SEMI-STABLE HIGGS SHEAVES AND BOGOMOLOV TYPE INEQUALITY

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Abstract. In this paper, we study semistable Higgs sheaves over compact Kähler manifolds, we prove that there is an approximate admissible Hermitian-Einstein structure on a semi-stable reflexive Higgs sheaf and consequently, the Bogomolove type inequality holds on a semi-stable reflexive Higgs sheaf.

1. Introduction

Let (M,ω) be a compact Kähler manifold, and E be a holomorphic vector bundle on M. Donaldson-Uhlenbeck-Yau theorem states that the ω-stability of E implies the existence of ω-Hermitian-Einstein metric on E. Hitchin [17] and Simpson [32] proved that the theorem holds also for Higgs bundles. We [25] proved that there is an approximate Hermitian-Einstein structure on a semi-stable Higgs bundle, which confirms a conjecture due to Kobayashi [19] (also see [18]). There are many interesting and important works related ([21, 17, 32, 4, 6, 12, 5, 1, 3, 7, 22, 23, 29, 27, 28], etc.). Among all of them, we recall that, Bando and Siu [6] introduced the notion of admissible Hermitian metrics on torsion-free sheaves, and proved the Donaldson-Uhlenbeck-Yau theorem on stable reflexive sheaves.

Let $\mathcal{E}$ be a torsion-free coherent sheaf, and Σ be the set of singularities where $\mathcal{E}$ is not locally free. A Hermitian metric H on the holomorphic bundle $\mathcal{E}|_{M\setminus\Sigma}$ is called *admissible* if

(1) $|F_H|_{H,\omega}$ is square integrable;

(2) $|\Lambda_\omega F_H|_H$ is uniformly bounded.

Here F_H is the curvature tensor of Chern connection D_H with respect to the Hermitian metric H, and Λ_ω denotes the contraction with the Kähler metric ω.

Higgs bundle and Higgs sheaf are studied by Hitchin ([17]) and Simpson ([32], [33]), which play an important role in many different areas including gauge theory, Kähler and hyperkähler geometry, group representations, and nonabelian Hodge theory. A Higgs sheaf on (M,ω) is a pair $(\mathcal{E},\phi)$ where $\mathcal{E}$ is a coherent sheaf on M and the Higgs field $\phi\in\Omega^{1,0}(\mathrm{End}(\mathcal{E}))$ is a holomorphic section such that $\phi\wedge\phi=0$. If the sheaf $\mathcal{E}$ is torsion-free (resp. reflexive, locally free), then we say the Higgs sheaf $(\mathcal{E},\phi)$ is torsion-free (resp. reflexive, locally free). A torsion-free Higgs sheaf $(\mathcal{E},\phi)$ is said to be ω-stable (respectively, ω-semi-stable), if for every ϕ-invariant coherent proper sub-sheaf $\mathcal{F}\hookrightarrow\mathcal{E}$, it holds:

$$\mu_\omega(\mathcal{F})=\frac{\deg_\omega(\mathcal{F})}{\mathrm{rank}(\mathcal{F})}<(\leq)\mu_\omega(\mathcal{E})=\frac{\deg_\omega(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}, \tag{1.1}$$

where $\mu_\omega(\mathcal{F})$ is called the ω-slope of $\mathcal{F}$.

Given a Hermitian metric H on the locally free part of the Higgs sheaf $(\mathcal{E},\phi)$, we consider the Hitchin-Simpson connection

$$\overline{\partial}_\phi:=\overline{\partial}_{\mathcal{E}}+\phi,\quad D^{1,0}_{H,\phi}:=D^{1,0}_H+\phi^{*H},\quad D_{H,\phi}=\overline{\partial}_\phi+D^{1,0}_{H,\phi}, \tag{1.2}$$

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where D_H is the Chern connection with respect to the metric H and ϕ^{*H} is the adjoint of ϕ with respect to H. The curvature of the Hitchin-Simpson connection is

$$F_{H,\phi} = F_H + [\phi, \phi^{*H}] + D_H^{1,0}\phi + \overline{\partial}_{\mathcal{E}}\phi^{*H}, \tag{1.3}$$

where F_H is the curvature of the Chern connection D_H. A Hermitian metric H on the Higgs sheaf $(\mathcal{E}, \phi)$ is said to be admissible Hermitian-Einstein if it is admissible and satisfies the following Einstein condition on $M \setminus \Sigma$, i.e

$$\sqrt{-1}\Lambda_\omega(F_H + [\phi, \phi^{*H}]) = \lambda \mathrm{Id}_{\mathcal{E}}, \tag{1.4}$$

where λ is a constant given by $\lambda = \frac{2\pi}{\mathrm{Vol}(M,\omega)}\mu_\omega(\mathcal{E})$. Hitchin ([17]) and Simpson ([32]) proved that a Higgs bundle admits a Hermitian-Einstein metric if and only if it's Higgs poly-stable. Biswas and Schumacher [8] studied the Donaldson-Uhlenbeck-Yau theorem for reflexive Higgs sheaves.

In this paper, we study the semi-stable Higgs sheaves. We say a torsion-free Higgs sheaf $(\mathcal{E}, \phi)$ admits an approximate admissible Hermitian-Einstein structure if for every positive δ, there is an admissible Hermitian metric H_δ such that

$$\sup_{x \in M\setminus\Sigma} |\sqrt{-1}\Lambda_\omega(F_{H_\delta} + [\phi, \phi^{*H_\delta}]) - \lambda \mathrm{Id}_{\mathcal{E}}|_{H_\delta}(x) < \delta. \tag{1.5}$$

The approximate Hermitian-Einstein structure was introduced by Kobayashi ([19]) on a holomorphic vector bundle, it is the differential geometric counterpart of the semi-stability. Kobayashi [19] proved there is an approximate Hermitian-Einstein structure on a semi-stable holomorphic vector bundle over an algebraic manifold, which he conjectured should be true over any Kähler manifold. The conjecture was confirmed in [18, 25]. In this paper, we proved our theorem holds for a semi-stable reflexive Higgs sheaf over a compact Kähler manifold.

Theorem 1.1. *A reflexive Higgs sheaf $(\mathcal{E}, \phi)$ on an n-dimensional compact Kähler manifold (M, ω) is semi-stable, if and only if it admits an approximate admissible Hermitian-Einstein structure. Specially, for a semi-stable reflexive Higgs sheaf $(\mathcal{E}, \phi)$ of rank r, we have the following Bogomolov type inequality*

$$\int_M (2c_2(\mathcal{E}) - \frac{r-1}{r}c_1(\mathcal{E}) \wedge c_1(\mathcal{E})) \wedge \frac{\omega^{n-2}}{(n-2)!} \geq 0. \tag{1.6}$$

The Bogomolov inequality was first obtained by Bogomolov ([9]) for semi-stable holomorphic vector bundles over complex algebraic surfaces, it had been extended to certain classes of generalized vector bundles, including parabolic bundles and orbibundles. By constructing a Hermitian-Einstein metric, Simpson proved the Bogomolov inequality for stable Higgs bundles on compact Kähler manifolds. Recently, Langer ([20]) proved the Bogomolov type inequality for semi-stable Higgs sheaves over algebraic varieties by using an algebraic-geometric method. His method can not be applied to the Kähler manifold case. We use analytic method to study the Bogomolov inequality for semi-stable reflexive Higgs sheaves over compact Kähle manifolds, new idea is needed.

We now give an overview of our proof. As in [6], we make a regularization on the reflexive sheaf $\mathcal{E}$, i.e. take blowing up with smooth centers finite times $\pi_i : M_i \to M_{i-1}$, where $i = 1, \cdots, k$ and $M_0 = M$, such that the pull-back of $\mathcal{E}^*$ to M_k modulo torsion is locally free and

$$\pi = \pi_1 \circ \cdots \circ \pi_k : M_k \to M \tag{1.7}$$

is biholomorphic outside Σ. In the following, we denote M_k by $\tilde{M}$, the exceptional divisor $\pi^{-1}\Sigma$ by $\tilde{\Sigma}$, and the holomorphic vector bundle $(\pi^*\mathcal{E}^*/torsion)^*$ by E. Since $\mathcal{E}$ is locally free outside Σ, and the holomorphic bundle E is isomorphic to $\mathcal{E}$ on $\tilde{M} \setminus \tilde{\Sigma}$, the pull-back field

$\pi^*\phi$ is a holomorphic section of $\Omega^{1,0}(\mathrm{End}(E))$ on $\tilde{M}\setminus\tilde{\Sigma}$. By Hartogs' extension theorem, the holomorphic section $\pi^*\phi$ can be extended to the whole $\tilde{M}$ as a Higgs field of E. In the following, we also denote the extended Higgs field $\pi^*\phi$ by ϕ for simplicity. So we get a Higgs bundle (E,ϕ) on $\tilde{M}$ which is isomorphic to the Higgs sheaf $(\mathcal{E},\phi)$ outside the exceptional divisor $\tilde{\Sigma}$.

It is well known that $\tilde{M}$ is also Kähler ([15]). Fix a Kähler metric η on $\tilde{M}$ and set

$$\omega_\epsilon = \pi^*\omega + \epsilon\eta \tag{1.8}$$

for any small $0<\epsilon\leq 1$. Let $K_\epsilon(t,x,y)$ be the heat kernel with respect to the Kähler metric ω_ϵ. Bando and Siu (Lemma 3 in [6]) obtained a uniform Sobolev inequality for $(\tilde{M},\omega_\epsilon)$, using Cheng and Li's estimate ([11]), they got a uniform upper bound of the heat kernels $K_\epsilon(t,x,y)$. Given a smooth Hermitian metric $\hat{H}$ on the bundle E, it is easy to see that there exists a constant $\hat{C}_0$ such that

$$\int_{\tilde{M}}(|\Lambda_{\omega_\epsilon}F_{\hat{H}}|_{\hat{H}}+|\phi|^2_{\hat{H},\omega_\epsilon})\frac{\omega_\epsilon^n}{n!}\leq \hat{C}_0, \tag{1.9}$$

for all $0<\epsilon\leq 1$. This also gives a uniform bound on $\int_{\tilde{M}}|\Lambda_{\omega_\epsilon}(F_{\hat{H}}+[\phi,\phi^{*\hat{H}}])|_{\hat{H}}\frac{\omega_\epsilon^n}{n!}$.

We study the following evolution equation on Higgs bundle (E,ϕ) with the fixed initial metric $\hat{H}$ and with respect to the Kähler metric ω_ϵ,

$$\begin{cases} H_\epsilon(t)^{-1}\dfrac{\partial H_\epsilon(t)}{\partial t} = -2(\sqrt{-1}\Lambda_{\omega_\epsilon}(F_{H_\epsilon(t)}+[\phi,\phi^{*H_\epsilon(t)}])-\lambda_\epsilon \mathrm{Id}_E), \\ H_\epsilon(0)=\hat{H}, \end{cases} \tag{1.10}$$

where $\lambda_\epsilon = \frac{2\pi}{\mathrm{Vol}(\tilde{M},\omega_\epsilon)}\mu_{\omega_\epsilon}(E)$. Simpson ([32]) proved the existence of long time solution of the above heat flow. By the standard parabolic estimates and the uniform upper bound of the heat kernels $K_\epsilon(t,x,y)$, we know that $|\Lambda_{\omega_\epsilon}(F_{H_\epsilon(t)}+[\phi,\phi^{*H_\epsilon(t)}])|_{H_\epsilon(t)}$ has a uniform L^1 bound for $t\geq 0$ and a uniform L^∞ bound for $t\geq t_0>0$. As in [6], taking the limit as $\epsilon\to 0$, we have a long time solution $H(t)$ of the following evolution equation on $M\setminus\Sigma\times[0,+\infty)$, i.e. $H(t)$ satisfies:

$$\begin{cases} H(t)^{-1}\dfrac{\partial H(t)}{\partial t} = -2(\sqrt{-1}\Lambda_{\omega}(F_{H(t)}+[\phi,\phi^{*H(t)}])-\lambda \mathrm{Id}_{\mathcal{E}}), \\ H(0)=\hat{H}. \end{cases} \tag{1.11}$$

Here $H(t)$ can be seen as a Hermitian metric defined on the locally free part of $\mathcal{E}$, i.e. on $M\setminus\Sigma$.

In order to get the admissibility of Hermitian metric $H(t)$ for positive time $t>0$, we should show that $|\phi|_{H(t),\omega}\in L^\infty$ for $t>0$. In fact, we can prove that $|\phi|_{H(t),\omega}$ has a uniform L^∞ bound for $t\geq t_0>0$. In [24], by using the maximum principle, we proved this uniform L^∞ bound of $|\phi|_{H(t),\omega}$ along the evolution equation for the Higgs bundle case. In the Higgs sheaf case, since the equation (1.11) has singularity on Σ, we can not use the maximum principle directly. So we need new argument to get a uniform L^∞ bound of $|\phi|_{H(t),\omega}$, see section 3 for details.

The key part in the proof of Theorem 1.1 is to prove the existence of admissible approximate Hermitian-Einstein structure on a semi-stable reflexive Higgs sheaf. The Bogomolov type inequality (1.6) is an application. In fact, we prove that if the reflexive Higgs sheaf $(\mathcal{E},\phi)$ is semi-stable, along the evolution equation (1.11), we must have

$$\sup_{x\in M\setminus\Sigma}|\sqrt{-1}\Lambda_\omega(F_{H(t)}+[\phi,\phi^{*H(t)}])-\lambda\mathrm{Id}_{\mathcal{E}}|_{H(t)}(x)\to 0, \tag{1.12}$$

as $t\to+\infty$. We prove (1.12) by contradiction, if not, we can construct a saturated Higgs subsheaf such that its ω-slope is greater than $\mu_\epsilon(\mathcal{E})$. Since the singularity set Σ is a complex analytic subset with co-dimension at least 3, it is easy to show that $(M\setminus\Sigma,\omega)$ satisfies all three assumptions that Simpson ([32]) imposes on the non-compact base Kähler manifold. Let's recall Simpson's argument for a Higgs bundle in the case where the base Kähler manifold is non-compact. Simpson assumes that there exists a good initial Hermitian metric K satisfying $\sup_{M\setminus\Sigma}|\Lambda_\omega F_{K,\phi}|_K<\infty$, then he defines the analytic stability for $(\mathcal{E},\phi,K)$ by using the Chern-Weil formula with respect to the metric K (Lemma 3.2 in [32]). Under the K-analytic stability condition, he constructs a Hermitian-Einstein metric for the Higgs bundle by limiting the evolution equation (1.11).

Here, we have to pay more attention to the analytic stability (or semi-stability) of $(\mathcal{E},\phi)$. Let $\mathcal{F}$ be a saturated sub-sheaf of $\mathcal{E}$, we know that $\mathcal{F}$ can be seen as a sub-bundle of $\mathcal{E}$ outside a singularity set $V=\Sigma_{\mathcal{F}}\cup\Sigma$ of codimension at least 2, then $\hat{H}$ induces a Hermitian metric $\hat{H}_{\mathcal{F}}$ on $\mathcal{F}$. Bruasse (Proposition 4.1 in [10]) had proved the following Chern-Weil formula

$$\deg_\omega(\mathcal{F})=\int_{M\setminus V}c_1(\mathcal{F},\hat{H}_{\mathcal{F}})\wedge\frac{\omega^{n-1}}{(n-1)!},\tag{1.13}$$

where $c_1(\mathcal{F},\hat{H}_{\mathcal{F}})$ is the first Chern form with respect to the induced metric $\hat{H}_{\mathcal{F}}$. By (1.13), we see that the stability (semi-stability) of the reflexive Higgs sheaf $(\mathcal{E},\phi)$ is equivalent to the analytic stability (semi-stability) with respect to the metric $\hat{H}$ in Simpson's sense. But, we are not clear whether the above Chern-Weil formula is still valid if the metric $\hat{H}$ is replaced by an admissible metric $H(t)$ $(t>0)$. So, the stability (or semi-stability) of the reflexive Higgs sheaf $(\mathcal{E},\phi)$ may not imply the analytic stability (or semi-stability) with respect to the metric $H(t)$ $(t>0)$. The admissible metric $H(t)$ $(t>0)$ can not be chosen as a good initial metric in Simpson's sense. On the other hand, the initial metric $\hat{H}$ may not satisfy the curvature finiteness condition (i.e. $|\Lambda_\omega F_{\hat{H},\phi}|_{\hat{H}}$ may not be L^∞ bounded), so we should modify Simpson's argument in our case, see the proof of Proposition 4.1 in section 4 for details.

If the reflexive Higgs sheaf $(\mathcal{E},\phi)$ is ω-stable, it is well known that the pulling back Higgs bundle (E,ϕ) is ω_ϵ-stable for sufficiently small ϵ. By Simpson's result ([32]), there exists an ω_ϵ-Hermitian-Einstein metric H_ϵ for every small ϵ. In [6], Bando and Siu point out that it is possible to get an ω-Hermitian-Einstein metric H on the reflexive Higgs sheaf $(\mathcal{E},\phi)$ as a limit of ω_ϵ-Hermitian-Einstein metric H_ϵ of Higgs bundle (E,ϕ) on $\tilde{M}$ as $\epsilon\to 0$. In the end of this paper, we solve this problem.

Theorem 1.2. *Let H_ϵ be an ω_ϵ-Hermitian-Einstein metric on the Higgs bundle (E,ϕ), by choosing a subsequence and rescaling it, H_ϵ must converge to an ω-Hermitian-Einstein metric H in local C^∞-topology outside the exceptional divisor $\tilde{\Sigma}$ as $\epsilon\to 0$.*

This paper is organized as follows. In Section 2, we recall some basic estimates for the heat flow (1.10) and give proofs for local uniform C^0, C^1 and higher order estimates for reader's convenience. In section 3, we give a uniform L^∞ bound for the norm of the Higgs field along the heat flow (1.11). In section 4, we prove the existence of admissible approximate Hermitian-Einstein structure on the semi-stable reflexive Higgs sheaf and complete the proof of Theorem 1.1. In section 5, we prove Theorem 1.2.

2. Analytic preliminaries and basic estimates

Let (M,ω) be a compact Kähler manifold of complex dimension n, and $(\mathcal{E},\phi)$ be a reflexive Higgs sheaf on M with the singularity set Σ. There exists a bow-up $\pi:\tilde{M}\to M$ such that the pulling back Higgs bundle (E,ϕ) on $\tilde{M}$ is isomorphic to $(\mathcal{E},\phi)$ outside the exceptional divisor $\tilde{\Sigma}=\pi^{-1}\Sigma$. It is well known that $\tilde{M}$ is also Kähler ([15]). Fix a Kähler metric η on $\tilde{M}$ and set $\omega_\epsilon=\pi^*\omega+\epsilon\eta$ for $0<\epsilon\le 1$. Let $K_\epsilon(x,y,t)$ be the heat kernel with respect to the Kähler metric ω_ϵ. Bando and Siu (Lemma 3 in [6]) obtained a uniform Sobolev inequality for $(\tilde{M}, \omega_\epsilon)$. Combining Cheng and Li's estimate ([11]) with Grigor'yan's result (Theorem 1.1 in [16]), we have the following uniform upper bound of the heat kernels, furthermore, we also have a uniform lower bound of the Green functions.

Proposition 2.1. (Proposition 2 in [6]**)** *Let K_ϵ be the heat kernel with respect to the metric ω_ϵ, then for any $\tau > 0$, there exists a constant $C_K(\tau)$ which is independent of ϵ, such that*

$$0 \leq K_\epsilon(x, y, t) \leq C_K(\tau)(t^{-n} \exp{(-\frac{(d_{\omega_\epsilon}(x, y))^2}{(4+\tau)t})} + 1) \tag{2.1}$$

for every $x, y \in \tilde{M}$ and $0 < t < +\infty$, where $d_{\omega_\epsilon}(x, y)$ is the distance between x and y with respect to the metric ω_ϵ. There also exists a constant C_G such that

$$G_\epsilon(x, y) \geq -C_G \tag{2.2}$$

for every $x, y \in \tilde{M}$ and $0 < \epsilon \leq 1$, where G_ϵ is the Green function with respect to the metric ω_ϵ.

Let $H_\epsilon(t)$ be the long time solutions of the heat flow (1.10) on the Higgs bundle (E, ϕ) with the fixed smooth initial metric $\hat{H}$ and with respect to the Kähler metric ω_ϵ. By (1.9), there is a constant $\hat{C}_1$ independent of ϵ such that

$$\int_{\tilde{M}} |\sqrt{-1}\Lambda_{\omega_\epsilon}(F_{\hat{H}} + [\phi, \phi^{*\hat{H}}]) - \lambda_\epsilon \mathrm{Id}_E|_{\hat{H}} \frac{\omega_\epsilon^n}{n!} \leq \hat{C}_1. \tag{2.3}$$

For simplicity, we set:

$$\Phi(H_\epsilon(t), \omega_\epsilon) = \sqrt{-1}\Lambda_{\omega_\epsilon}(F_{H_\epsilon(t)} + [\phi, \phi^{*H_\epsilon(t)}]) - \lambda_\epsilon \mathrm{Id}_E. \tag{2.4}$$

The following estimates are essentially proved by Simpson (Lemma 6.1 in [32], see also Lemma 4 in [25]). Along the heat flow (1.10), we have:

$$(\Delta_\epsilon - \frac{\partial}{\partial t})\mathrm{tr}\,(\Phi(H_\epsilon(t), \omega_\epsilon)) = 0, \tag{2.5}$$

$$(\Delta_\epsilon - \frac{\partial}{\partial t})|\Phi(H_\epsilon(t), \omega_\epsilon)|^2_{H_\epsilon(t)} = 2|D_{H_\epsilon, \phi}(\Phi(H_\epsilon(t), \omega_\epsilon))|^2_{H_\epsilon(t), \omega_\epsilon}, \tag{2.6}$$

and

$$(\Delta_\epsilon - \frac{\partial}{\partial t})|\Phi(H_\epsilon(t), \omega_\epsilon)|_{H_\epsilon(t)} \geq 0. \tag{2.7}$$

Then, for $t > 0$,

$$\int_{\tilde{M}} |\Phi(H_\epsilon(t), \omega_\epsilon)|_{H_\epsilon(t)} \frac{\omega_\epsilon^n}{n!} \leq \int_{\tilde{M}} |\Phi(\hat{H}, \omega_\epsilon)|_{\hat{H}} \frac{\omega_\epsilon^n}{n!} \leq \hat{C}_1, \tag{2.8}$$

$$\max_{x \in \tilde{M}} |\Phi(H_\epsilon(t), \omega_\epsilon)|_{H_\epsilon(t)}(x) \leq \int_{\tilde{M}} K_\epsilon(x, y, t)|\Phi(\hat{H}, \omega_\epsilon)|_{\hat{H}} \frac{\omega_\epsilon^n}{n!}, \tag{2.9}$$

and

$$\max_{x \in \tilde{M}} |\Phi(H_\epsilon(t+1), \omega_\epsilon)|_{H_\epsilon(t+1)}(x) \leq \int_{\tilde{M}} K_\epsilon(x, y, 1)|\Phi(H_\epsilon(t), \omega_\epsilon)|_{H_\epsilon(t)} \frac{\omega_\epsilon^n}{n!}. \tag{2.10}$$

By the upper bound of the heat kernels (2.1), we have

$$\max_{x \in \tilde{M}} |\Phi(H_\epsilon(t), \omega_\epsilon)|_{H_\epsilon(t)}(x) \leq C_K(\tau)\hat{C}_1(t^{-n} + 1), \tag{2.11}$$

and

$$\max_{x \in \tilde{M}} |\Phi(H_\epsilon(t+1), \omega_\epsilon)|_{H_\epsilon(t+1)}(x) \leq 2C_K(\tau) \int_{\tilde{M}} |\Phi(H_\epsilon(t), \omega_\epsilon)|_{H_\epsilon(t)} \frac{\omega_\epsilon^n}{n!}. \tag{2.12}$$

Set

$$\exp(S_\epsilon(t)) = h_\epsilon(t) = \hat{H}^{-1}H_\epsilon(t), \tag{2.13}$$

where $S_\epsilon(t) \in \mathrm{End}(E)$ is self-adjoint with respect to $\hat{H}$ and $H_\epsilon(t)$. By the heat flow (1.10), we have:

$$\frac{\partial}{\partial t}\log\det(h_\epsilon(t)) = \mathrm{tr}\,(h_\epsilon^{-1}\frac{\partial h_\epsilon}{\partial t}) = -2\mathrm{tr}\,(\Phi(H_\epsilon(t),\omega_\epsilon)), \tag{2.14}$$

and

$$\int_{\tilde{M}} \mathrm{tr}\,(S_\epsilon(t))\frac{\omega_\epsilon^n}{n!} = \int_{\tilde{M}} \log\det(h_\epsilon(t))\frac{\omega_\epsilon^n}{n!} = 0 \tag{2.15}$$

for all $t \geq 0$.

In the following, we denote:

$$B_{\omega_1}(\delta) = \{x \in \tilde{M} | d_{\omega_1}(x,\Sigma) < \delta\}, \tag{2.16}$$

where d_{ω_1} is the distance function with respect to the Kähler metric ω_1. Since $\hat{H}$ is a smooth Hermitian metric on E, $\phi \in \Omega^{1,0}_{\tilde{M}}(\mathrm{End}(E))$ is a smooth field, and $\pi^*\omega$ is degenerate only along Σ, there exist constants $\hat{c}(\delta^{-1})$ and $\hat{b}_k(\delta^{-1})$ such that

$$\begin{aligned} &\{|\Lambda_{\omega_\epsilon}F_{\hat{H}}|_{\hat{H}} + |\phi|^2_{\hat{H},\omega_\epsilon}\}(y) \leq \hat{c}(\delta^{-1}),\\ &\{|\nabla^k_{\hat{H}}F_{\hat{H}}|^2_{\hat{H},\omega_\epsilon} + |\nabla^{k+1}_{\hat{H}}\phi|^2_{\hat{H},\omega_\epsilon}\} \leq \hat{b}_k(\delta^{-1}), \end{aligned} \tag{2.17}$$

for all $y \in \tilde{M} \setminus B_{\omega_1}(\frac{\delta}{2})$, all $0 \leq \epsilon \leq 1$ and all $k \geq 0$.

In order to get a uniform local C^0-estimate of $h_\epsilon(t)$, We first prove that $|\Phi(H_\epsilon(t),\omega_\epsilon)|_{H_\epsilon(t)}$ is uniform locally bounded, i.e. we obtain the following Lemma.

Lemma 2.2. *There exists a constant $\tilde{C}_1(\delta^{-1})$ such that*

$$|\Phi(H_\epsilon(t),\omega_\epsilon)|_{H_\epsilon(t)}(x) \leq \tilde{C}_1(\delta^{-1}) \tag{2.18}$$

for all $(x,t) \in (\tilde{M} \setminus B_{\omega_1}(\delta)) \times [0,\infty)$, and all $0 < \epsilon \leq 1$.

Proof. Using the inequality (2.9), we have

$$|\Phi(H_\epsilon(t),\omega_\epsilon)|_{H_\epsilon(t)}(x) \leq \Big(\int_{M\setminus B_\epsilon(\frac{\delta}{2})} + \int_{B_\epsilon(\frac{\delta}{2})}\Big)K_\epsilon(x,y,t)|\Phi(\hat{H},\omega_\epsilon)|_{\hat{H}}(y)\frac{\omega_\epsilon^n(y)}{n!}. \tag{2.19}$$

Noting $\int_{\tilde{M}} K_\epsilon(x,y,t)\frac{\omega_\epsilon^n}{n!} = 1$ and using (2.17), we have

$$\begin{aligned} &\int_{\tilde{M}\setminus B_\epsilon(\frac{\delta}{2})} K_\epsilon(x,y,t)|\Phi(\hat{H},\omega_\epsilon)|_{\hat{H}}(y)\frac{\omega_\epsilon^n}{n!}\\ &\leq (\hat{c}(\delta^{-1}) + \lambda_\epsilon\sqrt{r})\int_{\tilde{M}} K_\epsilon(x,y,t)\frac{\omega_\epsilon^n(y)}{n!}\\ &\leq \hat{c}_1(\delta^{-1}). \end{aligned} \tag{2.20}$$

where $\hat{c}_1(\delta^{-1})$ is a constant independent of ϵ. Since $\pi^*\omega$ is degenerate only along Σ, there exists a constant $\tilde{a}(\delta)$ such that

$$\tilde{a}(\delta)\omega_1 < \pi^*\omega < \omega_\epsilon < \omega_1 \tag{2.21}$$

on $\tilde{M} \setminus B_{\omega_1}(\frac{\delta}{4})$, for all $0 < \epsilon \leq 1$. Let $x \in \tilde{M} \setminus B_{\omega_1}(\delta)$ and $y \in \partial(B_{\omega_1}(\frac{\delta}{2}))$, it is clear that

$$d_{\omega_\epsilon}(x,y) \geq d_{\pi^*\omega}(x,y) > \sqrt{\tilde{a}(\delta)}d_{\omega_1}(x,y) \geq \frac{\delta\sqrt{\tilde{a}(\delta)}}{2}. \tag{2.22}$$

Let $a(\delta)=\frac{\delta\sqrt{\tilde{a}(\delta)}}{2}$. If $x\in\tilde{M}\setminus B_{\omega_1}(\delta)$ and $y\in B_{\omega_1}(\frac{\delta}{2})$, we have

$$d_{\omega_\epsilon}(x,y)\geq a(\delta) \tag{2.23}$$

for all $0\leq\epsilon\leq 1$. Then,

$$\begin{aligned}
&\int_{B_{\omega_1}(\frac{\delta}{2})}K_\epsilon(x,y,t)|\Phi(\hat{H},\omega_\epsilon)|_{\hat{H}}(y)\frac{\omega_\epsilon^n(y)}{n!}\\
&\leq C_k(\tau)\int_{B_{\omega_1}(\frac{\delta}{2})}(t^{-n}\exp(-\frac{d_{\omega_\epsilon}(x,y)}{(4+\tau)t})+1)|\Phi(\hat{H},\omega_\epsilon)|_{\hat{H}}(y)\frac{\omega_\epsilon^n(y)}{n!}\\
&\leq C_k(\tau)\int_{B_{\omega_1}(\frac{\delta}{2})}(t^{-n}\exp(-\frac{a(\delta)}{(4+\tau)t})+1)|\Phi(\hat{H},\omega_\epsilon)|_{\hat{H}}\frac{\omega_\epsilon^n}{n!}\\
&\leq C_k(\tau)\big(\frac{a(\delta)}{4+\tau}n\big)^{-n}\exp(-n)\int_{B_{\omega_1}(\frac{\delta}{2})}|\Phi(\hat{H},\omega_\epsilon)|_{\hat{H}}\frac{\omega_\epsilon^n}{n!}\\
&\leq C_k(\tau)\hat{C}_1\big(\frac{a(\delta)}{4+\tau}n\big)^{-n}\exp(-n),
\end{aligned} \tag{2.24}$$

for all $(x,t)\in(\tilde{M}\setminus B_{\omega_1}(\delta))\times[0,\infty)$. It is obvious that (2.19), (2.20) and (2.24) imply (2.18). □

By a direct calculation, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\log(\mathrm{tr}\,h_\epsilon(t)+\mathrm{tr}\,h_\epsilon^{-1}(t))\\
&=\frac{\mathrm{tr}\,(h_\epsilon(t)\cdot h_\epsilon^{-1}(t)\frac{\partial h_\epsilon(t)}{\partial t})-\mathrm{tr}\,(h_\epsilon^{-1}(t)\frac{\partial h_\epsilon(t)}{\partial t}\cdot h_\epsilon^{-1}(t))}{\mathrm{tr}\,h_\epsilon(t)+\mathrm{tr}\,h_\epsilon^{-1}(t)}\\
&\leq 2|\Phi(H_\epsilon(t),\omega_\epsilon)|_{H_\epsilon(t)},
\end{aligned} \tag{2.25}$$

and

$$\log(\frac{1}{2r}(\mathrm{tr}\,h_\epsilon(t)+\mathrm{tr}\,h_\epsilon(t)^{-1}))\leq|S_\epsilon(t)|_{\hat{H}}\leq r^{\frac{1}{2}}\log(\mathrm{tr}\,h_\epsilon(t)+\mathrm{tr}\,h_\epsilon(t)^{-1}), \tag{2.26}$$

where $r=\mathrm{rank}(E)$. By (2.8) and (2.18), we have

$$\int_{\tilde{M}}\log(\mathrm{tr}\,h_\epsilon(t)+\mathrm{tr}\,h_\epsilon^{-1}(t))-\log(2r)\frac{\omega_\epsilon^n}{n!}\leq\hat{C}_1 t, \tag{2.27}$$

and

$$\log(\mathrm{tr}\,h_\epsilon(t)+\mathrm{tr}\,h_\epsilon^{-1}(t))-\log(2r)\leq 2\tilde{C}_1(\delta^{-1})T \tag{2.28}$$

for all $(x,t)\in(\tilde{M}\setminus B_{\omega_1}(\delta))\times[0,T]$. Then, we have the following local C^0-estimate of $h_\epsilon(t)$.

Lemma 2.3. *There exists a constant $\overline{C}_0(\delta^{-1},T)$ which is independent of ϵ such that*

$$|S_\epsilon(t)|_{\hat{H}}(x)\leq\overline{C}_0(\delta^{-1},T) \tag{2.29}$$

for all $(x,t)\in(\tilde{M}\setminus B_{\omega_1}(\delta))\times[0,T]$, and all $0<\epsilon\leq 1$.

In the following lemma, we derive a local C^1-estimate of $h_\epsilon(t)$.

Lemma 2.4. *Let $T_\epsilon(t)=h_\epsilon^{-1}(t)\partial_{\hat{H}}h_\epsilon(t)$. Assume that there exists a constant $\overline{C}_0$ such that*

$$\max_{(x,t)\in(\tilde{M}\setminus B_{\omega_1}(\delta))\times[0,T]}|S_\epsilon(t)|_{\hat{H}}(x)\leq\overline{C}_0, \tag{2.30}$$

for all $0<\epsilon\leq 1$. *Then, there exists a constant* $\overline{C}_1$ *depending only on* $\overline{C}_0$ *and* δ^{-1} *such that*

$$\max_{(x,t)\in(\tilde{M}\setminus B_{\omega_1}(\frac{3}{2}\delta))\times[0,T]}|T_\epsilon(t)|_{\hat{H},\omega_\epsilon}\leq\overline{C}_1 \tag{2.31}$$

for all $0<\epsilon\leq 1$.

Proof. By a direct calculation, we have

$$\begin{aligned}
&(\Delta_\epsilon-\frac{\partial}{\partial t})\mathrm{tr}\, h_\epsilon(t)\\
=&\ 2\mathrm{tr}\,(-\sqrt{-1}\Lambda_{\omega_\epsilon}\overline{\partial}h_\epsilon(t)\cdot h_\epsilon^{-1}(t)\cdot\partial_{\hat{H}}h_\epsilon(t))+2\mathrm{tr}\,(h_\epsilon(t)\Phi(\hat{H},\omega_\epsilon))\\
&+2\sqrt{-1}\Lambda_{\omega_\epsilon}\mathrm{tr}\,\{h_\epsilon(t)\circ([\phi,\phi^{*H_\epsilon(t)}]-[\phi,\phi^{*\hat{H}}])\}\\
=&\ 2\mathrm{tr}\,(-\sqrt{-1}\Lambda_{\omega_\epsilon}\overline{\partial}h_\epsilon(t)\cdot h_\epsilon^{-1}(t)\cdot\partial_{\hat{H}}h_\epsilon(t))+2\mathrm{tr}\,(h_\epsilon(t)\Phi(\hat{H},\omega_\epsilon))\\
&+2\sqrt{-1}\Lambda_{\omega_\epsilon}\mathrm{tr}\,\{[\phi,h_\epsilon(t)]\wedge h_\epsilon^{-1}(t)[h_\epsilon(t),\phi^{*\hat{H}}]\}\\
\geq&\ 2\mathrm{tr}\,(-\sqrt{-1}\Lambda_{\omega_\epsilon}\overline{\partial}h_\epsilon(t)\cdot h_\epsilon^{-1}(t)\cdot\partial_{\hat{H}}h_\epsilon(t))+2\mathrm{tr}\,(h_\epsilon(t)\Phi(\hat{H},\omega_\epsilon)),
\end{aligned} \tag{2.32}$$

$$\frac{\partial}{\partial t}T_\epsilon(t)=\partial_{H_\epsilon(t)}(h_\epsilon^{-1}(t)\frac{\partial}{\partial t}h_\epsilon(t))=-2\partial_{H_\epsilon(t)}(\Phi(H_\epsilon(t),\omega_\epsilon)), \tag{2.33}$$

and

$$\begin{aligned}
&(\Delta_\epsilon-\frac{\partial}{\partial t})|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}\geq 2|\nabla_{H_\epsilon(t)}T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}\\
&-\check{C}_1(|\Lambda_{\omega_\epsilon}F_{H_\epsilon(t)}|_{H_\epsilon(t)}+|F_{\hat{H}}|_{H_\epsilon(t),\omega_\epsilon}+|\phi|^2_{H_\epsilon(t),\omega_\epsilon}+|Ric(\omega_\epsilon)|_{\omega_\epsilon})|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}\\
&-\check{C}_2|\nabla_{\hat{H}}(\Lambda_{\omega_\epsilon}F_{\hat{H}})|_{H_\epsilon(t),\omega_\epsilon}|T_\epsilon(t)|_{H_\epsilon(t),\omega_\epsilon}-|\nabla_{\hat{H}}\phi|^2_{H_\epsilon(t),\omega_\epsilon},
\end{aligned} \tag{2.34}$$

where constants $\check{C}_1,\check{C}_2$ depend only on the dimension n and the rank r.

By the local C^0-assumption (2.30), the local estimate (2.18) and the definition of ω_ϵ, it is easy to see that all coefficients in the right term of (2.34) are uniformly local bounded outside $\tilde{\Sigma}$. Then there exists a constant $\check{C}_3$ depending only on δ^{-1} and $\overline{C}_0$ such that

$$\begin{aligned}
(\Delta_\epsilon-\frac{\partial}{\partial t})|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}\geq&\ 2|\nabla_{H_\epsilon(t)}T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}\\
&-\check{C}_3|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}-\check{C}_3
\end{aligned} \tag{2.35}$$

on the domain $\tilde{M}\setminus B_{\omega_1}(\delta)\times[0,T]$.

Let φ_1, φ_2 be nonnegative cut-off functions satisfying:

$$\varphi_1(x)=\begin{cases}0, & x\in B_{\omega_1}(\frac{5}{4}\delta),\\ 1, & x\in\tilde{M}\setminus B_{\omega_1}(\frac{3}{2}\delta),\end{cases} \tag{2.36}$$

$$\varphi_2(x)=\begin{cases}0, & x\in B_{\omega_1}(\delta),\\ 1, & x\in\tilde{M}\setminus B_{\omega_1}(\frac{5}{4}\delta),\end{cases} \tag{2.37}$$

and $|d\varphi_i|^2_{\omega_1}\leq\frac{8}{\delta^2}$, $-\frac{c}{\delta^2}\omega_1\leq\sqrt{-1}\partial\bar{\partial}\varphi_i\leq\frac{c}{\delta^2}\omega_1$. By the inequality (2.21), there exists a constant $C_1(\delta^{-1})$ depending only on δ^{-1} such that

$$(|d\varphi_i|^2_{\omega_\epsilon}+|\Delta_\epsilon\varphi_i|)\leq C_1(\delta^{-1}), \tag{2.38}$$

for all $0<\epsilon\leq 1$.

We consider the following test function

$$f(\cdot,t)=\varphi_1^2|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}+W\varphi_2^2\mathrm{tr}\, h_\epsilon(t), \tag{2.39}$$

where the constant W will be chosen large enough later. From (2.32) and (2.34), we have

$$\begin{aligned}&(\Delta_\epsilon-\frac{\partial}{\partial t})f\\ =&\varphi_1^2(2|\nabla_{H_\epsilon(t)}T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}-\check{C}_3|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}-\check{C}_3+\Delta_{\omega_\epsilon}\varphi_1^2|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}\\ &+4\langle\varphi_1\nabla\varphi_1,\nabla|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}\rangle_{\omega_\epsilon}+W\Delta_{\omega_\epsilon}\varphi_2^2\mathrm{tr}\, h_\epsilon(t)+4W\langle\varphi_2\nabla\varphi_2,\nabla\mathrm{tr}\, h_\epsilon(t)\rangle_{\omega_\epsilon}\\ &+2W\varphi_2^2(\mathrm{tr}\,(\sqrt{-1}\Lambda_{\omega_\epsilon}h_\epsilon^{-1}(t)\partial_{\hat{H}}h_\epsilon(t)\bar{\partial}h_\epsilon(t)))+\mathrm{tr}\,(h_\epsilon(t)(\Phi(\hat{H},\omega_\epsilon))).\end{aligned}\tag{2.40}$$

We use

$$\begin{aligned}2\langle\varphi_1\nabla\varphi_1,\nabla|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}\rangle_{\omega_\epsilon}&\geq-4\varphi_1|\nabla\varphi_1|_{\omega_\epsilon}|T_\epsilon(t)|_{H_\epsilon(t),\omega_\epsilon}|\nabla_{H_\epsilon(t)}T_\epsilon(t)|_{H_\epsilon(t),\omega_\epsilon}\\ &\geq-\varphi_1^2|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}-4|\nabla\varphi_1|^2_{\omega_\epsilon}|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon},\end{aligned}\tag{2.41}$$

$$W\langle\varphi_2\nabla\varphi_2,\nabla\mathrm{tr}\, h_\epsilon(t)\rangle_{\omega_\epsilon}\geq-\varphi_2^2|\nabla\mathrm{tr}\, h_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}-W^2|\nabla\varphi_2|^2_{\omega_\epsilon},\tag{2.42}$$

and

$$\begin{aligned}&|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}\\ =&\mathrm{tr}\,(\sqrt{-1}\Lambda_{\omega_\epsilon}h_\epsilon^{-1}(t)\partial_{\hat{H}}h_\epsilon(t)H_\epsilon^{-1}(t)\overline{(h_\epsilon^{-1}(t)\partial_{\hat{H}}h_\epsilon(t))}^T H_\epsilon(t))\\ =&\mathrm{tr}\,(\sqrt{-1}\Lambda_{\omega_\epsilon}h_\epsilon^{-1}(t)\partial_{\hat{H}}h_\epsilon(t)h_\epsilon^{-1}(t)\bar{\partial}h_\epsilon(t))\\ \leq&e^{\overline{C}_0}\mathrm{tr}\,(\sqrt{-1}\Lambda_{\omega_\epsilon}h_\epsilon^{-1}(t)\partial_{\hat{H}}h_\epsilon(t)\bar{\partial}h_\epsilon(t)),\end{aligned}\tag{2.43}$$

and choose

$$W=(\check{C}_3+4C_1(\delta^{-1})+2r)e^{\overline{C}_0}+1.\tag{2.44}$$

Then there exists a positive constant $\tilde{C}_0$ depending only on $\overline{C}_0$ and δ^{-1} such that

$$(\Delta_\epsilon-\frac{\partial}{\partial t})f\geq\varphi_1^2|\nabla_{H_\epsilon(t)}T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}+\varphi_2^2|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}-\tilde{C}_0\tag{2.45}$$

on $\tilde{M}\times[0,T]$. Let $f(q,t_0)=\max_{\tilde{M}\times[0,T]}\eta$, by the definition of φ_i and the uniform local C^0-assumption of $h_\epsilon(t)$, we can suppose that:

$$(q,t_0)\in\tilde{M}\setminus B_{\omega_1}(\frac{5}{4}\delta)\times(0,T].$$

By the inequality (2.45), we have

$$|T_\epsilon(t_0)|^2_{H_\epsilon(t_0),\omega_\epsilon}(q)\leq\tilde{C}_0.\tag{2.46}$$

So there exists a constant $\overline{C}_1$ depending only on $\overline{C}_0$ and δ^{-1}, such that

$$|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}(x)\leq\overline{C}_1\tag{2.47}$$

for all $(x,t)\in\tilde{M}\setminus B_{\omega_1}(\frac{3}{2}\delta)\times[0,T]$ and all $0<\epsilon\leq1$.

□

One can get the local uniform C^∞ estimates of $h_\epsilon(t)$ by the standard Schauder estimate of the parabolic equation after getting the local C^0 and C^1 estimates. But by applying the parabolic Schauder estimates, one can only get the uniform C^∞ estimates of $h_\epsilon(t)$ on $\tilde{M}\setminus B_{\omega_1}(\delta)\times[\tau,T]$, where $\tau>0$ and the uniform estimates depend on τ^{-1}. In the following, we first use the maximum principle to get a local uniform bound on the curvature $|F_{H_\epsilon(t)}|_{H_\epsilon(t),\omega_\epsilon}$, then we apply the elliptic estimates to get local uniform C^∞ estimates. The benefit of our argument is that we can get uniform C^∞ estimates of $h_\epsilon(t)$ on $\tilde{M}\setminus B_{\omega_1}(\delta)\times[0,T]$. In the following, for simplicity, we denote

$$\Xi_{\epsilon,j}=|\nabla^j_{H_\epsilon(t)}(F_{H_\epsilon(t)}+[\phi,\phi^{*H_\epsilon(t)}])|^2_{H_\epsilon(t),\omega_\epsilon}(x)+|\nabla^{j+1}_{H_\epsilon(t)}\phi|^2_{H_\epsilon(t),\omega_\epsilon}\tag{2.48}$$

for $j=0,1,\cdots$. Here $\nabla_{H_\epsilon(t)}$ denotes the covariant derivative with respect to the Chern connection $D_{H_\epsilon(t)}$ of $H_\epsilon(t)$ and the Riemannian connection ∇_{ω_ϵ} of ω_ϵ.

Lemma 2.5. *Assume that there exists a constant $\overline{C}_0$ such that*

$$\max_{(x,t)\in(\tilde{M}\setminus B_{\omega_1}(\delta))\times[0,T]}|S_\epsilon(t)|_{\hat{H}}(x)\leq\overline{C}_0, \tag{2.49}$$

for all $0<\epsilon\leq 1$. Then, for every integer $k\geq 0$, there exists a constant $\overline{C}_{k+2}$ depending only on $\overline{C}_0$, δ^{-1} and k, such that

$$\max_{(x,t)\in(\tilde{M}\setminus B_{\omega_1}(2\delta))\times[0,T]}\Xi_{\epsilon,k}\leq\overline{C}_{k+2} \tag{2.50}$$

for all $0<\epsilon\leq 1$. Furthermore, there exist constants $\hat{C}_{k+2}$ depending only on $\overline{C}_0$, δ^{-1} and k, such that

$$\max_{(x,t)\in(\tilde{M}\setminus B_{\omega_1}(2\delta))\times[0,T]}|\nabla_{\hat{H}}^{k+2}h_\epsilon|_{\hat{H},\omega_\epsilon}\leq\hat{C}_{k+2} \tag{2.51}$$

for all $0<\epsilon\leq 1$.

Proof. By computing, we have the following inequalities (see Lemma 2.4 and Lemma 2.5 in ([24]) for details):

$$\begin{aligned}&(\Delta_\epsilon-\frac{\partial}{\partial t})|\nabla_{H_\epsilon(t)}\phi|^2_{H_\epsilon(t),\omega_\epsilon}-2|\nabla_{H_\epsilon(t)}\nabla_{H_\epsilon(t)}\phi|^2_{H_\epsilon(t),\omega_\epsilon}\\ \geq&-C_7(|F_{H_\epsilon(t)}|_{H_\epsilon(t),\omega_\epsilon}+|Rm(\omega_\epsilon)|_{\omega_\epsilon}+|\phi|^2_{H_\epsilon(t),\omega_\epsilon})|\nabla_{H_\epsilon(t)}\phi|^2_{H_\epsilon(t),\omega_\epsilon}\\ &-C_7|\phi|_{H_\epsilon(t),\omega_\epsilon}|\nabla Ric(\omega_\epsilon)|_{\omega_\epsilon}|\nabla_{H_\epsilon(t)}\phi|_{H_\epsilon(t),\omega_\epsilon},\end{aligned} \tag{2.52}$$

$$\begin{aligned}&(\Delta_\epsilon-\frac{\partial}{\partial t})|F_{H_\epsilon(t)}+[\phi,\phi^{*H_\epsilon(t)}]|^2_{H_\epsilon(t),\omega_\epsilon}-2|\nabla_{H_\epsilon(t)}(F_{H_\epsilon(t)}+[\phi,\phi^{*H_\epsilon(t)}])|^2_{H_\epsilon(t),\omega_\epsilon}\\ \geq&-C_8(|F_{H_\epsilon(t)}+[\phi,\phi^{*H_\epsilon(t)}]|^2_{H_\epsilon(t),\omega_\epsilon}+|\nabla_{H_\epsilon(t)}\phi|^2_{H_\epsilon(t),\omega_\epsilon})^{\frac{3}{2}}\\ &-C_8(|\phi|^2_{H_\epsilon(t),\omega_\epsilon}+|Rm(\omega_\epsilon)|_{\omega_\epsilon})(|F_{H_\epsilon(t)}+[\phi,\phi^{*H_\epsilon(t)}]|^2_{H_\epsilon(t),\omega_\epsilon}+|\nabla_{H_\epsilon(t)}\phi|^2_{H_\epsilon(t),\omega_\epsilon}),\end{aligned} \tag{2.53}$$

then

$$\begin{aligned}(\Delta_\epsilon-\frac{\partial}{\partial t})\Xi_{\epsilon,0}\geq&\,2\Xi_{\epsilon,1}-C_8(\Xi_{\epsilon,0})^{\frac{3}{2}}\\ &-C_8(|\phi|^2_{H_\epsilon(t),\omega_\epsilon}+|Rm(\omega_\epsilon)|_{\omega_\epsilon})(\Xi_{\epsilon,0})-C_8|\nabla Ric(\omega_\epsilon)|^2_{\omega_\epsilon},\end{aligned} \tag{2.54}$$

where C_7, C_8 are constants depending only on the complex dimension n and the rank r. Furthermore, we have

$$\begin{aligned}&(\Delta_\epsilon-\frac{\partial}{\partial t})\Xi_{\epsilon,j}\\ \geq&\,2\Xi_{\epsilon,j+1}-\acute{C}_j(\Xi_{\epsilon,j})^{\frac{1}{2}}\{\sum_{i+k=j}((\Xi_{\epsilon,i})^{\frac{1}{2}}+|\phi|^2_{H_\epsilon(t),\omega_\epsilon}+|Rm(\omega_\epsilon)|_{\omega_\epsilon}+|\nabla Ric(\omega_\epsilon)|_{\omega_\epsilon})\\ &\cdot((\Xi_{\epsilon,k})^{\frac{1}{2}}+|\phi|^2_{H_\epsilon(t),\omega_\epsilon}+|Rm(\omega_\epsilon)|_{\omega_\epsilon}+|\nabla Ric(\omega_\epsilon)|_{\omega_\epsilon})\},\end{aligned} \tag{2.55}$$

where $\acute{C}_j$ is a positive constant depending only on the complex dimension n, the rank r and j. Direct computations yield the following inequality (see (2.5) in ([24]) for details):

$$\begin{aligned}&(\Delta_\epsilon-\frac{\partial}{\partial t})|\phi|^2_{H_\epsilon(t),\omega_\epsilon}\geq 2|\nabla_{H_\epsilon(t)}\phi|^2_{H_\epsilon(t),\omega_\epsilon}\\ &+2|\Lambda_{\omega_\epsilon}[\phi,\phi^{*H_\epsilon(t)}]|^2_{H_\epsilon(t)}-2|Ric(\omega_\epsilon)|_{\omega_\epsilon}|\phi|^2_{H_\epsilon(t),\omega_\epsilon}.\end{aligned} \tag{2.56}$$

From the local C^0-assumption (2.30), we see that $|\phi|_{H_\epsilon(t),\omega_\epsilon}$ is also uniformly bounded on $\tilde{M}\setminus B_{\omega_1}(\delta)\times[0,T]$. By Lemma 2.4, we have $|T_\epsilon(t)|_{H_\epsilon(t),\omega_\epsilon}$ is uniformly bounded on $\tilde{M}\setminus B_{\omega_1}(\frac{3}{2}\delta)\times[0,T]$. We choose a constant $\hat{C}$ depending only on δ^{-1} and $\overline{C}_0$ such that

$$\frac{1}{2}\hat{C}\leq\hat{C}-(|\phi|^2_{H_\epsilon(t),\omega_\epsilon}+|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon})(x)\leq\hat{C} \tag{2.57}$$

on $\tilde{M}\setminus B_{\omega_1}(\frac{3}{2}\delta)\times[0,T]$. We consider the test function:

$$\zeta(x,t)=\rho^2\frac{\Xi_{\epsilon,0}(x,t)}{\hat{C}-(|\phi|^2_{H_\epsilon(t),\omega_\epsilon}+|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon})(x)}, \tag{2.58}$$

where ρ is a cut-off function satisfying:

$$\rho(x)=\begin{cases}0, & x\in B_{\omega_1}(\frac{13}{8}\delta),\\ 1, & x\in\tilde{M}\setminus B_{\omega_1}(\frac{7}{4}\delta),\end{cases} \tag{2.59}$$

and $|d\rho|^2_{\omega_1}\leq\frac{8}{\delta^2}$, $-\frac{c}{\delta^2}\omega_1\leq\sqrt{-1}\partial\bar{\partial}\rho\leq\frac{c}{\delta^2}\omega_1$. We suppose $(x_0,t_0)\in\tilde{M}\setminus B_{\omega_1}(\frac{3}{2}\delta)\times(0,T]$ is a maximum point of ζ. Using (2.35), (2.52), (2.54), (2.56) and the fact $\nabla\zeta=0$ at the point (x_0,t_0), we have

$$\begin{aligned}
0\geq\ &(\Delta_\epsilon-\frac{\partial}{\partial t})\zeta|_{(x_0,t_0)}\\
=\ &\frac{1}{\hat{C}-(|\phi|^2_{H_\epsilon(t),\omega_\epsilon}+|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon})}(\Delta_\epsilon-\frac{\partial}{\partial t})(\rho^2\Xi_{\epsilon,0})\\
&-\rho^2\frac{\Xi_{\epsilon,0}}{(\hat{C}-(|\phi|^2_{H_\epsilon(t),\omega_\epsilon}+|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}))^2}(\Delta_\epsilon-\frac{\partial}{\partial t})(\hat{C}-(|\phi|^2_{H_\epsilon(t),\omega_\epsilon}+|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}))\\
&-\frac{2}{\hat{C}-(|\phi|^2_{H_\epsilon(t),\omega_\epsilon}+|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon})}\nabla(\zeta)\cdot\nabla(\hat{C}-(|\phi|^2_{H_\epsilon(t),\omega_\epsilon}+|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}))\\
\geq\ &\frac{\Xi_{\epsilon,0}}{(\hat{C}-(|\phi|^2_{H_\epsilon(t),\omega_\epsilon}+|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}))^2}\{\rho^2\frac{2\Xi_{\epsilon,0}-\check{C}_3|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon}-\check{C}_3}{\hat{C}-(|\phi|^2_{H_\epsilon(t),\omega_\epsilon}+|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon})}\\
&-\rho^2\frac{2|Ric(\omega_\epsilon)|_{\omega_\epsilon}|\phi|^2_{H_\epsilon(t),\omega_\epsilon}}{\hat{C}-(|\phi|^2_{H_\epsilon(t),\omega_\epsilon}+|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon})}\\
&-C_8\rho^2\Xi^{\frac{1}{2}}_{\epsilon,0}-C_8\rho^2(|\phi|^2_{H_\epsilon(t),\omega_\epsilon}+|Rm(\omega_\epsilon)|_{\omega_\epsilon})-8|d\rho|^2_{\omega_\epsilon}+\Delta_{\omega_\epsilon}\rho^2\}\\
&-C_8\frac{\rho^2|\nabla Ric(\omega_\epsilon)|^2_{\omega_\epsilon}}{\hat{C}-(|\phi|^2_{H_\epsilon(t),\omega_\epsilon}+|T_\epsilon(t)|^2_{H_\epsilon(t),\omega_\epsilon})}.
\end{aligned} \tag{2.60}$$

So there exist positive constants $\dot{C}_2$ and $\overline{C}_2$ depending only on $\overline{C}_0$ and δ^{-1}, such that

$$\zeta(x_0,t_0)\leq\dot{C}_2, \tag{2.61}$$

and

$$\Xi_{\epsilon,0}(x,t)\leq\overline{C}_2 \tag{2.62}$$

for all $(x,t)\in\tilde{M}\setminus B_{\omega_1}(\frac{7}{4}\delta)\times[0,T]$.

Furthermore, we choose two suitable cut-off functions ρ_1, ρ_2, a suitable constant A which depends only on $\overline{C}_0$ and δ^{-1}, and a test function

$$\zeta_1(x,t)=\rho_1^2\Xi_{\epsilon,1}+A\rho_2^2\Xi_{\epsilon,0}. \tag{2.63}$$

Running a similar argument as above, we can show that there exist constants $\overline{C}_3$ and $\dot{C}_3$ depending only on $\overline{C}_0$ and δ^{-1} such that

$$\Xi_{\epsilon,1}(x,t)\leq\overline{C}_3, \tag{2.64}$$

and

$$|\nabla_{\hat{H}}F_{H_\epsilon(t)}|^2_{\hat{H},\omega_\epsilon}\leq\dot{C}_3 \tag{2.65}$$

for all $(x,t)\in\tilde{M}\setminus B_{\omega_1}(\frac{15}{8}\delta)\times[0,T]$.

Recalling the equality

$$\overline{\partial}\partial_{\hat{H}}h_\epsilon(t)=h_\epsilon(t)(F_{H_\epsilon(t)}-F_{\hat{H}})+\overline{\partial}h_\epsilon(t)\wedge(h_\epsilon(t))^{-1}\partial_{\hat{H}}h_\epsilon(t) \tag{2.66}$$

and noting that Kähler metrics ω_ϵ are uniform locally quasi-isometry to $\pi^*\omega$ outside the exceptional divisor $\tilde{\Sigma}$, by standard elliptic estimates, because we have local uniform bounds on h_ϵ, T_ϵ, F_{H_ϵ} and $F_{\hat{H}}$, we get a uniform $C^{1,\alpha}$-estimate of h_ϵ on $\tilde{M}\setminus B_{\omega_1}(\frac{61}{32}\delta)\times[0,T]$.

We can iterate this procedure by induction and then obtain local uniform bounds for $\Xi_{\epsilon,k}$, $|\nabla^k_{\hat{H}}F_{H_\epsilon(t)}|^2_{\hat{H},\omega_\epsilon}$, and $\|h_\epsilon\|_{C^{k+1,\alpha}}$ on $\tilde{M}\setminus B_{\omega_1}(2\delta)\times[0,T]$ for any $k\geq1$.

□

From the above local uniform C^∞-bounds on H_ϵ, we get the following Lemma.

Lemma 2.6. *By choosing a subsequence, $H_\epsilon(t)$ converges to $H(x,t)$ locally in C^∞ topological on $\tilde{M}\setminus\tilde{\Sigma}\times[0,\infty)$ as $\epsilon\to0$ and $H(t)$ satisfies (1.11).*

3. Uniform estimate of the Higgs field

In this section, we prove that the norm $|\phi|_{H(t),\omega}$ is uniformly bounded along the heat flow (1.11) for $t\geq t_0>0$.

Firstly, we know $|\phi|^2_{\hat{H},\omega_\epsilon}\in L^1(\tilde{M},\omega_\epsilon)$ and the L^1-norm is uniformly bounded. In fact,

$$\begin{aligned}\int_{\tilde{M}}|\phi|^2_{\hat{H},\omega_\epsilon}\frac{\omega_\epsilon^n}{n!}&=\int_{\tilde{M}}\operatorname{tr}(\sqrt{-1}\Lambda_{\omega_\epsilon}(\phi\wedge\phi^{*\hat{H}}))\frac{\omega_\epsilon^n}{n!}\\&=\int_{\tilde{M}}\operatorname{tr}(\phi\wedge\phi^{*\hat{H}})\wedge\frac{\omega_\epsilon^{n-1}}{(n-1)!}\leq\check{C}_\phi<\infty,\end{aligned} \tag{3.1}$$

where $\check{C}_\phi$ is a positive constant independent of ϵ. Moreover, we will show the L^{1+2a}-norm of $|\phi|^2_{\hat{H},\omega_\epsilon}$ is also uniformly bounded, for any $0\leq2a<\frac{1}{2}$. Let's recall Lemma 5.5 in [31] (see also Lemma 5.8 in [26]).

Lemma 3.1. ([31]) *Let (M,ω) be a compact Kähler manifold of complex dimension n, and $\pi:\tilde{M}\to M$ be a blow-up along a smooth complex sub-manifold Σ of complex codimension k where $k\geq2$. Let η be a Kähler metric on $\tilde{M}$, and consider the family of Kähler metric $\omega_\epsilon=\pi^*\omega+\epsilon\eta$. Then for any $0\leq2a<\frac{1}{k-1}$, we have $\frac{\eta^n}{\omega_\epsilon^n}\in L^{2a}(\tilde{M},\eta)$, and the $L^{2a}(\tilde{M},\eta)$-norm of $\frac{\eta^n}{\omega_\epsilon^n}$ is uniformly bounded independent of ϵ, i.e. there is a positive constant C^* such that*

$$\int_{\tilde{M}}(\frac{\eta^n}{\omega_\epsilon^n})^{2a}\frac{\eta^n}{n!}\leq C^* \tag{3.2}$$

for all $0<\epsilon\leq1$.

Since $\phi \in \Omega^{1,0}(\mathrm{End}(E))$ is a smooth section and $\omega_\epsilon = \pi^*\omega + \epsilon\eta$, there exists a uniform constant $\tilde{C}_\phi$ such that

$$\Big(\frac{|\phi|^2_{\hat{H},\omega_\epsilon}\frac{\omega_\epsilon^n}{n!}}{\frac{\eta^n}{n!}}\Big) = \frac{n\mathrm{tr}\,(\phi\wedge\phi^{*\hat{H}})\wedge\omega_\epsilon^{n-1}}{\eta^n} \le \tilde{C}_\phi \tag{3.3}$$

for all $0<\epsilon\le 1$. By (3.2), for any $0\le 2a<\frac{1}{2}$, there exists a uniform constant C_ϕ such that

$$\begin{aligned}&\int_{\tilde{M}} |\phi|^{2(1+2a)}_{\hat{H},\omega_\epsilon}\frac{\omega_\epsilon^n}{n!}\\ =&\int_{\tilde{M}}\Big(\frac{|\phi|^2_{\hat{H},\omega_\epsilon}\frac{\omega_\epsilon^n}{n!}}{\frac{\eta^n}{n!}}\Big)^{1+2a}\Big(\frac{\eta^n}{\omega_\epsilon^n}\Big)^{1+2a}\frac{\omega_\epsilon^n}{n!}\\ =&\int_{\tilde{M}}\Big(\frac{|\phi|^2_{\hat{H},\omega_\epsilon}\frac{\omega_\epsilon^n}{n!}}{\frac{\eta^n}{n!}}\Big)^{1+2a}\Big(\frac{\eta^n}{\omega_\epsilon^n}\Big)^{2a}\frac{\eta^n}{n!}\\ \le& C_\phi\end{aligned} \tag{3.4}$$

for all $0<\epsilon\le 1$. By limiting (3.4), we have the following lemma.

Lemma 3.2. *For any $0\le 2a<\frac{1}{2}$, we have $|\phi|^2_{\hat{H},\omega}\in L^{1+2a}(M\setminus\Sigma,\omega)$, i.e. there exists a constant C_ϕ such that*

$$\int_{M\setminus\Sigma}|\phi|^{2(1+2a)}_{\hat{H},\omega}\frac{\omega^n}{n!}\le C_\phi. \tag{3.5}$$

On $M\setminus\Sigma$, we get ((2.5) in [24] for details)

$$(\Delta-\frac{\partial}{\partial t})|\phi|^2_{H(t),\omega}\ge 2|\nabla_{H(t)}\phi|^2_{H(t),\omega}+2|\sqrt{-1}\Lambda_\omega[\phi,\phi^{*H(t)}]|^2_{H(t)}-2|Ric_\omega|_\omega|\phi|^2_{H(t),\omega}. \tag{3.6}$$

By a direct computation, we have

$$\begin{aligned}(\Delta-\frac{\partial}{\partial t})\log(|\phi|^2_{H(t),\omega}+e)&=\frac{1}{\log(|\phi|^2_{H(t),\omega}+e)}(\Delta-\frac{\partial}{\partial t})|\phi|^2_{H(t),\omega}-\frac{\nabla|\phi|^2_{H(t),\omega}\cdot\nabla|\phi|^2_{H(t),\omega}}{(|\phi|^2_{H(t),\omega}+e)^2}\\ &\ge\frac{1}{\log(|\phi|^2_{H(t),\omega}+e)}(\Delta-\frac{\partial}{\partial t})|\phi|^2_{H(t),\omega}-\frac{2|\nabla^{1,0}_{H(t)}\phi|^2_{H(t),\omega}\cdot|\phi|^2_{H(t),\omega}}{(|\phi|^2_{H(t),\omega}+e)^2}.\end{aligned} \tag{3.7}$$

Combining this with (3.6), we obtain

$$(\Delta-\frac{\partial}{\partial t})\log(|\phi|^2_{H(t),\omega}+e)\ge\frac{2|\Lambda_\omega[\phi,\phi^{*H(t)}]|^2_{H(t)}}{|\phi|^2_{H(t),\omega}+e}-2|Ric_\omega|_\omega \tag{3.8}$$

on $M\setminus\Sigma$. Based on Lemma 2.7 in [33], we obtain

$$|\sqrt{-1}\Lambda_\omega[\phi,\phi^{*H(t)}]|_{H(t)}=|[\phi,\phi^{*H(t)}]|_{H(t),\omega}\ge a_1|\phi|^2_{H(t),\omega}-a_2(|\phi|^2_{\hat{H},\omega}+1), \tag{3.9}$$

where a_1 and a_2 are positive constants depending only on r and n. Then, for any $0\le 2a<\frac{1}{2}$, we have

$$\begin{aligned}&2|\Lambda_\omega[\phi,\phi^{*H(t)}]|^2_{H(t)}\\ \ge&(|\Lambda_\omega[\phi,\phi^{*H(t)}]|_{H(t)}+e)^2-6e^2\\ \ge&(|\Lambda_\omega[\phi,\phi^{*H(t)}]|_{H(t)}+e)^{1+\frac{a}{2}}-6e^2\\ \ge& a_3(|\phi|^2_{H(t),\omega}+e)^{1+\frac{a}{2}}-a_4|\phi|^{2+a}_{\hat{H},\omega}-a_5,\end{aligned} \tag{3.10}$$

where a_3, a_4 and a_5 are positive constants depending only on a, r and n. Then it is clear that (3.8) implies:

$$(\Delta-\frac{\partial}{\partial t})\log(|\phi|^2_{H(t),\omega}+e)\geq a_3(|\phi|^2_{H(t),\omega}+e)^{\frac{a}{2}}-a_4|\phi|^{2+a}_{\hat{H},\omega}-a_5-2|Ric_\omega|_\omega, \tag{3.11}$$

on $M\setminus\Sigma$.

In the following, we denote:

$$f=\log(|\phi|^2_{H(t),\omega}+e). \tag{3.12}$$

For any $b>1$, we have:

$$\begin{aligned}(\Delta-\frac{\partial}{\partial t})f^b&=bf^{b-1}(\Delta-\frac{\partial}{\partial t})f+b(b-1)|\nabla f|^2_\omega f^{b-2}\\&\geq a_3bf^{b-1}(|\phi|^2_{H(t),\omega}+e)^{\frac{a}{2}}-a_4bf^{b-1}|\phi|^{2+a}_{\hat{H},\omega}-(a_5+2|Ric_\omega|_\omega)bf^{b-1}\\&\quad+b(b-1)|\nabla f|^2_\omega f^{b-2}.\end{aligned} \tag{3.13}$$

Choosing a cut-off function φ_δ with

$$\varphi_\delta(x)=\begin{cases}1, & x\in M\setminus B_{2\delta}(\Sigma),\\ 0, & x\in B_\delta(\Sigma),\end{cases} \tag{3.14}$$

where $B_\delta=\{x\in M|d_\omega(x,\Sigma)<\delta\}$, and integrating by parts, we have

$$\begin{aligned}&-\frac{\partial}{\partial t}\int_M\varphi_\delta^4f^b\frac{\omega^n}{n!}=\int_M\varphi_\delta^4(\Delta-\frac{\partial}{\partial t})f^b\frac{\omega^n}{n!}+\int_M4\varphi_\delta^3\nabla\varphi_\delta\nabla f^b\frac{\omega^n}{n!}\\
\geq&\int_M a_3b\varphi_\delta^4f^{b-1}(|\phi|^2_{H(t),\omega}+e)^{\frac{a}{2}}\frac{\omega^n}{n!}-\int_M a_4b\varphi_\delta^4f^{b-1}|\phi|^{2+a}_{\hat{H},\omega}\frac{\omega^n}{n!}\\
&-\int_M(a_5+2|Ric_\omega|_\omega)b\varphi_\delta^4f^{b-1}\frac{\omega^n}{n!}+\int_M b(b-1)\varphi_\delta^4|\nabla f|^2_\omega f^{b-2}\frac{\omega^n}{n!}\\
&-\int_M4b\varphi_\delta^3|\nabla\varphi_\delta|_\omega\cdot|\nabla f|_\omega f^{b-1}\frac{\omega^n}{n!}\\
\geq&\int_M a_3b\varphi_\delta^4f^{b-1}(|\phi|^2_{H(t),\omega}+e)^{\frac{a}{2}}\frac{\omega^n}{n!}-\int_M a_4b\varphi_\delta^4f^{b-1}(|\phi|^2_{\hat{H},\omega})^{1+\frac{a}{2}}\frac{\omega^n}{n!}\\
&-\int_M(a_5+2|Ric_\omega|_\omega)b\varphi_\delta^4f^{b-1}\frac{\omega^n}{n!}-\int_M\frac{4b}{b-1}\varphi_\delta^2|\nabla\varphi_\delta|^2_\omega f^b\frac{\omega^n}{n!}\\
\geq&\int_M a_3b\varphi_\delta^4f^{b-1}f^{(b-1)B}\frac{(|\phi|^2_{H(t),\omega}+e)^{\frac{a}{2}}}{f^{(b-1)B}}\frac{\omega^n}{n!}\\
&-a_4b\Big(\int_M(\varphi_\delta^3f^{b-1})^p\frac{\omega^n}{n!}\Big)^{\frac{1}{p}}\Big(\int_M\varphi_\delta^q(|\phi|^2_{\hat{H},\omega})^{1+2a}\frac{\omega^n}{n!}\Big)^{\frac{1}{q}}\\
&-\int_M(a_5+2|Ric_\omega|_\omega)b\varphi_\delta^4f^{b-1}\frac{\omega^n}{n!}\\
&-\frac{4b}{b-1}\Big(\int_M\varphi_\delta^4f^{2b}\frac{\omega^n}{n!}\Big)^{\frac{1}{2}}\Big(\int_M|\nabla\varphi_\delta|^4_\omega\frac{\omega^n}{n!}\Big)^{\frac{1}{2}},\end{aligned} \tag{3.15}$$

where $q=\frac{2(1+2a)}{2+a}$, $p=\frac{2(1+2a)}{3a}$ and $B=\frac{2(1+2a)}{3a}+\frac{2b}{b-1}$. We can see that there exists a constant $C(a,b)$ depending only on a and b such that

$$\frac{(|\phi|^2_{H(t),\omega}+e)^{\frac{a}{2}}}{(\log(|\phi|^2_{H(t),\omega}+e))^{(b-1)B}}\geq C(a,b). \tag{3.16}$$

Since the complex codimension of Σ is at least 3, we can choose the cut-off function φ_δ such that

$$\int_M |\nabla\varphi_\delta|_\omega^4 \frac{\omega^n}{n!} \sim O(\delta^{-4}\delta^6) = O(\delta^2). \tag{3.17}$$

By (3.5), we obtain

$$\begin{aligned} -\frac{\partial}{\partial t}\int_M \varphi_\delta^4 f^b \frac{\omega^n}{n!} \geq a_6 \int_M \varphi_\delta^4 f^{(b-1)B}\frac{\omega^n}{n!} - a_7\Big(\int_M \varphi_\delta^4 f^{(b-1)B}\frac{\omega^n}{n!}\Big)^{\frac{1}{B}} \\ - a_8\Big(\int_M \varphi_\delta^4 f^{(b-1)B}\frac{\omega^n}{n!}\Big)^{\frac{1}{B}} - a_9\Big(\int_M \varphi_\delta^4 f^{(b-1)B}\frac{\omega^n}{n!}\Big)^{\frac{b}{(b-1)B}}, \end{aligned} \tag{3.18}$$

where a_i are positive constants depending only on $r, n, a, b, |Ric_\omega|_\omega, \mathrm{Vol}(M,\omega)$ and C_ϕ for $i = 6, 7, 8, 9$.

Lemma 3.3. *For any $b > 1$, there exists a constant $\hat{C}_b$ depending only on $r, n, b, |Ric_\omega|_\omega, \mathrm{Vol}(M,\omega)$ and C_ϕ such that*

$$\int_{M\setminus\Sigma} (\log(|\phi|^2_{H(t),\omega} + e))^b \frac{\omega^n}{n!} \leq \hat{C}_b \tag{3.19}$$

for all $t \geq 0$.

Proof. Suppose that $\int_M \varphi_\delta^4 f^b \frac{\omega^n}{n!}(t^*) = \max_{t\in[0,T]} \int_M \varphi_\delta^4 f^b \frac{\omega^n}{n!}(t)$ with $t^* > 0$. Choosing $a = \frac{1}{8}$ in (3.20), at point t^*, we have

$$\begin{aligned} 0 &\geq -\frac{\partial}{\partial t}|_{t=t^*}\int_M \varphi_\delta^4 f^b \frac{\omega^n}{n!} \\ &\geq a_6 \int_M \varphi_\delta^4 f^{(b-1)B}\frac{\omega^n}{n!} - a_7\Big(\int_M \varphi_\delta^4 f^{(b-1)B}\frac{\omega^n}{n!}\Big)^{\frac{1}{B}} \\ &\quad - a_8\Big(\int_M \varphi_\delta^4 f^{(b-1)B}\frac{\omega^n}{n!}\Big)^{\frac{1}{B}} - a_9\Big(\int_M \varphi_\delta^4 f^{(b-1)B}\frac{\omega^n}{n!}\Big)^{\frac{b}{(b-1)B}}. \end{aligned} \tag{3.20}$$

This inequality implies that there exists a constant $\tilde{C}_b$ depending only on $r, n, b, |Ric_\omega|_\omega, \mathrm{Vol}(M,\omega)$ and C_ϕ such that

$$\int_M \varphi_\delta^4 f^{(b-1)B}\frac{\omega^n}{n!}(t^*) \leq \tilde{C}_b. \tag{3.21}$$

So we have

$$\max_{t\in[0,T]}\int_M \varphi_\delta^4 f^b \frac{\omega^n}{n!}(t) \leq \tilde{C}_b + \int_M (\log(|\phi|^2_{\hat{H},\omega} + e))^b \frac{\omega^n}{n!}. \tag{3.22}$$

Noting that the last term in the above inequality is also bounded, and letting $\delta \to 0$, we obtain the estimate (3.19) .

□

By the heat equation (1.11), we have

$$|\frac{\partial}{\partial t}\log(|\phi|^2_{H(t),\omega} + e)| = \Big|\frac{\frac{\partial}{\partial t}|\phi|^2_{H(t),\omega}}{|\phi|^2_{H(t),\omega} + e}\Big| = \Big|\frac{2\langle[\Phi(H(t),\omega),\phi],\phi\rangle_{H(t)}}{|\phi|^2_{H(t),\omega} + e}\Big| \leq 2|\Phi(H(t),\omega)|_{H(t)}, \tag{3.23}$$

then

$$\Delta(\log(|\phi|^2_{H(t),\omega} + e)) \geq -2|Ric_\omega|_\omega - 2|\Phi(H(t),\omega)|_{H(t)}. \tag{3.24}$$

By (2.11), we have

$$\max_{x\in M\setminus\Sigma} |\Phi(H(t),\omega)|_{H(t)}(x) \leq C_K(\tau)\hat{C}_1(t^{-n} + 1). \tag{3.25}$$

So there exists a positive constant $C^*(t_0^{-1})$ depending only on t_0^{-1} and $|Ric_\omega|_\omega$ such that

$$\Delta(\log(|\phi|^2_{H(t),\omega}+e)) \geq -C^*(t_0^{-1}) \tag{3.26}$$

on $M\setminus\Sigma$, for $t\geq t_0>0$. Then, we have

$$\begin{aligned} -C^*(t_0^{-1})\int_M \varphi_\delta^2 f\frac{\omega^n}{n!} &\leq \int_M \varphi_\delta^2 f\Delta f\frac{\omega^n}{n!} \\ &= \int_M div(\varphi_\delta^2 f\nabla f)\frac{\omega^n}{n!} - \int_M \nabla(\varphi_\delta^2 f)\cdot\nabla f\frac{\omega^n}{n!} \\ &= -\int_M |\nabla(\varphi_\delta f)|^2_\omega\frac{\omega^n}{n!} + \int_M |\nabla\varphi_\delta|^2_\omega f^2\frac{\omega^n}{n!} \end{aligned} \tag{3.27}$$

for $t\geq t_0>0$. By (3.17) and (3.19), we obtain

$$\begin{aligned} &\int_{M\setminus\Sigma}|\nabla f|^2_\omega\frac{\omega^n}{n!} = \lim_{\delta\to 0}\int_{M\setminus B_{2\delta}(\Sigma)}|\nabla f|^2_\omega\frac{\omega^n}{n!} \\ \leq& \lim_{\delta\to 0}\int_M |\nabla(\varphi_\delta f)|^2_\omega\frac{\omega^n}{n!} \\ \leq& \lim_{\delta\to 0}\int_M C^*(t_0^{-1})\varphi_\delta^2 f + |\nabla\varphi_\delta|^2_\omega f^2\frac{\omega^n}{n!} \\ \leq& C^*(t_0^{-1})\cdot\hat{C}_b \end{aligned} \tag{3.28}$$

for $t\geq t_0>0$. This implies $f\in W^{1,2}(M,\omega)$ and f satisfies the elliptic inequality $\Delta f\geq -C^*(t_0^{-1})$ globally on M in weakly sense for $t\geq t_0>0$. By the standard elliptic estimate (see Theorem 8.17 in [14]), we can show that $f\in L^\infty(M)$ for all $t\geq t_0>0$, and the L^∞-norm depending on $C^*(t_0^{-1})$, the L^b-norm (i.e. $\hat{C}_b$) and the geometry of (M,ω), i.e. we have the following proposition.

Proposition 3.4. *Along the heat flow (1.11), there exists a positive constant $\hat{C}_\phi$ depending only on r,n,t_0^{-1},C_ϕ and the geometry of (M,ω) such that*

$$\sup_{M\setminus\Sigma}|\phi|^2_{H(t),\omega}\leq\hat{C}_\phi \tag{3.29}$$

for all $t\geq t_0>0$.

Recalling the Chern-Weil formula in [32] (Proposition 3.4) and using Fatou's lemma, we have

$$\begin{aligned} &4\pi^2\int_M(2c_2(\mathcal{E})-c_1(\mathcal{E})\wedge c_1(\mathcal{E}))\wedge\frac{\omega^{n-2}}{(n-2)!} \\ =&\lim_{\epsilon\to 0}4\pi^2\int_{\tilde{M}}(2c_2(E)-c_1(E)\wedge c_1(E))\wedge\frac{\omega_\epsilon^{n-2}}{(n-2)!} \\ =&\lim_{\epsilon\to 0}\int_{\tilde{M}}\mathrm{tr}\,(F_{H_\epsilon(t),\phi}\wedge F_{H_\epsilon(t),\phi})\wedge\frac{\omega_\epsilon^{n-2}}{(n-2)!} \\ =&\lim_{\epsilon\to 0}\int_{\tilde{M}}(|F_{H_\epsilon(t),\phi}|^2_{H_\epsilon(t),\omega_\epsilon}-|\Lambda_{\omega_\epsilon}F_{H_\epsilon(t),\phi}|^2_{H_\epsilon(t)})\frac{\omega_\epsilon^n}{n!} \\ \geq&\int_{M\setminus\Sigma}(|F_{H(t),\phi}|^2_{H(t),\omega}-|\sqrt{-1}\Lambda_\omega F_{H(t),\phi}|^2_{H(t)})\frac{\omega^n}{n!} \end{aligned} \tag{3.30}$$

for $t>0$. Here, over a non-projective compact complex manifold, the Chern classes of a coherent sheaf can be defined by the classes of Atiyah-Hirzenbruch ([2], see [16] for details).

The L^∞ estimate of $|\phi|^2_{H(t),\omega}$, (2.11) and the above inequality imply that $|F_{H(t)}|_{H(t),\omega}$ is square integrable and $|\Lambda_\omega F_{H(t)}|_{H(t)}$ is uniformly bounded, i.e. we have the following corollary.

Corollary 3.5. *Let $H(t)$ be a solution of the heat flow (1.11), then $H(t)$ must be an admissible Hermitian metric on $\mathcal{E}$ for every $t > 0$.*

4. Approximate Hermitian-Einstein structure

Let $H_\epsilon(t)$ be the long time solution of (1.10) and $H(t)$ be the long time solution of (1.11). We set:

$$\exp S(t) = h(t) = \hat{H}^{-1}H(t), \tag{4.1}$$

$$\exp S(t_1,t_2) = h(t_1,t_2) = H^{-1}(t_1)H(t_2), \tag{4.2}$$

$$\exp S_\epsilon(t_1,t_2) = h_\epsilon(t_1,t_2) = H_\epsilon^{-1}(t_1)H_\epsilon(t_2). \tag{4.3}$$

By Lemma 3.1 in [32], we have

$$\Delta_{\omega_\epsilon} \log(\operatorname{tr} h + \operatorname{tr} h^{-1}) \geq -2|\Lambda_{\omega_\epsilon}(F_{H,\phi})|_H - 2|\Lambda_\omega(F_{K,\phi})|_K, \tag{4.4}$$

where $\exp S = h = K^{-1}H$. By the uniform lower bound of Green functions G_ϵ (2.11) and the inequalities (2.26) , we have

$$\|S_\epsilon(t_1,t_2)\|_{L^\infty(\tilde{M})} \leq C_1\|S_\epsilon(t_1,t_2)\|_{L^1(\tilde{M},\omega_\epsilon)} + C_2(t_0^{-1}) \tag{4.5}$$

for $0 < t_0 \leq t_1 \leq t_2$, where C_1 is a constant depending only on the rank r and $C_2(t_0^{-1})$ is a constant depending only on C_K, C_G and t_0^{-1}. By limiting, we also have

$$\|S(t_1,t_2)\|_{L^\infty(M\setminus\Sigma)} \leq C_1\|S(t_1,t_2)\|_{L^1(M\setminus\Sigma,\omega)} + C_2(t_0^{-1}) \tag{4.6}$$

for $0 < t_0 \leq t_1 \leq t_2$. On the other hand, (2.25) and (2.26) imply that

$$\begin{aligned} & r^{-\frac{1}{2}}\|S(t_1,t_2)\|_{L^1(M\setminus\Sigma,\omega)} - \mathrm{Vol}(M,\omega)\log(2r) \\ \leq & \int_{t_1}^{t_2}\int_{M\setminus\Sigma} |\sqrt{-1}\Lambda_\omega F_{H(s),\phi} - \lambda \mathrm{Id}_{\mathcal{E}}|_{H(s)}\frac{\omega^n}{n!}ds \\ \leq & \hat{C}_1(t_2-t_1). \end{aligned} \tag{4.7}$$

So, we know that the metrics $H(t_1)$ and $H(t_2)$ are mutually bounded each other on $\mathcal{E}|_{M\setminus\Sigma}$. $(\mathcal{E}|_{M\setminus\Sigma},\phi)$ can be seen as a Higgs bundle on the non-compact Kähler manifold $(M\setminus\Sigma,\omega)$. Let's recall Donaldson's functional defined on the space $\mathscr{P}_0$ of Hermitian metrics on the Higgs bundle $(\mathcal{E}|_{M\setminus\Sigma},\phi)$ (see Section 5 in [32] for details),

$$\mu_\omega(K,H) = \int_{M\setminus\Sigma} \operatorname{tr}(S\sqrt{-1}\Lambda_\omega F_{K,\phi}) + \langle\Psi(S)(D''_\phi S), D''_\phi S\rangle_K \frac{\omega^n}{n!}, \tag{4.8}$$

where $\Psi(x,y) = (x-y)^{-2}(e^{y-x}-(y-x)-1)$, $\exp S = K^{-1}H$. Since we have known that $|\Lambda_\omega F_{H(t),\phi}|_{H(t)}$ is uniformly bounded for $t \geq t_0 > 0$, it is easy to see that $H(t)$ (for every $t > 0$) belongs to the definition space $\mathscr{P}_0$. By Lemma 7.1 in [32], we have a formula for the derivative with respect to t of Donaldson's functional,

$$\frac{d}{dt}\mu(H(t_1),H(t)) = -2\int_{M\setminus\Sigma} |\Phi(H(t),\phi)|^2_{H(t)}\frac{\omega^n}{n!}. \tag{4.9}$$

Proposition 4.1. *Let $H(t)$ be the long time solution of (1.11). If the reflexive Higgs sheaf $(\mathcal{E},\phi)$ is ω-semi-stable, then*

$$\int_{M\setminus\Sigma}|\sqrt{-1}\Lambda_\omega F_{H(t),\phi}-\lambda \mathrm{Id}_{\mathcal{E}}|^2_{H(t)}\frac{\omega^n}{n!}\to 0, \tag{4.10}$$

as $t\to+\infty$.

Proof. We prove (4.10) by contradiction. If not, by the monotonicity of $\|\Lambda_\omega(F_{H(t),\phi})-\lambda\mathrm{Id}\|_{L^2}$, we can suppose that

$$\lim_{t\to+\infty}\int_M|\sqrt{-1}\Lambda_\omega F_{H(t),\phi}-\lambda \mathrm{Id}_{\mathcal{E}}|^2_{H(t)}\frac{\omega^n}{n!}=C^*>0. \tag{4.11}$$

By (4.9), we have

$$\mu_\omega(H(t_0),H(t))=-\int_{t_0}^t\int_{M\setminus\Sigma}|\Lambda_\omega F_{H(s),\phi}-\lambda \mathrm{Id}_{\mathcal{E}}|^2_{H(s)}\frac{\omega^n}{n!}ds\le -C^*(t-t_0) \tag{4.12}$$

for all $0<t_0\le t$. Then it is clear that (4.7) implies

$$\liminf_{t\to+\infty}\frac{-\mu_\omega(H(t_0),H(t))}{\|S(t_0,t)\|_{L^1(M\setminus\Sigma,\omega)}}\ge r^{-\frac{1}{2}}\frac{C^*}{\hat{C}_1}. \tag{4.13}$$

By the definition of Donaldson's functional (4.8), we must have a sequence $t_i\to+\infty$ such that

$$\|S(1,t_i)\|_{L^1(M\setminus\Sigma,\omega)}\to+\infty. \tag{4.14}$$

On the other hand, it is easy to check that

$$|S(t_1,t_3)|_{H(t_1)}\le r(|S(t_1,t_2)|_{H(t_1)}+|S(t_2,t_3)|_{H(t_2)}) \tag{4.15}$$

for all $0\le t_1,t_2,t_3$. Then, by (4.6), we have

$$\lim_{i\to\infty}\|S(t_0,t_i)\|_{L^1(M\setminus\Sigma,\omega)}\to+\infty, \tag{4.16}$$

and

$$\begin{aligned}\|S(t_0,t)\|_{L^\infty(M\setminus\Sigma)}&\le r\|S(1,t)\|_{L^\infty(M\setminus\Sigma)}+r\|S(t_0,1)\|_{L^\infty(M\setminus\Sigma)}\\&\le r^2C_3(\|S(t_0,t)\|_{L^1}+\|S(t_0,1)\|_{L^1})+r\|S(t_0,1)\|_{L^\infty(M\setminus\Sigma)}+rC_4\end{aligned} \tag{4.17}$$

for all $0<t_0\le t$, where C_3 and C_4 are uniform constants depending only on r, C_K and C_G.

Set $u_i(t_0)=\|S(t_0,t_i)\|^{-1}_{L^1}S(t_0,t_i)\in S_{H(t_0)}(\mathcal{E}|_{M\setminus\Sigma})$, where $S_{H(t_0)}(\mathcal{E}|_{M\setminus\Sigma})=\{\eta\in\Omega^0(M\setminus\Sigma,\mathrm{End}(\mathcal{E}|_{M\setminus\Sigma}))|\quad \eta^{*H(t_0)}=\eta\}$, then $\|u_i(t_0)\|_{L^1}=1$. By (2.15) and (4.5), we have

$$\int_{M\setminus\Sigma}\mathrm{tr}\, S(t_0,t_i)\frac{\omega^n}{n!}=0, \tag{4.18}$$

so

$$\int_{M\setminus\Sigma}\mathrm{tr}\, u_i(t_0)\frac{\omega^n}{n!}=0. \tag{4.19}$$

By the inequalities (4.13), (4.14), (4.17), and the Lemma 5.4 in [32], we can see that, by choosing a subsequence which we also denote by $u_i(t_0)$, we have $u_i(t_0)\to u_\infty(t_0)$ weakly in L^2_1, where the limit $u_\infty(t_0)$ satisfies: $\|u_\infty(t_0)\|_{L^1}=1$, $\int_M\mathrm{tr}\,(u_\infty(t_0))\frac{\omega^n}{n!}=0$ and

$$\|u_\infty(t_0)\|_{L^\infty}\le r^2C_3. \tag{4.20}$$

Furthermore, if $\Upsilon : R\times R \to R$ is a positive smooth function such that $\Upsilon(\lambda_1,\lambda_2) < (\lambda_1-\lambda_2)^{-1}$ whenever $\lambda_1>\lambda_2$, then

$$\begin{aligned}&\int_{M\setminus\Sigma}\operatorname{tr}(u_\infty(t_0)\sqrt{-1}\Lambda_\omega(F_{H(t_0),\phi}))+\langle\Upsilon(u_\infty(t_0))(\overline{\partial}_\phi u_\infty(t_0)),\overline{\partial}_\phi u_\infty(t_0)\rangle_{H(t_0)}\frac{\omega^n}{n!}\\&\le -r^{-\frac{1}{2}}\frac{C^*}{\hat{C}_1}.\end{aligned}\tag{4.21}$$

Since $\|u_\infty(t_0)\|_{L^\infty}$ and $\|\Lambda_\omega(F_{H(t_0),\phi})\|_{L^1}$ are uniformly bounded (independent of t_0), (4.21) implies that: there exists a uniform constant $\check{C}$ independent of t_0 such that

$$\int_{M\setminus\Sigma}|\overline{\partial}_\phi u_\infty(t_0)|^2_{H(t_0)}\frac{\omega^n}{n!}\le\check{C}.\tag{4.22}$$

From Lemma 2.2, we see that $\hat{H}$ and $H(t_0)$ are locally mutually bounded each other. By choosing a subsequence, we have $u_\infty(t_0)\to u_\infty$ weakly in local L^2_1 outside Σ as $t_0\to 0$, where u_∞ satisfies

$$\int_M\operatorname{tr}(u_\infty)\frac{\omega^n}{n!}=0,\quad \textit{and}\quad \|u_\infty\|_{L^1}=1.\tag{4.23}$$

Since $|\sqrt{-1}\Lambda_{\omega_\epsilon}F_{H_\epsilon(t),\phi}|_{H_\epsilon(t)}\in L^\infty$ for $t>0$, by the uniform upper bound of the heat kernels (2.1), we have

$$\begin{aligned}&\int_{B_{\omega_1}(\delta)\setminus\Sigma}|\sqrt{-1}\Lambda_\omega F_{H(t),\phi}|_{H(t)}\frac{\omega^n}{n!}\\
=&\lim_{\epsilon\to0}\int_{B_{\omega_1}(\delta)}|\sqrt{-1}\Lambda_{\omega_\epsilon}F_{H_\epsilon(t),\phi}|_{H_\epsilon(t)}\frac{\omega_\epsilon^n}{n!}\\
\le&\lim_{\epsilon\to0}\int_{B_{\omega_1}(\delta)}\int_{\tilde{M}}K_\epsilon(x,y,t)|\sqrt{-1}\Lambda_{\omega_\epsilon}F_{\hat{H},\phi}|_{\hat{H}}(y)\frac{\omega_\epsilon^n(y)}{n!}\cdot\frac{\omega_\epsilon^n(x)}{n!}\\
=&\lim_{\epsilon\to0}\int_{B_{\omega_1}(\delta)}\Big(\big(\int_{B_{\omega_1}(2\delta)}+\int_{\tilde{M}\setminus B_{\omega_1}(2\delta)}\big)K_\epsilon(x,y,t)|\sqrt{-1}\Lambda_{\omega_\epsilon}F_{\hat{H},\phi}|_{\hat{H}}(y)\frac{\omega_\epsilon^n(y)}{n!}\Big)\frac{\omega_\epsilon^n(x)}{n!}\\
\le&\lim_{\epsilon\to0}\int_{\tilde{M}}\int_{B_{\omega_1}(2\delta)}K_\epsilon(x,y,t)|\sqrt{-1}\Lambda_{\omega_\epsilon}F_{\hat{H},\phi}|_{\hat{H}}(y)\frac{\omega_\epsilon^n(y)}{n!}\cdot\frac{\omega_\epsilon^n(x)}{n!}\\
&+\int_{B_{\omega_1}(\delta)}\Big(\int_{\tilde{M}\setminus B_{\omega_1}(2\delta)}C_K(\tau)t^{-n}\exp\big(-\frac{d_{\omega_\epsilon}(x,y)}{(4+\tau)t}\big)|\sqrt{-1}\Lambda_{\omega_\epsilon}F_{\hat{H},\phi}|_{\hat{H}}(y)\frac{\omega_\epsilon^n(y)}{n!}\Big)\frac{\omega_\epsilon^n(x)}{n!}\\
\le&\int_{B_{\omega_1}(2\delta)\setminus\Sigma}|\sqrt{-1}\Lambda_\omega F_{\hat{H},\phi}|_{\hat{H}}\frac{\omega^n}{n!}\\
&+C_K(\tau)t^{-n}\exp\big(-\frac{a(\delta)}{(4+\tau)t}\big)\mathrm{Vol}_{\omega_1}(B_{\omega_1}(\delta))\int_M|\sqrt{-1}\Lambda_\omega F_{\hat{H},\phi}|_{\hat{H}}\frac{\omega^n}{n!}.\end{aligned}\tag{4.24}$$

By (4.24) and the uniform bound of $\|u_\infty(t_0)\|_{L^\infty}$, we have

$$\lim_{t_0\to0}\int_M\operatorname{tr}(u_\infty(t_0)\sqrt{-1}\Lambda_\omega F_{H(t_0),\phi})\frac{\omega^n}{n!}=\int_M\operatorname{tr}(u_\infty\sqrt{-1}\Lambda_\omega F_{\hat{H},\phi})\frac{\omega^n}{n!}.\tag{4.25}$$

Let's denote

$$S_{\hat{H}}(\mathcal{E}|_{M\setminus\Sigma})=\{\eta\in\Omega^0(M\setminus\Sigma,\mathrm{End}(\mathcal{E}|_{M\setminus\Sigma}))|\quad \eta^{*\hat{H}}=\eta\}.\tag{4.26}$$

and

$$\hat{u}_\infty(t_0)=(h(t_0))^{\frac{1}{2}}\cdot u_\infty(t_0)\cdot(h(t_0))^{-\frac{1}{2}}.\tag{4.27}$$

It is easy to check that: $\hat{u}_\infty(t_0) \in S_{\hat{H}}(\mathcal{E}|_{M\setminus\Sigma})$ and $|\hat{u}_\infty(t_0)|_{\hat{H}} = |u_\infty(t_0)|_{H(t_0)}$. Furthermore, we have:

Lemma 4.2. *For any compact domain $\Omega \subset M \setminus \Sigma$ and any positive smooth function $\Upsilon : R \times R \to R$, we have*

$$\lim_{t_0\to 0}\int_\Omega |\langle \Upsilon(u_\infty(t_0))(\overline{\partial}_\phi u_\infty(t_0)), \overline{\partial}_\phi u_\infty(t_0)\rangle_{H(t_0)} - \langle \Upsilon(\hat{u}_\infty(t_0))(\overline{\partial}_\phi \hat{u}_\infty(t_0)), \overline{\partial}_\phi \hat{u}_\infty(t_0)\rangle_{\hat{H}}|\frac{\omega^n}{n!} = 0. \tag{4.28}$$

Proof. At each point x on Ω, we choose a unitary basis $\{e_i\}_{i=1}^r$ with respect to the metric $H(t_0)$, such that $u_\infty(t_0)(e_i) = \lambda_i e_i$. Then, $\{\hat{e}_i = (h(t_0))^{\frac{1}{2}} e_i\}$ is a unitary basis with respect to the metric $\hat{H}$ and $\hat{u}_\infty(t_0)(\hat{e}_i) = \lambda_i \hat{e}_i$. Set:

$$\overline{\partial}_\phi u_\infty(t_0)(e_i) = (\overline{\partial}_\phi u_\infty(t_0))_i^j e_j, \quad \overline{\partial}_\phi \hat{u}_\infty(t_0)(\hat{e}_i) = (\overline{\partial}_\phi \hat{u}_\infty(t_0))_i^j \hat{e}_j, \tag{4.29}$$

then

$$|\overline{\partial}_\phi u_\infty(t_0)|^2_{H(t_0),\omega} = \sum_{i,j=1}^r \langle (\overline{\partial}_\phi u_\infty(t_0))_i^j, (\overline{\partial}_\phi u_\infty(t_0))_i^j\rangle_\omega, \tag{4.30}$$

$$\langle \Upsilon(u_\infty(t_0))(\overline{\partial}_\phi u_\infty(t_0)), \overline{\partial}_\phi u_\infty(t_0)\rangle_{H(t_0)} = \sum_{i,j=1}^r \langle \Upsilon(\lambda_i,\lambda_j)(\overline{\partial}_\phi u_\infty(t_0))_i^j, (\overline{\partial}_\phi u_\infty(t_0))_i^j\rangle_\omega, \tag{4.31}$$

$$\Upsilon(\hat{u}_\infty(t_0))(\overline{\partial}_\phi \hat{u}_\infty(t_0))(\hat{e}_i) = \sum_{j=1}^r \Upsilon(\lambda_i,\lambda_j)(\overline{\partial}_\phi \hat{u}_\infty(t_0))_i^j \hat{e}_j, \tag{4.32}$$

and

$$\langle \Upsilon(\hat{u}_\infty(t_0))(\overline{\partial}_\phi \hat{u}_\infty(t_0)), \overline{\partial}_\phi \hat{u}_\infty(t_0)\rangle_{\hat{H}} = \sum_{i,j=1}^r \langle \Upsilon(\lambda_i,\lambda_j)(\overline{\partial}_\phi \hat{u}_\infty(t_0))_i^j, (\overline{\partial}_\phi \hat{u}_\infty(t_0))_i^j\rangle_\omega. \tag{4.33}$$

By the definition, we have

$$\begin{aligned}\overline{\partial}_\phi \hat{u}_\infty(t_0) &= (h(t_0))^{\frac{1}{2}} \circ \overline{\partial}_\phi u_\infty(t_0) \circ (h(t_0))^{-\frac{1}{2}} + \overline{\partial}_\phi (h(t_0))^{\frac{1}{2}} \circ u_\infty(t_0) \circ (h(t_0))^{-\frac{1}{2}} \\ &\quad - (h(t_0))^{\frac{1}{2}} \circ u_\infty(t_0) \circ (h(t_0))^{-\frac{1}{2}} \circ \overline{\partial}_\phi (h(t_0))^{\frac{1}{2}} \circ (h(t_0))^{-\frac{1}{2}} \\ &= (h(t_0))^{\frac{1}{2}} \circ \overline{\partial}_\phi u_\infty(t_0) \circ (h(t_0))^{-\frac{1}{2}} + \overline{\partial}_\phi (h(t_0))^{\frac{1}{2}} \circ (h(t_0))^{-\frac{1}{2}} \hat{u}_\infty(t_0) \\ &\quad - \hat{u}_\infty(t_0) \circ \overline{\partial}_\phi (h(t_0))^{\frac{1}{2}} \circ (h(t_0))^{-\frac{1}{2}},\end{aligned} \tag{4.34}$$

and

$$(\overline{\partial}_\phi \hat{u}_\infty(t_0))_i^j = (\overline{\partial}_\phi u_\infty(t_0))_i^j + (\lambda_i - \lambda_j)\{\overline{\partial}_\phi (h(t_0)^{\frac{1}{2}} \circ (h(t_0))^{-\frac{1}{2}}\}_i^j, \tag{4.35}$$

where $\overline{\partial}_\phi (h(t_0)^{\frac{1}{2}} \circ (h(t_0)^{-\frac{1}{2}})(\hat{e}_i) = (\overline{\partial}_\phi (h(t_0)^{\frac{1}{2}} \circ (h(t_0)^{-\frac{1}{2}})_i^j \hat{e}_j$. By (4.20), (4.31), (4.33) and (4.35), we have

$$\begin{aligned}&|\langle \Upsilon(\hat{u}_\infty(t_0))(\overline{\partial}_\phi \hat{u}_\infty(t_0)), \overline{\partial}_\phi \hat{u}_\infty(t_0)\rangle_{\hat{H}} - \langle \Upsilon(u_\infty(t_0))(\overline{\partial}_\phi u_\infty(t_0)), \overline{\partial}_\phi u_\infty(t_0)\rangle_{H(t_0)}| \\ &\le 8(r^2C_3)^2(B^*(\Upsilon))(|\overline{\partial}_\phi u_\infty(t_0)|_{H(t_0)}|\overline{\partial}_\phi (h(t_0)^{\frac{1}{2}} \circ (h(t_0))^{-\frac{1}{2}}|_{\hat{H}} + |\overline{\partial}_\phi (h(t_0)^{\frac{1}{2}} \circ (h(t_0))^{-\frac{1}{2}}|^2_{\hat{H}}),\end{aligned} \tag{4.36}$$

where $B^*(\Upsilon) = \max_{[-r^2C_3, r^2C_3]^2} \Upsilon$. Since $H(t)$ are smooth on $M \setminus \Sigma \times [0,1]$ and $h(t) \to \mathrm{Id}_\mathcal{E}$ locally in C^∞-topology as $t \to 0$, it is easy to check that

$$\sup_{x\in\Omega}(|(h(t_0))^{-\frac{1}{2}}\overline{\partial}_\phi (h(t_0))^{\frac{1}{2}}|_{\hat{H},\omega} + |\overline{\partial}_\phi (h(t_0))^{\frac{1}{2}}(h(t_0))^{-\frac{1}{2}}|_{\hat{H},\omega}) \le C_\Omega(t_0), \tag{4.37}$$

where $C_\Omega(t_0) \to 0$ as $t_0 \to 0$. On the other hand, $|\overline{\partial}_\phi u_\infty(t_0)|_{H(t_0),\omega}$ are uniform bounded in L^2, so (4.36) and (4.37) imply (4.28).

□

By (4.21), (4.25) and (4.28), we have that given any compact domain $\Omega \subset M \setminus \Sigma$ and any positive number $\tilde{\epsilon} > 0$,

$$\int_{M\setminus\Sigma} \mathrm{tr}\,(u_\infty\sqrt{-1}\Lambda_\omega F_{\hat{H}},\phi)\frac{\omega^n}{n!} + \int_\Omega \langle\Upsilon(\hat{u}_\infty(t_0))(\overline{\partial}_\phi \hat{u}_\infty(t_0)), \overline{\partial}_\phi \hat{u}_\infty(t_0)\rangle_{\hat{H}}\frac{\omega^n}{n!} \leq -r^{-\frac{1}{2}}\frac{C^*}{\hat{C}_1} + \tilde{\epsilon} \tag{4.38}$$

for small t_0. As we know that $\hat{u}_\infty(t_0) \to u_\infty$ in $L^2(\Omega)$, $|\hat{u}_\infty(t_0)|_{\hat{H}}$ is uniformly bounded in L^∞ and $|\overline{\partial}_\phi \hat{u}_\infty(t_0)|_{\hat{H},\omega}$ is uniformly bounded in $L^2(\Omega)$. By the same argument as that in Simpson's paper (Lemma 5.4 in [32]), we have

$$\int_{M\setminus\Sigma} \mathrm{tr}\,(u_\infty\sqrt{-1}\Lambda_\omega F_{\hat{H},\phi})\frac{\omega^n}{n!} + \|\Upsilon^{\frac{1}{2}}(u_\infty)(\overline{\partial}_\phi u_\infty)\|^2_{L^q(\Omega)} \leq -r^{-\frac{1}{2}}\frac{C^*}{\hat{C}_1} + 2\tilde{\epsilon} \tag{4.39}$$

for any $q < 2$ and any $\tilde{\epsilon}$. Since $\tilde{\epsilon}$, $q < 2$ and Ω are arbitrary, we get

$$\int_{M\setminus\Sigma} \mathrm{tr}\,(u_\infty\sqrt{-1}\Lambda_\omega F_{\hat{H},\phi}) + \langle\Upsilon(u_\infty)(\overline{\partial}_\phi u_\infty), \overline{\partial}_\phi u_\infty\rangle_{\hat{H}}\frac{\omega^n}{n!} \leq -r^{-\frac{1}{2}}\frac{C^*}{\hat{C}_1}. \tag{4.40}$$

By the above inequality and the Lemma 5.5 in [32], we can see that the eigenvalues of u_∞ are constant almost everywhere. Let $\lambda_1 < \cdots < \lambda_l$ denote the distinct eigenvalue of u_∞. Since $\int_M \mathrm{tr}\, u_\infty \frac{\omega^n}{n!} = 0$ and $\|u_\infty\|_{L^1} = 1$, we must have $l \geq 2$. For any $1 \leq \alpha < l$, define function $P_\alpha : R \to R$ such that

$$P_\alpha = \begin{cases} 1, & x \leq \lambda_\alpha, \\ 0, & x \geq \lambda_{\alpha+1}. \end{cases} \tag{4.41}$$

Set $\pi_\alpha = P_\alpha(u_\infty)$, Simpson (p887 in [32]) proved that:

(1) $\pi_\alpha \in L_1^2(M \setminus \Sigma, \omega, \hat{H})$;
(2) $\pi_\alpha^2 = \pi_\alpha = \pi_\alpha^{*\hat{H}}$;
(3) $(\mathrm{Id}_\mathcal{E} - \pi_\alpha)\bar{\partial}\pi_\alpha = 0$;
(4) $(\mathrm{Id}_\mathcal{E} - \pi_\alpha)[\phi, \pi_\alpha] = 0$.

By Uhlenbeck and Yau's regularity statement of L_1^2-subbundle ([35]), π_α represent a saturated coherent Higgs sub-sheaf E_α of $(\mathcal{E}, \phi)$ on the open set $M \setminus \Sigma$. Since the singularity set Σ is co-dimension at least 3, by Siu's extension theorem ([34]), we know that E_α admits a coherent analytic extension $\hat{E}_\alpha$. By Serre's result ([30]), we get the direct image i_*E_α under the inclusion $i : M \setminus \Sigma \to M$ is coherent. So, every E_α can be extended to the whole M as a saturated coherent Higgs sub-sheaf of $(\mathcal{E}, \phi)$, which will also be denoted by E_α for simplicity. By the Chern-Weil formula (1.13) (Proposition 4.1 in [10]) and the above condition (4), we have

$$\begin{aligned} \deg_\omega(E_\alpha) &= \int_{M\setminus\Sigma} \mathrm{tr}\,(\pi_\alpha\sqrt{-1}\Lambda_\omega F_{\hat{H}}) - |\overline{\partial}\pi_\alpha|^2_{\hat{H},\omega}\frac{\omega^n}{n!} \\ &= \int_{M\setminus\Sigma} \mathrm{tr}\,(\pi_\alpha\sqrt{-1}\Lambda_\omega F_{\hat{H},\phi}) - |D''_\phi\pi_\alpha|^2_{K,\omega}\frac{\omega^n}{n!}. \end{aligned} \tag{4.42}$$

Set

$$\nu = \lambda_l \deg_\omega(\mathcal{E}) - \sum_{\alpha=1}^{l-1}(\lambda_{\alpha+1} - \lambda_\alpha)\deg_\omega(E_\alpha). \tag{4.43}$$

Since $u_\infty = \lambda_l \mathrm{Id}_{\mathcal{E}} - \sum_{\alpha=1}^{l-1}(\lambda_{\alpha+1}-\lambda_\alpha)\pi_\alpha$ and $\int_{M\setminus\Sigma} \mathrm{tr}\, u_\infty \frac{\omega^n}{n!} = 0$, we have

$$\lambda_l \mathrm{rank}(\mathcal{E}) - \sum_{\alpha=1}^{l-1}(\lambda_{\alpha+1}-\lambda_\alpha)\mathrm{rank}(E_\alpha) = 0, \tag{4.44}$$

then

$$\nu = \sum_{\alpha=1}^{l-1}(\lambda_{\alpha+1}-\lambda_\alpha)\mathrm{rank}(E_\alpha)\left(\frac{\deg_\omega(\mathcal{E})}{\mathrm{rank}(\mathcal{E})} - \frac{\deg_\omega(E_\alpha)}{\mathrm{rank}(E_\alpha)}\right). \tag{4.45}$$

By the argument similar to the one used in Simpson's paper (P888 in [32]) and the inequality (4.40), we have

$$\begin{aligned} \nu =& \int_M \mathrm{tr}\,(u_\infty \sqrt{-1}\Lambda_\omega F_{\hat{H},\phi}) \\ &+ \langle \sum_{\alpha=1}^{l-1}(\lambda_{\alpha+1}-\lambda_\alpha)(dP_\alpha)^2(u_\infty)(D''_\phi u_\infty), D''_\phi u_\infty\rangle_{\hat{H}} \\ \leq& - r^{-\frac{1}{2}}\frac{C^*}{\hat{C}_1}. \end{aligned} \tag{4.46}$$

On the other hand, (4.45) and the semi-stability imply $\nu \geq 0$, so we get a contradiction.

□

Proof of Theorem 1.1 By (2.12), we have

$$\sup_{x\in M\setminus\Sigma} |\sqrt{-1}\Lambda_\omega(F_{H(t+1),\phi}) - \lambda \mathrm{Id}_{\mathcal{E}}|^2_{H(t+1)}(x) \leq C_K \int_{M\setminus\Sigma} |\sqrt{-1}\Lambda_\omega(F_{H(t),\phi}) - \lambda\mathrm{Id}_{\mathcal{E}}|^2_{H(t)}\frac{\omega^n}{n!}. \tag{4.47}$$

If the reflexive Higgs sheaf $(\mathcal{E},\phi)$ is ω-semi-stable, (4.10) implies

$$\sup_{x\in M\setminus\Sigma} |\sqrt{-1}\Lambda_\omega(F_{H(t),\phi}) - \lambda\mathrm{Id}_{\mathcal{E}}|^2_{H(t+1)}(x) \to 0, \tag{4.48}$$

as $t\to+\infty$. By corollary 3.5, we know that every $H(t)$ is an admissible Hermitian metric. Then we get an approximate Hermitian-Einstein structure on a semi-stable reflexive Higgs sheaf.

By choosing a subsequence $\epsilon \to 0$, we have $H_\epsilon(t)$ converge to $H(t)$ in local C^∞-topology. Applying Fatou's lemma we obtain

$$\begin{aligned} &4\pi^2 \int_M (2c_2(\mathcal{E}) - \frac{r-1}{r}c_1(\mathcal{E})\wedge c_1(\mathcal{E}))\wedge \frac{\omega^{n-2}}{(n-2)!} \\ =& \lim_{\epsilon\to 0} 4\pi^2 \int_{\tilde{M}} (2c_2(E) - \frac{r-1}{r}c_1(E)\wedge c_1(E))\wedge\frac{\omega_\epsilon^{n-2}}{(n-2)!} \\ =& \lim_{\epsilon\to 0}\int_{\tilde{M}} \mathrm{tr}\,(F^\perp_{H_\epsilon(t),\phi}\wedge F^\perp_{H_\epsilon(t),\phi})\wedge\frac{\omega_\epsilon^{n-2}}{(n-2)!} \\ =& \lim_{\epsilon\to 0}\int_{\tilde{M}} |F^\perp_{H_\epsilon(t),\phi}|^2_{H_\epsilon(t),\omega_\epsilon} - |\Lambda_{\omega_\epsilon}F^\perp_{H_\epsilon(t),\phi}|^2_{H_\epsilon(t)}\frac{\omega_\epsilon^n}{n!} \\ \geq& \int_{M\setminus\Sigma} |F^\perp_{H(t),\phi}|^2_{H(t),\omega}\frac{\omega^n}{n!} \\ &- \int_{M\setminus\Sigma} |\sqrt{-1}\Lambda_\omega F_{H(t),\phi} - \lambda\mathrm{Id}_{\mathcal{E}} - \frac{1}{r}\mathrm{tr}\,(\sqrt{-1}\Lambda_\omega F_{H(t),\phi} - \lambda\mathrm{Id}_{\mathcal{E}})\mathrm{Id}_{\mathcal{E}}|^2_{H(t)}\frac{\omega^n}{n!} \end{aligned} \tag{4.49}$$

for $t > 0$, where $F_{H,\phi}^{\perp}$ is the trace free part of $F_{H,\phi}$. Let $t \to +\infty$, then (4.10) implies the following Bogomolov type inequality

$$\int_M (2c_2(\mathcal{E}) - \frac{r-1}{r}c_1(\mathcal{E}) \wedge c_1(\mathcal{E})) \wedge \frac{\omega^{n-2}}{(n-2)!} \geq 0. \tag{4.50}$$

Now we prove that the existence of an approximate Hermitian-Einstein structure implies the semistability of $(\mathcal{E}, \phi)$. Let s be a θ-invariant holomorphic section of a reflexive Higgs sheaf $(\mathcal{G}, \theta)$ on a compact Kähler manifold (M, ω), i.e. there exists a holomorphic 1-form η on $M \setminus \Sigma_{\mathcal{G}}$ such that $\theta(s) = \eta \otimes s$, where $\Sigma_{\mathcal{G}}$ is the singularity set of $\mathcal{G}$. Given a Hermitian metric H on $\mathcal{G}$, by computing, we have

$$\begin{aligned} &\sqrt{-1}\Lambda_\omega \langle s, -[\theta, \theta^{*H}]s\rangle_H \\ = &-\sqrt{-1}\Lambda_\omega \langle \theta^{*H}s, \theta^{*H}s\rangle_H - \sqrt{-1}\Lambda_\omega\langle \theta s, \theta s\rangle_H \\ = &-\sqrt{-1}\Lambda_\omega \langle \theta^{*H}s - \langle\theta^{*H}s, s\rangle_H \frac{s}{|s|_H^2}, \theta^{*H}s - \langle \theta^{*H}s, s\rangle_H \frac{s}{|s|_H^2}\rangle_H \\ &-\sqrt{-1}\Lambda_\omega \langle\langle \theta^{*H}s, s\rangle_H \frac{s}{|s|_H^2}, \langle\theta^{*H}s, s\rangle_H \frac{s}{|s|_H^2}\rangle_H - \sqrt{-1}\Lambda_\omega\langle \phi s, \phi s\rangle_H \\ = &|\theta^{*H}s - \langle \theta^{*H}s, s\rangle_H \frac{s}{|s|_H^2}|_{H,\omega}^2 \geq 0, \end{aligned} \tag{4.51}$$

where we have used $\theta(s) = \eta \otimes s$ in the third equality. Then, we have the following Weitzenböck formula

$$\begin{aligned} \frac{1}{2}\Delta_\omega |s|_H^2 &= \sqrt{-1}\Lambda_\omega \partial\bar{\partial}|s|_H^2 \\ &= |D_H^{1,0}s|_{H,\omega}^2 + \sqrt{-1}\Lambda_\omega \langle s, F_H s\rangle_H \\ &= |D_H^{1,0}s|_{H,\omega}^2 - \langle s, \sqrt{-1}\Lambda_\omega F_{H,\theta}s\rangle_H - \sqrt{-1}\Lambda_\omega\langle s, [\theta, \theta^{*H}]s\rangle_H \\ &\geq |D_H^{1,0}s|_{H,\omega}^2 - \langle s, \sqrt{-1}\Lambda_\omega F_{H,\theta}s\rangle_H \end{aligned} \tag{4.52}$$

on $M \setminus \Sigma_{\mathcal{G}}$.

We suppose that the reflexive Higgs sheaf $(\mathcal{G}, \theta)$ admits an approximate admissible Hermitian-Einstein structure, i.e. for every positive δ, there is an admissible Hermitian metric H_δ such that

$$\sup_{x \in M \setminus \Sigma_{\mathcal{G}}} |\sqrt{-1}\Lambda_\omega F_{H_\delta, \theta} - \lambda(\mathcal{G})\mathrm{Id}|_{H_\delta}(x) < \delta. \tag{4.53}$$

If $\deg_\omega \mathcal{G}$ is negative, i.e. $\lambda(\mathcal{G}) < 0$, by choosing δ small enough, we have

$$\Delta_\omega |s|_{H_\delta}^2 \geq 2|D_H^{1,0}s|_{H_\delta,\omega}^2 - \lambda(\mathcal{G})|s|_{H_\delta}^2 \tag{4.54}$$

on $M \setminus \Sigma_{\mathcal{G}}$. Since every H_δ is admissible, by Theorem 2 in [6], we know that $|s|_{H_\delta} \in L^\infty(M)$. Then, the inequality (4.54) can be extended globally to the compact manifold M. So, we must have

$$s \equiv 0. \tag{4.55}$$

Assume that $(\mathcal{E}, \phi)$ admits an approximate Hermitian-Einstein structure and $\mathcal{F}$ is a saturated Higgs subsheaf of $(\mathcal{E}, \phi)$ with rank p. Let $\mathcal{G} = \wedge^p \mathcal{E} \otimes \det(\mathcal{F})^{-1}$, and θ be a Higgs filed naturally induced on $\mathcal{G}$ by the Higgs field ϕ. One can check that $(\mathcal{G}, \theta)$ is also a reflexive Higgs sheaf which admits an approximate Hermitian-Einstein structure with constant

$$\lambda(\mathcal{G}) = \frac{2p\pi}{\mathrm{Vol}(M, \omega)}(\mu_\omega(\mathcal{E}) - \mu_\omega(\mathcal{F})). \tag{4.56}$$

The inclusion $\mathcal{F} \hookrightarrow \mathcal{E}$ induces a morphism $\det(\mathcal{F}) \to \wedge^p \mathcal{E}$ which can be seen as a nontrivial θ-invariant holomorphic section of $\mathcal{G}$. From above, we have $\lambda(\mathcal{G}) \geq 0$, so the reflexive sheaf $(\mathcal{E}, \phi)$ is ω-semistable. This completes the proof of Theorem 1.1.

□

5. Limit of ω_ϵ-Hermitian-Einstein metrics

Assume that the reflexive Higgs sheaf $(\mathcal{E}, \phi)$ is ω-stable. It is well known that the pulling back Higgs bundle (E, ϕ) is ω_ϵ-stable for sufficiently small ϵ. By Simpson's result ([32]), there exists an ω_ϵ-Hermitian-Einstein metric H_ϵ for every sufficiently small ϵ. In this section, we prove that, by choosing a subsequence and rescaling it, H_ϵ converges to an ω-Hermitian-Einstein metric H in local C^∞-topology outside the exceptional divisor Σ.

As above, let $\hat{H}$ be a fixed smooth Hermitian metric on the bundle E over $\tilde{M}$. By taking a constant on H_ϵ, we can suppose that

$$\int_{\tilde{M}} \operatorname{tr} \hat{S}_\epsilon \frac{\omega_\epsilon^n}{n!} = \int_{\tilde{M}} \log\det(\hat{h}_\epsilon) \frac{\omega_\epsilon^n}{n!} = 0. \tag{5.1}$$

where $\exp(\hat{S}_\epsilon) = \hat{h}_\epsilon = \hat{H}^{-1} H_\epsilon$.

Let $H_\epsilon(t)$ be the long time solutions of the heat flow (1.10) on the Higgs bundle (E, ϕ) with the fixed initial metric $\hat{H}$ and with respect to the Kähler metric ω_ϵ. We set:

$$\exp(\tilde{S}_\epsilon(t)) = \tilde{h}_\epsilon(t) = H_\epsilon(t)^{-1} H_\epsilon. \tag{5.2}$$

By (2.15), (5.1) and noting that $\exp(\hat{S}_\epsilon) = \exp(S_\epsilon(t)) \exp(\tilde{S}_\epsilon(t))$, we have

$$\int_{\tilde{M}} \operatorname{tr} \tilde{S}_\epsilon(t) \frac{\omega_\epsilon^n}{n!} = \int_{\tilde{M}} \log\det(\tilde{h}_\epsilon(t)) \frac{\omega_\epsilon^n}{n!} = 0 \tag{5.3}$$

for all $t \geq 0$. We first give a uniform L^1 estimate of $\hat{S}_\epsilon$.

Lemma 5.1. *There exists a constant $\hat{C}$ which is independent of ϵ, such that*

$$\|\hat{S}_\epsilon\|_{L^1(\tilde{M}, \omega_\epsilon, \hat{H})} := \int_{\tilde{M}} |\hat{S}_\epsilon|_{\hat{H}} \frac{\omega_\epsilon^n}{n!} \leq \hat{C} \tag{5.4}$$

for all $0 < \epsilon \leq 1$.

Proof. We prove (5.4) by contradiction. If not, there exists a subsequence $\epsilon_i \to 0$ such that

$$\lim_{i \to \infty} \|\hat{S}_{\epsilon_i}\|_{L^1(\tilde{M}, \omega_{\epsilon_i}, \hat{H})} \to \infty. \tag{5.5}$$

By (2.26), (2.27) and (4.15), we also have

$$\lim_{i \to \infty} \|\tilde{S}_{\epsilon_i}(t)\|_{L^1(\tilde{M}, \omega_{\epsilon_i}, H_{\epsilon_i}(t))} \to \infty, \tag{5.6}$$

for all $t > 0$. By (4.4), the uniform lower bound of Green functions G_ϵ (2.11) and the inequalities (2.26), we have

$$\|\tilde{S}_\epsilon(1)\|_{L^\infty(\tilde{M}, H_\epsilon(1))} \leq \grave{C}_1 \|\tilde{S}_\epsilon(1)\|_{L^1(\tilde{M}, \omega_\epsilon, H_\epsilon(1))} + \grave{C}_2, \tag{5.7}$$

where $\grave{C}_1$ and $\grave{C}_2$ are uniform constants independent of ϵ and t. Using the inequality (4.15) again, we have

$$\begin{aligned} \|\tilde{S}_\epsilon(t)\|_{L^\infty(\tilde{M}, H_\epsilon(t))} \leq\ & r^2 \grave{C}_1 (\|\tilde{S}_\epsilon(t)\|_{L^1(\tilde{M}, \omega_\epsilon, H_\epsilon(t))} + \|S_\epsilon(t,1)\|_{L^1(\tilde{M}, \omega_\epsilon, H_\epsilon(1))}) \\ & + r \|S_\epsilon(t,1)\|_{L^\infty(\tilde{M}, H_\epsilon(1))} + r \grave{C}_2 \end{aligned} \tag{5.8}$$

for all $t>0$.

Set $\tilde{u}_i(t)=\|\tilde{S}_{\epsilon_i}(t)\|^{-1}_{L^1(\tilde{M},\omega_{\epsilon_i},H_{\epsilon_i}(t))}\tilde{S}_{\epsilon_i}(t)$, then $\|\tilde{u}_i(t)\|_{L^1(\tilde{M},\omega_\epsilon,H_\epsilon(t))}=1$. By (5.3) and (5.8), we have $\int_{\tilde{M}}\operatorname{tr}u_i(t)\frac{\omega_\epsilon^n}{n!}=0$ and $\|\tilde{u}_i(t)\|_{L^\infty(\tilde{M},H_{\epsilon_i}(t))}\leq C(t)$. Since $H_\epsilon(t)\to H(t)$ locally in C^∞-topology and ω_ϵ are locally uniform bounded outside $\tilde{\Sigma}$, by the Lemma 5.4 in [32], we can show that, by choosing a subsequence which we also denote by $\tilde{u}_i(t)$, we have $\tilde{u}_i(t)\to\tilde{u}(t)$ weakly in $L^2_{1,loc}(\tilde{M}\setminus\tilde{\Sigma},\omega,H(t))$, where the limit $\tilde{u}(t)$ satisfies: $\|\tilde{u}(t)\|_{L^1(\tilde{M}\setminus\tilde{\Sigma},\omega,H(t))}=1$, $\int_{\tilde{M}\setminus\tilde{\Sigma}}\operatorname{tr}(\tilde{u}(t))\frac{\omega^n}{n!}=0$. By (5.8), we have

$$\|\tilde{u}(t)\|_{L^\infty(\tilde{M}\setminus\tilde{\Sigma},\omega,H(t))}\leq r^2\grave{C}_1. \tag{5.9}$$

Furthermore, if $\Upsilon:R\times R\to R$ is a positive smooth function such that $\Upsilon(\lambda_1,\lambda_2)<(\lambda_1-\lambda_2)^{-1}$ whenever $\lambda_1>\lambda_2$, then

$$\begin{aligned}&\int_{\tilde{M}\setminus\tilde{\Sigma}}\operatorname{tr}(\tilde{u}(t)\sqrt{-1}\Lambda_\omega(F_{H(t),\phi}))+\langle\Upsilon(\tilde{u}(t))(\overline{\partial}_\phi\tilde{u}(t)),\overline{\partial}_\phi\tilde{u}(t)\rangle_{H(t)}\frac{\omega^n}{n!}\\&\leq 0.\end{aligned} \tag{5.10}$$

Since $M\setminus\Sigma$ is biholomorphic to $\tilde{M}\setminus\tilde{\Sigma}$, and $\mathcal{E}$ is locally free on $M\setminus\Sigma$, $\tilde{u}(t)$ can be seen as an L^2_1 section of $\operatorname{End}(\mathcal{E})$. By the same argument as that in section 4 (the proof of (4.40)), we can show that, by choosing a subsequence $t\to 0$, we have $\tilde{u}(t)\to\tilde{u}_0$ weakly in local L^2_1, where $\tilde{u}_0$ satisfies

$$\int_M\operatorname{tr}(\tilde{u}_0)\frac{\omega^n}{n!}=0,\quad\|\tilde{u}_0\|_{L^1(M\setminus\Sigma,\omega,\hat{H})}=1,\quad\|\tilde{u}(t)\|_{L^\infty(M\setminus\Sigma,\hat{H})}\leq r^2\grave{C}_1. \tag{5.11}$$

and

$$\int_{M\setminus\Sigma}\operatorname{tr}(\tilde{u}_0\sqrt{-1}\Lambda_\omega F_{\hat{H},\phi})+\langle\Upsilon(\tilde{u}_0)(\overline{\partial}_\phi\tilde{u}_0),\overline{\partial}_\phi\tilde{u}_0\rangle_{\hat{H}}\frac{\omega^n}{n!}\leq 0. \tag{5.12}$$

Now, by Simpson's trick (P888 in [32]), we can construct a saturated Higgs subsheaf $\mathcal{F}$ of $(\mathcal{E},\phi)$ with $\mu_\omega(\mathcal{F})\geq\mu_\omega(\mathcal{E})$, which contradicts with the stability of $(\mathcal{E},\phi)$.

□

Proof of Theorem 1.2 Since $\|\hat{S}_\epsilon\|_{L^1(\tilde{M},\omega_\epsilon,\hat{M})}$ are uniformly bounded, by (2.26), (2.27) and (4.15), there also exists a uniform constant $\grave{C}_3$ such that

$$\|\tilde{S}_\epsilon(1)\|_{L^1(\tilde{M},\omega_\epsilon,H_\epsilon(1))}\leq\grave{C}_3. \tag{5.13}$$

By (5.7), we have

$$\|\tilde{S}_\epsilon(1)\|_{L^\infty(\tilde{M},H_\epsilon(1))}\leq\grave{C}_1\grave{C}_3+\grave{C}_2 \tag{5.14}$$

for all $0<\epsilon\leq 1$. By the local estimate (2.29) in Lemma 2.3, we see that there exists a constant $\tilde{C}_0(\delta^{-1})$ independent of ϵ such that

$$|\hat{S}_\epsilon|_{\hat{H}}(x)\leq\tilde{C}_0(\delta^{-1}) \tag{5.15}$$

for all $x\in\tilde{M}\setminus B_{\omega_1}(\delta)$ and all $0<\epsilon\leq 1$. Since H_ϵ satisfies the ω_ϵ-Hermitian-Einstein equation (1.4), by the same argument as that in Lemmas 2.4 and 2.5 in section 2, we have uniform higher-order estimates for h_ϵ, i.e. there exist constants $\tilde{C}_k(\delta^{-1})$ independent of ϵ, such that

$$\|\hat{h}_\epsilon\|_{C^{k+1,\alpha},\tilde{M}\setminus B_{\omega_1}(2\delta)}\leq\tilde{C}_{k+1}(\delta^{-1}) \tag{5.16}$$

for all $k \geq 0$ and all $0 < \epsilon \leq 1$. So by choosing a subsequence, we have H_ϵ converges to a Hermitian metric H on $M \setminus \Sigma$ in locally C^∞-topology, and H satisfies the Hermitian-Einstein equation, i.e.

$$\sqrt{-1}\Lambda_\omega(F_H + [\phi, \phi^{*H}]) = \lambda \mathrm{Id}_{\mathcal{E}}. \tag{5.17}$$

By (5.14), we see that the metrics $H(1)$ and H are mutually bounded each other on $\mathcal{E}|_{M\setminus\Sigma}$. On the other hand, we have shown that $|\phi|_{H(1),\omega} \in L^\infty(M)$ in section 3, then $|\phi|_{H,\omega}$ also belongs to $L^\infty(M)$. This implies that $|\Lambda_\omega(F_H)|_H$ is uniform bounded on $M \setminus \Sigma$. By (3.30), it is easy to see that $|F_H|_{H,\omega}$ is square integrable. So we know that the metric H is an admissible Hermitian-Einstein metric on the Higgs sheaf $(\mathcal{E}, \phi)$. This completes the proof of Theorem 1.2. □

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