ON THE FLATNESS AND THE PROJECTIVITY OVER HOPF SUBALGEBRAS OF HOPF ALGEBRAS OVER DISCRETE VALUATION RINGS

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ABSTRACT. We study the flatness and the projectivity of Hopf algebras, defined over a Dedekind ring, over their Hopf subalgebras. We give a criterion for the faithful flatness and use it to show the faithful flatness of an arbitrary flat Hopf algebra upon its finite normal Hopf subalgebra. For the projectivity of a projective Hopf algebras we need some finiteness condition in terms of the module of integral. In particular we show the the module of integral has rank one.

1. INTRODUCTION

For Hopf algebras defined over a field, a conjecture of Kaplansky states that "a Hopf algebra is free as a module over any Hopf subalgebra". Although this was quickly shown to be false in the infinite dimensional case, the finite dimensional case is true and was proven by Nichols and Zoeller [10]. Schneider [18] showed that any Hopf algebra is free over its finite normal Hopf subalgebras.

This work is devoted to the study of the same questions for Hopf algebras defined over a DVR: when a Hopf algebra, defined over a DVR, is faithfully flat/projective over a Hopf subalgebra. There are already many works devoted to Hopf algebras defined over a ring base. For instance Schneider [16] generalized Nichols-Zoeller's result to Hopf algebras over a local ring. One of our aims is to generalize to the case of Hopf algebras over a DVR Schneider's result: a Hopf algebra is projective over a normal Hopf subalgebra. We haven't proved the complete generalization but still need an extra-condition on the finiteness of the Hopf algebra: the existence of integrals. We show that the module of integrals on a projective (over the base DVR) Hopf algebra is free of rank one. Further we show that a projective Hopf algebra possesses a (left) integral if and only if it is projective as a (right) comodule over itself. These results are used to proved the projectivity of a projective Hopf algebra over a finite normal Hopf subalgebra.

Our approach is to rely on existing results for Hopf algebra over field and to lift information on fibers to the global base. This method has been utilized for the study of Hopf algebras over rings. For instance Pareigis [12] has used this method to show the uniqueness of integrals on finite flat Hopf algebras over a Dedekind ring. Our new input is a cohomological lemma (Lemma 3.1) relating the flatness over the global base with the flatness on the fibers. We also make use of the correspondence between normal Hopf subalgebras and co-normal quotient Hopf algebras, due originally to Takeuchi and Schneider as well as various equivalences of the category of Hopf modules.

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The paper is organized as follows. In Section 2 we recall the Takeuchi-Schneider correspondence between normal Hopf subalgebras and co-normal quotients Hopf algebas of a given flat Hopf algebra over a Dedekind ring. This correspondence seems to be well-known by experts but we cannot find any reference suitable to our aim. In Section 2 we provide the key technical lemma.

Proposition 3.2. *Let B be a flat Hopf algebras over R and A be a special Hopf subalgebra of B. Suppose that A is R-finite. Then B is left faithfully flat over A if and only if B_k is left faithfully flat over A_k for k being the fraction field and any residue field of R.*

As a consequence we show that a flat Hopf algebra over a Dedekind ring is flat over any normal Hopf subalgebra (Theorem 3.6). To prove the projectivity over normal Hopf subalgebras we will need some supplementary results.

In Section 4 we study Hopf algebras equipped with an integral. The base ring here is assume to be a DVR. First we notice that the module of integrals is free of rank one over R.

Proposition 4.2. *Suppose H possesses a nonzero integral on H. Then space of integrals on H is a free R-module of rank one.*

The proof is a simple utilization of the theory of rational modules as developed in [21] and [3]. Next we show that if H is moreover R-projective then H possesses an integral if and only if it decomposes into direct sum of its R-finite comodules hence is itself a projective comodule.

Proposition 4.6. *Let H be an R-projective Hopf algebra. Then H possesses a left integral if and only if it is a direct sum of its R-finite subcomodules.*

Theorem 4.8. *Let H be an R-projective Hopf algebra. Then H possesses an integral if and only if H is projective in $\mathcal{M}^H$.*

At the end of this section we discuss the divisible ideal in a Hopf algebra. In particular we show that the projectivity assumption is quite natural when study integrals, at least when the base ring is a complete DVR.

In Section 5 we show that an R-projective Hopf with bijective antipode and possessing integrals is projective over any of its R-finite normal Hopf subalgebras.

Theorem 5.7. *Let B be a Hopf over R with bijective antipode and A be an R-finite normal Hopf subalgebra of B. Assume that $C = B/A^+B$ is projective in $\mathcal{M}^C$. Then B is right projective over A. This is the case if B is R-projective an possesses an integral.*

The proof is similar to that of [15, Theorem 3.1], we use several equivalences of module-, comodule- and Hopf module categories to provide a splitting to the multiplication map $B \otimes A \to B$. This technique was utilized in [4, Section 2].

2. Flatness and co-flatness

In this section we recall the Takeuchi-Schneider correspondence between normal Hopf subalgebras and conormal quotient Hopf algebras.

Let R be a Dedekind ring with fraction field K. In what follows, the tensor product, when not indicated, is understood as the tensor product over R. We shall frequently make use of the following facts: a torsion-free R-module is flat, a finite R-flat module is R-projective. For an R-module M, the torsion submodule is denoted by M_{tor}, it consists of elements, each is annihilated by a non-zero element of R. The quotient M/M_{tor} is torsion-free, hence flat. The saturation of a submodule N of M is the preimage in M of the torsion submodule of M/N, i.e. the miminal pure submodule of M, which contains N. N is called saturated if it is equal to its saturation, this is the same as being a pure submodule.

A coalgebra (resp. Hopf algebra) over R is called R-flat (resp. R-projective, R-finite) if it is flat (resp. projective, finitely generated) as R-module.

Definition 2.1. Let A, B be R-flat coalgebras, and $f : A \to B$ a homomorphism of coalgebras.

(i) If f is injective, we shall say that A is a Hopf subalgebra of B and usually identify A with a subset of B by means of f. Notice that the map $f \otimes f : A \otimes A \to B \otimes B$ is injective (as f is injective, A is R-torsion free, hence flat), thus the coproduct of A is the restriction of that of B.
(ii) f is called pure if it is pure as a homomorphism of R-modules (in our case, this is equivalent to requiring that $B/f(A)$ is R-flat).
(iii) If f is injective and pure, we shall say that A is a pure or saturated subcoalgebra of B.

Definition 2.2. Let $f : A \longrightarrow B$ be a homomorphism of R-flat Hopf algebras.

(i) f is called normal if f is pure and for all $a \in A, b \in B$ we have
$$\sum b_1 f(a) S(b_2) \in f(A) \quad \text{and} \quad \sum S(b_1) f(a) b_2 \in f(A).$$
(ii) f is called conormal if for all $a \in \mathsf{Ker}(f)$, we have
$$\sum a_2 \otimes S(a_1)a_3, \sum a_2 \otimes a_1 S(a_3) \in \mathsf{Ker}(f) \otimes A.$$
(iii) A Hopf subalgebra is called normal if the inclusion map is normal (in particular, it is pure), a Hopf ideal is called normal if the induced quotient map is conormal (in particular, the Hopf ideal is saturated as an R-submodule).

Let $f : A \longrightarrow B$ be a homomorphism of flat Hopf algebras. Let
$$A^{co(B)} := \{a \in A | \sum a_1 \otimes f(a_2) = a \otimes 1\},$$
$${}^{co(B)}A := \{a \in A | \sum f(a_1) \otimes a_2 = 1 \otimes a\}.$$
These are saturated R-submodules of A.

For a Hopf algebra of A, we denote $A^+ := \mathsf{ker}\varepsilon_A$, the augmented Hopf ideal of A. The proof of the following lemma is similar to that of [18, Lemma 1.3, p. 3342].

Lemma 2.3. *Let $f : A \longrightarrow B$ be a homomorphism of R-flat Hopf algebras.*

(i) *If f normal then $I := Bf(A)^+B = Bf(A)^+ = f(A)^+B$ is a normal Hopf ideal in B and $B \longrightarrow B/I$ is conormal.*
(ii) *If f is conormal then $A^{co(B)} = {}^{co(B)}A$ is normal Hopf subalgebra of A.*

2.1. Let C be an R-flat coalgebra. Then the category $\mathcal{M}^C$ of right C-comodules is abelian. Similarly, the category ${}^C\mathcal{M}$ of left C-comodules is abelian. Let $M \in \mathcal{M}^C$ and $N \in {}^C\mathcal{M}$. The co-tensor product $M\square_C N$ is defined as an equalizer:

$$0 \longrightarrow M\square_C N \longrightarrow M \otimes N \xrightarrow{\rho_M\otimes N - M\otimes\rho_N} M \otimes C \otimes N, \tag{1}$$

where ρ_M, ρ_N are respectively the coactions of C on M and N. We say that M is (left) co-flat over C if the functor $M\square - : {}^C\mathcal{M} \to \mathcal{M}_R$ is exact (this functor is left exact if M is R-flat).

The special case of importance is when an R-flat coalgebra B is a left and right C-comodule by means of a coalgebra map $B \to C$. Then for any $M \in \mathcal{M}^C$, $M\square_C B$ is a right B-comodule in the natural way. In fact, by the R-flatness of B we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \longrightarrow & M\square_C B & \longrightarrow & M\otimes B & \xrightarrow{\rho_M\otimes B - M\otimes\rho_B} & M\otimes C\otimes B \\ & \downarrow & & \downarrow{\scriptstyle M\otimes\Delta} & & \downarrow{\scriptstyle M\otimes C\otimes\Delta} \\ 0 \longrightarrow & (M\square_C B)\otimes B & \longrightarrow & M\otimes B\otimes B & \longrightarrow & M\otimes C\otimes B\otimes B, \end{array}$$

where ρ_M is the coaction of C on M and ρ_B is the left coaction of C on B given by

$$\rho_B : B \xrightarrow{\Delta} B \otimes B \longrightarrow C \otimes B.$$

Hence we have left exact functor $-\square B : \mathcal{M}^C \to \mathcal{M}^B$, called the induced functor. This functor is left adjoint to the restriction functor $\mathcal{M}^B \to \mathcal{M}^C$:

$$\mathsf{Hom}^C(M, N) \cong \mathsf{Hom}^B(M, N\square_C B), \quad \text{for } M \in \mathcal{M}^B,\ N \in \mathcal{M}^C. \tag{2}$$

The isomorphism is given by composing a morphism on the left hand side with the map $\rho_M : M \to M\square_C B \subset M \otimes B$:

$$f \mapsto g := (f \otimes \mathsf{id})\rho_M.$$

2.2. Let A be an algebra flat over R and C be an R-flat coalgebra. A right (C, A)-bimodule is an R-module M equipped with a left action of A, say r, and a right coaction of C, say ρ, such that

$$\rho(a \cdot m) = a\rho(m), \text{ for } a \in A, m \in M.$$

We define left (C, A)-module by switching the left and the right actions.

Let M be a right (C, A)-bimodule. The for any left C-comodule N, there exists a natural action of A on $M\square_C N$ given by the action of A on M. Dually, for any right A-module P, C coacts on $P \otimes_A M$ through the coation on M.

Assume now that P is A-flat. Then we have a canonical isomorphism

$$P \otimes_A (M\square_C N) \xrightarrow{\sim} (P \otimes_A M)\square_C N, \tag{3}$$

obtained by tensoring the exact sequence (1) with P over A. We notice that, by the flatness of P over A, these two spaces are subspaces of $P \otimes_A M \otimes N$.

Dually, if N is co-flat over C, then (3) holds for any A-module P. Indeed, let $P_1 \to P_0 \to P \to 0$ be an A-free resolution of P, then by means of the co-flatness of N over C we have the exactness of the lower sequence in the diagram below forcing the rightmost vertical morphism to be bijective:

$$\begin{array}{ccccccc} P_1 \otimes_A (M\square_C N) & \longrightarrow & P_0 \otimes_A (M\square_C N) & \longrightarrow & P \otimes_A (M\square_C N) & \longrightarrow & 0 \\ \cong\downarrow & & \cong\downarrow & & \downarrow & & \\ (P_1 \otimes_A M)\square_C N & \longrightarrow & (P_0 \otimes_A M)\square_C N & \longrightarrow & (P \otimes_A M)\square_C N & \longrightarrow & 0 \end{array}$$

2.3. There is a mirrored version of all claims in 2.2, in which "left" and "right" are interchanged. We shall frequently use them later on. For instance, we have

$$(N\square_C M) \otimes_A P \xrightarrow{\sim} N\square_C(M \otimes_A P). \tag{4}$$

2.4. Let $f : A \to B$ be a homomorphism of R-flat Hopf algebras. A right Hopf (B, A)-module is an R-module M equipped with a right B-comodule structure and a right A-module structure, such that the comodule structure map $\rho_M : M \longrightarrow M \otimes B$ is A-linear, where A acts diagonally on $M \otimes B$. Explicitly we have

$$\rho(ma) = \sum_{m,a} m_0 a_1 \otimes m_1 f(a_2), \quad m \in M, a \in A.$$

Morphisms of Hopf modules are those maps of the underlying R-modules which are both A-linear and B-colinear. Denote by $\mathcal{M}_A^B$ the category of Hopf (B, A)-modules.

We shall be interested in two cases: either f is a normal inclusion of Hopf algebras or f is a conormal quotient map of Hopf algebras.

2.5. Assume that A is a normal Hopf subalgebra of a flat Hopf algebra B (recall that we assume A be saturated in B as an R- module). Then $C := B/A^+B$ is R-flat Hopf algebra, and the quotient map $\pi : B \to C$ is conormal (Lemma 2.3(i)). The normality of A in B also ensures that B is a *left* (C, A)- bimodule with respect to the natural (co)actions.

For $N \in \mathcal{M}^C$, there is a Hopf (B, A)-module structure on $N\square_C B$, where the coaction of B is the induced coation, as in 2.1, and the action of A is induced from the action on B, as in 2.3. Thus we have functor

$$\Psi : \mathcal{M}^C \longrightarrow \mathcal{M}_A^B, \quad \Psi(N) = N\square_C B.$$

We construct the left adjoint to Ψ as follows:

$$\Phi(M) := M \otimes_A R, \quad M \in \mathcal{M}_A^B.$$

Lemma 2.4. *The functor Ψ is right adjoint to the functor Φ:*

$$\mathsf{Hom}^C(M \otimes_A R, N) \cong \mathsf{Hom}_A^B(M, N\square_C B).$$

The adjunctions are $\rho_M : M \to M\square_C B$ and $\varepsilon_B : N\square_C B \to N$.

Proof. According to 2.1 we have a functorial isomorphism

$$\mathsf{Hom}^C(M, N) \cong \mathsf{Hom}^B(M, N\square_C B),$$

given by composing a morphism on the right hand side with the map $\rho_M : M \to M\square_C B$: $f \mapsto g := (f \otimes B)\rho_M$. Thus we see that if g is A-linear, i.e., for $a \in A$, $m \in M$ we have

$$\sum f(m_1 a_1) \otimes m_2 a_2 = \sum f(m_1) \otimes m_2 a,$$

then, applying ε on the second tensor factor, we get

$$f(ma) = \varepsilon(a) f(m).$$

That is, $f : M \to N$ factors a the composition of $\bar{f} : M \otimes_A R \to N$ and the quotient map $q_M : M \to M \otimes_A R$. The converse also holds by the same reason. Thus the two maps g and $\bar{f}$ are related by the following diagram

$$\begin{array}{ccc} M & \xrightarrow{q_M} & M \otimes_A R \\ {\scriptstyle g}\downarrow & \overset{f}{\searrow} & \downarrow{\scriptstyle \bar{f}} \\ N\Box_C B & \xrightarrow[\varepsilon_B]{} & N. \end{array}$$

□

The following proposition generalizes [11, Theorem 1].

Proposition 2.5. *Suppose that B faithfully flat over A. Then the functors Φ and Ψ establish an equivalence between categories $\mathcal{M}_A^B$ and $\mathcal{M}^C$.*

Proof. For each $M \in \mathcal{M}^B$, we have an isomorphism

$$\gamma_M : M \otimes B \cong M \otimes B, \quad \gamma_M(m \otimes b) = \sum m_{(0)} \otimes m_{(1)} b,$$

with inverse $m \otimes b \longmapsto \sum m_{(0)} \otimes S(m_{(1)}) b$.

If $M \in \mathcal{M}_A^B$, consider $M \otimes A$ as a B-comodule by the diagonal coaction, we have the following diagram:

(5)
$$\begin{array}{ccccccc} M \otimes A \otimes B & \xrightarrow{r_M \otimes B - M \otimes l_B} & M \otimes B & \longrightarrow & M \otimes_A B & \longrightarrow & 0 \\ {\scriptstyle \gamma_{M\otimes A}}\downarrow & & {\scriptstyle \gamma_M}\downarrow & & \downarrow{\scriptstyle \gamma_{A,M}} & & \\ (M \otimes A) \otimes B & \xrightarrow[r_M \otimes B - M \otimes \varepsilon_A \otimes B]{} & M \otimes B & \xrightarrow[q_M \otimes B]{} & (M \otimes_A R) \otimes B & \longrightarrow & 0, \end{array}$$

where, r_M, l_B denote the actions of A and ε_A denotes the counit of A. Consequently we get an isomorphism

$$\gamma_{A,M} : M \otimes_A B \xrightarrow{\sim} (M \otimes_A R) \otimes B = \Phi(M) \otimes B.$$

In particular, for $M = B$ we have

(6)
$$\gamma_{A,B} : B \otimes_A B \xrightarrow{\sim} C \otimes B.$$

Therefore, for $M \in \mathcal{M}_A^B$ and $N \in \mathcal{M}^C$, since B is flat over R and over A, and using (3), we have

(7)
$$\Psi(N) \otimes_A B = (N \Box_C B) \otimes_A B \overset{(3)}{\cong} N \Box_C (B \otimes_A B) \overset{\gamma_{A,B}}{\cong} N \Box_C (C \otimes B) \cong N \otimes B,$$

$$\sum_i n_i \otimes b_i \otimes b \longmapsto \sum_i n_i \otimes b_{i(1)} \otimes b_{i(2)} b \longmapsto \sum_i n_i \otimes b_i b.$$

Suppose that $\bar{f} : M \otimes_A R \longrightarrow N$ in $\mathcal{M}^C$ corresponds to $g : M \longrightarrow N \Box_C B$ in $\mathcal{M}_A^B$. As shown in the proof of Lemma 2.4, both morphisms come from a morphism $f : M \to N$ in $\mathcal{M}^C$.

Tensoring the diagram there with B and compose it with γ we get the following commutative diagram:

$$\begin{array}{ccccc} M \otimes B & \xrightarrow{\gamma_M} & M \otimes B & \xrightarrow{q_M \otimes b} & (M \otimes_A R) \otimes B \\ \downarrow{\scriptstyle g\otimes \mathrm{id}_B} & & \downarrow{\scriptstyle g\otimes \mathrm{id}_B} & & \downarrow{\scriptstyle \bar{f}\otimes \mathrm{id}_B} \\ (N\Box_C B) \otimes B & \xrightarrow[\gamma_{N\Box_C B}]{} & (N\Box_C B) \otimes B & \xrightarrow[N\otimes \varepsilon_B \otimes B]{} & N \otimes B. \end{array}$$

By means of the right commutative diagram in (5) we obtain the following commutative diagram:

$$\begin{array}{ccc} M \otimes_A B & \xrightarrow{\gamma_A} & (M \otimes_A R) \otimes B \\ \downarrow{\scriptstyle g\otimes \mathrm{id}_B} & & \downarrow{\scriptstyle \bar{f}\otimes \mathrm{id}_B} \\ (N\Box_C B) \otimes_A B & \xrightarrow{\simeq} & N \otimes B, \end{array}$$

where the lower horizontal map is nothing but the isomorphism in (7). Since B faithfully flat over A and R, $\bar{f}$ is an isomorphism iff g is. Consequently, the adjunctions $M \longrightarrow \Psi\Phi(M) = (M \otimes_A R)\Box_C B$, and $\Phi\Psi(N) = (N\Box_C B) \otimes_A R \longrightarrow N$, are isomorphisms. Thus Φ and Ψ are equivalences. □

2.6. Consider now the dual situation. Let $B \to C$, $b \mapsto \overline{b}$, be a conormal quotient map of flat Hopf algebras over R. Let

$$A := B^{co(C)}.$$

If $T \in \mathcal{M}_A$, then $T \otimes_A B$ is in $\mathcal{M}_B^C$, where the C-comodule structure is given by that on B:

$$t \otimes b \mapsto \sum_b t \otimes b_1 \otimes \overline{b}_2, \quad \text{for } b \in B, t \in T.$$

This yields a functor $- \otimes_A B : \mathcal{M}_A \longrightarrow \mathcal{M}_B^C$.

Lemma 2.6. *The functor* $(-)^{co(C)} : \mathcal{M}_B^C \longrightarrow \mathcal{M}_A, Q \longmapsto Q^{co(C)}$ *is right adjoint to* $- \otimes_A B$*:*

$$\mathsf{Hom}_B^C(T \otimes_A B, Q) \cong \mathsf{Hom}_A(T, Q^{co(C)}).$$

The adjunctions are given by

$$\eta : T \longrightarrow (T \otimes_A B)^{co(C)}, \quad t \mapsto t \otimes 1,$$

$$\zeta : Q^{co(C)} \otimes_A B \longrightarrow Q, \quad q \otimes b \mapsto qb.$$

Proof. Let $T \in \mathcal{M}_A$ and $Q \in \mathcal{M}_B^C$. Then $f : T \longrightarrow Q^{co(C)}$ in $\mathcal{M}_A$ corresponds to $g : T \otimes_A B \longrightarrow Q$ in $\mathcal{M}_A^B$ by means of the following commutative diagram:

(8)
$$\begin{array}{ccc} T & \longrightarrow & T \otimes_A B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ Q^{co(C)} & \hookrightarrow & Q, \end{array}$$

where the upper horizontal map is given explicitly by $t \mapsto t \otimes 1$. In fact, the C-colinearity of g forces the image of f to be in $Q^{co(C)}$ and vice-versa. □

Proposition 2.7. [11, Theorem 2] *Suppose that B faithfully co-flat over C. The above functors establish an equivalence between categories $\mathcal{M}_A$ and$\mathcal{M}_B^C$.*

Proof. For any $P \in \mathcal{M}_B^C$ we have an isomorphism

$$\theta_P : P \otimes B \xrightarrow{\sim} P \otimes B, \quad p \otimes b \mapsto \sum_b pb_1 \otimes b_2,$$

with the converse map given by $p\otimes b \mapsto \sum_b pS(b_1)\otimes b_2$. Since B is R-flat we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & Q^{co(C)} \otimes B & \longrightarrow & Q \otimes B & \xrightarrow{(\rho_Q - Q\otimes u)\otimes B} & (Q \otimes C) \otimes B \\
& \downarrow{\scriptstyle \theta_{C,Q}} & & \downarrow{\scriptstyle \theta_Q} & & \downarrow{\scriptstyle \theta_{Q\otimes C}} \\
0 \longrightarrow & Q\square_C B & \longrightarrow & Q \otimes B & \xrightarrow[\rho_Q - \rho_B]{} & Q \otimes C \otimes B,
\end{array}$$

where $u : R \to C$ denotes the unit map for C and in the definition of $\theta_{Q\otimes C}$ the action of B on $Q \otimes C$ is diagonal. This forces the first vertical map to be an isomorphism. Thus we have an isomorphism

$$\theta_{C,Q} : Q^{co(C)} \otimes B \xrightarrow{\sim} Q\square_C B, \quad q \otimes b \mapsto \sum_b qb_1 \otimes b_2,$$

for any C-comodule Q. In particular we have

$$\theta_{C,B} : A \otimes B \xrightarrow{\sim} B\square_C B, \quad a \otimes b \mapsto \sum_b ab_1 \otimes b_2.$$

The inverse map is given by $b \otimes b' \mapsto \sum_{b'} bS(b'_1) \otimes b'_2$.

Tensoring (8) with B and twist it with the map θ we have commutative diagram

$$\begin{array}{ccc}
T \otimes B & \longrightarrow & (T \otimes_A B) \otimes B \\
\downarrow{\scriptstyle f\otimes B} & & \downarrow{\scriptstyle g\otimes B} \\
Q^{co(C)} \otimes B & \hookrightarrow & Q \otimes B,
\end{array}$$

where the upper and lower horizontal map are given explicitly by $t \otimes b \mapsto \sum_b (t \otimes b_1) \otimes b_2$ and $q \otimes b \mapsto \sum_b qb_1 \otimes b_2$, respectively. By means of the equality (3), we have

(9) $$T \otimes B = T \otimes_A (A \otimes B) \cong T \otimes_A (B\square_C B) = (T \otimes_A B)\square_C B \subset (T \otimes_A B) \otimes B,$$

with the composed map given by $t \otimes b \mapsto \sum_b t \otimes b_1 \otimes b_2$. Thus the above diagram reduces to the following commutative diagram with horizontal morphisms being isomorphisms:

$$\begin{array}{ccc}
T \otimes B & \longrightarrow & (T \otimes_A B)\square_C B \\
\downarrow{\scriptstyle f\otimes B} & & \downarrow{\scriptstyle g\square_C B} \\
Q^{co(C)} \otimes B & \xrightarrow[\theta_{C,Q}]{} & Q\square_C B.
\end{array}$$

Consequently, f is an isomorphism iff g is, as B is faithfully co-flat over C. This implies the adjunctions $T \longrightarrow (T \otimes_A B)^{coC}$, $t \mapsto t \otimes 1$; and $Q^{co(C)} \otimes_A B \longrightarrow Q$, $q \otimes b \mapsto qb$ are isomorphisms, where $t \in T, q \in Q$ and $b \in B$. Thus the functors are equivalences. □

The faithful flatness of B over A implies the faithful co-flatness of B over C [14, Lemma 2.4.1]. We show here the converse.

Proposition 2.8. *Let B be a Hopf algebra. Let $\mathcal{S}(B)$ be the set of all normal Hopf subalgebras A such that B faithfully flat over A and $\mathcal{I}(B)$ be the set of all normal Hopf ideals I such that B is right faithfully co-flat over B/I. Then*

$$\Phi : \mathcal{S}(B) \longrightarrow \mathcal{I}(B), \quad A \mapsto BA^+$$

and

$$\Psi : \mathcal{I}(B) \longrightarrow \mathcal{S}(B), \quad I \mapsto B^{co(B/I)}$$

are mutually inverse maps.

Proof. Let $A \in \mathcal{S}(B)$. Set $C := B \otimes_A R$ then C is a conormal quotient Hopf algebra of B, by Lemma 2.3(i). By means of the isomorphisms in (7), the functor $-\square_C B$ is faithfully flat, i.e., B is co-flat over C. Moreover, setting $N = R$ in (7) and noticing that $A \subset B^{co(C)}$, we conclude that $A = B^{co(C)}$.

Conversely, assume that $I \in \mathcal{I}(B)$ and let $B \to C = B/I$ be the corresponding conomal quotient map. Then $A := B^{co(C)}$ is a normal Hopf subalgebra of B, by Lemma 2.3(ii). As B is co-flat over C, the isomorphism in (9) shows that the functor $- \otimes_A B$ is faithfully exact, i.e. B is faithfully flat over A. Moreover, setting $T = R$ in (9) and noticing that the map $R \otimes_A B \to C$ is surjective as $A^+ \subset I$, we conclude that $R \otimes_A B \cong C$. □

3. The faithfully flatness over Hopf subalgebras

We continue to use the assumptions of the last section. The proof of the following lemma is similar to that of [5, Thm 4.1.1].

Lemma 3.1. *Let B be a flat Hopf algebras over R and A be a Hopf subalgebra of B. Then B is left flat over A if and only if B_k is left flat over A_k for the fraction field and any residue field of R.*

Proof. We prove the "only if" claim. To show that B is left flat over A, it suffices to check that $\mathrm{Tor}_1^A(M, B) = 0$ for any right A-module M (see [20, Exer. 3.2.1, p.69]). Choose a left projective resolution P_* of B over A. Since A is R-flat, so are the terms of P^* and since a submodule of an R-flat module is again flat, the resolution $P_* \to B$ remains a resolution after base change. On the other hand, the projectivity is preserved by base change, therefore, for any residue field $k := k_p$, $p \in \mathsf{Spec(R)}$, $P_* \otimes k$ is a left projective resolution of $B \otimes k$ over $A \otimes k$. We have

$$(M \otimes k) \otimes_{(A\otimes k)} (P_* \otimes k) \cong (M \otimes_A P_*) \otimes k,$$

implying

$$H_i((M \otimes_A P_*) \otimes k) \cong \mathrm{Tor}_i^{A\otimes k}(M \otimes k, B \otimes k), \text{ for all } i \geq 0.$$

First assume that M is R-flat. Since $M \otimes_A P_*$ is flat over R, we can apply the universal coefficient theorem, (see, e.g., [20, Thm 3.6.1]). Thus, for each $i \geq 1$, we have an exact sequence

$$0 \to H_i(M \otimes_A P_*) \otimes k \to H_i((M \otimes_A P_*) \otimes k) \to \mathrm{Tor}_1^R(H_{i-1}(M \otimes_A P_*), k) \to 0.$$

That is, for all $i \geq 1$,

$$0 \to \mathrm{Tor}_i^A(M, B) \otimes k \to \mathrm{Tor}_i^{A\otimes k}(M \otimes k, B \otimes k) \to \mathrm{Tor}_1^R(\mathrm{Tor}_{i-1}^A(M, B), k) \to 0.$$

By assumption $A\otimes k \longrightarrow B\otimes k$ is flat. So $\mathrm{Tor}_i^{A\otimes k}(M\otimes k, B\otimes k) = 0$, for all $i \geq 1$. Consequently

$$\mathrm{Tor}_1^R(\mathrm{Tor}_{i-1}^A(M, B), k) = \mathrm{Tor}_i^A(M, B) \otimes k = 0, \text{ for all } i \geq 1.$$

This holds for any residue field and the fraction field of R, hence $\mathrm{Tor}_i^A(M,B)$ is flat over R for $i \geq 0$ and we conclude that $\mathrm{Tor}_i^A(M,B) = 0$ for all $i \geq 1$.

Let now M be an arbitrary right A-module. Note that R-torsion submodule M_τ of M is also a right A-submodule. Then the quotient module M/M_τ is R-flat and from the exact sequence

$$\mathrm{Tor}_1^A(M_\tau, B) \to \mathrm{Tor}_1^A(M,B) \to \mathrm{Tor}_1^A(M/M_\tau, B) \to \dots$$

it suffices to show $\mathrm{Tor}_1^A(M,B) = 0$ for M being R-torsion.

For each non-zero ideal $p \subset R$, the submodule M_p of elements annihilated by elements of p, is also an A-submodule. As M is torsion, it is the direct limit of M_p. Since the Tor-functor commutes with direct limits, one can replace M by some M_p. Since each non- zero ideal $p \subset R$ is a product of finitely many prime ideals, each M_p has a filtration, each grade module of which is annihilated by a certain non-zero prime ideal. Thus using induction we can reduce to the case M is annihilated by a prime ideal p. In this case $M = M \otimes k_p$ is an $A \otimes k_p$- module, where $k_p := R/p$ and we have

$$M \otimes_A P_* = M \otimes_{A\otimes k_p} (P_* \otimes k_p).$$

Since $P_* \otimes_R k_p$ is an $A \otimes_R k_p$-projective resolution of $B \otimes_R k_p$, we see that

$$\mathrm{Tor}_i^A(M,B) = \mathrm{Tor}_i^{A\otimes k_p}(M, B \otimes k_p) = 0,$$

as $B \otimes k_p$ is flat over $A \otimes k_p$. □

The next theorem is a generalization of the well-known faithful flatness theorem for flat commutative Hopf algebras over a Dedekind ring [5].

Proposition 3.2. *Let B be a flat Hopf algebras over R and A be a special Hopf subalgebra of B. Suppose that A is R-finite. Then B is left faithfully flat over A if and only if B_k is left faithfully flat over A_k for k being the fraction field and any residue field of R.*

Proof. Assume that B is faithfully flat over A. For the fraction field or a residue field k of R, and for any right A_k-module M which satisfies $M \otimes_{A_k} B_k = 0$, we have $M \otimes_{A_k} B_k \cong M \otimes_A B = 0$, hence $M = 0$. Thus B_k is faithfully flat over A_k.

Conversely, let M be an right A-module such that $M \otimes_A B = 0$. Then we have

$$M_k \otimes_{A_k} B_k \cong (M \otimes_A B) \otimes_R k = 0,$$

for any residue field k of R. Since $A_k \to B_k$ is faithfully flat, we have $M_k = 0$. If M is finite over A then it is R-finite, this implies that $M = 0$, according to [9, Thm 4.8]. In the general case, M always contains a non-zero finite submodule and since B is flat over A, we see that $M = 0$ if $M \otimes_A B = 0$. Thus B is faithfully flat over A. □

Remarks 3.3. It is not known if one can drop the finiteness condition on A in the previous proposition. This assumption is needed for proving the faithfulness of B over A (flatness is fine by Lemma 3.1).

As a corollary of Proposition 3.2 and Nichols-Zoeller's theorem [10] we have following

Proposition 3.4. *Let R be a Dedekind ring and B be an R-finite flat Hopf algebra and A is a special Hopf subalgebra of B. Then B is faithfully flat over A.*

Proof. Since B/A is R-flat we have $A_k \subset B_k$ for any residue field and the fraction K. B_k is free hence, in particular, faithfully flat over A_k, therefore Proposition 3.2 ensures that B is faithfully flat over A. □

It is well known that every finitely presented flat A-module is projective over A, (see [20, Thm 3.2.7]). Hence we obtain immediately

Corollary 3.5. *Assume that B is finite over R. Then for any special Hopf subalgebra A, B is left projective over A.*

Theorem 3.6. *Let B be a flat Hopf algebra over Dedekind ring R and A is a normal Hopf subalgebra of B. Assume that A is R-finite. Then B is faithfully flat over A. Consequently B is faithfully co-flat over $C := B/A^+B$ and we have equivalences $\mathcal{M}^C \cong \mathcal{M}_A^B$ and $\mathcal{M}_A \cong \mathcal{M}_B^C$.*

Proof. As B/A flat over R, we have A_k is a normal finite Hopf subalgebra of B_k for any residue k and fraction field K. According to[18, Lemma 2.2 and Theorem 2.4], B_k free over A_k. The assertion follows by Proposition 3.2. □

Questions 3.7. Let R be a DVR. Suppose that B is R-projective and A is finite normal Hopf subalgebra of B. Under which circumstance will B be projective or free over A.

If R is local and B is R-finite free, then it is known that B will be free over any R-projective Hopf subalgebra, see [16, Remark 2.1].

4. Module of integrals on a Hopf algebra

We study in this section the module of integrals on an R-projective Hopf algebra, where R is a discrete valuation ring. We show that this module is either 0 or is projective of rank 1 over R. We also prove some finiteness property of Hopf algebras with integrals.

4.1. In what follows, the Hopf algebra H is assumed to be projective over a DVR R with dual algebra H^*. The convolution product on H^* is denote by a dot "$\cdot$". Any right comodule $M \in \mathcal{M}^H$ is a left module over H^* by the action

$$H^* \otimes M \to M, \quad f \cdot m = \sum_m m_0 \otimes f(m_1), \quad f \in H^*, m \in M.$$

One shows that $\mathcal{M}^H$ is a full subcategory of ${}_{H^*}\mathcal{M}$ of left H^*-modules. An H^*-module M is called rational if the action of H is induced from a coaction of H in the way described above. Each $M \in {}_{H^*}\mathcal{M}$ contains a unique maximal rational submodule denoted by $\mathsf{Rat}(M)$ - the sum of all rational submodules of M. In particular, the rational module $\mathsf{Rat}(H^*)$ can be characterized by finiteness condition (see [21, 5.3, 5.4] or [3, 1.4]):

$$\begin{aligned} \mathsf{Rat}(H^*) &= \{f \in H^* | f \cdot H^* \text{ is } R\text{-finite}\} \\ &= \{f \in H^* | f \cdot H \text{ is } R\text{-finite}\}. \end{aligned}$$

One checks that $\mathsf{Rat}(H^*)$ is a right Hopf module and hence ([3, Lemma 3.3])

$$\mathsf{Rat}(H^*) \cong \mathsf{Rat}(H^*)^{co(H)} \otimes H.$$

Notice that by definition, $\mathsf{Rat}(H^*)^{co(H)}$ is the *module of left integrals* on H:

$$I_H^l := \mathsf{Hom}^H(H, R) = \mathsf{Rat}(H^*)^{co(H)}.$$

Lemma 4.1. *The R-module $\mathsf{Rat}(H^*)$ is flat and it is saturated in H^*.*

Proof. The first claim is obvious. We prove the second claim. For any $r \in R$ and $f \in H^*$, suppose that $rf \in \mathsf{Rat}(H^*)$. Then $(rf)\cdot H \subset H$ is an R-finite submodule. Since H is projective, the saturation of this submodule in H is also R-finite, thus $f \cdot H$ is also R-finite. □

Since H is projective over R, it is free and $H = \bigoplus_\alpha R_\alpha$. Then $H^* \cong \prod_\alpha R_\alpha$. In the case k is residue of R, it is finitely generated as R-module, so we have $(H^*)_k \cong (H_k)^*$. The above lemma shows that the natural map

$$\mathsf{Rat}(H^*)_k \hookrightarrow (H^*)_k \cong (H_k)^*$$

is injective, its image is by construction contained in $\mathsf{Rat}((H_k)^*)$. This map also induces the inclusion on the co-invariants:

$$(I^l_H)_k \to I^l_{H_k}.$$

Thus, if $(I^l_H)_k \neq 0$, it has to be one-dimensional. Similar argument is valid for the ring of quotients K. The inclusion $(H^*)_K \to (H_K)^*$ is injective (generally not surjective!), giving rise to the inclusion $\mathsf{Rat}(H^*)_K \hookrightarrow \mathsf{Rat}((H_K)^*)$, and hence the inclusion

$$(\mathsf{Rat}(H^*)^{co(H)})_K \hookrightarrow \mathsf{Rat}((H_K)^*)^{co(H_K)}.$$

Thus, if $(I^l_H)_K \neq 0$, it has to be one-dimensional.

Proposition 4.2. *Suppose H possesses a nonzero integral on H. Then space of integrals on H is a free R-module of rank one.*

Proof. Assume H possesses a non-zero integral φ. Since R is a PID, one can assume that the integral is a surjective map. Therefore $(I^l_H)_k \neq 0$, hence it is one-dimensional. The same hold for $(I^l_H)_K$. It is now well-known that I_H is isomorphic to R, for example, by the lemma below. □

Lemma 4.3. [19, Theorem 3.1] *A flat module M of (finite) constant rank over a local ring R is finitely generated (and thus free).*

4.2. In what follows we shall prove the projectivity of H as a comodule on itself and some finiteness properties of H assuming that H possesses a non-zero integral.

Lemma 4.4. *For any $M \in \mathcal{M}^H$ which is R-finite, $\mathsf{Hom}^H(H, M) \cong M \otimes \mathsf{Hom}^H(H, R)$. Hence the functor $\mathsf{Hom}^H(H, -)$ is exact when restricted to R-finite comodules.*

Proof. First assume that M is projective. Then using the isomorphism of H-comodules

$$N \otimes H \cong N \otimes H, \quad n \otimes h \mapsto \sum_n n_0 \otimes n_1 h,$$

where on the source H coacts diagonally and on the target H coacts through the coproduct on itself, we have isomorphisms:

$$M \otimes \mathsf{Hom}^H(H, R) \cong \mathsf{Hom}^H((M^*) \otimes H, R) \cong \mathsf{Hom}^H(M^* \otimes H, R) \cong \mathsf{Hom}^H(H, M),$$

which is explicitly given as follows:

$$\begin{aligned} m\otimes\varphi &\mapsto \varphi_m : \eta\otimes h \mapsto \eta(m)\varphi(h), \quad \eta\in M^* \\ &\mapsto \widehat{\varphi}_m : \eta\otimes h \mapsto \sum_\eta \eta_0(m)\varphi(\eta_1 h) = \sum_m \eta(m_0)\varphi(S(m_1)h) \\ &\mapsto \overline{\varphi}_m : h \mapsto \sum_m m_0\varphi(S(m_1h)). \end{aligned}$$

This construction yields the map

$$\theta_M : M\otimes \mathsf{Hom}^H(H,R) \xrightarrow{\sim} \mathsf{Hom}^H(H,M), \qquad (10)$$
$$m\otimes\varphi \mapsto \overline{\varphi}_m : h\mapsto \sum_m m_0\varphi(S(m_1h)),$$

for any H-comodule M, which is functorial in M. We will show that it is an isomorphism for any R-finite module M. The injectivity can be argued as follows. Consider an R-finite free resolution of M: $0\to M_1\to M_0\to M\to 0$, then we have the following commutative diagram with exact rows ($I^l_H = \mathsf{Hom}^H(H,R)$):

$$\begin{array}{ccccccccc} 0 & \to & M_1\otimes I^l_H & \to & M_0\otimes I^l_H & \to & M\otimes I^l_H & \to & 0 \\ & & \cong\downarrow & & \cong\downarrow & & \downarrow & & \\ 0 & \to & \mathsf{Hom}^H(H,M_1) & \to & \mathsf{Hom}^H(H,M_0) & \to & \mathsf{Hom}^H(H,M), & & \end{array}$$

which shows that the right vertical map is injective.

To show the bijectivity of (10) we first assume that M is annihilated by the uniformizer $\pi\in R$, i.e. $M = M_k$. Then

$$\mathsf{Hom}^H(H,M) = \mathsf{Hom}^{H_k}(H_k,M) \cong M\otimes_k \mathsf{Hom}^H(H_k,k) \cong M\otimes_R \mathsf{Hom}^H(H,R).$$

For a arbitrary comodule M, the R-torsion part $M_{\rm tor}$ of M is a subcomodule. Since M is R-finite, $M_{\rm tor}$ has finite length over R. We will use induction on the length of $M_{\rm tor}$ to show that the map in (10) is an isomorphism. If the length of $M_{\rm tor}$ is 0, then M is torsion free, hence R-projective, so θ_M is bijective by the discussion above. For general M, consider the exact sequence $0\to\pi M\to M\to M_k\to 0$. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \pi M\otimes I^l_H & \to & M\otimes I^l_H & \to & M_k\otimes I^l_H & \to & 0 \\ & & \cong\downarrow & & \hookdownarrow & & \cong\downarrow & & \\ 0 & \to & \mathsf{Hom}^H(H,\pi M) & \to & \mathsf{Hom}^H(H,M) & \to & \mathsf{Hom}^H(H,M_k). & & \end{array}$$

The left vertical map is bijective by the induction hypothesis, the right vertical map is bijective according to the discussion above. This forces the middle vertical map to be bijective. ☐

Lemma 4.5. *Let H be an R-projective Hopf algebra. Then H decomposes into a direct sum of its R-finite subcomodules if and only if H_K decomposes into a direct sum of its finite dimensional subcomodules.*

Proof. Assume that Hence H_K is decomposed into a direct sum of its finite dimensional subcomodules: $H_K = \bigoplus_\alpha H_\alpha$. Since H_α is finite dimensional over K, there exists $a\in R$ such that $aH_\alpha\subset H\subset H_K$. Thus we can find a K- basis of H_α which consists of elements of H. The H-comodule generated by elements of this basis lies in H but also in H_α as H_α is also an H-comodule (as we have $H_\alpha \otimes_R H \cong H_\alpha \otimes_K H_K$). Thus it is a subcomodule of $H \cap H_\alpha$. Its saturation in H is also contained in $H \cap H_\alpha$. But as H is R-projective, this saturation is also R-finite, and by construction, its rank over R is the same as the dimension of H_α over K. We conclude that it is equal to $H \cap H_\alpha$, in other words, this intersection is saturated in H. Consequently, we have $H = \bigoplus H \cap H_\alpha$. □

Proposition 4.6. *Let H be an R-projective Hopf algebra. Then H possesses a left integral if and only if it is a direct sum of its R-finite subcomodules.*

Proof. Let $\varphi : H \to R$ be a non-zero integral on H, we can assume that φ is surjective. Then it yields an integral on $H_K := H \otimes_R K$. Hence H_K decomposes into a direct sum of its finite dimensional subcomodules. According to the lemma above, H also decomposes into a direct sum of its finite subcomodules.

Conversely, assume that H possesses such a decomposition. Then H_K also possesses a decomposition into sum of finite subcomodules. Consequently H_K possesses a left integral φ. Then we can decomposes H_K in such a way that $1 \in H_K$ is contained in one direct summand and the integral of H_K when restricted to other summand becomes zero. The discussion above yields a decomposition of H, which is compatible with that of H_K. Hence there exists $a \in R$ such that the value of $a\varphi$ on H lies in R. □

Corollary 4.7. *Let H be an R-projective Hopf algebra. Then H possesses a left integral if and only if so does H_K.*

Theorem 4.8. *Let H be an R-projective Hopf algebra. Then H possesses an integral if and only if H is projective in $\mathcal{M}^H$.*

Proof. Assume that $\varphi : H \to R$ is a non-zero left integral. We have shown in the previous proposition that H is a direct sum of its subcomodules: $H = \bigoplus H_\alpha$, where each H_α is R-finite projective. Then we have

$$\mathsf{Hom}^H(H, M) \cong \prod_\alpha \mathsf{Hom}^H(H_\alpha, M).$$

Therefore H is projective iff H_α is projective for each α.

But we know that H is projective with respect to R-finite subcomodules by Lemma 4.4. Consequently H_α is also projective with respect to R-finite subcomodules. The category $\mathcal{M}^H$ is a locally noetherian and Grothendieck category, and the functor $\mathsf{Hom}^H(H_\alpha, -)$ commutes with inductive limit ([6, Proposition 3.7.4]). This means H_α is projective in $\mathcal{M}^H$.

Conversely, assume that H is projective in $\mathcal{M}^H$. By the local finiteness, H is a union of its R-finite subcomodules. Hence there exists a surjective map $\bigoplus H_\alpha \to H$, where each H_α is an R-finite subcomodule of H. The projectivity of H implies that H is a direct summand of that direct sum: $\bigoplus H_\alpha = H \oplus H'$, for some comodule H'. Tensoring with K we obtain $\bigoplus H_{\alpha,K} = H_K \oplus H'_K$. Each $H_{\alpha,K}$ is a direct sum of finite indecomposable comodules, hence we can assume $H_{\alpha,K}$ is itself indecomposable. Then Azumaya's theorem [1, Thm. 12.6] implies that H_K itself contains a finite direct summand. Consequently H_K possesses an integral. Hence by Corollary (4.7), H possesses an integral. □

4.3. In this last subsection we discuss the space of integrals on non-projective H. Since R is a DVR, we can always choose an integral which is a surjective map. On the other hand notice that an R-linear map $M \to R$ has to be zero if M is a divisible module, i.e. if any element in M is divisible by the uniformizer π. On the other hand, it is well-known that the maximal divisible submodule of a flat module over R is a direct summand.

Proposition 4.9. *Let H be an R-flat Hopf algebra possessing a left integral. Let I be the maximal divisible R-submodule of H. Then I is a normal Hopf ideal I which is annihilated by any integral on H. If R is a complete DVR then the quotient module H/I is Mittag-Leffler as an R-module, in particular it will be projective if it is countable generated.*

Proof. We have $H = R \oplus H^+$, hence $I \subset H^+$. Define $I_n := \pi^n H^+$. Then $I = \bigcap_n I_n$. On the other hand, I_n is a non-saturated normal Hopf ideal (i.e. I_n satisfies the condition (ii) of Definition (2.2)). Consequently $I := \bigcap_n I_n$ is a normal Hopf ideal (as it is saturated in H).

Assume that R is a complete DVR. To see that it is Mittag-Leffler as an R-module, we show that each of its finite submodule has a finite saturation. This follows immediately from Lemma (4.10) below. Indeed, if M is a finite submodule of H/I and M_{sat} its saturation in H/I, then M_{sat} has no divisible submodule, hence is finite. Thus H/I is a union of its saturated submodules, hence is Mittag- Leffler. □

Notice that a Mittag-Leffler R-module which is countably generated is projective. But there exist R-flat Mittag-Leffler Hopf algebras which is not R-projective. An example is the ring of infinite series in one parameter $R[[T]]$ with the coproduct give by

$$\Delta(T) = T \otimes 1 + 1 \otimes T.$$

Lemma 4.10. *Let R be a complete DVR and M be an R- flat module of finite rank (i.e. M_K is finite dimensional over K). If M does not contain divisible submodule then M is finite.*

Proof. This is a special case of Kaplansky's result [7, Thm 12] stating that any torsion-free module of countable rank is decomposable into a direct sum of rank one module. A rank one module over R is either R itself of K. □

5. Projectivity over Hopf subalgebras

Our aim in this section is to give a condition for an R-flat Hopf algebra to be projective over a finite normal Hopf subalgebra. We shall keep the settings and assumptions of the previous section. Thus let R be a DVR and assume that B is an R-flat Hopf algebra and A is an R-finite normal Hopf subalgebra of B. Denote $C := B \otimes_A R = B/A^+B$.

5.1. We first give a description of C using the integral-elements:

$$_{\varepsilon}A := \{b \in A | ab = \varepsilon(a)b, \forall a \in A\}.$$

According to [12], if $_{\varepsilon}A \neq 0$ then it is free of rank 1 as an R-module, moreover, the antipode of A is bijective. Choose a generator Λ of $_{\varepsilon}A$. For $b \in B$ denote $\bar{b}$ its coset in C. By definition of Λ, the map

$$f_B : C \longrightarrow B, \quad f(\bar{b}) := \Lambda b,$$

is well-defined. The following result is a generalization of [8, Lemma 3.2].

Proposition 5.1. *The map f_B is right B-linear, right C-colinear and bijective. Thus $C \cong \Lambda B$ as B-modules, where Λ is a non-zero integral element of A.*

Proof. It follows from definition that f_B is surjective and right B-linear. To see that it is C-colinear we notice that for $a \in A$, $\bar{a} = \varepsilon(a) \cdot 1 \in C$. Hence

$$\sum_{\Lambda} \Lambda_1 \otimes \overline{\Lambda_2} = \Lambda \otimes 1.$$

Consequently f_B is right C-colinear. (It is also left C-colinear by the same argument).

It remains to show that f_B is injective. Since f_B is injective if its generic fiber $(f_B)_K = f_{B_K}$ is injective, we shall work with B_K in stead of B. That mean we are working with Hopf algebras over a field. In this case we know that B is free as a left A-module, cf. [18]. So we can choose a basis $\{b_j\}_{j\in B}$ of B over A. Assume that $b = \sum_j a_j b_j$ and $\Lambda b = 0$. We have $\Lambda b = \sum_j \varepsilon_A(a_j)\Lambda b_j = 0$. Hence $\varepsilon_A(a_j)\Lambda = 0$ for all $j \in J$, so $\varepsilon_A(a_j) = 0$ implies $a_j \in A^+B$ for all j. So f_B is injective. □

Corollary 5.2. *If B is R-projective then $C = B \otimes_A R$ is also R-projective.*

Proof. It is well-known that over a DVR, projective modules are free and a submodule of a free module is again free, see, e.g. [13, Section 9.1]. □

Corollary 5.3. *Let A be an R-finite normal Hopf subalgebra of an R-projective Hopf algebra B. Then B possesses a left integral if and only if $C := B/A^+B$ does.*

Proof. We know that C is also R-projective, so we can use the results of the previous section. Assume that C possesses a left integral. Then we have a decomposition of right C-comodules

$$C = \bigoplus_{\alpha} C_\alpha,$$

where C_α are all R-finite. The induced functor $C_\alpha \mapsto B_\alpha := C_\alpha \square_C B$ induces a direct decomposition of B-comodules

$$B = \bigoplus_{\alpha} B_\alpha.$$

According to (7), we have an isomorphism of right B-modules

$$B_\alpha \otimes_A B \cong C_\alpha \otimes B.$$

Since B is faithfully flat over A and since C_α are R-finite, we conclude that B_α are also finite as right A-modules, hence are finite as R-modules. Thus, according to Proposition 4.6, B is projective as a (right) comodule over itself.

Conversely, assume that B possesses a left integral. Then B decomposes into direct sum of its finite subcomodules B_α. As B and hence its direct summand are projective in $\mathcal{M}^B$ and since B is faithfully co-flat over C, we conclude, by means of the natural isomorphism (2), that each B_α is projective in $\mathcal{M}^C$. Notice that the B_α are R-finite projective. Consequently C_K also possesses a finite dimensional projective comodule. This implies that C_K possesses an integral, hence so does C, by Corollary (4.7). □

5.2. To put an end to the work, we need some basis properties of Hopf modules in $\mathcal{M}_B^C$ (see [2, 1.2, 1.4]).

Remarks 5.4. (i) For any right B-module M, $M \otimes C$ is in $\mathcal{M}_B^C$ by letting C coacts on itself and B acts diagonally:

$$\rho : M \otimes C \to (M \otimes C) \otimes C, \quad m \otimes \overline{b} \mapsto \sum_b m \otimes \overline{b_1} \otimes \overline{b_2},$$

$$\mu : (M \otimes C) \otimes B \to M \otimes C, \quad (m \otimes \overline{b}) \otimes b' \mapsto \sum_{b'} mb'_1 \otimes \overline{bb'_2}.$$

(ii) The module $C \otimes B$ is in $\mathcal{M}_B^C$ with B acting on itself and C coating diagonally:

$$\mu : (C \otimes B) \otimes B \longrightarrow C \otimes B, \quad \overline{b} \otimes b' \otimes b'' \mapsto \overline{b} \otimes b'b'';$$

$$\rho : (C \otimes B \longrightarrow (C \otimes B) \otimes C, \quad \overline{b} \otimes b' \mapsto \sum_{b,b'} \overline{b_1} \otimes b'_1 \otimes \overline{b_2}\, \overline{b'_2}.$$

(iii) If the antipode S of B has an inverse then

$$C \otimes B \longrightarrow B \otimes C, \quad \overline{b} \otimes b' \mapsto \sum_{b'} b'_1 \otimes \overline{bb'_2}$$

is an isomorphism in $\mathcal{M}_B^C$ with inverse map

$$B \otimes C \longrightarrow C \otimes B, \quad b \otimes \overline{b'} \mapsto \sum_{b} \overline{b'} S^{-1}(\overline{b_2}) \otimes b_1.$$

Lemma 5.5. *Assume that the antipode of B is bijective and C is projective in $\mathcal{M}^C$. Then $B \otimes C$ considered as an object of $\mathcal{M}_B^C$ as in 5.4(i) is projective.*

Proof. Consider $C \otimes B$ as an object in $\mathcal{M}_B^C$ as in 5.4(ii) above, we have an isomorphism

$$\mathsf{Hom}_B^C(C \otimes B, M) \longrightarrow \mathsf{Hom}^C(C, M), \quad g \mapsto g(- \otimes B)$$

with the inverse given by $f \mapsto r_M \circ (f \otimes B)$. Now by the isomorphism in 5.4(iii), we have

$$\mathsf{Hom}_B^C(B \otimes C, M) \cong \mathsf{Hom}_B^C(C \otimes B, M) \cong \mathsf{Hom}^C(C, M),$$

for any $M \in \mathcal{M}_B^C$. □

Lemma 5.6. *Assume that $B \otimes C$ is projective in $\mathcal{M}_B^C$. Then B is right projective over A.*

Proof. Consider $B \otimes B$ as an object in $\mathcal{M}_B^C$ letting B coact diagonally and C coact by its coaction on B. Then the quotient map $B \otimes B \longrightarrow B \otimes C$ is in $\mathcal{M}_B^C$, hence it splits if $B \otimes C$ is projective in $\mathcal{M}_B^C$. Since $\mathcal{M}_B^C \cong \mathcal{M}_A$ by means of the functor $(-)^{co(C)} : \mathcal{M}_B^C \longrightarrow \mathcal{M}_A$, we obtain a split surjective:

$$(B \otimes B)^{co(C)} \longrightarrow (B \otimes C)^{co(C)}$$

of right A-modules. On the other hand, we have

$$(B \otimes B)^{co(C)} \cong (B \otimes B)\square_C R \cong B \otimes (B \square_C R) \cong B \otimes B^{co(C)} \cong B \otimes A,$$

where, on the rightmost term A acts diagonally, and

$$(B \otimes C)^{co(C)} \cong (B \otimes C)\square_C R \cong B \otimes (C \square_C R) \cong B,$$

where A acts on the rightmost term by the right action, furthermore, the induced map

$$B \otimes A \to B$$

is given by $b \otimes a \mapsto \varepsilon(a)b$. Now we have an isomorphism in $\mathcal{M}_A$:

$$(B) \otimes A \cong B \otimes A, \quad b \otimes a \mapsto \sum_a ba_1 \otimes a_2.$$

Combine this map with the previous one we obtain the map $B \otimes A \to B$, $b \otimes a \mapsto ba$. As this map splits in $\mathcal{M}_A$, we conclude that B is a direct summand of the free A-module $B \otimes A$, hence B is projective as a right A-module. □

The above lemmas bring together:

Theorem 5.7. *Let B be a Hopf over R with bijective antipode and A be an R-finite normal Hopf subalgebra of B. Assume that $C = B/A^+B$ is projective in $\mathcal{M}^C$. Then B is right projective over A. This is the case if B is R-projective an possesses an integral.*

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