On the k-Semispray of Nonlinear Connections in k-Tangent Bundle Geometry

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Abstract

In this paper we present a method by which is obtained a sequence of k-semisprays and two sequences of nonlinear connections on the k-tangent bundle T^kM, starting from a given one. Interesting particular cases appear for Lagrange and Finsler spaces of order k.

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1 Introduction

Classical Mechanics have been entirely geometrized in terms of symplectic geometry and in this approach there exists certain dynamical vector field on the tangent bundle TM of a manifold M whose integral curves are the solutions of the Euler-Lagrange equations. This vector field is usually called *spray* or *second-order differential equation (SODE)*. Sometimes it is called *semispray* and the term *spray* is reserved to homogeneous second-order differential equations ([7], [15]). Let us remember that a SODE on TM is a vector field on TM such that $JC = C$, where J is the almost tangent structure and C is the canonical Liouville field ([5], [6]).

In [2], [3], [4] J. Grifone studies the relationship among SODEs, nonlinear connections and the autonomous Lagrangian formalism. In paper [12] Gh. Munteanu and Gh. Pitiş also studied the relation between sprays and nonlinear connectiosn on TM. This study was extended to the non-autonomous case by M. de León and P. Rodgrigues ([5]). Also, important results for singular non-autonomous case was obtained in [13]. In this paper, following the ideas of papers [10], [11], [12] and [13] we will extend the study of the relationship between sprays and nonlinear connections to the k-tangent bundle of a manifold M. The study of the geometry of this k-tangent bundle was by introduced by R. Miron ([7], [8], [9]). For this case the k-spray represent a system of ordinary differential equations of $k+1$ order.

2 The k-Semispray of a Nonlinear Connection

Let M be a real n-dimensional manifold of class C^∞ and (T^kM, π^k, M) the bundle of accelerations of order k. It can be identified with the k-osculator bundle or k-tangent bundle ([7], [9]).

A point $u \in T^kM$ will be written by $u = (x, y^{(1)}, ..., y^{(k)})$, $\pi^k(u) = x$, $x \in M$. The canonical coordinates of u are $(x^i, y^{(1)i}, ..., y^{(k)i})$, $i = \overline{1, n}$, where $y^{(1)i} = \dfrac{1}{1!}\dfrac{dx^i}{dt}$, ..., $y^{(2)i} = \dfrac{1}{k!}\dfrac{d^k x^i}{dt^k}$. A transformation of local coordinates $(x^i, y^{(1)i}, ..., y^{(k)i}) \to (\widetilde{x}^i, \widetilde{y}^{(1)i}, ..., \widetilde{y}^{(k)i})$ on $(k+1)n$-dimensional manifold T^kM is given by

$$(1)\qquad \begin{cases} \widetilde{x}^i = \widetilde{x}^i(x^1, ..., x^n),\ rang\left(\dfrac{\partial \widetilde{x}^i}{\partial x^j}\right) = n, \\ \widetilde{y}^{(1)i} = \dfrac{\partial \widetilde{x}^i}{\partial x^j} y^{(1)j}, \\ 2\widetilde{y}^{(2)i} = \dfrac{\partial \widetilde{y}^{(1)i}}{\partial x^j} y^{(1)j} + 2\dfrac{\partial \widetilde{y}^{(1)i}}{\partial y^{(1)j}} y^{(2)j}, \\ \dotfill \\ k\widetilde{y}^{(k)i} = \dfrac{\partial \widetilde{y}^{(k-1)i}}{\partial x^j} y^{(1)j} + 2\dfrac{\partial \widetilde{y}^{(k-1)i}}{\partial y^{(1)j}} y^{(2)j} + \cdots + k\dfrac{\partial \widetilde{y}^{(k-1)i}}{\partial y^{(k-1)j}} y^{(k)j}. \end{cases}$$

A local coordinates change (1) transforms the natural basis $\left\{\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial y^{(1)i}}, \cdots, \dfrac{\partial}{\partial y^{(k)i}}\right\}_u$ of the tangent space T_uT^kM by the rule:

(2)

$$\begin{cases} \dfrac{\partial}{\partial x^i} = \dfrac{\partial \widetilde{x}^j}{\partial x^i}\dfrac{\partial}{\partial \widetilde{x}^j} + \dfrac{\partial \widetilde{y}^{(1)j}}{\partial x^i}\dfrac{\partial}{\partial \widetilde{y}^{(1)j}} + \cdots + \dfrac{\partial \widetilde{y}^{(k)j}}{\partial x^i}\dfrac{\partial}{\partial \widetilde{y}^{(k)j}}, \\ \dfrac{\partial}{\partial y^{(1)i}} = \dfrac{\partial \widetilde{y}^{(1)j}}{\partial y^{(1)i}}\dfrac{\partial}{\partial \widetilde{y}^{(1)j}} + \cdots + \dfrac{\partial \widetilde{y}^{(k)j}}{\partial y^{(1)i}}\dfrac{\partial}{\partial \widetilde{y}^{(k)j}}, \\ \vdots \\ \dfrac{\partial}{\partial y^{(k)i}} = \dfrac{\partial \widetilde{y}^{(k)j}}{\partial y^{(k)i}}\dfrac{\partial}{\partial \widetilde{y}^{(k)j}}. \end{cases}$$

The distribution $V_1 : u \in T^kM \to V_{1,u} \subset T_uT^kM$ generated by the tangent vectors $\left\{\dfrac{\partial}{\partial y^{(1)i}}, \cdots, \dfrac{\partial}{\partial y^{(k)i}}\right\}_u$ is a vertical distribution on the bundle T^kM. Its local dimension is kn. Similarly, the distribution $V_2 : u \in T^kM \to V_{2,u} \subset T_uT^kM$ generated by $\left\{\dfrac{\partial}{\partial y^{(2)i}}, \cdots, \dfrac{\partial}{\partial y^{(k)i}}\right\}_u$ is a subdistribution of V_1 of local dimension $(k-1)n$. So, by this procedure on obtains a sequence of integrable distributions $V_1 \supset V_2 \supset \cdots \supset V_k$. The last distribution V_k is generated by $\left\{\dfrac{\partial}{\partial y^{(k)i}}\right\}_u$ and $\dim V_k = n$ ([7]).

Hereafter, we consider the open submanifold

$$\widetilde{T^kM} = T^kM \setminus \{\mathbf{0}\} = \left\{(x, y^{(1)}, ..., y^{(k)}) \in T^kM | rank\ ||y^{(1)i}|| = 1\right\},$$

where $\mathbf{0}$ is the null section of the projection $\pi^k : T^kM \to M$.

The following operators in algebra of functions $\mathcal{F}(T^kM)$

$$
\begin{aligned}
&\overset{1}{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(k)i}},\\
&\overset{2}{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(k-1)i}} + 2y^{(2)i}\frac{\partial}{\partial y^{(k)i}},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\overset{k}{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i}\frac{\partial}{\partial y^{(2)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k)i}}
\end{aligned}
\tag{3}
$$

are k vector fields, globally defined on T^kM and linearly independent on the manifold $\widetilde{T^kM} = T^kM \setminus \{\mathbf{0}\}$, $\overset{1}{\Gamma}$ belongs of distribution V_k, $\overset{2}{\Gamma}$ belongs of distribution V_{k-1}, ..., $\overset{k}{\Gamma}$ belongs of distribution V_1 (see [7]). $\overset{1}{\Gamma}$, $\overset{2}{\Gamma}$, ..., $\overset{k}{\Gamma}$ are called *Liouville vector fields.*

In applications we shall use also the following nonlinear operator, which is not a vector field,

$$
\Gamma = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}}. \tag{4}
$$

Under a coordinates transformation (1) on T^kM, Γ changes as follows:

$$
\Gamma = \widetilde{\Gamma} + \left\{ y^{(1)i}\frac{\partial \widetilde{y}^{(k)j}}{\partial x^i} + \cdots + ky^{(k)i}\frac{\partial \widetilde{y}^{(k)j}}{\partial y^{(k-1)i}} \right\} \frac{\partial}{\partial \widetilde{y}^{(k)j}}. \tag{5}
$$

A *k-tangent structure* J on T^kM is defined as usually ([7]) by the following $\mathcal{F}(T^kM)$-linear mapping $J : \mathcal{X}(T^kM) \to \mathcal{X}(T^kM)$:

$$
\begin{aligned}
J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^{(1)i}}, J\left(\frac{\partial}{\partial y^{(1)i}}\right) = \frac{\partial}{\partial y^{(2)i}}, ...,\\
J\left(\frac{\partial}{\partial y^{(k-1)i}}\right) = \frac{\partial}{\partial y^{(k)i}}, J\left(\frac{\partial}{\partial y^{(k)i}}\right) = 0.
\end{aligned}
\tag{6}
$$

J is a tensor field of type $(1,1)$, globally defined on T^kM.

Definition 2.1 *([7]) A k-semispray on T^kM is a vector field $S \in \mathcal{X}(T^kM)$ with the property*

$$
JS = \overset{k}{\Gamma}. \tag{7}
$$

Obviously, there not always exists a k-semispray, globally defined on T^kM. Therefore the notion of local k-semispray is necessary. For example, if M is a paracompact manifold then on T^kM there exists local k-semisprays ([7]).

Theorem 2.1 *([7]) i) A k-semispray S can be uniquely written in local coordinates in the form:*

$$
\begin{aligned}
S \quad = \quad & y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}} - \\
- \quad & (k+1)G^i(x, y^{(1)}, ..., y^{(k)})\frac{\partial}{\partial y^{(k)i}}\,.
\end{aligned}
\tag{8}
$$

ii) With respect to (1) the coefficients $G^i(x, y^{(1)}, ..., y^{(k)})$ change as follows:

$$
(9) \qquad \begin{aligned}
(k+1)\widetilde{G}^i &= (k+1)G^j \frac{\partial \widetilde{x}^i}{\partial x^j} - \\
&- \left(y^{(1)j} \frac{\partial \widetilde{y}^{(k)i}}{\partial x^j} + \cdots + k y^{(k)j} \frac{\partial \widetilde{y}^{(k)i}}{\partial y^{(k-1)j}} \right).
\end{aligned}
$$

iii) If the functions $G^i(x, y^{(1)}, ..., y^{(k)})$ are given on every domain of local chart of T^kM, so that (9) holds, then the vector field S from (8) is a k-semispray.

Let us consider a curve $c : I \to M$, represented in a local chart (U, φ) by $x^i = x^i(t)$, $t \in I$. Thus, the mapping $\widetilde{c} : I \to T^kM$, given on $(\pi^k)^{-1}(U)$, by

$$
(10) \qquad x^i = x^i(t), y^{(1)i}(t) = \frac{1}{1!}\frac{dx^i}{dt}(t), \ldots, y^{(k)i}(t) = \frac{1}{k!}\frac{d^k x^i}{dt^k}(t), t \in I
$$

is a curve in T^kM, called the *k-extension* to T^kM of the curve c.

A curve $c : I \to M$ is called *k-path* of a k-semispray S (from (8)) if its k-extension $\widetilde{c}$ is an integral curve for S, that is

(11)

$$
\left\{ \frac{dx^i}{dt} = y^{(1)i}, \; \frac{dy^{(1)i}}{dt} = 2y^{(2)i}, \; ..., \frac{dy^{(k-1)i}}{dt} = k y^{(k)i}, \frac{dy^{(k)i}}{dt} = -(k+1)G^i \right. .
$$

Definition 2.2 *The k-semispray S is called k-spray if the functions $\left(G^i(x, y^{(1)}, ..., y^{(k)})\right)$ are $(k+1)$-homogeneous, that is*

$$
G^i(x, \lambda y^{(1)}, ..., \lambda^k y^{(k)}) = \lambda^{k+1} G^i(x, y^{(1)}, ..., y^{(k)}), \; \forall \lambda > 0.
$$

Like in the case of tangent bundle, an Euler Theorem holds. That is, a function $f \in \mathcal{F}(\widetilde{T^kM})$ is r-homogeneous if and only if

$$
\mathcal{L}_{\overset{k}{\Gamma}} f = rf.
$$

Then a k-semispray S is a k-spray if and only if

$$
(12) \qquad y^{(1)h}\frac{\partial G^i}{\partial y^{(1)h}} + 2y^{(2)h}\frac{\partial G^i}{\partial y^{(2)h}} + \cdots + k y^{(k)h}\frac{\partial G^i}{\partial y^{(k)h}} = (k+1)G^i.
$$

Definition 2.3 *A vector subbundle NT^kM of the tangent bundle $(TT^kM, d\pi^k, M)$ which is supplementary to the vertical subbundle V_1T^kM,*

$$
(13) \qquad TT^kM = NT^kM \oplus V_1T^kM
$$

is called a nonlinear connection on T^kM.

The fibres of NT^kM determine a horizontal distribution $N : u \in T^kM \to N_uT^kM \subset T_uT^kM$ supplementary to the vertical distribution V_1, that is

$$T_uT^kM = N_uT^kM \oplus V_{1,u}T^kM,\ \forall u \in T^kM. \tag{14}$$

The dimension of horizontal distribution N is n.

If the base manifold M is paracompact then on T^kM there exists the nonlinear connections ([7]).

There exists a unique local basis, adapted to the horizontal distribution N, $\left\{\dfrac{\delta}{\delta x^i}\right\}_{i=\overline{1,n}}$, such that $d\pi^k\left(\dfrac{\delta}{\delta x^i}|_u\right) = \dfrac{\partial}{\partial x^i}|_{\pi^k(u)}$, $i = 1, ..., n$. More over, on each domain of local chart of T^kM there exists the functions $\underset{(1)}{N}{}^i_j$, $\underset{(2)}{N}{}^i_j$, ..., $\underset{(k)}{N}{}^i_j$ such that

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{(1)}{N}{}^j_i \frac{\partial}{\partial y^{(1)j}} - \cdots - \underset{(k)}{N}{}^j_i \frac{\partial}{\partial y^{(k)j}}\,. \tag{15}$$

The functions $\underset{(1)}{N}{}^i_j$, $\underset{(2)}{N}{}^i_j$, ...,$\underset{(k)}{N}{}^i_j$ are called *the primal coefficients* of the nonlinear connection N and under a coordinates transformation (1) on T^kM this coefficients are changing by the rule:

(16)

$$\begin{cases} \underset{(1)}{\widetilde{N}}{}^i_m \dfrac{\partial\widetilde{x}^m}{\partial x^j} &= \dfrac{\partial\widetilde{x}^i}{\partial x^m}\underset{(1)}{N}{}^m_j - \dfrac{\partial\widetilde{y}^{(1)i}}{\partial x^j}, \\ \underset{(2)}{\widetilde{N}}{}^i_m \dfrac{\partial\widetilde{x}^m}{\partial x^j} &= \dfrac{\partial\widetilde{x}^i}{\partial x^m}\underset{(2)}{N}{}^m_j + \dfrac{\partial\widetilde{y}^{(1)i}}{\partial x^m}\underset{(1)}{N}{}^m_j - \dfrac{\partial\widetilde{y}^{(2)i}}{\partial x^j}, ..., \\ \underset{(k)}{\widetilde{N}}{}^i_m \dfrac{\partial\widetilde{x}^m}{\partial x^j} &= \dfrac{\partial\widetilde{x}^i}{\partial x^m}\underset{(k)}{N}{}^m_j + \dfrac{\partial\widetilde{y}^{(1)i}}{\partial x^m}\underset{(k-1)}{N}{}^m_j + \cdots + \dfrac{\partial\widetilde{y}^{(k-1)i}}{\partial x^m}\underset{(1)}{N}{}^m_j - \dfrac{\partial\widetilde{y}^{(k)i}}{\partial x^j}. \end{cases}$$

Conversely, if on each local chart of T^kM a set of functions $\underset{(1)}{N}{}^i_j$, ..., $\underset{(k)}{N}{}^i_j$ is given so that, according to (1), the equalities (16) hold, then there exists on T^kM a unique nonlinear connection N which has as coefficients just the given set of function ([7]).

The local adapted basis $\left\{\dfrac{\delta}{\delta x^i}, \dfrac{\delta}{\delta y^{(1)i}}, \cdots, \dfrac{\delta}{\delta y^{(k)i}}\right\}_{i=\overline{1,n}}$ is given by (15) and

$$\begin{aligned} &\frac{\delta}{\delta y^{(1)i}} = \frac{\partial}{\partial y^{(1)i}} - \underset{(1)}{N}{}^j_i \frac{\partial}{\partial y^{(2)j}} - \cdots - \underset{(k-1)}{N}{}^j_i \frac{\partial}{\partial y^{(k)j}}, ..., \\ &\frac{\delta}{\delta y^{(k-1)i}} = \frac{\partial}{\partial y^{(k-1)i}} - \underset{(1)}{N}{}^j_i \frac{\partial}{\partial y^{(k)j}}, \frac{\delta}{\delta y^{(k)i}} = \frac{\partial}{\partial y^{(k)j}} \end{aligned} \tag{17}$$

and *the dual basis* (or *the adapted cobasis*) of adapted basis is

$\left\{\delta x^i, \delta y^{(1)i}, \dots, \delta y^{(k)i}\right\}_{i=\overline{1,n}}$, where $\delta x^i = dx^i$ and

$$(18)\qquad \begin{cases} \delta y^{(1)i} &= dy^{(1)i} + \underset{(1)}{M}{}^i_j dx^j, \\ \delta y^{(2)i} &= dy^{(2)i} + \underset{(1)}{M}{}^i_j dy^{(1)j} + \underset{(2)}{M}{}^i_j dx^j, ..., \\ \delta y^{(k)i} &= dy^{(k)i} + \underset{(1)}{M}{}^i_j dy^{(k-1)j} + \cdots + \underset{(k)}{M}{}^i_j dx^j \end{cases}$$

and

$$(19)\qquad \begin{cases} \underset{(1)}{M}{}^i_j &= \underset{(1)}{N}{}^i_j, \\ \underset{(2)}{M}{}^i_j &= \underset{(2)}{N}{}^i_j + \underset{(1)}{N}{}^i_m \underset{(1)}{M}{}^m_j,\ ..., \\ \underset{(k)}{M}{}^i_j &= \underset{(k)}{N}{}^i_j + \underset{(k-1)}{N}{}^i_m \underset{(1)}{M}{}^m_j + \cdots + \underset{(1)}{N}{}^i_m \underset{(k-1)}{M}{}^m_j. \end{cases}$$

Conversely, if the adapted cobasis $\left\{\delta x^i, \delta y^{(1)i}, \dots, \delta y^{(k)i}\right\}_{i=\overline{1,n}}$ is given in the form (18), then the adapted basis $\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \cdots, \frac{\delta}{\delta y^{(k)i}}\right\}_{i=\overline{1,n}}$ is expressed in the form (17), where

$$(20)\qquad \begin{cases} \underset{(1)}{N}{}^i_j &= \underset{(1)}{M}{}^i_j, \\ \underset{(2)}{N}{}^i_j &= \underset{(2)}{M}{}^i_j - \underset{(1)}{N}{}^i_m \underset{(1)}{M}{}^m_j,\ ..., \\ \underset{(k)}{N}{}^i_j &= \underset{(k)}{M}{}^i_j - \underset{(k-1)}{N}{}^i_m \underset{(1)}{M}{}^m_j - \cdots - \underset{(1)}{N}{}^i_m \underset{(k-1)}{M}{}^m_j. \end{cases}.$$

The functions $\underset{(1)}{M}{}^i_j$, $\underset{(2)}{M}{}^i_j$, ..., $\underset{(k)}{M}{}^i_j$ are called *the dual coefficients* of the nonlinear connection N.

A nonlinear connection N is complete determined by a system of functions $\underset{(1)}{M}{}^i_j$, ..., $\underset{(k)}{M}{}^i_j$ which is given on each domain of local chart on T^kM, so that, according to (1), the relations hold:

(21)

$$\begin{cases} \underset{(1)}{M}{}^m_j \dfrac{\partial \widetilde{x}^i}{\partial x^m} &= \dfrac{\partial \widetilde{x}^m}{\partial x^j} \underset{(1)}{\widetilde{M}}{}^i_m + \dfrac{\partial \widetilde{y}^{(1)i}}{\partial x^j}, \\ \underset{(2)}{M}{}^m_j \dfrac{\partial \widetilde{x}^i}{\partial x^m} &= \dfrac{\partial \widetilde{x}^m}{\partial x^j} \underset{(2)}{\widetilde{M}}{}^i_m + \dfrac{\partial \widetilde{y}^{(1)m}}{\partial x^j} \underset{(1)}{\widetilde{M}}{}^i_m + \dfrac{\partial \widetilde{y}^{(2)i}}{\partial x^j}, ..., \\ \underset{(k)}{M}{}^m_j \dfrac{\partial \widetilde{x}^i}{\partial x^m} &= \dfrac{\partial \widetilde{x}^m}{\partial x^j} \underset{(k)}{\widetilde{M}}{}^i_m + \dfrac{\partial \widetilde{y}^{(1)m}}{\partial x^j} \underset{(k-1)}{\widetilde{M}}{}^i_m + \cdots + \dfrac{\partial \widetilde{y}^{(k-1)m}}{\partial x^j} \underset{(1)}{\widetilde{M}}{}^i_m + \dfrac{\partial \widetilde{y}^{(k)i}}{\partial x^j}. \end{cases}$$

Let $c : I \to M$ be a parametrized curve on the base manifold M, given by $x^i = x^i(t)$, $t \in I$. If we consider its k-extension $\widetilde{c}$ to T^kM, then we say that c is *an autoparallel curve* for the nonlinear connection N if its k-extension $\widetilde{c}$ is an horizontal curve, that is $\dfrac{d\widetilde{c}}{dt}$ belongs to the horizontal distribution.

From (18) and

$$\frac{d\widetilde{c}}{dt}=\frac{dx^i}{dt}\frac{\delta}{\delta x^i}+\frac{\delta y^{(1)i}}{dt}\frac{\delta}{\delta y^{(1)i}}+\cdots+\frac{\delta y^{(k)i}}{dt}\frac{\delta}{\delta y^{(k)i}} \tag{22}$$

it result that the autoparallels curves of the nonlinear connection N with the dual coefficients $\underset{(1)}{M}{}^i_j, ..., \underset{(k)}{M}{}^i_j$ are characterized by the system of differential equations ([7]):

$$\begin{cases} y^{(1)i}=\dfrac{dx^i}{dt},\ y^{(2)i}=\dfrac{1}{2!}\dfrac{d^2x^i}{dt^2},\ ...,\ y^{(k)i}=\dfrac{1}{k!}\dfrac{d^kx^i}{dt^k}, \\ \dfrac{\delta y^{(1)i}}{dt}=\dfrac{dy^{(1)i}}{dt}+\underset{(1)}{M}{}^i_j\dfrac{dx^j}{dt}=0, \\ \dfrac{\delta y^{(2)i}}{dt}=\dfrac{dy^{(2)i}}{dt}+\underset{(1)}{M}{}^i_j\dfrac{dy^{(1)j}}{dt}+\underset{(2)}{M}{}^i_j\dfrac{dx^j}{dt}=0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{\delta y^{(k)i}}{dt}=\dfrac{dy^{(k)i}}{dt}+\underset{(1)}{M}{}^i_j\dfrac{dy^{(k-1)j}}{dt}+\cdots+\underset{(k)}{M}{}^i_j\dfrac{dx^j}{dt}=0. \end{cases} \tag{23}$$

Now, let be $S=\overset{1}{S}$ a k-semispray with the coefficients $\overset{1}{G}{}^i=G^i(x,y^{(1)},...,y^{(k)})$ like in (8). Then the set of functions

$$\begin{cases} \underset{(1)}{M}{}^i_j=\dfrac{\partial G^i}{\partial y^{(k)j}}, \\ \underset{(2)}{M}{}^i_j=\dfrac{1}{2}\left(S\underset{(1)}{M}{}^i_j+\underset{(1)}{M}{}^i_m\underset{(1)}{M}{}^m_j\right), \\ \dots\dots\dots\dots\dots\dots \\ \underset{(k)}{M}{}^i_j=\dfrac{1}{k}\left(S\ \underset{(k-1)}{M}{}^i_j+\underset{(1)}{M}{}^i_m\underset{(k-1)}{M}{}^m_j\right) \end{cases} \tag{24}$$

gives the dual coefficients of a nonlinear connection N determined only by the k-semispray S (see the book [7] of Radu Miron).

Other result, obtained by Ioan Bucătaru ([1]), give a second nonlinear connection N^* on T^kM determined only by the k-semispray S. That is, the following set of functions

$$\underset{(1)}{M}{}^{*i}_j=\frac{\partial G^i}{\partial y^{(k)j}},\ \underset{(2)}{M}{}^{*i}_j=\frac{\partial G^i}{\partial y^{(k-1)j}},\ ...,\ \underset{(k)}{M}{}^{*i}_j=\frac{\partial G^i}{\partial y^{(1)j}} \tag{25}$$

is the set of dual coefficients of a nonlinear connection N^*.

Let us consider the set of functions $(\overset{2}{G^i}(x, y^{(1)}, ..., y^{(k)}))$, given on every domain of local chart by
(26)

$$\overset{2}{G^i} = \frac{1}{k+1}\overset{k}{\Gamma}\overset{1}{G^i} = \frac{1}{k+1}y^{(1)h}\frac{\partial \overset{1}{G^i}}{\partial y^{(1)h}} + \frac{2}{k+1}y^{(2)h}\frac{\partial \overset{1}{G^i}}{\partial y^{(2)h}} + \cdots + \frac{k}{k+1}y^{(k)h}\frac{\partial \overset{1}{G^i}}{\partial y^{(k)h}}.$$

Using (5) we obtain that the functions $\overset{2}{G^i}$ verifies (9). So, the functions $\overset{2}{G^i}$ represent the coefficients of a k-semispray $\overset{2}{S}$,

$$\begin{aligned} \overset{2}{S} &= y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}} - \\ &- (k+1)\overset{2}{G^i}(x, y^{(1)}, ..., y^{(k)})\frac{\partial}{\partial y^{(k)i}}. \end{aligned} \tag{27}$$

Obviously, there exists two nonlinear connections on T^kM, which depend only by the k-semispray $\overset{2}{S}$:
$\overset{2}{N}$ with the dual coefficients

$$\begin{cases} \underset{(1)}{\overset{2}{M}}{}^i_j = \dfrac{\partial G^i}{\partial y^{(k)j}}, \\ \underset{(2)}{\overset{2}{M}}{}^i_j = \dfrac{1}{2}\left(S\underset{(1)}{\overset{2}{M}}{}^i_j + \underset{(1)}{\overset{2}{M}}{}^i_m \underset{(1)}{\overset{2}{M}}{}^m_j\right), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \underset{(k)}{\overset{2}{M}}{}^i_j = \dfrac{1}{k}\left(S\ \underset{(k-1)}{\overset{2}{M}}{}^i_j + \underset{(1)}{\overset{2}{M}}{}^i_m \underset{(k-1)}{\overset{2}{M}}{}^m_j\right) \end{cases} \tag{28}$$

and $\overset{2}{N}{}^*$ with the dual coefficients

$$\underset{(1)}{\overset{2}{M}}{}^{*i}_j = \frac{\partial \overset{2}{G^i}}{\partial y^{(k)j}},\ \underset{(2)}{\overset{2}{M}}{}^{*i}_j = \frac{\partial \overset{2}{G^i}}{\partial y^{(k-1)j}},\ ...,\ \underset{(k)}{\overset{2}{M}}{}^{*i}_j = \frac{\partial G^i}{\partial y^{(1)j}}. \tag{29}$$

By this method is obtained a sequence of k-semisprays $\left(\overset{m}{S}\right)_{m\geq 1}$ and two sequence of nonlinear connections, $\left(\overset{m}{N}\right)_{m\geq 1}$, $\left(\overset{m}{N}{}^*\right)_{m\geq 1}$.

From (11), (23) and (26) we have the following results:

Proposition 2.1 *If c is an autoparallel curve for nonlinear connection $\overset{1}{N}{}^*$, then c is a k-path of k-semispray $\overset{2}{S}$.*

Theorem 2.2 *The following assertions are equivalent:*

i) the k-semispray $\overset{1}{S}$ *is a k-spray;*

ii) the k-paths of $\overset{1}{S}$ *and* $\overset{2}{S}$ *coincide.*

Theorem 2.3 *If* $\overset{1}{S}$ *is a k-spray then* $\underset{(1)}{\overset{1}{M}}{}^{i}_{j}$, ..., $\underset{(k)}{\overset{1}{M}}{}^{i}_{j}$ *(or* $\underset{(1)}{\overset{1}{M}}{}^{*i}_{j}$, ..., $\underset{(k)}{\overset{1}{M}}{}^{*i}_{j}$*) are homogeneous functions of degree* 1, 2, ..., k, *respectively. The same property have the primal coefficients* $\underset{(1)}{\overset{1}{N}}{}^{i}_{j}$, ..., $\underset{(k)}{\overset{1}{N}}{}^{i}_{j}$ *(or* $\underset{(1)}{\overset{1}{N}}{}^{*i}_{j}$, ..., $\underset{(k)}{\overset{1}{N}}{}^{*i}_{j}$*).*

We remark that the converse of this proposition is generally not valid and we have the result:

Theorem 2.4 *If* $\overset{1}{S}$ *is a k-spray then the sequence* $\left(\overset{m}{S}\right)_{m\geq 1}$ *is constant and the sequences* $\left(\overset{m}{N}\right)_{m\geq 1}$, $\left(\overset{m}{N^*}\right)_{m\geq 1}$ *are constant.*

3 The k-Semispray of a Nonlinear Connection in a Lagrange Space of Order k

A Lagrangian of order k is a mapping $L : T^kM \to \mathbf{R}$. L is called *differentiable* if it is of C^∞-class on $\widetilde{T^kM}$ and continuous on the null section of the projection $\pi^k : T^kM \to M$.

The Hessian of a differentiable Lagrangian L, with respect to the variables $y^{(k)i}$ on $\widetilde{T^kM}$ is the matrix $||2g_{ij}||$, where

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^{(k)i}\partial y^{(k)j}}. \tag{30}$$

We have that g_{ij} is a d-tensor field on the manifold $\widetilde{T^kM}$, covariant of order 2, symmetric (see [7]).

If

$$rank\ ||g_{ij}|| = n, \quad \text{on} \quad \widetilde{T^kM} \tag{31}$$

we say that $L(x, y^{(1)}, ..., y^{(k)})$ is a *regular* (or *nondegenerate*) Lagrangian.

The existence of the regular Lagrangians of order k is proved for the case of paracompacts manifold M in the book [7] of Radu Miron.

Definition 3.1 *([7]) We call a Lagrange space of order k a pair* $L^{(k)n} = (M, L)$, *formed by a real n-dimensional manifold M and a regular differentiable Lagrangian of order k,* $L : (x, y^{(1)}, ..., y^{(k)}) \in T^kM \to L(x, y^{(1)}, ..., y^{(k)}) \in \mathbf{R}$, *for which the quadratic form* $\Psi = g_{ij}\xi^i\xi^j$ *on* $\widetilde{T^kM}$ *has a constant signature.*

L is called *the fundamental function* and g_{ij} *the fundamental* (or *metric*) *tensor field* of the space $L^{(k)n}$.

It is known that for any regular Lagrangian of order k, $L(x, y^{(1)}, ..., y^{(k)})$, there exists a k-semispray S_L determined only by the Lagrangian L (see [7]). The coefficients of S_L are given by

$$(k+1)G^i = \frac{1}{2} g^{ij} \left\{ \Gamma\left(\frac{\partial L}{\partial y^{(k)j}}\right) - \frac{\partial L}{\partial y^{(k-1)j}} \right\}. \tag{32}$$

This k-semispray S_L depending only by L will be called *canonical*. If L is globally defined on T^kM, then S_L has the same property on $\widetilde{T^kM}$.

From (24) and (25) it result that there exists two nonlinear connections: *Miron's connection* N and *Bucătaru's connection* N^* which depending only by the Lagrangian L. For this reason, both are called *canonical*.

So, the coefficients of k-semisprays $\overset{m}{S}$ and the coefficients of nonlinear connections $\overset{m}{N}$, $\overset{m}{N}{}^*$ depend only by the Lagrangian L, for any $m \geq 1$, but their expressions is not attractive for us.

Interesting results appear for Finsler spaces of order k.

Definition 3.2 *([8]) A Finsler space of order k is a pair $F^{(k)n} = (M, F)$ formed by a real differentiable manifold M of dimension n and a function $F : T^kM \to \mathbf{R}$ having the following properties:*

i) F is differentiable on $\widetilde{T^kM}$ and continuous on null section $0 : M \to T^kM$;

ii) F is positive;

iii) F is k-homogeneous;

iv) the Hessian of F^2 with elements

$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^{(k)i} \partial y^{(k)j}} \tag{33}$$

is positively defined on $\widetilde{T^kM}$.

The function F is called *the fundamental function* and the d-tensor field g_{ij} is called *fundamental*(or *metric*) *tensor field* of the Finsler space of order k, $F^{(k)n}$.

The class of spaces $F^{(k)n}$ is a subclass of spaces $L^{(k)n}$.

Taking into account the k-homogeneity of the fundamental function F and $2k$-homogeneity of F^2 we get:

1. the coefficients G^i of the canonical k-semispray S_{F^2}, determined only by the fundamental function F,

$$(k+1)G^i = \frac{1}{2} g^{ij} \left\{ \Gamma\left(\frac{\partial F^2}{\partial y^{(k)j}}\right) - \frac{\partial F^2}{\partial y^{(k-1)j}} \right\}, \tag{34}$$

is $(k+1)$-homogeneous functions, that is S_{F^2} is a k-spray;

2. the dual coefficients of *the Cartan nonlinear connection* N associated to Finsler space of order k, $F^{(k)n}$ (see [8]),

$$
(35) \qquad
\begin{aligned}
&\underset{(1)}{M}{}^i_j = \frac{1}{2(k+1)} \frac{\partial}{\partial y^{(k)j}} \left\{ g^{im} \left[\Gamma \left(\frac{\partial F^2}{\partial y^{(k)m}} \right) - \frac{\partial F^2}{\partial y^{(k-1)m}} \right] \right\}, \\
&\underset{(2)}{M}{}^i_j = \frac{1}{2} \left(S_{F^2} \underset{(1)}{M}{}^i_j + \underset{(1)}{M}{}^i_m \underset{(1)}{M}{}^m_j \right), \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\underset{(k)}{M}{}^i_j = \frac{1}{k} \left(S_{F^2} \underset{(k-1)}{M}{}^i_j + \underset{(1)}{M}{}^i_m \underset{(k-1)}{M}{}^m_j \right),
\end{aligned}
$$

are homogeneous functions of degree 1, 2, ..., k, respectively, and the primal coefficients has the same property;

3. the dual coefficients of Bucătaru's connection N^* associated to Lagrangian F^2 are also homogeneous functions of degree 1, 2, ..., k, respectively, and the primal coefficients has the same property.

Using the previous results, we obtain the results:

Theorem 3.1 *If $F^{(k)n} = (M, F)$ is a Finsler space of order k, then:*

a) the sequence $\left(\overset{m}{S}\right)_{m \geq 1}$ is constant, $\overset{1}{S}$ being the canonical k-spray S_{F^2};

b) the sequences of nonlinear connections $\left(\overset{m}{N}\right)_{m \geq 1}$, $\left(\overset{m}{N}{}^*\right)_{m \geq 1}$ are constants, $\overset{1}{N}$ being the Cartan nonlinear connection of $F^{(k)n}$ and $\overset{1}{N}{}^*$ being the Bucătaru's connection for $L = F^2$.

4 Conclusions

In this paper was studied the the relation between semisprays and nonlinear connections on the k-tangent bundle T^kM of a manifold M. This results was generalized by the author from the 2-tangent bundle T^2M ([11]). More that, the relationship between SOPDEs and nonlinear connections on the tangent bundle of k^1-velocities of a manifold M (i.e. the Whitney sum of k copies of TM, $T^1_kM = TM \oplus \cdots \oplus TM$) was studied by F. Munteanu in [14] (2006) and by N. Roman-Roy, M. Salgado, S. Vilarino in [15] (2011).

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