Some New Symmetric Relations and the Prediction of Left and Right Handed Neutrino Masses using Koide's Relation

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Masses of the three generations of charged leptons are known to completely satisfy the Koide's mass relation. But the question remains if such a relation exists for neutrinos? In this paper, by considering SeeSaw mechanism as the mechanism generating tiny neutrino masses, we show how neutrinos satisfy the Koide's mass relation, on the basis of which we systematically give exact values of not only left but also right handed neutrino masses.

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I. INTRODUCTION

Despite being the most successful model of particle physics, Standard Model (SM) fails to answer many questions like why the parameters of SM are the way they are? Is there any relation among these parameters? Why the Koide's relation for the charged lepton is 2/3? Yoshio Koide [1, 2] pointed out that a very simple relationship exists for the pole masses (given in the Table I) of the three generations of charged leptons,

$$(m_e + m_\mu + m_\tau) = \frac{2}{3}(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2. \tag{1}$$

which is surprisingly precise to a good degree of accuracy. This precision inspired Koide to propose models [3–5] in an attempt to explain the underlying physics. Various attempts have been made to extend this formula to other particles. In [6] some speculations related to the extension of Eq. [1] to quark and leptons are given along with its relations to recent theoretical developments. Different ideas following the implementation of this formula can also be found [7–10]. A geometric interpretation for Koide's relation was given in [11] in which the square root of the mass of leptons $\sqrt{m_l}$ is used to construct a vector $\overrightarrow{V}$, such that

$$\overrightarrow{V} = \left(\sqrt{m_e}, \quad \sqrt{m_\mu}, \quad \sqrt{m_\tau}\right), \tag{2}$$

then Koide's formula can be considered equivalent to the angle between the vector $(1, 1, 1)$ and $\overrightarrow{V}$ which is $\frac{\pi}{4}$, which will be considered, in details, in Sect. III. The questions that follow from the above interpretation are: why the vector is $(1, 1, 1)$ and why is the angle $\frac{\pi}{4}$? The aim of this paper is to give a meaning to the geometric interpretation and to extend Koide's formula to neutrinos such that the masses of left handed and right handed neutrinos can be predicted.

Table I: Mass of leptons

lepton	mass (MeV)
e	$0.510998928 \pm 0.000000011$
μ	$105.6583715 \pm 0.0000035$
τ	1776.82 ± 0.16

The plan of the paper is as follow: In Sect.2 we find an analytical formula to achieve the masses of neutrinos using the data provided by experiments. Sect.3 is served to give a meaning to the geometrical interpretation given by Foot[11]. In Sect.4 we devise a formula to find the value of right handed neutrino mass terms. Considering a relation between left and right handed neutrinos we can solve the analytical formulas for neutrino masses, details of which are given in Sect.5. Sect.6 the results and propositions are given. The last section is the summary and conclusion.

II. ANALYTICAL FORMULA FOR NEUTRINO MASSES

The neutrino mass term has the form

$$\mathcal{L} = \frac{1}{2}\begin{pmatrix} \bar{\nu}_l & \bar{\nu}_R^c \end{pmatrix} \mathcal{M} \begin{pmatrix} \nu_l^c \\ \nu_R \end{pmatrix} + h.c. \tag{3}$$

Supposing that the mass matrix $\mathcal{M}_{mass}$ can be diagonalized as follow

$$\begin{pmatrix} 0 & 0 & 0 & m_{\mathtt{D}_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{\mathtt{D}_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{\mathtt{D}_3} \\ m_{\mathtt{D}_1} & 0 & 0 & M_1 & 0 & 0 \\ 0 & m_{\mathtt{D}_2} & 0 & 0 & M_2 & 0 \\ 0 & 0 & m_{\mathtt{D}_3} & 0 & 0 & M_3 \end{pmatrix} \tag{4}$$

where M_1, M_2 and M_3 are the Majorana mass coefficients.

The eigenvalues of the matrix are:

$$\begin{aligned} &\frac{1}{2}M_1 \pm \frac{1}{2}\sqrt{M_1^2 + 4m_{\mathtt{D}_1}^2}, \\ &\frac{1}{2}M_2 \pm \frac{1}{2}\sqrt{M_2^2 + 4m_{\mathtt{D}_2}^2}, \\ &\frac{1}{2}M_3 \pm \frac{1}{2}\sqrt{M_3^2 + 4m_{\mathtt{D}_3}^2}. \end{aligned} \tag{5}$$

When $M_i \gg m_{\mathtt{D}_i} (i = 1, 2, 3)$, the neutrino masses would be

$$\frac{m_{\mathtt{D}_1}^2}{M_1}, \quad \frac{m_{\mathtt{D}_2}^2}{M_2}, \quad \frac{m_{\mathtt{D}_3}^2}{M_3}, \tag{6}$$

which is just the SeeSaw mechanism. The strict form of $\mathcal{M}_{mass}$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & m_{\mathtt{D}_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{\mathtt{D}_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{\mathtt{D}_3} \\ m_{\mathtt{D}_1} & 0 & 0 & M_1 & m_{_1} & m_{_2} \\ 0 & m_{\mathtt{D}_2} & 0 & m_{_1} & M_2 & m_{_3} \\ 0 & 0 & m_{\mathtt{D}_3} & m_{_2} & m_{_3} & M_3 \end{pmatrix}. \tag{7}$$

$m_{\mathtt{D}_1}, m_{\mathtt{D}_2}, m_{\mathtt{D}_3}$ are the Dirac masses. The constants $M_1, M_2, M_3, m_{_1}, m_{_2}, m_{_3}$ are unknown so the neutrino masses can not be calculated directly. The case in which the mass matrix has the most general form involves so many parameters and becomes so complicated such that it cannot be solved therefore, we take a simpler form.

There is no exact data available about the neutrino masses but the cosmological measurements [12] give a boundary of active neutrino masses

$$\sum_i m_i < 0.17\text{eV}. \tag{8}$$

Also the neutrino mass differences [14] are given by experimental measurements of solar, atmospheric, accelerator and reactor neutrinos.

$$\begin{aligned} |\Delta m_{21}^2| &= (7.53 \pm 0.18) \cdot 10^{-5}\text{eV}^2, \\ |\Delta m_{32}^2| &= (2.44 \pm 0.06) \cdot 10^{-3}\text{eV}^2. \end{aligned} \tag{9}$$

If we denote neutrino masses as m_{ν_1}, m_{ν_2}, and m_{ν_3} then with the help of Eq.(9) we can write

$$\begin{aligned} |m_{\nu_1}^2 - m_{\nu_2}^2| &= |\Delta m_{21}^2|, \\ |m_{\nu_3}^2 - m_{\nu_2}^2| &= |\Delta m_{32}^2|. \end{aligned} \tag{10}$$

Putting the values of $|\Delta m_{21}^2|$ and $|\Delta m_{32}^2|$ in Eq.(10) we get two analytical formulas, such that

$$\begin{aligned} m_{\nu_2}^2 &= m_{\nu_1}^2 + 7.53 \times 10^{-5}\text{eV}^2, \\ m_{\nu_3}^2 &= m_{\nu_2}^2 + 2.44 \times 10^{-3}\text{eV}^2. \end{aligned} \tag{11}$$

Following the Koide's formula for leptons, we can write a relation for neutrinos as

$$k_{\nu_L}^2 = \frac{(m_{\nu_1} + m_{\nu_2} + m_{\nu_3})}{(\sqrt{m_{\nu_1}} + \sqrt{m_{\nu_2}} + \sqrt{m_{\nu_3}})^2}, \tag{12}$$

Using Eq.(11), Eq. (12) can be rewritten as

$$k_{\nu_L}^2 = \frac{(m_{\nu_1} + \sqrt{m_{\nu_1}^2 + 7.53 \times 10^{-5}\mathrm{eV}^2} + \sqrt{m_{\nu_1}^2 + 251.53 \times 10^{-5}\mathrm{eV}^2})}{(\sqrt{m_{\nu_1}} + (m_{\nu_1}^2 + 7.53 \times 10^{-5}\mathrm{eV}^2)^{1/4} + (\mathrm{m}_{\nu_1}^2 + 251.53 \times 10^{-5}\mathrm{eV}^2)^{1/4})^2} \tag{13}$$

The above equation can be solved to find the value of m_{ν_1} if we can somehow constrain the value of $k_{\nu_L}^2$.

III. MEANING OF FOOT'S GEOMETRICAL INTERPRETATION

In this section we present Foot's geometrical interpretation explaining what the vector $\overrightarrow{u} = (1, \quad 1, \quad 1)$ means.

The lepton masses have an equal status in Koide's relation which indicates a presence of some underneath symmetry. With the help of this symmetry we can give a Koide's-like relation for Dirac neutrino mass terms.

The neutrino mass matrix is given by Eq.(7). We consider that there exists a symmetry such that the Dirac mass term gives invariable result for the three generations of neutrinos which would mean that, in the original neutrino mass matrix, three generation neutrinos have the same mass coefficient, that is,

$$m_{\mathtt{D}_1} = m_{\mathtt{D}_2} = m_{\mathtt{D}_3}, \tag{14}$$

we can write a vector

$$\overrightarrow{U} = \left(\sqrt{m_{\mathtt{D}_1}}, \quad \sqrt{m_{\mathtt{D}_2}}, \quad \sqrt{m_{\mathtt{D}_3}}\right), \tag{15}$$

having characteristic

$$\frac{\overrightarrow{U}}{|\overrightarrow{U}|} = \frac{(1, \quad 1, \quad 1)}{|\,(1, \quad 1, \quad 1)\,|}, \tag{16}$$

which appears in Eq.(2) of Foot's paper [11]. The lepton masses can form a vector $\overrightarrow{V} = \left(\sqrt{m_e}, \quad \sqrt{m_\mu}, \quad \sqrt{m_\tau}\right)$. The angle between $\overrightarrow{U}$ and $\overrightarrow{V}$ is

$$\cos\theta = \frac{\left(\sqrt{m_e}, \quad \sqrt{m_\mu}, \quad \sqrt{m_\tau}\right)\left(\sqrt{m_{\mathtt{D}_1}}, \quad \sqrt{m_{\mathtt{D}_2}}, \quad \sqrt{m_{\mathtt{D}_3}}\right)}{|\left(\sqrt{m_e}, \quad \sqrt{m_\mu}, \quad \sqrt{m_\tau}\right)||\left(\sqrt{m_{\mathtt{D}_1}}, \quad \sqrt{m_{\mathtt{D}_2}}, \quad \sqrt{m_{\mathtt{D}_3}}\right)|} \tag{17}$$

$$= \frac{\left(\sqrt{m_e}, \quad \sqrt{m_\mu}, \quad \sqrt{m_\tau}\right)(1, \quad 1, \quad 1)}{|\left(\sqrt{m_e}, \quad \sqrt{m_\mu}, \quad \sqrt{m_\tau}\right)||(1, \quad 1, \quad 1)|}$$

$$= \frac{1}{\sqrt{3}}\frac{\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}}{\sqrt{m_e + m_\mu + m_\tau}}. \tag{18}$$

Using Eq.(1), Eq.(18) gives $\cos\theta = \frac{\sqrt{2}}{2}$, making $\theta = \frac{\pi}{4}$. This relation can be expressed by vectors, as given by Fig[1].

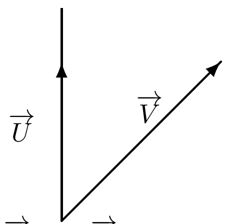

Fig 1. The vectors $\overrightarrow{U}$ and $\overrightarrow{V}$ forms an angle $\frac{\pi}{4}$.

For Dirac neutrino mass terms, using Eq.(18), we can get

$$m_{\mathtt{D}_1} + m_{\mathtt{D}_2} + m_{\mathtt{D}_3} = \frac{1}{3}(\sqrt{m_{\mathtt{D}_1}} + \sqrt{m_{\mathtt{D}_2}} + \sqrt{m_{\mathtt{D}_3}})^2 \tag{19}$$

Because we have

$$k_l^2 = \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2},$$
$$k_\nu^2 = \frac{m_{\mathtt{D}_1} + m_{\mathtt{D}_2} + m_{\mathtt{D}_3}}{(\sqrt{m_{\mathtt{D}_1}} + \sqrt{m_{\mathtt{D}_2}} + \sqrt{m_{\mathtt{D}_3}})^2}, \tag{20}$$

we can write a new symmetric relation

$$k_l^2 + k_\nu^2 = 1, \tag{21}$$

which is similar to

$$\sin^2\alpha + \cos^2\alpha = 1. \tag{22}$$

IV. ANALYTICAL FORMULA FOR RIGHT HANDED NEUTRINO MASSES

We know the matrix given in Eq.(4) has 6 eigenvalues, when $M_i \gg m_{\mathtt{D}_i} (i = 1, 2, 3)$ three would be given by Eqs.(5-6) and the rest of them would be equal to $M_i (i = 1, 2, 3)$. There is another way to find this second set of eigenvalues i.e. by using Eq.(6), approximately similar relations can be found in [15] and others.

The Dirac neutrino masses satisfy Eq.(19) and also as discussed in Section (III), Eq.(14), the Dirac mass term gives invariable result for three generations of neutrinos. We can have Dirac neutrino masses to be proportional to the electroweak scale i.e. $\lambda_{EW} \approx 246$GEV. According to the SM gauge symmetries the right handed neutrino spinor in Eq.(3) is uncharged indicating M_i for $i = 1, 2, 3$ a free parameter. While the Dirac masses are forbidden by electroweak gauge symmetry and can appear only after spontaneously breaking down through Higgs mechanism, as in the case of charged leptons, which employs Dirac masses naturally to be of the order of vacuum expectation value of Higgs field in standard model, which is $v = 246GeV$ then $v/\sqrt{2} \approx 174GeV$. If we consider Dirac neutrino particle masses to be of the order of electroweak scale, knowing the masses of neutrinos, we can get the masses for Majorana neutrinos, which is to say using expression in Eq(6), right handed neutrino masses can be written as

$$M_1 = \frac{m_{\mathtt{D}_1}^2}{m_{\nu_1}}, \quad M_2 = \frac{m_{\mathtt{D}_2}^2}{m_{\nu_2}}, \quad M_3 = \frac{m_{\mathtt{D}_3}^2}{m_{\nu_3}}. \tag{23}$$

V. RELATION BETWEEN LEFT AND RIGHT HANDED NEUTRINOS MASSES

Eq.(21) gives a relation between leptons and Dirac neutrinos masses. A similar kind of relation must exist for the right and left handed neutrino masses, such that

$$k_{\nu_R}^2 + k_{\nu_L}^2 = 1, \tag{24}$$

where $k_{\nu_R}^2$ would be

$$k_{\nu_R}^2 = \frac{(M_1 + M_2 + M_3)}{(\sqrt{M_1} + \sqrt{M_2} + \sqrt{M_3})^2}, \tag{25}$$

and $k_{\nu_L}^2$ is given by Eq(12). It is interesting to note that when we assume the above relation in Eq.(24) exist we can deduce the below mentioned important relations. Using Eq.(23), Eq.(25) can be re-written as

$$k_{\nu_R}^2 = \frac{(\frac{m_{\mathtt{D}_1}^2}{m_{\nu_1}} + \frac{m_{\mathtt{D}_2}^2}{m_{\nu_2}} + \frac{m_{\mathtt{D}_3}^2}{m_{\nu_3}})}{(\sqrt{\frac{m_{\mathtt{D}_1}^2}{m_{\nu_1}}} + \sqrt{\frac{m_{\mathtt{D}_2}^2}{m_{\nu_2}}} + \sqrt{\frac{m_{\mathtt{D}_3}^2}{m_{\nu_3}}})^2}. \tag{26}$$

$$= \frac{(\frac{m_{\mathtt{D}_1}^2}{m_{\nu_1}} + \frac{m_{\mathtt{D}_2}^2}{\sqrt{m_{\nu_1}^2+7.53\times10^{-5}\mathrm{eV}^2}} + \frac{m_{\mathtt{D}_3}^2}{\sqrt{m_{\nu_1}^2+251.53\times10^{-5}\mathrm{eV}^2}})}{(\sqrt{\frac{m_{\mathtt{D}_1}^2}{m_{\nu_1}}} + \sqrt{\frac{m_{\mathtt{D}_2}^2}{\sqrt{m_{\nu_1}^2+7.53\times10^{-5}\mathrm{eV}^2}}} + \sqrt{\frac{m_{\mathtt{D}_3}^2}{\sqrt{m_{\nu_1}^2+251.53\times10^{-5}\mathrm{eV}^2}}})^2}, \tag{27}$$

where we used Eq.(11) to make Eq.(26) dependent only on one unknown parameter m_{ν_1}. Running an analysis on the value of m_{ν_1} using Eq.(13), in such a way that $k_{\nu_L}^2$ gives a value, which when added to $k_{\nu_R}^2$ will satisfy Eq. (24) completely. Since $k_{\nu_R}^2$ is dependent on the value of left handed neutrinos, we can obtain the left handed neutrino masses , i.e. m_{ν_2} and m_{ν_3} masses for the value of m_{ν_1}, by using Eq. (11). The only value which satisfies Eq.(24) up to three decimal points gives left handed neutrino masses to be:

$$\begin{aligned} m_{\nu_1} &= 0.00107\mathrm{eV}, \\ m_{\nu_2} &= 0.00874\mathrm{eV}, \\ m_{\nu_3} &= 0.05016\mathrm{eV}. \end{aligned} \tag{28}$$

The above values of ν_2 and ν_3 neutrino masses are not only in accordance with the previous predictions [9, 13] up to three decimal points, but are also more precise. These masses follow normal mass hierarchy i.e. $m_{\nu_1} < m_{\nu_2} < m_{\nu_3}$.

Once we have obtained the left handed neutrino masses we can use it in Eq.(23) to obtain the right handed neutrino masses to be

$$\begin{aligned} M_1 &= 2.8295 \times 10^{16}\mathrm{GeV}, \\ M_2 &= 3.4627 \times 10^{15}\mathrm{GeV}, \\ M_3 &= 6.0353 \times 10^{14}\mathrm{GeV}. \end{aligned} \tag{29}$$

which are approximately of 10^{16} GeV. If we consider the case with inverted hierarchy of left handed neutrinos there are no such values that Eq.(24) can be satisfied.

VI. CONCLUSION AND SUMMARY

In this paper we showed how neutrino masses can satisfy Koide's relation. We discussed the Koide's mass relation and gave the Dirac mass terms a family symmetry. We consider the Dirac mass terms invariable and used them to give a meaning to the geometrical interpretation of Koide's formula given in [11] which in turn leads to a new Koide's like relation for Dirac neutrino mass terms given by Eq.(19). Koide's relation and this new Koide like relation for Dirac mass terms, if added together equals one which lead us to Eq.(22). A similar kind of plane must exist for the left and right handed neutrinos because according to the seesaw mechanics the extreme masses of neutrinos is because of the interaction between them. This relation can solve the analytical formula for the left handed neutrinos in Eq.(13) giving mass of three generations of neutrinos which are precise up to three decimal points with the previously proposed values.

In our paper we define the Yukawa coupling to be 1 to take $m_{\mathtt{D}}$ to be 174 GeV but we noticed that it does not affect the results, especially for the left handed-neutrino masses, even if changed over a wide range. Our model proposes the following points;

1. **Inverted Hierarchy**: This model rules out the presence of neutrinos following inverted mass hierarchy as there are no such values such that $m_{\nu_3} < m_{\nu_1} < m_{\nu_2}$ and the Eq.(24) is satisfied.

2. **SeeSaw Mechanism**: Since mass of the left handed neutrinos are dependent on the mass of the Dirac neutrino and the right-handed neutrinos masses, the Koide like relation for the neutrino would be

$$k_L{'}^2 = \frac{k_\nu{}^2}{k_{\nu_R}{}^2}, \tag{30}$$

where

$$k_{\nu_R}^2 = \frac{1}{2}, \tag{31}$$

so we can have,

$$k_L{'}^2 = \frac{2}{3}. \tag{32}$$

The above relation indicates that the SeeSaw mechanism is the underlying mechanism and thus explains why neutrinos have such extreme masses. This relation gives the answer why neutrinos do not satisfy Koide's relation giving 2/3 as we have for leptons. The reason is that the neutrino masses are the ratio of Dirac and Majorana neutrinos so the ratio of the Koide's formula for these would give the same 2/3 as for leptons. This formula is to justify the 2/3 value and not to be used as the Koide's formula for neutrinos which give the value approximately equal to 1/2 that can be calculated by using the mass values in Eq.(26).

So far there is not much experimental evidence to explain the mechanism by which neutrinos gain mass and also with the present experimental energy range it is not possible to test the speculation about the existence of Majorana neutrinos but as mentioned above the values of neutrinos obtained in this paper are in accordance with the masses of neutrinos predicted by several other studies.

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CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this article.

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