven and damped harmonic oscillator described in Eq.(68), the parameter θ_+ has a solution as Eq.(73) of

$$\begin{aligned} \theta_+(t) &= -e^{2\gamma t}\dot{\rho}, \\ \theta_0(t) &= \ln \rho \end{aligned}$$

with the auxiliary equation of

$$\ddot{\rho} + 2\gamma\dot{\rho} + \rho = 0. \quad (90)$$

As θ_- is decoupled from θ_+ and θ_0 in Eq.(87), we can simply set

$$K(t) = e^{-2\gamma t}e^{-2\theta_0} = \frac{e^{-2\gamma t}}{\rho^2}$$

to keep θ_- as a constant. Therefore the wave function in real space representation will be

$$\Psi(X, t) = \frac{1}{\sqrt{2\pi\hbar\rho}}e^{-is(t)}e^{-iE_0\int_0^t \frac{e^{-2\gamma\tau}}{\rho^2}d\tau}e^{ie^{2\gamma t}\dot{X}_c(X-X_c)}e^{ie^{2\gamma t}\dot{\rho}(X-X_c)^2/2}e^{-i\frac{\sqrt{2E_0}}{\rho}(X-X_c)}, \quad (91)$$

where we have set $\theta_- = 0$ for simplicity and X_c satisfies the classical dynamical equation of Eq.(69). As Eq.(90) is the same as Eq.(74), above wave function of Eq.(91) is not an optimal solution because it has a coefficient of $1/\sqrt{\rho}$ which becomes divergent when $\rho = 0$. Although above wave function is just a solution of (35) based on the plane wave basis, it can show a detailed evolution of different phase components during an evolution of a waves function starting from a plane wave.

Above method in this case can be use to solve the problem of a free particle in an infinite square well with a modulating boundary, which is often used as a model to study the quantum piston [44]. The position of the modulating boundary can be described by a time-dependent function of $L(t)$ and a time-dependent scale transformation of $x \rightarrow x/L(t)$ can change the problem to a normal time-independent infinite square well with width of 1. Clearly, the Lie transformation corresponding to above scale transformation in real space is

$$\hat{U}(t) = e^{-i \ln L(t) (\hat{X} \hat{P})},$$

which gives $\hat{U}f(X) = f(X/L)$. Therefore, we can choose a Lie transformation of

$$\hat{U}_2 = e^{-i\theta_+ \hat{X}^2/2} e^{-i\theta_0 (\hat{X} \hat{P} + \hat{P} \hat{X})/2} \quad (92)$$

to solve this problem with $\theta_0 = \ln L(t)$ and $\theta_- = 0$. Therefore, our problem becomes a free particle in an infinite square well with an unit width described by an initial Hamiltonian of

$$\hat{\mathcal{H}}_U = \hat{\mathcal{H}}(0) = \frac{1}{2} \hat{P}^2, \quad (93)$$

which is a specific case of Eq.(77) for $\Omega = 0$, and its eigenstates are well known

$$\varphi_n(X, t) = \sqrt{2} \sin(n\pi X) e^{-i(\pi^2 n^2 t/2)}.$$

Since the wave function of Eq.(91) is bad at $\rho = 0$ under the condition of $K_3 = 1$ and $\theta_- = 0$, we can use the invariant method instead to solve this problem. By using Lie transformation of \hat{U}_2 , we can construct an invariant Hamiltonian by inverse Lie transformation of \hat{U}_2 as

$$\hat{\mathcal{H}}(t) = \hat{U}_2 \hat{\mathcal{H}}(0) \hat{U}_2^{-1} = \frac{1}{2} \left(e^{\theta_0} \hat{P} + \theta_+ e^{\theta_0} \hat{X} \right)^2, \quad (94)$$

where we have set $\alpha = 0, \beta = 0$ for a no driving case first. The condition of above Hamiltonian to be an invariant for a free particle of Hamiltonian (93) can determine the transformation parameter θ_+ as

$$\theta_+ = -\frac{\dot{L}}{L}, \quad \ddot{L} = 0.$$

Alternatively, θ_+ can also be obtained by using the invariant operator derived from Eq.(66) for the free particle Hamiltonian (93) as

$$\hat{I}_1(t) = \frac{1}{2} \left(\rho \hat{P} - \frac{\dot{\rho}}{a} \hat{X} \right)^2,$$

where $a = 1, b = 0, c = 0$ and $\Omega = 0$ in this case. Compare above invariant operator with Eq.(94), we have the same result of

$$\rho = L(t), \quad \theta_+ = -e^{-\theta_0} \frac{\dot{\rho}}{a} = -\frac{\dot{L}}{L}.$$

Therefore, the final wave function will be

$$\Psi(X, t) = \hat{U}_2(t) \varphi_n(X, t) = e^{i\frac{\dot{L}}{2L} X^2} \sqrt{\frac{2}{L(t)}} \sin \left[\frac{n\pi X}{L(t)} \right] e^{-in^2 \pi^2 t/2},$$

where the width of the well should have a constant expanding speed with the constraint of $\ddot{L} = 0$. Fig.15 demonstrates the probability density of a free particle in an infinite well for a linear expansion of the well width, that is $L(t) = L_0 + vt$, where v is the expanding speed of the well width. The evolutions of the wave function shown in Fig.15(a) and (b) demonstrate that the free particles in an infinite well with a constant expanding width, i.e $L(t) = L_0 + vt$, only exhibit the adiabatic evolutions no matter how fast the expanding speed is. This result is due to a fact that the invariant operator of Eq.(94) for the free particle in the infinite well give a constant ‘‘energy’’ during the well expanding. The limit of $\ddot{L} = 0$ on above solution excludes an acceleration boundary case but it can be easily generalized by using this

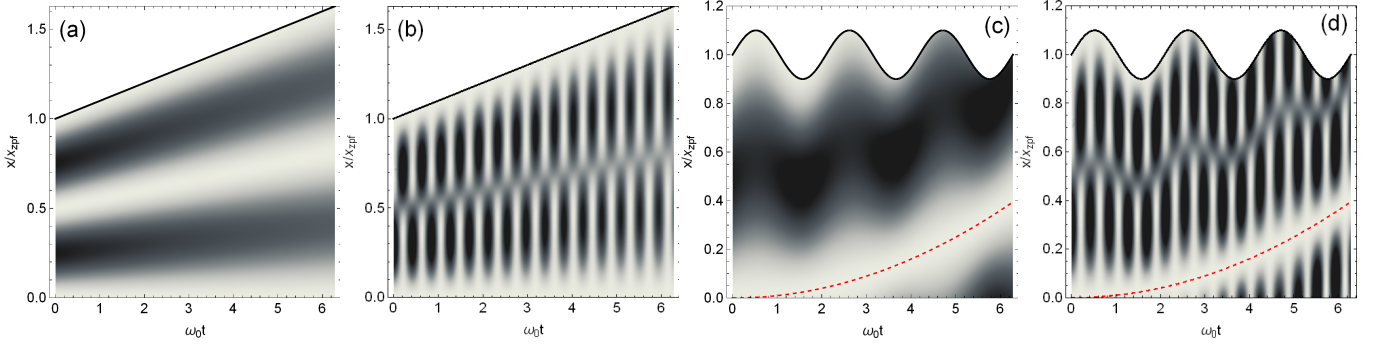


FIG. 15: The evolution of the probability density of a free particle in an infinite well with varying boundaries starting from (a) the ground state φ_1 and (b) the superposition state of $(\varphi_1 + \varphi_2)/\sqrt{2}$ with the parameters $L_0 = 1, v = 0.1$. The corresponding cases for the driven free particles in the varying infinite well are shown by (c) and (d), respectively. The parameters are $L_0 = 1, \lambda = 0.1, \omega = 3$ and $f_0 = 0.02$.

method. For a driven free particle in an infinite well with varying boundary $L(t)$ beyond $\ddot{L} = 0$, we can construct a more general invariant operator for Eq.(77) with $\Omega \neq 0$ as Eq.(80) instead,

$$\hat{I}(t) = \hat{U}\hat{\mathcal{H}}_U(0)\hat{U}^{-1} = \frac{1}{2} \left[e^{\theta_0} (\hat{P} + \beta) + \theta_+ e^{\theta_0} (\hat{X} - \alpha) \right]^2 + \frac{1}{2} \Omega^2 e^{-2\theta_0} (\hat{X} - \alpha)^2,$$

where the parameters α and β are classical variables defined by transformation of \hat{U}_1 . In this case the transformation parameters can be determined by

$$\theta_+ = -\frac{\dot{L}}{L},$$

$$\Omega = \pm \sqrt{\frac{\ddot{L}}{L} \frac{L^2}{\sqrt{1-L^4}}},$$

under the condition of $\theta_0 = \ln L$ and $\theta_- = 0$. As $\Omega \neq 0$ in this case, a further harmonic potential is introduced in order to valid the invariant operator of Eq.(80). Therefore the quadratic potential for the free particle which can be used to solve a general well expansion under driving is

$$\hat{\mathcal{H}}(0) = \frac{1}{2} (\hat{P} - P_c)^2 + \frac{1}{2} \frac{\ddot{L}}{L} \frac{L^4}{1-L^4} (\hat{X} - X_c)^2,$$

where the classical position follows the equation of acceleration of free particle of $\ddot{X}_c = f(t)$ with $f(t)$ being the driving force. We find that above auxiliary harmonic potential is undefined for $L = 1$ due to the initial scaled width of the well is $L(0) = 1$, and which is repulsive for $L < 1$ but attractive for $L > 1$ when $\ddot{L} > 0$. Therefore, the wave function starting with the initial state of $\varphi_n(X, 0) = \sqrt{2} \sin(n\pi X)$ can be determined by the Lie transformation as

$$\Psi(X, t) = e^{i \int_0^t X_c(t') f(t') dt'} e^{-in^2 \pi^2 t/2} e^{i \frac{\dot{L}}{2L} (X - X_c)^2} \sqrt{\frac{2}{L(t)}} \sin \left[\frac{n\pi}{L(t)} (X - X_c) \right].$$

Fig.15(c) (d) demonstrate the probability density of the free particle located in an infinite well with an oscillating boundary, $L(t) = L_0 + \lambda \sin(\omega t)$, and under a constant driving f_0 . We can clearly see a density modulation between the varying boundary (thick black lines) and the dynamical motion (red dashed lines) of the free particle in the infinite square well. Under the frame work of Lie transformation method, the other types of driving or width modulations for the free particles can also be similarly investigated.

4. The position-momentum case

If we set $K_1(t) = K_2(t) = 0$ and $K_3(t) = K(t)$, then we get a reduced Hamiltonian as

$$\hat{\mathcal{H}}_U = \frac{1}{2} K(t) (\hat{X} \hat{P} + \hat{P} \hat{X}), \quad (95)$$

and the classical parametric equation to result in this case is

$$\begin{aligned}\dot{\theta}_+ &= a\theta_+^2 - 2c\theta_+ + b, \\ \dot{\theta}_- &= ae^{-2\theta_0} + 2K(t)\theta_-, \\ \dot{\theta}_0 &= c - a\theta_+ - K(t),\end{aligned}\tag{96}$$

where the parameters determined by this equations are also divergent at a long time. The transformations with above parameters θ s turns the original Hamiltonian into a famous $\hat{X}\hat{P}$ Hamiltonian, which has been intensively studied due to its close relation with the Riemann theorem [45–47]. Now we should solve the wave function of Hamiltonian (95) by

$$i\frac{\partial}{\partial t}\varphi(X,t) = \frac{1}{2}K(t)\left(\hat{X}\hat{P} + \hat{P}\hat{X}\right)\varphi(X,t).$$

By using variable separation method, the wave function will be

$$\varphi(X,t) = \frac{1}{\sqrt{2\pi\hbar}}e^{-iE\int_0^t K(\tau)d\tau} \frac{1}{\sqrt{\frac{X}{X_0}}}e^{i\xi\ln\left(\frac{X}{X_0}\right)},\tag{97}$$

where $X_0 = X(0)$ and the solution is on the condition of $X/X_0 > 0$. Surely, a solution for $X/X_0 < 0$ can also be constructed [46, 48]. The parameter ξ is the eigenvalue determined by the eigen equation of the posmom operator

$$\frac{1}{2}(\hat{X}\hat{P} + \hat{P}\hat{X})g(X) = \xi g(X),$$

The eigen wave function $g_\xi(X)$ [48],

$$g_\xi(X) = \frac{1}{\sqrt{2\pi\hbar}} \frac{e^{i\xi\ln|X|}}{\sqrt{|X|}},\tag{98}$$

can also form a complete continuous basis to expand any initial state. If we consider a simple case for $X > 0, X_0 > 0$, the wave function will be

$$\Psi(X,t) = A\sqrt{X_0}e^{-i\xi\ln X_0}e^{-i\xi\int_0^t K(\tau)d\tau}e^{-is}e^{-i\alpha\hat{P}}e^{-i\beta X}e^{-i\theta_+X^2/2}e^{-i\theta_0(\hat{X}\hat{P}+\hat{P}\hat{X})/2}e^{-i\theta_- \hat{P}^2/2} \frac{1}{\sqrt{X}}e^{i\xi\ln X}.$$

Fig.16 demonstrate the solutions of a driven and damped parametric oscillator with $c(t) = 1$ starting with the eigenstate of Eq.(98). Surely, we can further simplify it by setting $\theta_- = 0$, and θ_+ can be solved by the differential equation of

$$\dot{\theta}_+ = a\theta_+^2 - 2c\theta_+ + b,$$

and then the parameter θ_0 can be determined analytically. The solution in this case is omitted here.

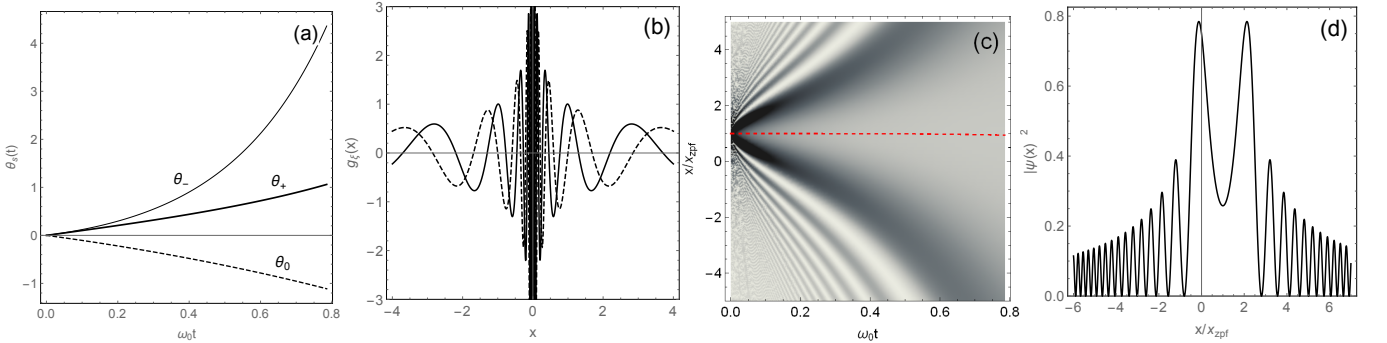


FIG. 16: (a) The temporal evolution of the transformation parameters θ s. (b) The real part (solid line) and imaginary part (dashed line) of the wave function of Eq.(98) for the posmom operator with $\xi = 6$. (c) The evolution of the probability density of the driven and damped oscillator starting from eigenstate of posmom operator of $\xi = 0.5$ (the red dashed line is the classical orbit). (d) The density distribution of the wave function at time $t = 0.2$. The other parameters are $X_0 = 1, \gamma = 0.1, \Omega = 2$ and $f_0 = 1$.

E. The general solutions with invariant parameters

Generally, we can solve the driven and damped parametric oscillator by using the invariant parameters K s defined by Eq.(64). In this case, the evolution of a wave packet starting from a specific initial state can be completely controlled by the dynamical parametric equations of Eq.(43),

$$\begin{aligned}\dot{\alpha} &= c\alpha - a\beta + d, \\ \dot{\beta} &= b\alpha - c\beta + e, \\ \dot{s} &= \frac{1}{2}b\alpha^2 - \frac{1}{2}a\beta^2 + e\alpha + f,\end{aligned}\tag{99}$$

and by Eq.(59) combined with Eq.(63) or Eq.(64) as

$$\begin{aligned}\dot{K}_1 &= 2cK_1 - 2aK_3, \\ \dot{K}_2 &= 2bK_3 - 2cK_2, \\ \dot{K}_3 &= bK_1 - aK_2, \\ \dot{\theta}_+ &= a\theta_+^2 - 2c\theta_+ - K_2e^{-2\theta_0} + b, \\ \dot{\theta}_- &= ae^{-2\theta_0} - K_2\theta_-^2 + 2K_3\theta_- - K_1, \\ \dot{\theta}_0 &= K_2\theta_- - a\theta_+ + c - K_3.\end{aligned}\tag{100}$$

Therefore, above classical parameters govern the quantum dynamics of the wave function by the following transfor-

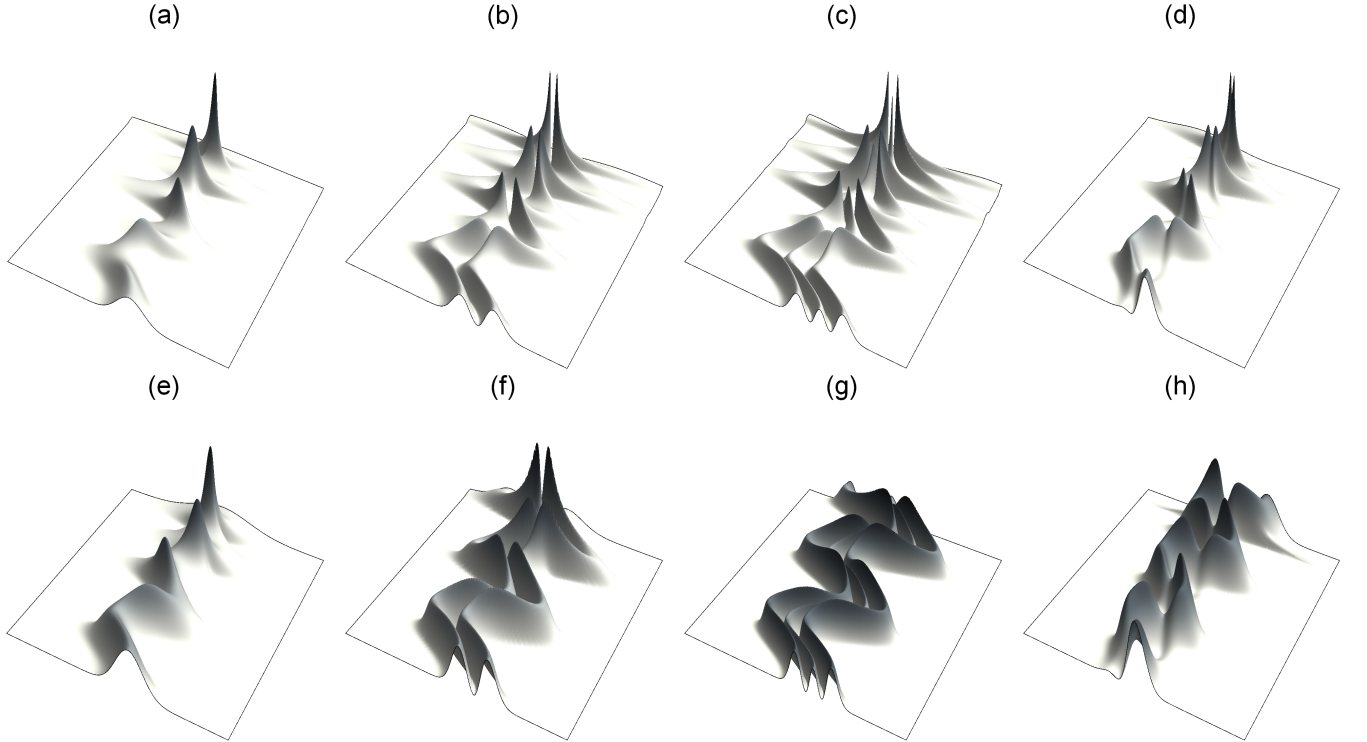


FIG. 17: The evolution of the wave packet of driven parametric oscillator starting from initial states of (a)(e) $|0\rangle$, (b)(f) $|1\rangle$, (c)(g) $|2\rangle$ and (d)(h) $(|0\rangle + |1\rangle)/\sqrt{2}$. The driving amplitude $f_0 = 0.2$ and the driving frequency $\Omega = 0.8$. The other parameters are the same as that of Fig.13.

mations

$$\Psi(X, t) = e^{-is(t)} e^{-i\alpha(t)\hat{P}} e^{-i\beta(t)\hat{X}} e^{-i\theta_+\hat{X}^2/2} e^{-i\theta_0(\hat{X}\hat{P} + \hat{P}\hat{X})/2} e^{-i\theta_-\hat{P}^2/2} \psi(X, 0),$$

where the arbitrary initial state $\psi(X, 0)$ can be expanded by any basis as Eq.(41). As discussed in the previous section, the classical dynamics of Eq.(99) only induce an overall shifting to the initial wave packet and Eq.(100) will lead deformation of the wave packet by rotation, squeezing or dilation.

In Fig.17, we calculate the driven and damped harmonic oscillator of Eq.(68) with $a(t) = e^{-2\gamma t}$, $b(t) = e^{2\gamma t}$, $c = 0$ and $e(t) = -e^{2\gamma t} f_c(t)$ in the upper frames of (a)(b)(c)(d) and the driven parametric oscillator with varying mass of $a(t) = 1/M(t)$, the time-dependent frequency of $b(t) = M(t)\Omega^2(t)$, $c = 0$ and $e(t) = f_c(t)$ in the lower frames of (e)(f)(g)(h). The dynamics of the wave packets starting from different number states of harmonic oscillator exhibit a classical parametric control on the evolution of the wave packet, such as shifting, squeezing or dilation of the wave packet. The classical resonance of the driving in a forced damped harmonic oscillator with a force of $f_c = f_0 \cos(\Omega t + \phi)$ ($\Omega \sim 1$) happens only on Eq.(99) and thus doesn't directly induce quantum transitions between different quantum states as expected. Because Eq.(100) is a system of nonlinear dynamical equations, its solution will be very sensitive to the initial values of the parameters at certain parametric windows when it conducts chaotic dynamics. Therefore, the affiliated classical Eq.(99) and Eq.(100) can be used to define the quantum chaos on a quantum system if its parametric dynamical equation is chaotic. Generally, the initial parameters of α, β, s, K s and θ s can be randomly chosen, but for a stable control problem, their initial values often depend on the initial parameters of the original Hamiltonian, i.e. $a(0), b(0), c(0), d(0), e(0)$ and $f(0)$ (for example see Eq.(64)), or determined by its initial states.

In a conclusion, we intensively investigate the driven and damping parametric oscillator with a general time-dependent quadratic Hamiltonian under an algebraic transformation framework. The method provides a general procedure to solve a time-dependent system with a closed Lie algebra accompanied with its parametric dynamical equations of the wave function and reveals a clear connection between classical and quantum dynamics during a quantum state evolution. We show that the classical dynamics controls the evolution of the quantum state through the classical parameters of the wave function. This Lie transformation method reveals that the classical resonance are different from a quantum resonance because the classical resonance happens only on Eq.(99) or Eq.(47) and the quantum resonance means transition between different internal quantum states. Anyway, as the parameters of K s can be randomly chosen, we can often use it to design different parametric controls on a quantum system to get specific target states.

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Appendix A: The FUT results for Eq.(48)

The following transformation formula are used to derive Eq.(48):

$$\begin{aligned} e^{i\theta_+ \hat{X}^2/2} \hat{P} e^{-i\theta_+ \hat{X}^2/2} &= \hat{P} - \theta_+ \hat{X}, \\ e^{i\theta_- \hat{P}^2/2} \hat{X} e^{-i\theta_- \hat{P}^2/2} &= \hat{X} + \theta_- \hat{P}, \\ e^{i\theta_0 (\hat{X}\hat{P} + \hat{P}\hat{X})/2} \hat{P} e^{-i\theta_0 (\hat{X}\hat{P} + \hat{P}\hat{X})/2} &= \hat{P} e^{-\theta_0}, \\ e^{i\theta_0 (\hat{X}\hat{P} + \hat{P}\hat{X})/2} \hat{X} e^{-i\theta_0 (\hat{X}\hat{P} + \hat{P}\hat{X})/2} &= \hat{X} e^{\theta_0}. \end{aligned}$$

In order to give a more general derivation beyond Eq.(48), we derive the FUT transformation of

$$\hat{\mathcal{H}}_U = \hat{U}^{-1}(t) \hat{\mathcal{H}}(t) \hat{U}(t) - i\hat{U}^{-1}(t) \frac{\partial \hat{U}(t)}{\partial t}$$

for $e^{-i\theta_+ \hat{X}^2/2}$, $e^{-i\theta_- \hat{P}^2/2}$ and $e^{-i\theta_0 (\hat{X}\hat{P} + \hat{P}\hat{X})/2}$, respectively, on the scaled standard quadratic Hamiltonian of

$$\hat{\mathcal{H}} = \frac{1}{2} a \hat{P}^2 + \frac{1}{2} b \hat{X}^2 + \frac{1}{2} c (\hat{X}\hat{P} + \hat{P}\hat{X}). \quad (\text{A1})$$

(1) The first transformation is for

$$\hat{U}_+ = e^{-i\theta_+ \hat{X}^2/2}, \quad (\text{A2})$$

and the FUT on Hamiltonian (A1) gives

$$\hat{\mathcal{H}}_{U_+} = \frac{1}{2} a \hat{P}^2 + \frac{1}{2} (a\theta_+^2 - \theta_+ - 2c\theta_+ + b) \hat{X}^2 + \frac{1}{2} (c - a\theta_+) (\hat{X}\hat{P} + \hat{P}\hat{X}). \quad (\text{A3})$$

(2) The second is for

$$\hat{U}_0 = e^{-i\theta_0(\hat{X}\hat{P} + \hat{P}\hat{X})/2}, \quad (\text{A4})$$

and the FUT on Hamiltonian (A1) results in

$$\hat{\mathcal{H}}_{U_0} = \frac{1}{2}ae^{-2\theta_0}\hat{P}^2 + \frac{1}{2}be^{2\theta_0}\hat{X}^2 + \frac{1}{2}(c - \dot{\theta}_0)(\hat{X}\hat{P} + \hat{P}\hat{X}). \quad (\text{A5})$$

(3) The last one is

$$\hat{U}_- = e^{-i\theta_-\hat{P}^2/2}, \quad (\text{A6})$$

and FUT on the Hamiltonian (A1) leads to

$$\hat{\mathcal{H}}_{U_-} = \frac{1}{2}(a + 2c\theta_- + b\theta_-^2 - \dot{\theta}_-)\hat{P}^2 + \frac{1}{2}b\hat{X}^2 + \frac{1}{2}(b\theta_- + c)(\hat{X}\hat{P} + \hat{P}\hat{X}). \quad (\text{A7})$$

Then the combination of above transformations can give any results of successive FUTs on Hamiltonian (A1). For the present successive transformations of

$$\hat{U} = \hat{U}_+\hat{U}_0\hat{U}_- = e^{-i\theta_+\hat{X}^2/2}e^{-i\theta_0(\hat{X}\hat{P} + \hat{P}\hat{X})/2}e^{-i\theta_-\hat{P}^2/2}, \quad (\text{A8})$$

we can easily use above three FUT results to verify Eq.(48).

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