BEYOND GEVREY REGULARITY

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Abstract. We define and study classes of smooth functions which are less regular than Gevrey functions. To that end we introduce two-parameter dependent sequences which do not satisfy Komatsu's condition (M.2)', which implies stability under differential operators within the spaces of ultradifferentiable functions. Our classes therefore have particular behavior under the action of differentiable operators. On a more advanced level, we study microlocal properties and prove that

$$\mathrm{WF}_{0,\infty}(P(D)u) \subseteq \mathrm{WF}_{0,\infty}(u) \subseteq \mathrm{WF}_{0,\infty}(P(D)u) \cup \mathrm{Char}(P),$$

where u is a Schwartz distribution, $P(D)$ is a partial differential operator with constant coefficients and $\mathrm{WF}_{0,\infty}$ is the wave front set described in terms of new regularity conditions. For the analysis we introduce particular admissibility condition for sequences of cut-off functions, and a new technical tool called enumeration.

1. Introduction

We propose new regularity conditions for smooth functions which are weaker than the Gevrey regularity conditions. Instead of the Gevrey sequence $\{p!^t\}_{p\in\mathbf{N}}$, determined by parameter $t > 1$, we observe two-parameter dependent sequences of the form $\{p^{\tau p^\sigma}\}_{p\in\mathbf{N}}$, with $\tau > 0$ and $\sigma > 1$. When $\sigma = 1$ and $\tau > 1$ we recapture the Gevrey regularity as well as the analytic regularity for $\sigma = 1$ and $\tau = 1$.

Gevrey classes were initially introduced for the study of regularity properties of the fundamental solution of the heat operator, cf. [13], and thereafter used to describe regularities stronger than smoothness and weaker than analyticity. In particular, it turned out that the well-posedness of the Cauchy problem for weakly hyperbolic linear partial differential equations (PDEs) can be characterized by the Gevrey index t, while the same problem is ill-posed in the class of analytic functions, cf. [3, 28] and the references given there. Roughly speaking, fundamental solution ϕ may have C^∞-regularity property, which in this paper means that it is smooth without restrictions to the growth of its derivatives, $\mathcal{E}_t$-regularity (Gevrey regularity) if $\partial^\alpha\phi$ are bounded by $C^{\alpha+1}\alpha!^t$,

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$\alpha \in \mathbf{N}^d$, for some $C > 0$, $t > 1$, and $\mathcal{A}$-regularity if $\partial^\alpha \phi$ are bounded by $C^{\alpha+1}\alpha!$, $\alpha \in \mathbf{N}^d$, for some $C > 0$. Since there is a gap between the Gevrey and C^∞-regularity, new regularity conditions could be useful in local analysis of the solutions of PDEs, which is one motivation for our work. In particular, our condition describes hypoellipticity property standing between C^∞ hypoellipticity and Gevrey hypoellipticity.

Another motivation comes from microlocal analysis, where the notion of wave-front set plays a crucial role. We recall that

$$\mathrm{WF}(u) \subseteq \mathrm{WF}_t(u) \subseteq \mathrm{WF}_A(u)\,, \quad t > 1, \tag{1.1}$$

where u is a Schwartz distribution, WF is the classical (C^∞) wave front set, WF_t is the Gevrey wave front set, and WF_A is analytic wave front set, we refer to Subsection 1.1 for precise definitions, and to [12, 15] for details. We note that one can find examples of (ultra)distributions for which the inclusions in (1.1) are strict, and the same holds for other inclusions of wave front sets in this paper. Extension of (1.1) to Gevrey type ultradistributions is given in [28] and "stronger" singularities related to $t < 1$ are recently treated in [25].

Apart from the Gevrey wave front set, different types of wave front sets that modify the classical wave front set are introduced in the literature in connection to the equation under investigation, and we do not intend to survey the definitions here. However, let us briefly mention the Gabor wave front set, originally defined in [16] and further developed in [29], which is based on microlocal analysis on cones taken with respect to the whole of the phase space variables. Such approach is recently successfully applied to the study of Schrödinger equations in [2, 4, 5, 26, 32], see also the references therein. Note that the Gabor wave front set of a tempered distribution is characterized in terms of rapid decay of its Gabor coefficients on appropriate set. The idea to use Gabor coefficients and, consequently, methods of time-frequency analysis and modulation spaces in the study of wave front sets is introduced in [17,22,23], and extended in [6,7] to more general Banach and Fréchet spaces. We refer to [8–11] for details on modulation spaces and their role in time-frequency analysis. Since versions of Gabor wave front set can be adapted to analytic and Gevrey regularity (cf. [1, 30, 31]) it is natural to assume that the same holds in the framework of regularity proposed in this paper, which will be considered by the authors in a separate contribution.

Our approach gives a possibility to define wave-front sets which detect singularities that are "stronger" then the classical C^∞ singularities and at the same time "weaker" than any Gevrey type singularities, and to show that the usual properties (such as pseudo-local property), valid for wave-front sets quoted in (1.1), hold also in the context of our new regularity conditions. More precisely, one of the main results of the paper is the following (see Section 3 for the definition of $\mathrm{WF}_{\{\tau,\sigma\}}(u)$).

Theorem 1.1. *Let $\tau > 0$, $\sigma > 1$, and $u \in \mathcal{D}'(U)$. Then*

$$\mathrm{WF}_{\{2^{\sigma-1}\tau,\sigma\}}(P(D)u) \subseteq \mathrm{WF}_{\{2^{\sigma-1}\tau,\sigma\}}(u) \subseteq \mathrm{WF}_{\{\tau,\sigma\}}(P(D)u) \cup \mathrm{Char}(P(D)), \quad (1.2)$$

where $P(D)$ is a partial differential operator of order m with constant coefficients and $\mathrm{Char}(P(D))$ *is its characteristic set.*

In fact, the result of Theorem 1.1 holds true when $P(D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$, where $a_\alpha(x) \in \mathcal{E}_{\{\tau,\sigma\}}(\mathbf{R}^d)$ (see Section 2 for the definition). This extension requires nontrivial modifications of the proof of Theorem 1.1 and will be given in another paper. In particular, to handle the approximate solution (see Section 4) one should prove and use inverse closedness property of the corresponding algebra, cf. [18].

We refer to (1.4) for the definition of $\mathrm{Char}(P(D))$ and recall that if $\mathrm{Char}(P(D)) = \emptyset$ then $P(D)$ is called hypoelliptic.

In particular, with $\mathrm{WF}_{0,\infty}(u) = \bigcup_{\sigma>1} \bigcap_{\tau>0} \mathrm{WF}_{\{\tau,\sigma\}}(u)$ we have:

Corollary 1.1. *Let $u \in \mathcal{D}'(U)$ and $P(D)$ be a partial differential operator of order m with constant coefficients. Then*

$$\mathrm{WF}_{0,\infty}(P(D)u) \subseteq \mathrm{WF}_{0,\infty}(u) \subseteq \mathrm{WF}_{0,\infty}(P(D)u) \cup \mathrm{Char}(P(D)). \quad (1.3)$$

For the proof of Theorem 1.1 we perform a careful analysis of sequences of cut-off test functions which lead to specific *admissibility condition*. Moreover, we introduce a simple procedure called *enumeration* which is quite useful for the description of asymptotic behavior in microlocalization. In short, enumeration of a sequence "speeds up" or "slows down" the decay estimates of single terms while preserving the asymptotic behavior of the whole sequence.

Different values of parameters $\tau > 0$ and $\sigma > 1$ define different local regularity conditions which in turn implies that in many situations we obtain *strict inclusions* between the corresponding wave front sets. In particular, $\mathrm{WF}(u)$ is, in general, a strict subset of the intersections of our wave front sets, while the intersection of the Gevrey wave front sets, $\cap_{t>1} \mathrm{WF}_t$ contains the union of our wave front sets as a strict subset, see Corollary 3.1.

We note that our wave front sets are different from WF_L introduced in [15, Chapter 8.4] with respect to C^L regularity classes defined by an increasing sequence of positive numbers such that $p \leq L_p$ and $L_{p+1} \leq CL_p$, for some $C > 0$ and for every $p \in \mathbf{N}$. When $L_p = (p+1)^t$, $t > 1$, C^L is the Gevrey class. However, our defining sequence $\{p^{\tau p^\sigma}\}_{p\in\mathbf{N}}$ gives $L_p = p^{\tau p^{\sigma-1}}$, which does not satisfy $L_{p+1} \leq CL_p$, $p \in \mathbf{N}$, for any choice of $\tau > 0$, $\sigma > 1$. Therefore our approach describes another type of regularity than C^L regularity.

The paper is organized as follows. In Section 2 we observe sequences of the form $\{p^{\tau p^\sigma}\}_{p\in\mathbf{N}}$, $\tau > 0$ and $\sigma > 1$, which do not satisfy Komatsu's property (M.2)' (stability under differentiation) which is the basic one in the theory of ultradifferentiable functions, cf. [19]. Next, we use such sequences to define spaces of ultradifferentiable functions of regularity weaker than the Gevrey regularity, and study their main properties. In particular, we discuss stability under the action of ultradifferentiable operators.

In Section 3 we review the most common local regularity conditions and wave-front sets of (ultra)distributions. We introduce the notion of *enumeration* to motivate the definition of wave-front sets with respect to the regularity introduced in Section 2. Due to specific properties of our defining sequences, we had to modify Hörmander's construction from [15] by introducing a new *admissibility condition* for sequences of cut-off functions used in the microlocalization. Next, we describe local regularity via decay estimates on the Fourier transform side (Propositions 3.1 and 3.2) and discuss singular supports of (ultra)distributions.

Finally, in Section 4 we prove Theorem 1.1. Although we follow the general idea of the proof of [15, Theorem 8.6.1] we present here a detailed proof since our approach brings nontrivial changes and modifications into it.

We remark that some preliminary results of our investigations are given in [24], where test function spaces for Roumieu type ultradistributions were considered.

1.1. **Notation.** Sets of numbers are denoted in a usual way, e.g. $\mathbf{N}$ (resp. $\mathbf{Z}_+$) denotes the set of nonnegative (resp. positive) integers. For $x \in \mathbf{R}_+$ the floor and the ceiling functions are denoted by $\lfloor x \rfloor := \max\{m \in \mathbf{N} : m \leq x\}$ and $\lceil x \rceil := \min\{m \in \mathbf{N} : x \leq m\}$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{N}^d$ we write $\partial^\alpha = \partial^{\alpha_1} \dots \partial^{\alpha_d}$ and $|\alpha| = |\alpha_1| + \dots |\alpha_d|$. We will often use Stirling's formula:

$$N! = N^N e^{-N} \sqrt{2\pi N} e^{\frac{\theta_N}{12N}},$$

for some $0 < \theta_N < 1$, $N \in \mathbf{N} \setminus 0$. By $C^m(K)$, $m \in \mathbf{N}$, we denote the Banach space of m-times continuously differentiable functions on a compact set $K \subset\subset U$ with smooth boundary, where $U \subseteq \mathbf{R}^d$ is an open set, $C^\infty(K)$ denotes the set of smooth functions on K, C_K^∞ are smooth functions supported by K, and $\mathcal{A}(U)$ denotes the space of analytic functions on U. The closure of $U \subset \mathbf{R}^d$ is denoted by $\overline{U}$. A conic neighborhood of $\xi_0 \in \mathbf{R}^d \setminus 0$ is an open cone $\Gamma \subset \mathbf{R}^d$ such that $\xi_0 \in \Gamma$. The convolution is given by $(f * g)(x) = \int_{\mathbf{R}^d} f(x - y)g(y)dy$, whenever the integral makes sense. The Fourier transform $\widehat{f}$ of a locally integrable function f is normalized to be $\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbf{R}^d} f(x) e^{-2\pi i x \xi} dx$, $\xi \in \mathbf{R}^d$, and the definition extends to distributions by duality. Open ball of radius r, centered at $x_0 \in \mathbf{R}^d$ is denoted by $B_r(x_0)$.

For locally convex topological spaces X and Y, $X \hookrightarrow Y$ means that X is dense in Y and that the identity mapping from X to Y is continuous, and we use $\varprojlim$ and $\varinjlim$ to denote the projective and inductive limit topologies respectively. By X' we denote the strong dual of X and by $\langle \cdot, \cdot \rangle_X$ the dual pairing between X and X'.

We will observe $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ partial differential operators of order m with constant coefficients. Then $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$, $\xi \in \mathbf{R}^d \backslash \{0\}$, is the symbol of $P(D)$ and $P_m(\xi) = \sum_{|\alpha| = m} a_\alpha \xi^\alpha$, $\xi \in \mathbf{R}^d \backslash \{0\}$, is its principal symbol. The characteristic set of $P(D)$ is then given by

$$\operatorname{Char}(P(D)) = \{\xi \in \mathbf{R}^d \backslash \{0\} \mid P_m(\xi) = 0\}. \tag{1.4}$$

Let $x_0 \in U$ and $\xi_0 \notin \operatorname{Char}(P)$. Then there is an compact neighborhood $K \subset U$ of x_0 and a conic neighborhood Γ of ξ_0 such that $P_m(\xi) \neq 0$ for all $(x, \xi) \in K \times \Gamma$. Moreover, there exist $C_1, C_2 > 0$ such that

$$C_1 |\xi|^m \leq P_m(\xi) \leq C_2 |\xi|^m, \quad (x, \xi) \in K \times \Gamma. \tag{1.5}$$

As usual, $\mathcal{D}'(U)$ stands for Schwartz distributions, and $\mathcal{E}'(U)$ for compactly supported distributions. We refer to [19] for the definition and detailed study of different classes of ultradifferentiable functions and their duals, and to Remark 2.1 for the definition of Gevrey classes $\mathcal{E}_t(U)$, $\mathcal{D}_t(U)$, $t > 1$.

Let $t > 1$ and $(x_0, \xi_0) \in U \times \mathbf{R}^d \backslash \{0\}$ and $u \in \mathcal{D}'(U)$. Then the *Gevrey wave front set* $WF_t(u)$ can be defined as follows: $(x_0, \xi_0) \notin WF_t(u)$ if and only if there exists an open neighborhood Ω of x_0, a conic neighborhood Γ of ξ_0 and a bounded sequence $u_N \in \mathcal{E}'(U)$, such that $u_N = u$ on Ω and

$$|\widehat{u}_N(\xi)| \leq A \frac{h^N N!^t}{|\xi|^N}, \quad N \in \mathbf{Z}_+, \, \xi \in \Gamma, \tag{1.6}$$

for some $A, h > 0$. In fact, we may take $u_N = \phi u$ for some $\phi \in \mathcal{D}_t(U)$ which is equal to 1 in a neighborhood of x_0. If $t = 1$ in (1.6), then the corresponding wave-front set is called the *analytic wave front set* and denoted by $WF_A(u)$. We refer to [12,15,28] for the classical wave-front set.

2. Regularity classes $\mathcal{E}_{\tau,\sigma}$

In this section we first observe sequences $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{N}$, where $\tau > 0$ and $\sigma > 1$, and list their basic properties in Subsection 2.1. The flexibility obtained by introducing the two-parameter dependence enables us to introduce and study smooth functions which are less regular than the Gevrey functions, see Subsection 2.2. In Subsection 2.3 the action of ultradifferentiable operators on such classes is studied.

2.1. **The defining sequence $M_p^{\tau,\sigma}$.** Basic properties of our defining sequences are given in the following lemma. We refer to [24] for the proof.

Lemma 2.1. *Let $\tau > 0$, $\sigma > 1$ and $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{Z}_+$, $M_0^{\tau,\sigma} = 1$. Then the following properties hold:*

$(M.1)$ $(M_p^{\tau,\sigma})^2 \leq M_{p-1}^{\tau,\sigma} M_{p+1}^{\tau,\sigma}$, $p \in \mathbf{Z}_+$,

$\widetilde{(M.2)}'$ $M_{p+1}^{\tau,\sigma} \leq C^{p^\sigma} M_p^{\tau,\sigma}$, *for some* $C > 1$, $p \in \mathbf{N}$,

$\widetilde{(M.2)}$ $M_{p+q}^{\tau,\sigma} \leq C^{p^\sigma + q^\sigma} M_p^{\tau 2^{\sigma-1},\sigma} M_q^{\tau 2^{\sigma-1},\sigma}$, $p, q \in \mathbf{N}$, *for some* $C > 1$.

$(M.3)'$ $\displaystyle\sum_{p=1}^{\infty} \frac{M_{p-1}^{\tau,\sigma}}{M_p^{\tau,\sigma}} < \infty$.

If $\sigma = 1$ then $\widetilde{(M.2)}'$ and $\widetilde{(M.2)}$ are standard Komatsu's conditions $(M.2)'$ and $(M.2)$, respectively.

We will occasionally use Stirling's formula

$$\lfloor p^\sigma \rfloor !^{\tau/\sigma} \sim (2\pi)^{\tau/(2\sigma)} p^{\tau/2} e^{-(\tau/\sigma)p^\sigma} M_p^{\tau,\sigma}, \quad p \to \infty. \tag{2.1}$$

2.2. **Classes of ultradifferentiable functions.** Let $\tau > 0$, $\sigma > 1$, $h > 0$, and $K \subset\subset U$, where U is an open set in $\mathbf{R}^d$. A smooth function ϕ on U belongs to the space $\mathcal{E}_{\tau,\sigma,h}(K)$ if there exists $A > 0$ such that

$$|\partial^\alpha \phi(x)| \leq A h^{|\alpha|^\sigma} |\alpha|^{\tau|\alpha|^\sigma}, \quad \alpha \in \mathbf{N}^d, x \in K.$$

It is a Banach space with the norm given by

$$\|\phi\|_{\mathcal{E}_{\tau,\sigma,h}(K)} = \sup_{\alpha \in \mathbf{N}^d} \sup_{x \in K} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|^\sigma} |\alpha|^{\tau|\alpha|^\sigma}}, \tag{2.2}$$

and $\mathcal{E}_{\tau_1,\sigma_1,h_1}(K) \hookrightarrow \mathcal{E}_{\tau_2,\sigma_2,h_2}(K)$, $0 < h_1 \leq h_2$, $0 < \tau_1 \leq \tau_2$, $1 < \sigma_1 \leq \sigma_2$.

Let $\mathcal{D}^K_{\tau,\sigma,h}$ be the set of functions in $\mathcal{E}_{\tau,\sigma,h}(K)$ with support contained in K. Then, in the topological sense, we set

$$\mathcal{E}_{\{\tau,\sigma\}}(U) = \varprojlim_{K \subset\subset U} \varinjlim_{h \to \infty} \mathcal{E}_{\tau,\sigma,h}(K), \tag{2.3}$$

$$\mathcal{E}_{(\tau,\sigma)}(U) = \varprojlim_{K \subset\subset U} \varprojlim_{h \to 0} \mathcal{E}_{\tau,\sigma,h}(K), \tag{2.4}$$

$$\mathcal{D}_{\{\tau,\sigma\}}(U) = \varinjlim_{K \subset\subset U} \mathcal{D}^K_{\{\tau,\sigma\}} = \varinjlim_{K \subset\subset U} \big(\varinjlim_{h \to \infty} \mathcal{D}^K_{\tau,\sigma,h} \big), \tag{2.5}$$

$$\mathcal{D}_{(\tau,\sigma)}(U) = \varinjlim_{K \subset\subset U} \mathcal{D}^K_{(\tau,\sigma)} = \varinjlim_{K \subset\subset U} \varprojlim_{h \to 0} \mathcal{D}^K_{\tau,\sigma,h}. \tag{2.6}$$

We will use abbreviated notation τ, σ for $\{\tau,\sigma\}$ or (τ,σ) . It can be proved that the spaces $\mathcal{E}_{\tau,\sigma}(U)$, $\mathcal{D}^K_{\tau,\sigma}$ and $\mathcal{D}_{\tau,\sigma}(U)$ are nuclear, cf. [24].

Remark 2.1. From Lemma 2.1 it follows that the norms in (2.2) can be replaced by

$$\|\phi\|_{\widetilde{\mathcal{E}}_{\tau,\sigma,h}(K)} = \sup_{\alpha \in \mathbf{N}^d} \sup_{x \in K} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|^\sigma} \lfloor |\alpha|^\sigma \rfloor !^{\tau/\sigma}} < \infty, \quad h > 0. \tag{2.7}$$

If $\tau > 1$ and $\sigma = 1$, then $\mathcal{E}_{\tau,1}(U) = \mathcal{E}_\tau(U)$ are the Gevrey classes and $\mathcal{D}_{\tau,1}(U) = \mathcal{D}_\tau(U)$ are the corresponding subspaces of compactly supported functions in $\mathcal{E}_\tau(U)$. When $0 < \tau \leq 1$ and $\sigma = 1$ such spaces are contained in the corresponding spaces of quasianalytic functions. In particular, $\mathcal{D}_\tau(U) = \{0\}$ when $0 < \tau \leq 1$.

By the Borel Theorem (cf. [15, 21]), there exists a smooth function f such that

$$f^{(p)}(0) = p^{\tau p^\sigma}, p \in \mathbb{Z}_+,$$

and from the Whitney extension theorem we may conclude that $\mathcal{E}_{\tau,\sigma}(U) \neq \emptyset$. However, there does not exist any sequence $(M_p)_p$ of the Komatsu class so that the corresponding space of ultradifferentiable functions contain f. Moreover, the existence of compactly supported functions in $\mathcal{D}_{\tau,\sigma}(U)$ which are not in Gevrey classes $\mathcal{D}_t(U)$ for any $t > 1$, and of compactly supported function $\phi \in \mathcal{E}_{\tau,\sigma}(U)$ such that $0 \leq \phi \leq 1$ and $\int_{\mathbf{R}^d} \phi\, dx = 1$ is discussed in [24].

The basic embeddings between the introduced spaces with respect to σ and τ are given in the following proposition.

Proposition 2.1. *Let $\sigma_1 \geq 1$. Then for every $\sigma_2 > \sigma_1$ and $\tau > 0$*

$$\varinjlim_{\tau\to\infty} \mathcal{E}_{\tau,\sigma_1}(U) \hookrightarrow \varprojlim_{\tau\to 0^+} \mathcal{E}_{\tau,\sigma_2}(U). \tag{2.8}$$

Moreover, if $0 < \tau_1 < \tau_2$, then for every $\sigma \geq 1$ it holds

$$\mathcal{E}_{\{\tau_1,\sigma\}}(U) \hookrightarrow \mathcal{E}_{(\tau_2,\sigma)}(U) \hookrightarrow \mathcal{E}_{\{\tau_2,\sigma\}}(U), \tag{2.9}$$

and

$$\varinjlim_{\tau\to\infty} \mathcal{E}_{\{\tau,\sigma\}}(U) = \varinjlim_{\tau\to\infty} \mathcal{E}_{(\tau,\sigma)}(U), \quad \varprojlim_{\tau\to 0^+} \mathcal{E}_{\{\tau,\sigma\}}(U) = \varprojlim_{\tau\to 0^+} \mathcal{E}_{(\tau,\sigma)}(U).$$

Proof. For the proof of (2.8) we refer to [24, Proposition 2.1]. Since the second embedding in (2.9) is trivial, we proceed with the proof of the first one. Let $\phi \in \mathcal{E}_{\tau_1,\sigma,k}(K)$ for some $k > 0$. Since

$$||\phi||_{\mathcal{E}_{\tau_2,\sigma,h}(K)} \leq \sup_{\alpha\in\mathbf{N}^d} \frac{k^{|\alpha|^\sigma}|\alpha|^{\tau_1|\alpha|^\sigma}}{h^{|\alpha|^\sigma}|\alpha|^{\tau_2|\alpha|^\sigma}} ||\phi||_{\mathcal{E}_{\tau_1,\sigma,k}(K)}, \quad h, k > 0,$$

and $\sup_{\alpha\in\mathbf{N}^d} \frac{k^{|\alpha|^\sigma}|\alpha|^{\tau_1|\alpha|^\sigma}}{h^{|\alpha|^\sigma}|\alpha|^{\tau_2|\alpha|^\sigma}} \leq e^{\frac{\tau_2-\tau_1}{e\sigma}(k/h)^{\frac{\sigma}{\tau_2-\tau_1}}}$, then for any given $h > 0$ there exists $C > 0$ such that $||\phi||_{\mathcal{E}_{\tau_2,\sigma,h}(K)} \leq C||\phi||_{\mathcal{E}_{\tau_1,\sigma,k}(K)}$, and the proof is finished. □

We denote the corresponding projective (when $\tau \to 0^+$ or when $\sigma \to 1^+$) and inductive (when $\tau \to \infty$ or when $\sigma \to \infty$) limit spaces as follows:

$$\mathcal{E}_{0,\sigma}(U) := \varprojlim_{\tau\to 0^+} \mathcal{E}_{\tau,\sigma}(U), \quad \mathcal{E}_{\infty,\sigma}(U) := \varinjlim_{\tau\to\infty} \mathcal{E}_{\tau,\sigma}(U),$$

$$\mathcal{E}_{\tau,1}(U) := \varprojlim_{\sigma\to 1^+} \mathcal{E}_{\tau,\sigma}(U), \quad \mathcal{E}_{\tau,\infty}(U) := \varinjlim_{\sigma\to\infty} \mathcal{E}_{\tau,\sigma}(U),$$

$$\mathcal{E}_{0,1}(U) := \varprojlim_{\sigma\to 1^+} \mathcal{E}_{0,\sigma}(U), \quad \mathcal{E}_{0,\infty}(U) := \varinjlim_{\sigma\to\infty} \mathcal{E}_{0,\sigma}(U), \tag{2.10}$$

$$\mathcal{E}_{\infty,1}(U) := \varprojlim_{\sigma\to 1^+} \mathcal{E}_{\infty,\sigma}(U), \quad \mathcal{E}_{\infty,\infty}(U) := \varinjlim_{\sigma\to\infty} \mathcal{E}_{\infty,\sigma}(U), \tag{2.11}$$

Then Proposition 2.1 implies the following dense embeddings:

$$\varinjlim_{\tau\to\infty} \mathcal{E}_{\tau}(U) \hookrightarrow \mathcal{E}_{0,1}(U) \hookrightarrow \mathcal{E}_{\infty,1}(U)$$

$$\hookrightarrow \mathcal{E}_{0,\infty}(U) \hookrightarrow \mathcal{E}_{\infty,\infty}(U) \hookrightarrow C^{\infty}(U). \tag{2.12}$$

In fact, the first embedding $\varinjlim_{t\to\infty} \mathcal{E}_{\tau}(U) \hookrightarrow \mathcal{E}_{0,1}(U)$ in (2.12) follows directly from Proposition 2.1 when $\sigma_2 > \sigma_1 = 1$. The embedding $\mathcal{E}_{0,1}(U) \hookrightarrow \mathcal{E}_{\infty,1}(U)$ is obvious. Fix $\sigma_1 > 1$ and let $\sigma_2 > \sigma_1$. Then for some $\tau_0 > 0$

$$\mathcal{E}_{\tau_0,\sigma_1}(U) \hookrightarrow \mathcal{E}_{0,\sigma_2}(U) \hookrightarrow \mathcal{E}_{0,\infty}(U),$$

where the first embedding follows from (2.8) and the last one is trivial. This implies $\mathcal{E}_{\infty,1}(U) \hookrightarrow \mathcal{E}_{0,\infty}(U)$. Since the embeddings

$$\mathcal{E}_{0,\infty}(U) \hookrightarrow \mathcal{E}_{\infty,\infty}(U) \hookrightarrow C^{\infty}(U)$$

are trivial, (2.12) is proved.

2.3. **Continuity properties of ultradifferentiable operators on $\mathcal{E}_{\tau,\sigma}(U)$.** The space $\mathcal{E}_{\tau,\sigma}(U)$ can not be closed under the action of differential operator ∂^{α} for any given $\tau > 0$ and $\sigma > 1$ since then $M_p^{\tau,\sigma}$ does not satisfy Komatsu's condition (M.2)'. However, if we consider $\mathcal{E}_{\infty,\sigma}(U)$ instead, then $\widetilde{(M.2)}$ provides the continuity of certain ultradifferentiable operators.

Definition 2.1. Let $\tau > 0$ and $\sigma \geq 1$ and let $a_{\alpha}(x) \in \mathcal{E}_{(\tau,\sigma)}(U)$ (resp. $a_{\alpha}(x) \in \mathcal{E}_{\{\tau,\sigma\}}(U)$. Then $P(x,\partial) = \sum_{|\alpha|=0}^{\infty} a_{\alpha}(x)\partial^{\alpha}$ is ultradifferentiable operator of class (τ,σ) (resp. $\{\tau,\sigma\}$) on $U \subseteq \mathbf{R}^d$ if for every $K \subset\subset U$ there exists constant $L > 0$ such that for any $h > 0$ there exists $A > 0$ (resp. for every $K \subset\subset U$ there exists $h > 0$ such that for any $L > 0$ there exists $A > 0$) such that,

$$\sup_{x\in K} |\partial^{\beta} a_{\alpha}(x)| \leq A h^{|\beta|^{\sigma}} |\beta|^{\tau|\beta|^{\sigma}} \frac{L^{|\alpha|^{\sigma}}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^{\sigma}}}, \quad \alpha,\beta \in \mathbf{N}^d. \tag{2.13}$$

$P(x,\partial)$ is of the class τ,σ if it is of the class (τ,σ) or $\{\tau,\sigma\}$.

In particular $\tau, 1$ are ultradifferentiable operators of class $*$ where $* = \{p!^{\tau}\}$ or $(p!^{\tau})$ in Komastu's notation, cf. [20].

Theorem 2.1. *Let there be given $\tau > 0$, $\sigma > 1$ and let $P(x, \partial)$ be an ultradifferentiable operator of class (τ, σ) (resp. $\{\tau, \sigma\}$). Then $\mathcal{E}_{(\infty,\sigma)}(U)$ (resp. $\mathcal{E}_{\{\infty,\sigma\}}(U)$) is closed under the action of $P(x, \partial)$. In particular,*

$$P(x,\partial): \quad \mathcal{E}_{\tau,\sigma}(U) \longrightarrow \mathcal{E}_{\tau 2^{\sigma-1},\sigma}(U)\,, \tag{2.14}$$

is a continuous linear map.

Proof. Let $a_\alpha, \phi \in \mathcal{E}_{\tau,\sigma,h}(K)$, $\alpha \in \mathbf{N}^d$, $h > 0$. By (2.13) we have

$$|\partial^\beta(a_\alpha(x)\partial^\alpha\phi(x))| \le \sum_{\gamma\le\beta}\binom{\beta}{\gamma}|\partial^{\beta-\gamma}a_\alpha(x)||\partial^{\alpha+\gamma}\phi(x)|$$

$$\le A||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}h^{|\beta-\gamma|^\sigma}(|\beta-\gamma|)^{\tau|\beta-\gamma|^\sigma}\frac{L^{|\alpha|^\sigma}h^{|\alpha+\gamma|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}}(|\alpha+\gamma|)^{\tau|\alpha+\gamma|^\sigma}$$

$$\le A||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)}\frac{L^{|\alpha|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}}(|\alpha+\beta|)^{\tau|\alpha+\beta|^\sigma}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}h^{|\beta-\gamma|^\sigma+|\alpha+\gamma|^\sigma}$$

$$\le A||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)}(CL)^{|\alpha|^\sigma}C^{|\beta|^\sigma}|\beta|^{\tau 2^{\sigma-1}|\beta|^\sigma}C_{h,\beta}, \tag{2.15}$$

where we have used the fact that $M_p^{\tau,\sigma}$ satisfies $(M.1)'$ and $\widetilde{(M.2)}$, and put $C_{h,\beta} = \sum_{\gamma\le\beta}\binom{\beta}{\gamma}h^{|\beta-\gamma|^\sigma+|\alpha+\gamma|^\sigma}$. Since

$$\frac{1}{2^{\sigma-1}}(|\alpha|^\sigma+|\beta|^\sigma) \le |\beta-\gamma|^\sigma + |\alpha+\gamma|^\sigma \le 2^{\sigma-1}(|\alpha|^\sigma+|\beta|^\sigma), \quad \gamma\le\beta,$$

we have

$$C_{h,\beta} \le 2^{|\beta|}h^{\frac{1}{2^{\sigma-1}}|\alpha|^\sigma}h^{\frac{1}{2^{\sigma-1}}|\beta|^\sigma}, \quad 0<h<1,$$

and

$$C_{h,\beta} \le 2^{|\beta|}h^{2^{\sigma-1}|\alpha|}h^{2^{\sigma-1}|\beta|}, \quad h\ge 1.$$

Put $c_h = \max\{h^{\frac{1}{2^{\sigma-1}}}, h^{2^{\sigma-1}}\}$. Then (2.15) implies

$$|\partial^\beta(a_\alpha(x)\partial^\alpha\phi(x))| \le B||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)}(c_hCL)^{|\alpha|^\sigma}(2c_hC)^{|\beta|^\sigma}|\beta|^{\tau 2^{\sigma-1}|\beta|^\sigma}.$$

Choosing $h > 0$ (resp. $L > 0$) such that $LCc_h < 1/2$, after summation with respect to $\alpha \in \mathbf{N}^d$, and by taking suprema with respect to $\beta \in \mathbf{N}^d$ and $x \in K$ it follows that there exist $C' > 0$ such that

$$||P(x,\partial)\phi||_{\mathcal{E}_{\tau 2^{\sigma-1},\sigma,2Cc_h}(K)} \le C'||\phi||_{\mathcal{E}_{\tau,\sigma,h}(K)}$$

which completes the proof. □

It immediately follows that $\mathcal{E}_{(\infty,\sigma)}(U)$ (resp. $\mathcal{E}_{\{\infty,\sigma\}}(U)$) is closed under the action of $P(\partial) = \sum_{|\alpha|=0}^{\infty} a_\alpha\partial^\alpha$, where $|a_\alpha| \le A\frac{L^{|\alpha|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}}$, for some $L > 0$ and $A > 0$ (resp. every $L > 0$ there exists $A > 0$).

3. Microlocal analysis with respect to $\mathcal{E}_{\tau,\sigma}(U)$

In this section we define wave front sets which detect singularities that are "stronger" then classical C^∞ singularities and "weaker" then Gevrey type singularities.

In the study of regularity properties (as opposed to the singularity properties) of a function (or distribution) u we are interested in points (x_0,ξ_0) in which the decrease of $|\widehat{\phi_N u}(\xi)|$ ($\{\phi_N\}_{N\in\mathbf{N}}$ is appropriate sequence of cut-off functions, $\phi_N = 1$, $N\in\mathbf{N}$, in a neighborhood if x_0) is faster than $|\xi|^{-N}$ for any $N\in\mathbf{N}$, and, at the same time, slower than $e^{-|\xi|^{1/t}}$ for any $t>1$, when $|\xi|\to\infty$ and belongs to an open cone which contains ξ_0. In other words, u is micro-locally more regular than being $C^\infty-$regular, but less than being Gevrey regular.

As a motivation for the definition of wave-front sets in the context of the above mentioned regularity we observe the following conditions.

Lemma 3.1. *Let $t\geq 1$ and let $\{u_N\}_{N\in\mathbf{N}}$ be a sequence of functions in C_K^∞, such that some of the following conditions hold for every $N\in\mathbf{N}$, and $\xi\in\mathbf{R}^d\backslash\{0\}$:*

$$|\widehat{u}_N(\xi)|\leq A\,\frac{h^{N^t}\lfloor N^t\rfloor !}{|\xi|^{\lfloor N^t\rfloor}}, \tag{3.1}$$

$$|\widehat{u}_N(\xi)|\leq A\,\frac{h^N N!^t}{|\xi|^N}, \tag{3.2}$$

$$|\widehat{u}_N(\xi)|\leq A\,\frac{h^N N!^{1/t}}{|\xi|^{\lfloor N^{1/t}\rfloor}}, \tag{3.3}$$

for some (different) constants $A,h>0$. Then (3.1) $\Rightarrow$ (3.2) $\Rightarrow$ (3.3) .

As mentioned in the introduction, (3.2) is related to the Gevrey wave front WF_t, $t>1$, and if $t=1$ then (3.1) - (3.3) are related to the analytic wave front set WF_A.

The proof of Lemma 3.1 is just an application of the procedure which we call *enumeration* and which consists of a change of variables in indices which "speeds up" or "slows down" the decay estimates of single members of the corresponding sequences, while preserving their asymptotic behavior when $N\to\infty$. In other words, although estimates for terms of a sequence before and after enumeration are different, the asymptotic behavior of the whole sequence remains unchanged.

In other words, the conditions of the form (3.1), (3.2) or (3.3) are equivalent if one is obtained from another one after replacing N with positive, increasing sequence a_N such that $a_N\to\infty$, $N\to\infty$. We call this procedure *enumeration*, and write $N\to a_N$ and u_N instead of u_{a_N}.

Now, for the proof of Lemma 3.1, it is enough to note that after enumeration $N\to N^{1/t}$, $t>1$, (3.1) is equivalent to local analyticity, and it immediately follows that (3.1) $\Rightarrow$ (3.2). Next, after enumeration

$N \to N^t$, $t > 1$, (3.3) is equivalent to

$$|\widehat{u}_N(\xi)| \le A\,\frac{h^{N^t}\lfloor N^t \rfloor!^{1/t}}{|\xi|^N}, \quad N \in \mathbf{N}, \xi \in \mathbf{R}^d \backslash \{0\},$$

and (3.2) $\Rightarrow$ (3.3) follows from (2.1).

Next we introduce new regularity condition and discuss its relation to the conditions of Lemma 3.1.

Let $\tau > 0$, $\sigma \ge 1$ and let $\{u_N\}_{N\in\mathbf{N}}$ be a sequence of compactly supported smooth functions such that

$$|\widehat{u}_N(\xi)| \le A\,\frac{h^N N!^{1/\sigma}}{|\xi|^{\lfloor (N/\tau)^{1/\sigma}\rfloor}}, \quad N \in \mathbf{N},\ \xi \in \mathbf{R}^d \backslash \{0\} \tag{3.4}$$

for some constants $A, h > 0$. Note that from after enumeration $N \to \tau N^\sigma$, (2.1) implies that (3.4) is equivalent to

$$|\widehat{u}_N(\xi)| \le A\,\frac{h^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^N}, \quad N \in \mathbf{N},\ \xi \in \mathbf{R}^d \backslash \{0\},$$

and from $N!^\sigma \le C N^{\tau N^\sigma}$ it follows that (3.2) $\Rightarrow$ (3.4). Note that (3.3) $\Leftrightarrow$ (3.4) when $\tau = 1$, while (3.4) $\Rightarrow$ (3.3) when $\tau \in (0,1)$.

We conclude that (3.4) describes regularity weaker than (3.2) and stronger than (3.3).

After applying Stirling's formula and enumeration $N \to N/\tau$ to

$$|\widehat{u}_N(\xi)| \le A\,\frac{h^N N!^{\tau/\sigma}}{|\xi|^{\lfloor N^{1/\sigma}\rfloor}}, \quad N \in \mathbf{N},\ \xi \in \mathbf{R}^d \backslash \{0\} \tag{3.5}$$

where $A, h > 0$, we obtain

$$|\widehat{u}_N(\xi)| \le A\,\frac{h^{N/\tau}(N/\tau)^{\frac{\tau}{\sigma}(N/\tau)}}{|\xi|^{\lfloor (N/\tau)^{1/\sigma}\rfloor}} \le B\,\frac{k^N N!^{1/\sigma}}{|\xi|^{\lfloor (N/\tau)^{1/\sigma}\rfloor}}, \quad N \in \mathbf{N},\ \xi \in \mathbf{R}^d \backslash \{0\},$$

for some $A, B, h, k > 0$, so that (3.5) is equivalent to (3.4).

This discussion motivates the use of (3.4) (or (3.5)) in the definition of a new type of wave front sets of distributions, see Definition 3.2.

3.1. **τ, σ-admissible sequences and local regularity of Gevrey ultradistributions.** An essential tool in our study is the use of carefully chosen sequences of cut-off functions, defined as follows.

Definition 3.1. Let $\tau > 0$, $\sigma > 1$, and $\Omega \subseteq K \subset\subset U$, such that $\overline{\Omega}$ is strictly contained in K. A sequence $\{\chi_N\}_{N\in\mathbf{N}}$ of functions in C_K^∞ is said to be *τ, σ-admissible with respect to K* if

a) $\chi_N = 1$ in a neighborhood of Ω, for every $N \in \mathbf{N}$,
b) there exists a positive sequence C_β such that

$$\sup_{x\in K} |D^{\alpha+\beta}\chi_N(x)| \le C_\beta^{|\alpha|+1} \lfloor N^{1/\sigma}\rfloor^{|\alpha|}, \quad |\alpha| \le \lfloor (N/\tau)^{1/\sigma}\rfloor, \tag{3.6}$$

for every $N \in \mathbf{N}$ and $\beta \in \mathbf{N}^d$.

When $\tau = \sigma = 1$ we recover the sequence used by Hörmander in the study of the analytic behavior of distributions.

Although the following Lemma is a consequence of [15, Theorems 1.3.5 and 1.4.2] we give its proof since it contains an important construction which will be used in the sequel.

Lemma 3.2. *Let there be given $r > 0$, $\tau > 0$, $\sigma > 1$ and $x_0 \in \mathbf{R}^d$. There exists τ,σ-admissible sequence $\{\chi_N\}_{N\in\mathbf{N}}$ with respect to $\overline{B_{2r}(x_0)}$ such that $\chi_N = 1$ on $B_r(x_0)$, for every $N \in \mathbf{N}$.*

Proof. Fix $r > 0$. Let $d_k = \dfrac{r}{4\lfloor (N/\tau)^{1/\sigma}\rfloor}$, $k \leq \lfloor (N/\tau)^{1/\sigma}\rfloor$, $N \in \mathbf{N}$. Note that

$$\sum_{k=1}^{\lfloor (N/\tau)^{1/\sigma}\rfloor} d_k = \frac{r}{4} < \frac{r}{2},$$

for every $N \in \mathbf{N}$.

Since the infimum of distances between points in $\overline{B_{5r/4}(x_0)}$ and $\mathbf{R}^d \backslash B_{7r/4}(x_0)$ is $r/2$, from [15, Theorem 1.4.2] it follows that for every $N \in \mathbf{N}$ there exists a smooth function $\widetilde{\chi_N}$ such that $\operatorname{supp}\widetilde{\chi_N} \subseteq B_{7r/4}(x_0)$, $\widetilde{\chi}_N = 1$ on $B_{5r/4}(x_0)$, and

$$\sup_{x\in K} |D^\alpha \widetilde{\chi_N}(x)| \leq A^{|\alpha|} \prod_{k=1}^{|\alpha|} d_k = A^{|\alpha|} \lfloor (N/\tau)^{1/\sigma}\rfloor^{|\alpha|} \leq C^{|\alpha|} \lfloor N^{1/\sigma}\rfloor^{|\alpha|}, \tag{3.7}$$

for $|\alpha| \leq \lfloor (N/\tau)^{1/\sigma}\rfloor$, $N \in \mathbf{N}$, where $C > 0$ depends on τ and σ.

Next, let θ be a non-negative function such that $\theta \in C_0^\infty(B_{r/4}(x_0))$ and $\int \theta(x)dx = 1$. Then $\chi_N = \theta * \widetilde{\chi_N}$ clearly satisfies (3.6) for every $N \in \mathbf{N}$, if we let β derivatives act on θ and α derivatives act on $\widetilde{\chi_N}$. Hence $\{\chi_N\}_{N\in\mathbf{N}}$ is a τ,σ-admissible sequence with respect to $\overline{B_{2r}(x_0)}$ and the lemma is proved. □

Remark 3.1. Note that if $\alpha = 0$ in (3.6), then $\{\chi_N\}_{N\in\mathbf{N}}$ is a bounded sequence in $C^\infty(U)$. Moreover, by standard calculations we have that

$$|\widehat{\chi}_N(\xi)| \leq A_\beta^{|\alpha|+1} \lfloor N^{1/\sigma}\rfloor^{|\alpha|} \langle \xi \rangle^{-|\alpha|-|\beta|}, \quad |\alpha| \leq \lfloor (N/\tau)^{1/\sigma}\rfloor, \tag{3.8}$$

for every $N \in \mathbf{N}, \xi \in \mathbf{R}^d$, where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. Therefore, if $u \in \mathcal{D}'(U)$, the sequence $\{\chi_N u\}_{N\in\mathbf{N}}$ is bounded in $\mathcal{E}'(U)$.

Local regularity in $\mathcal{E}_{\{\tau,\sigma\}}(U)$ is in fact determined by (3.5) as follows.

Proposition 3.1. *Let $u \in \mathcal{D}'(U)$, and let $\{u_N\}_{N\in\mathbf{N}}$ be a bounded sequence in $\mathcal{E}'(U)$, $u_N = u$ on Ω and such that* (3.5) *holds for $\tau > 0$ and $\sigma > 1$. Then $u \in \mathcal{E}_{\{\tau,\sigma\}}(\Omega)$.*

We omit the proof since it uses standard arguments based on the Paley-Wiener theorem, the Fourier inversion formula and suitable decomposition of the domain of integration in combination with the property $\widetilde{(M.2)}'$. We refer to [15] and [28] for details.

For the opposite direction, if $u \in \mathcal{E}_{\{\tau,\sigma\}}(\Omega)$ then we use $\tau^{\sigma/(\sigma-1)}, \sigma$-admissible sequences instead.

Proposition 3.2. *Let $\Omega \subseteq K \subset\subset U$, $\overline{\Omega}$ strictly contained in K, $u \in \mathcal{D}'(U)$, and let $\{\chi_N\}_{N\in\mathbf{N}}$ be the $\tilde{\tau}, \sigma$-admissible sequence with respect to K, where $\tilde{\tau} = \tau^{\sigma/(\sigma-1)}$, $\tau > 0$, $\sigma > 1$. If $u \in \mathcal{E}_{\{\tau,\sigma\}}(\Omega)$, then $\{\chi_N u\}_{N\in\mathbf{N}}$ is bounded in $\mathcal{E}'(U)$, $\chi_N u = u$ on Ω, and*

$$|\widehat{\chi_N u}(\xi)| \leq A \frac{h^N N!^{\tilde{\tau}^{-1/\sigma}/\sigma}}{|\xi|^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor}}, \quad N \in \mathbf{N},\, \xi \in \mathbf{R}^d \backslash \{0\}. \tag{3.9}$$

That is, after enumeration $N \to \tilde{\tau} N$, $\{\chi_N u\}_{N\in\mathbf{N}}$ satisfies (3.5) *for some $A, h > 0$.*

The proof is rather technical and follows the same idea as in [15, Proposition 8.4.2]. We therefore omit it.

Note that for $\tau = \sigma = 1$ Proposition 3.2 coincides with the necessity part of [15, Proposition 8.4.2.].

3.2. **Singular support and $\mathrm{WF}_{\tau,\sigma}$ related to the classes $\mathcal{E}_{\tau,\sigma}$.** In this section we introduce wave front set $\mathrm{WF}_{\{\tau,\sigma\}}(u)$, and prove the corresponding results related to singular support. We also discuss the wave-front set $\mathrm{WF}_{(\tau,\sigma)}(u)$.

Definition 3.2. Let $\tau > 0$ and $\sigma > 1$, $u \in \mathcal{D}'(U)$, and $(x_0, \xi_0) \in U \times \mathbf{R}^d \backslash \{0\}$. Then $(x_0, \xi_0) \notin \mathrm{WF}_{\{\tau,\sigma\}}(u)$ (resp. $\mathrm{WF}_{(\tau,\sigma)}(u)$) if there exists open neighborhood $\Omega \subset U$ of x_0, a conic neighborhood Γ of ξ_0, and a bounded sequence $\{u_N\}_{N\in\mathbf{N}}$ in $\mathcal{E}'(U)$ such that $u_N = u$ on Ω and (3.5) holds for some constants $A, h > 0$ (resp. for every $h > 0$ there exists $A > 0$).

Remark 3.2. It follows immediately from the definition that $\mathrm{WF}_{\{\tau,\sigma\}}(u)$, $u \in \mathcal{D}'(U)$, is closed subset of $U \times \mathbf{R}^d \backslash \{0\}$. Note that for $\tau > 0$ and $\sigma > 1$

$$\mathrm{WF}_{\{\tau,\sigma\}}(u) \subseteq \mathrm{WF}_{\sigma}(u) \subseteq \mathrm{WF}_{\{1,1\}}(u) = \mathrm{WF}_A(u),$$

where $\mathrm{WF}_\sigma(u)$ is the Gevrey wave-front set. Moreover, when $0 < \tau < 1$ and $\sigma = 1$ we have $\mathrm{WF}_A(u) \subseteq \mathrm{WF}_{\{\tau,1\}}(u)$, and $\mathrm{WF}_{\{\tau,\sigma\}}(u) \neq \mathrm{WF}_L(u)$ for any choice of $\tau > 0$, $\sigma > 1$, where $\mathrm{WF}_L(u)$ is given in the introduction.

Since Proposition 3.2 does not hold when $0 < \tau < 1$ and $\sigma = 1$, we are not able to prove the usual relation between $\mathrm{WF}_{\{\tau,1\}}(u)$ and the singular support of u, see Theorem 3.1. This suggests that the singularities related to $WF_{\{\tau,1\}}$ should be studied by a different approach (see [25]).

The singular support of a distribution with respect to classes $\mathcal{E}_{\{\tau,\sigma\}}$ can be defined in a usual manner.

Definition 3.3. Let $\tau > 0$, $\sigma > 1$, $u \in \mathcal{D}'(U)$ and $x_0 \in U$. Then $x_0 \notin \mathrm{singsupp}_{\{\tau,\sigma\}}(u)$ if and only if there exists a neighborhood Ω of x_0 such that $u \in \mathcal{E}_{\{\tau,\sigma\}}(\Omega)$.

The following lemma is an essential result on microlocal regularity, which will be used in the proof of Theorem 3.1.

Lemma 3.3. *Let $\tau > 0$, $\sigma > 1$, $u \in \mathcal{D}'(U)$, $K \subset\subset U$, and let $\{\chi_N\}_{N\in\mathbf{N}}$ be a $\tilde\tau,\sigma$-admissible sequence with respect to K with $\tilde\tau = \tau^{\sigma/(\sigma-1)}$. Then $\{\chi_N u\}_{N\in\mathbf{N}}$ is a bounded sequence in $\mathcal{E}'(U)$, and if $\mathrm{WF}_{\{\tau,\sigma\}}(u) \cap (K \times F) = \emptyset$, where F is a closed cone, then there exist $A, h > 0$ such that*

$$|\widehat{\chi_N u}(\xi)| \leq A \frac{h^N N!^{\tilde\tau^{-1/\sigma}/\sigma}}{|\xi|^{\lfloor (N/\tilde\tau)^{1/\sigma}\rfloor}}, \quad N \in \mathbf{N}\,, \xi \in F\,. \tag{3.10}$$

The main ingredient of the proof is $\tilde\tau,\sigma$-admissibility of $\{\chi_N\}_{N\in\mathbf{N}}$ and carefully chosen enumeration applied to (3.5). Apart from this technical conditions we may use the same idea as for the proof of [15, Lemma 8.4.4.] and therefore omit the details.

As a consequence of Propositions 3.1, 3.2, and Lemma 3.3 we obtain the following Theorem.

Theorem 3.1. *Let $\tau > 0$, $\sigma > 1$, $u \in \mathcal{D}'(U)$, and let $\pi_1 : \mathbf{R}^d \times \mathbf{R}^d \backslash \{0\} \to \mathbf{R}^d$ be the standard projection given with $\pi_1(x,\xi) = x$. Then*

$$\mathrm{singsupp}_{\{\tau,\sigma\}}(u) = \pi_1(\mathrm{WF}_{\{\tau,\sigma\}}(u)).$$

Proof. Fix $x_0 \notin \pi_1(\mathrm{WF}_{\{\tau,\sigma\}}(u))$ and let K be its compact neighborhood so that $\mathrm{WF}_{\{\tau,\sigma\}}(u) \cap (K \times \mathbf{R}^d \backslash \{0\}) = \emptyset$. By Lemma 3.3 there exists a bounded sequence $\{u_N\}_{N\in\mathbf{N}}$ in $\mathcal{E}'(U)$ such that $u_N = u$ on some open set Ω and, after enumeration $N \to \tilde\tau N$,

$$|\widehat{u}_N(\xi)| \leq A\, \frac{h^N N!^{\tau/\sigma}}{|\xi|^{\lfloor N^{1/\sigma}\rfloor}}, \quad N \in \mathbf{N},\, \xi \in \mathbf{R}^d \backslash \{0\}. \tag{3.11}$$

holds for some $A, h > 0$. From Proposition 3.1 it follows that $u \in \mathcal{E}_{\{\tau,\sigma\}}(\Omega)$, that is, $x_0 \notin \mathrm{singsupp}_{\{\tau,\sigma\}}(u)$.

Conversely, if $x_0 \notin \mathrm{singsupp}_{\{\tau,\sigma\}}(u)$, then there exist neighborhood Ω of x_0 such that $u \in \mathcal{E}_{\{\tau,\sigma\}}(\Omega)$. By Proposition 3.2, there exists a bounded sequence $\{u_N\}_{N\in\mathbf{N}}$ in $\mathcal{E}'(U)$ such that $u_N = u$ on Ω and (3.11) holds, which implies the desired equality. □

To conclude the section we discuss intersections and unions of wave-front sets $\mathrm{WF}_{\tau,\sigma}$, $\tau > 0$, $\sigma > 1$. It turns out that, from the microlocal point of view, the regularity related to complements of these unions and intersections in intimately related to the regularity properties in the classes given by (2.10) and (2.11).

Let there be given $u \in \mathcal{D}'(U)$. Then we put

$$\mathrm{WF}_{0,1}(u) = \bigcap_{\sigma>1} \bigcap_{\tau>0} \mathrm{WF}_{\tau,\sigma}(u), \tag{3.12}$$

$$\mathrm{WF}_{\infty,1}(u)=\bigcap_{\sigma>1}\bigcup_{\tau>0}\mathrm{WF}_{\tau,\sigma}(u), \tag{3.13}$$

$$\mathrm{WF}_{0,\infty}(u)=\bigcup_{\sigma>1}\bigcap_{\tau>0}\mathrm{WF}_{\tau,\sigma}(u), \tag{3.14}$$

$$\mathrm{WF}_{\infty,\infty}(u)=\bigcup_{\sigma>1}\bigcup_{\tau>0}\mathrm{WF}_{\tau,\sigma}(u). \tag{3.15}$$

Remark 3.3. Recall (cf. Proposition 2.9),

$$\mathcal{E}_{\{\tau,\sigma\}}(U)\hookrightarrow\mathcal{E}_{(\rho,\sigma)}(U)\hookrightarrow\mathcal{E}_{\{\rho,\sigma\}}(U), \tag{3.16}$$

when $0<\tau<\rho$ and $\sigma>1$. Since the inclusions are strict, Definition 3.2 implies

$$\mathrm{WF}_{\{\rho,\sigma\}}(u)\subseteq\mathrm{WF}_{(\rho,\sigma)}(u)\subseteq\mathrm{WF}_{\{\tau,\sigma\}}(u)\,,u\in\mathcal{D}'(U).$$

Moreover, $\bigcap\limits_{\tau>0}\mathrm{WF}_{\{\tau,\sigma\}}(u)=\bigcap\limits_{\tau>0}\mathrm{WF}_{(\tau,\sigma)}(u)$ and $\bigcup\limits_{\tau>0}\mathrm{WF}_{\{\tau,\sigma\}}(u)=\bigcup\limits_{\tau>0}\mathrm{WF}_{(\tau,\sigma)}(u)$. For that reason it is sufficient to consider intersections and unions of $\mathrm{WF}_{\{\tau,\sigma\}}(u)$ in (3.12)-(3.15).

First we prove the following technical result.

Lemma 3.4. *Let $u\in\mathcal{D}'(U)$, and $\sigma_2>\sigma_1\geq1$. Then*

$$\bigcup_{\tau>0}\mathrm{WF}_{\tau,\sigma_2}(u)\subseteq\bigcap_{\tau>0}\mathrm{WF}_{\tau,\sigma_1}(u)\,.$$

Proof. Let $(x_0,\xi_0)\notin\bigcap_{\tau>0}\mathrm{WF}_{\{\tau,\sigma_1\}}(u)$. Then there exists $\tau_0>0$ such that $(x_0,\xi_0)\notin\mathrm{WF}_{\{\tau_0,\sigma_1\}}(u)$. Hence there exists open conic neighborhood $\Omega\times\Gamma$ of (x_0,ξ_0) and a bounded sequence $\{u_N\}_{N\in\mathbf{N}}$ in $\mathcal{E}'(U)$ such that $u_N=u$ on Ω such that, after enumeration $N\to N^{\sigma_1}$ (see also Lemma 2.1),

$$|\widehat{u}_N(\xi)|\leq A\frac{h^{N^{\sigma_1}}N^{\tau_0N^{\sigma_1}}}{|\xi|^N},\quad N\in\mathbf{N},\xi\in\Gamma, \tag{3.17}$$

for some constants $A,h>0$.

We need to prove that for every $\tau>0$, $(x_0,\xi_0)\notin\mathrm{WF}_{\{\tau,\sigma_2\}}(u)$. This follows easily from (3.17), noting that (see the proof of the [24, Proposition 2.1.]) for every $\tau>0$ and $h>0$ there exists $A_1>0$ such that

$$h^{N^{\sigma_1}}N^{\tau_0N^{\sigma_1}}\leq A_1h^{N^{\sigma_2}}N^{\tau N^{\sigma_2}},\quad N\in\mathbf{N},$$

and the Lemma is proved. □

As a consequence of Lemma 3.4 we obtain the following result which relates our regularity with C^∞ and $\mathcal{E}_t$-regularity in terms of the corresponding wave-front sets.

Corollary 3.1. *Let $u \in \mathcal{D}'(U)$. Then, in the notation of* (3.12)-(3.15), *we have*

$$\mathrm{WF}(u) \subseteq \mathrm{WF}_{0,1}(u) \subseteq \mathrm{WF}_{\infty,1}(u) \subseteq \mathrm{WF}_{0,\infty}(u) \subseteq \mathrm{WF}_{\infty,\infty}(u) \subseteq \bigcap_{\tau>1} \mathrm{WF}_\tau(u)\,, \quad (3.18)$$

where WF *and* WF_τ *are the classical and the Gevrey wave-front sets, respectively.*

Proof. Note that the last inclusion follows from Lemma 3.4 for $\sigma_2 > \sigma_1 = 1$ by taking unions and intersections with respect to $\tau > 1$. The only nontrivial inclusion is $\mathrm{WF}_{\infty,1}(u) \subseteq \mathrm{WF}_{0,\infty}(u)$. Assume that $(x_0, \xi_0) \notin \mathrm{WF}_{0,\infty}(u)$, that is, $(x_0, \xi_0) \notin \bigcap_{\tau>0} \mathrm{WF}_{\tau,\sigma}(u)$, for every $\sigma > 1$. Fix some $\sigma = \sigma_1 > 1$ and let $\sigma_2 > \sigma_1$. By Lemma 3.4 it follows that $(x_0, \xi_0) \notin \bigcup_{\tau>0} \mathrm{WF}_{\tau,\sigma_2}(u)$. Hence there exists $\sigma > 1$ such that for every $\tau > 0$ $(x_0, \xi_0) \notin \mathrm{WF}_{\tau,\sigma}(u)$ and therefore $(x_0, \xi_0) \notin \mathrm{WF}_{\infty,1}(u)$. □

To end the section, we relate $\mathrm{WF}_{0,\infty}(u)$ to the regularity in $\mathcal{E}_{\infty,1}$, see (2.11). Let $\mathrm{singsupp}_{\infty,1}(u)$ denote the singular support of $u \in \mathcal{D}'(U)$ related to the classe $\mathcal{E}_{\infty,1}$ (as appropriate union and intersection of the corresponding singular supports in $\mathcal{E}_{\tau,\sigma}(U)$) Recall that, for every $\sigma > 1$, the space $\mathcal{E}_{\infty,\sigma}$ is closed under the action of ultradifferentiable operators of the class τ, σ (see Subsection2.3, Theorem 2.1). Then, arguing in the similar way as in the proof of Theorem 3.1 one can prove that

$$\pi_1(\mathrm{WF}_{0,\infty}(u)) = \mathrm{singsupp}_{\infty,1}(u). \quad (3.19)$$

4. Proof of Theorem 1.1

Note that Corollary 1.1 follows directly from Theorem 1.1 and Remark 3.3. The first embedding in (1.2) immediately follows form the next Lemma.

Lemma 4.1. *Let $u \in \mathcal{D}'(U)$, $\tau > 0, \sigma > 1$. Then*

$$\mathrm{WF}_{\{\tau,\sigma\}}(\partial_j u) \subseteq \mathrm{WF}_{\{\tau,\sigma\}}(u),$$

for all $1 \leq j \leq d$.

Proof. Let $(x_0, \xi_0) \notin \mathrm{WF}_{\tau,\sigma}(u)$. Then there exists a conical neighborhood $\Omega \times \Gamma$ of (x_0, ξ_0) and a bounded sequence $\{u_N\}$ in $u \in \mathcal{E}'(U)$ such that $u_N = u$ on Ω, and such that after the enumeration $N \to N^\sigma$ we obtain

$$|\widehat{u}_N(\xi)| \leq A \frac{h^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^N}, \quad N \in \mathbf{N},\, \xi \in \Gamma, \quad (4.1)$$

for some $A, h > 0$. Then, for $x_0 \in \Omega$,

$$|\widehat{\partial_j u}_{N+1}(\xi)| \leq A|\xi| \frac{h^N (N+1)^{\tau(N+1)^\sigma}}{|\xi|^{N+1}} \leq A_1 \frac{h_1^N N^{\tau N^\sigma}}{|\xi|^N}, \quad (4.2)$$

$N \in \mathbf{N}$, $\xi \in \Gamma$, $j \in \{1, \dots, d\}$, $\widetilde{(M.2)}'$ is used for the second inequality, and the inclusion follows. $\square$

Therefore it remains to prove that

$$\mathrm{WF}_{\{2^{\sigma-1}\tau,\sigma\}}(u) \subseteq \mathrm{WF}_{\{\tau,\sigma\}}(P(D)u) \cup \mathrm{Char}(P(D)).$$

The following inequality, which holds for $\tau > 0$, $\sigma > 1$ and for some $C > 0$, will be frequently used:

$$\lfloor N^{1/\sigma} \rfloor^{\lfloor (N/\tau)^{1/\sigma} \rfloor} \leq N^{N\tau^{-1/\sigma}/\sigma} \leq C^N N!^{\tau^{-1/\sigma}/\sigma}. \tag{4.3}$$

Assume that $(x_0, \xi_0) \notin \mathrm{WF}_{\{\tau,\sigma\}}(P(D)u) \cup \mathrm{Char}(P(D))$. Then there exists a compact set K containing x_0 and a closed cone Γ containing ξ_0 such that $P_m(x, \xi) \neq 0$ when $(x, \xi) \in K \times \Gamma$ and $(K \times \Gamma) \cap \mathrm{WF}_{\{\tau,\sigma\}}(P(D)u) = \emptyset$.

Let $\tilde{\tau} = \tau^{\frac{\sigma}{\sigma-1}}$ and let $\{\chi_N\}_{N \in \mathbf{N}}$, be a $\tilde{\tau}, \sigma$-admissible sequence with respect to K.

Put $u_N = \chi_{2^\sigma N} u$, $N \in \mathbf{N}$, so that

$$\widehat{u}_N(\xi) = \int u(x) \chi_{2^\sigma N}(x) e^{-ix\xi} dx, \quad \xi \in \mathbf{R}^d, \ N \in \mathbf{N}.$$

The easy part of the proof is the estimate of $|\widehat{u}_N(\xi)|$, $N \in \mathbf{N}$, for "small" values of $\xi \in \Gamma$, that is when $|\xi| \leq \lfloor N^{1/\sigma} \rfloor$. In fact, since $\{u_N\}_{N \in \mathbf{N}}$ is bounded in $\mathcal{E}'(U)$, Paley-Wiener theorems (see [20]), and the fact that $e^{-ix\cdot\xi} \in C^\infty(\mathbf{R}^d_x)$, for every $\xi \in \mathbf{R}^d$, implies that $|\widehat{u}_N(\xi)| = |\langle u_N, e^{-i\cdot\xi} \rangle| \leq C \langle \xi \rangle^M$, for some $C, M > 0$ independent of N. Hence, from (4.3) we have

$$|\xi|^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor} |\widehat{u_N}(\xi)| \leq \lfloor N^{1/\sigma} \rfloor^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor} |\widehat{u_N}(\xi)| \leq A C^N N^{\frac{\tilde{\tau}^{-1/\sigma}}{\sigma} N},$$

where $A, C > 0$ do not depend on N. After enumeration $N \to \tilde{\tau} N$ we obtain

$$|\widehat{u_N}(\xi)| \leq A \frac{C^N N^{\frac{\tilde{\tau}^{1-1/\sigma}}{\sigma} N}}{|\xi|^{\lfloor N^{1/\sigma} \rfloor}} \leq A \frac{h^N N!^{\frac{\tau}{\sigma}}}{|\xi|^{\lfloor N^{1/\sigma} \rfloor}},$$

which estimates $|\widehat{u_N}(\xi)|$ when $\xi \in \Gamma$, $|\xi| \leq \lfloor N^{1/\sigma} \rfloor$, $N \in \mathbf{N}$.

It remains to estimate $|\widehat{u}_N(\xi)|$, when $\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma} \rfloor$ and for $N \in \mathbf{N}$ large enough (so that $N \to \infty$ implies $|\xi| \to \infty$).

As in the proof of [15, Theorem 8.6.1], in Subsection 4.1 we use the technique of approximate solution (see also [27, Theorem 1, Section 1.6]) to obtain

$$\chi_{2^\sigma N}(x) e^{-ix\cdot\xi} = P^T(D) \left(\frac{e^{-ix\cdot\xi}}{P_m(\xi)} w_N(x, \xi) \right) + e_N(x, \xi) e^{-ix\xi} \tag{4.4}$$

$x \in K$, $\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma} \rfloor$, that is, the following representation holds:

$$\begin{aligned}\widehat{u}_N(\xi) &= \int u(x) e_N(x,\xi) e^{-ix\xi} dx + \int u(x) P^T(D) \left(\frac{e^{-ix\cdot\xi} w_N(x,\xi)}{P_m(\xi)} \right) dx \\ &= \int u(x) e_N(x,\xi) e^{-ix\xi} dx + \int P(D) u(x) \left(\frac{e^{-ix\cdot\xi} w_N(x,\xi)}{P_m(\xi)} \right) dx, \quad (4.5)\end{aligned}$$

where

$$w_N(x,\xi) = \sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m} \binom{|a|}{a_1, a_2 \ldots a_m} (R_1^{a_1} R_2^{a_2} \ldots R_m^{a_m} \chi_{2^\sigma N})(x,\xi), \quad (4.6)$$

$$e_N(x,\xi) = \sum_{k=1}^{m} \sum_{\mathfrak{S}=\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m+1}^{\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m+k} \binom{|a|}{a_1, \ldots, a_m} (R_1^{a_1} \ldots R_m^{a_m} \chi_{2^\sigma N})(x,\xi), \quad (4.7)$$

$x \in K$, $\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma} \rfloor$, and we put $\mathfrak{S} = a_1 + 2a_2 + \cdots + m a_m$.

The derivation of (4.5) and the calculation of $w_N(x,\xi)$ and $e_N(x,\xi)$ is done in Subsections 4.1 and 4.2, so we continue with the estimation of the first term in (4.5).

Estimated number of terms in $e_N(x,\xi)$ given in Subsection 4.1, and the estimates of $D^\beta(R_1^{a_1} \ldots R_m^{a_m} \chi_{2^\sigma N})$ given by (4.30) (Subsection 4.3) imply

$$\begin{aligned}|\langle u(x), e_N(x,\xi) e^{-ix\cdot\xi} \rangle| &\leq A \sum_{|\alpha| \leq M} |D_x^\alpha (e_N(x,\xi) e^{-ix\xi})| \\ &\leq A \sum_{|\alpha| \leq M} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |D_x^{\alpha-\beta} e^{-ix\xi}| |D_x^\beta e_N(x,\xi)| \\ &\leq A |\xi|^M |\xi|^{-\lfloor 2^{\frac{1-\sigma}{\sigma}} (N/\tilde{\tau})^{1/\sigma} \rfloor - M} C^N N!^{\frac{\tilde{\tau}^{-1/\sigma}}{\sigma}} \\ &= A \frac{C^N N!^{\frac{\tilde{\tau}^{-1/\sigma}}{\sigma}}}{|\xi|^{\lfloor 2^{\frac{1-\sigma}{\sigma}} (N/\tilde{\tau})^{1/\sigma} \rfloor}}, x \in K, \xi \in \Gamma, \qquad (4.8)\end{aligned}$$

for suitable constants $A, C > 0$ and $|\xi|$ large enough. After enumeration $N \to \tilde{\tau} 2^{\sigma-1} N$, (4.8) is equivalent to

$$|\langle u(x), e_N(x,\xi) e^{-ix\cdot\xi} \rangle| \leq A \frac{C^N N!^{\frac{\tau 2^{\sigma-1}}{\sigma}}}{|\xi|^{\lfloor N^{1/\sigma} \rfloor}}, \quad x \in K, \xi \in \Gamma,$$

which estimates the first term on the righthand side of (4.5). In fact, we will use a slightly weaker estimate which is obtained from (4.8) after enumeration

$$N \to N + \lceil \tilde{\tau} 2^{\sigma-1} (M+d+1)^\sigma \rceil. \qquad (4.9)$$

It remains to estimate the second term on the righthand side of (4.5) for $|\xi| > \lfloor N^{1/\sigma} \rfloor$. This is the hardest part of the proof. By the Lemma 3.3 there exists a bounded sequence $\{f_N\}_{N \in \mathbf{N}}$ in $\mathcal{E}'(U)$ such that $f_N = f = P(D)u$ in a neighborhood of K and there exists a cone V such that $\overline{\Gamma} \subset V$ and

$$|\mathcal{F}(f_N)(\eta)| \leq A \frac{h^N N!^{\frac{\tilde{\tau}-1/\sigma}{\sigma}}}{|\eta|^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor}}, \quad \eta \in V. \tag{4.10}$$

Since $\{\chi_{2^\sigma N}(x)\}_{N\in\mathbf{N}}$ is bounded in $C_0^\infty(U)$, by the Paley-Wiener theorem (see also Remark 3.1) it follows that for every $\tilde{M} > 0$ there exists $C > 0$ which does not depend on N so that $|\widehat{\chi}_{2^\sigma N}(\eta)| \leq C\langle\eta\rangle^{-\tilde{M}}$, $N \in \mathbf{N}$. From $\operatorname{supp}\chi_N \subseteq K$, $N \in \mathbf{N}$, it follows that

$$\pi_1(\operatorname{supp} w_N(x,\xi)) \subseteq K, \quad N \in \mathbf{N},$$

and since $f_N = f$ in a neighborhood of K, we have $w_N f = w_N f_{N'}$ in $\mathcal{D}'(U)$, where we put $N' = N - \lceil 2^{\sigma-1}\tilde{\tau}(M+d+1)^\sigma \rceil$. Therefore (and since $\mathcal{F}(g_1 \cdot g_2)(\xi) = (\mathcal{F}(g_1) * \mathcal{F}(g_2))(\xi))$)

$$\begin{aligned}\langle f(\cdot)e^{-i\xi\cdot}, w_N(\cdot,\xi)/P_m(\xi)\rangle &= \frac{1}{P_m(\xi)}\mathcal{F}_{x\to\xi}(f_{N'}(x)w_N(x,\xi))(\xi)\\ &= \frac{1}{P_m(\xi)}\int_{\mathbf{R}^d}\mathcal{F}(f_{N'})(\xi-\eta)\mathcal{F}_{x\to\eta}(w_N(x,\xi))(\eta)\,d\eta = I_1 + I_2,\end{aligned}$$

where

$$I_1 = \frac{1}{P_m(\xi)}\int_{|\eta|<\varepsilon|\xi|}\mathcal{F}(f_{N'})(\xi-\eta)\mathcal{F}_{x\to\eta}(w_N(x,\xi))(\eta,\xi)\,d\eta, \tag{4.11}$$

$$I_2 = \frac{1}{P_m(\xi)}\int_{|\eta|\geq\varepsilon|\xi|}\mathcal{F}(f_{N'})(\xi-\eta)\mathcal{F}_{x\to\eta}(w_N(x,\xi))(\eta,\xi)\,d\eta, \tag{4.12}$$

and $0 < \varepsilon < 1$ is chosen so that $\xi - \eta \in V$ when $\xi \in \Gamma$, $\xi > \lfloor N^{1/\sigma} \rfloor$, and $|\eta| < \varepsilon|\xi|$.

Since $|\eta| < \varepsilon|\xi|$ implies $|\xi-\eta| \geq (1-\varepsilon)|\xi|$, by using the computation of $\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)$ from Subsection 4.4, we estimate I_1 as follows:

$$\begin{aligned}|I_1| &\leq \frac{1}{|P_m(\xi)|}\int_{|\eta|<\varepsilon|\xi|}|\mathcal{F}(f_{N'})(\xi-\eta)||\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)|\,d\eta\\ &\leq \int_{|\eta|<\varepsilon|\xi|} A\frac{h^{N'}N'!^{\tau/\sigma}}{|\xi-\eta|^{\lfloor N'^{1/\sigma}\rfloor}}|\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)|\,d\eta\\ &\leq A\frac{h^{N'}N'!^{\tau/\sigma}}{((1-\varepsilon)|\xi|)^{\lfloor N'^{1/\sigma}\rfloor}}\int_{|\eta|<\varepsilon|\xi|}|\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)|\,d\eta\\ &\leq A_1\frac{h_1^{N'}N'!^{\tau/\sigma}}{|\xi|^{\lfloor N'^{1/\sigma}\rfloor}}C^{\lfloor (N/\tau)^{1/\sigma}\rfloor}\int_{\mathbf{R}^d}|\widehat{\chi}_{2^\sigma N}(\eta)|\,d\eta\\ &\leq A_2\frac{h_2^{N'}N'!^{\tau/\sigma}}{|\xi|^{\lfloor N'^{1/\sigma}\rfloor}}, \quad \xi\in\Gamma, |\xi| > \lfloor N^{1/\sigma}\rfloor.\end{aligned} \tag{4.13}$$

We used the Paley-Wiener theorem for $\{\widehat{\chi}_{2^\sigma N}\}$ and trivial inequality $|P_m(\xi)| \geq 1$ when $|\xi| > \lfloor N^{1/\sigma} \rfloor$.

It remains to estimate I_2. Note that $|\eta| \geq \varepsilon|\xi|$ implies $|\xi-\eta| \leq (1+1/\varepsilon)|\eta|$, and by Paley-Wiener type estimates we have $|\mathcal{F}(f_{N'})(\eta)| \leq C\langle\eta\rangle^M$, where $C>0$ does not depend on N'. Therefore

$$\begin{aligned}|I_2| &\leq \frac{1}{|P_m(\xi)|}\int_{|\eta|\geq\varepsilon|\xi|} |\mathcal{F}f_{N'}(\xi-\eta)||\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)|\,d\eta \\ &\leq A\int_{|\eta|\geq\varepsilon|\xi|}\langle\xi-\eta\rangle^M\langle\eta\rangle^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde\tau)^{1/\sigma}\rfloor+d+1}\frac{|\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)|}{\langle\eta\rangle^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde\tau)^{1/\sigma}\rfloor+d+1}}\,d\eta \\ &\leq C^{N+1}\frac{\sup_{\eta\in\mathbf{R}^d}\langle\eta\rangle^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde\tau)^{1/\sigma}\rfloor+M+d+1}}{|\xi|^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde\tau)^{1/\sigma}\rfloor}}|\mathcal{F}_{x\to\eta}(w_N(x,\xi))(\eta,\xi)|,\end{aligned}$$

when $\xi\in\Gamma$, $|\xi|>\lfloor N^{1/\sigma}\rfloor$.

To finish the proof, we show that if $\xi\in\Gamma$, $|\xi|>\lfloor N^{1/\sigma}\rfloor$ then there exists $h>0$ such that

$$\sup_{\eta\in\mathbf{R}^d}\langle\eta\rangle^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde\tau)^{1/\sigma}\rfloor+M+d+1}|\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)| \leq h^{N+1}N!^{1/\sigma}. \quad (4.14)$$

Since $N' = N - \lceil 2^{\sigma-1}\tilde\tau(M+d+1)^\sigma\rceil$, it follows that

$$(N/\tilde\tau)^{1/\sigma} = \Big(\frac{N'+\lceil 2^{\sigma-1}\tilde\tau(M+d+1)^\sigma\rceil}{\tau}\Big)^{1/\sigma} \geq 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde\tau)^{1/\sigma}+M+d+1. \quad (4.15)$$

If $\mathfrak{S} \leq \lfloor(N/\tilde\tau)^{1/\sigma}\rfloor - m$, $|\beta| = \lfloor(N/\tilde\tau)^{1/\sigma}\rfloor$ then

$$\mathfrak{S}+|\beta| < 2\lfloor(N/\tilde\tau)^{1/\sigma}\rfloor \leq \lfloor 2(N/\tilde\tau)^{1/\sigma}\rfloor, \quad (4.16)$$

From (4.16), when $x\in K$ and $\xi\in\Gamma$, $|\xi|>\lfloor N^{1/\sigma}\rfloor$ it follows that

$$\begin{aligned}|D^\beta w_N(x,\xi)| &\leq \sum_{\mathfrak{S}=0}^{\lfloor(\frac{N}{\tilde\tau})^{\frac{1}{\sigma}}\rfloor-m}\binom{|a|}{a_1,a_2\ldots a_m}\sup_{x\in K}|(D^\beta R_1^{a_1}R_2^{a_2}\ldots R_m^{a_m}\chi_{2^\sigma N})(x,\xi)| \\ &\leq \sum_{\mathfrak{S}=0}^{\lfloor(\frac{N}{\tilde\tau})^{\frac{1}{\sigma}}\rfloor-m}\binom{|a|}{a_1,a_2\ldots a_m}|\xi|^{-\mathfrak{S}}C^{\mathfrak{S}+|\beta|+1}\lfloor N^{1/\sigma}\rfloor^{\mathfrak{S}+|\beta|} \\ &\leq \lfloor N^{1/\sigma}\rfloor^{|\beta|}\sum_{\mathfrak{S}=0}^{\lfloor(\frac{N}{\tilde\tau})^{\frac{1}{\sigma}}\rfloor-m}\binom{|a|}{a_1,a_2\ldots a_m}C^{\mathfrak{S}+|\beta|+1} \leq C'^{\lfloor(N/\tilde\tau)^{1/\sigma}\rfloor+1}\lfloor N^{1/\sigma}\rfloor^{|\beta|}.\end{aligned}$$

Since $\pi_1(\operatorname{supp} w_N(x,\xi)) \subseteq K$ and $|\beta| = \lfloor(N/\tilde\tau)^{1/\sigma}\rfloor$, we obtain

$$|\eta|^{\lfloor(N/\tilde\tau)^{1/\sigma}\rfloor}|\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)| \leq C'^{\lfloor(N/\tilde\tau)^{1/\sigma}\rfloor+1}\lfloor N^{1/\sigma}\rfloor^{\lfloor(N/\tilde\tau)^{1/\sigma}\rfloor} \leq C''^{N+1}N^{\frac{\tilde\tau^{-1/\sigma}}{\sigma}N}, \quad (4.17)$$

where we used the first part of (4.3). Now (4.15) and (4.17) gives

$$\sup_{\eta\in\mathbf{R}^d}\langle\eta\rangle^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde\tau)^{1/\sigma}\rfloor+M+d+1}|\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)|$$
$$\leq \sup_{\eta\in\mathbf{R}^d}\langle\eta\rangle^{\lfloor (N/\tilde\tau)^{1/\sigma}\rfloor}|\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)| \leq C''^{N+1}N^{\frac{\tilde\tau^{-1/\sigma}}{\sigma}N}, \quad (4.18)$$

and (4.14) follows. Therefore

$$|I_2|\leq A\frac{h^N N^{\frac{\tilde\tau^{-1/\sigma}}{\sigma}N}}{|\xi|^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N'/\tilde\tau)^{1/\sigma}\rfloor}}, \quad (4.19)$$

for suitable constants $A,h>0$. After enumeration given by (4.9), and using $(M.2)'$ property of the sequence $N^{\frac{\tilde\tau^{-1/\sigma}}{\sigma}N}$, we conclude that (4.19) is equivalent to

$$|I_2|\leq A\frac{h^N N!^{\frac{\tilde\tau^{-1/\sigma}}{\sigma}}}{|\xi|^{\lfloor 2^{\frac{1-\sigma}{\sigma}}(N/\tilde\tau)^{1/\sigma}\rfloor}}, \quad (4.20)$$

for some $A,h>0$. After enumeration $N\to\tilde\tau 2^{\sigma-1}N$ we finally obtain

$$|\widehat{u}_N(\xi)|\leq A\frac{h^N N!^{\frac{\tau 2^{\sigma-1}}{\sigma}}}{|\xi|^{\lfloor N^{1/\sigma}\rfloor}},$$

for some $A,h>0$, and the proof is finished.

4.1. **Derivation of the representation of $\widehat{u}_N(\xi)$.** Formally, we are searching for $v(x,\xi)$ so that

$$\widehat{u}_N(\xi)=\int u(x)\chi_{2^\sigma N}(x)e^{-ix\xi}dx=\int u(x)P^T(D)v(x,\xi)dx,$$

$\xi\in\Gamma$, $|\xi|>\lfloor N^{1/\sigma}\rfloor$, where $P^T(D)=\sum_{|\alpha|\leq m}(-1)^{|\alpha|}a_\alpha D^\alpha$ is the transpose operator of $P(D)$, and $v(x,\xi)$ is the solution of the equation

$$P^T(D)v(x,\xi)=\chi_{2^\sigma N}(x)e^{-ix\xi},\quad x\in K,\xi\in\Gamma,|\xi|>\lfloor N^{1/\sigma}\rfloor. \quad (4.21)$$

If $v(x,\xi)$ is of the form $v(x,\xi)=\dfrac{e^{-ix\xi}w(x,\xi)}{P_m(\xi)}$, for some $w(\cdot,\xi)\in C^\infty(K)$, where $x\in K$, $\xi\in\Gamma$, $|\xi|>\lfloor N^{1/\sigma}\rfloor$, then (4.21) becomes

$$(I-R(\xi))w(x,\xi)=\chi_{2^\sigma N}(x)\quad x\in K,\xi\in\Gamma,|\xi|>\lfloor N^{1/\sigma}\rfloor, \quad (4.22)$$

where $R(\xi)=\sum_{j=1}^m R_j(\xi)$, $R_j(\xi)=p_j(\xi)\sum_{|\alpha|\le j} a_\alpha D^\alpha$, and $p_j(\xi)$ are homogeneous functions of order $-j$. In fact, formal calculation gives

$$\begin{aligned}
&e^{ix\xi}P^T(D)(\frac{w(x,\xi)e^{-ix\xi}}{P_m(\xi)})\\
&\quad = e^{ix\xi}\frac{1}{P_m(\xi)}\sum_{|\alpha|\le m}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}(-1)^{|\alpha|}a_\alpha D^{\alpha-\beta}(e^{-ix\xi})D^\beta w(x,\xi)\\
&\qquad = \sum_{|\alpha|\le m}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}(-1)^{|\alpha|}a_\alpha\Big(\frac{(-\xi)^{\alpha-\beta}}{P_m(\xi)}\Big)D^\beta w(x,\xi),
\end{aligned}$$

for $x\in K$ and $\xi\in\Gamma$, $|\xi|>\lfloor N^{1/\sigma}\rfloor$. Since $\dfrac{(-\xi)^{\alpha-\beta}}{P_m(\xi)}$ is homogeneous of order $|\alpha|-|\beta|-m$ with respect to ξ, it follows that (4.21) would imply (4.22).

Now, successive applications of the operator R in (4.22) give

$$R^{k-1}(\xi)w(x,\xi)-R^k(\xi)w(x,\xi)=R^{k-1}(\xi)\chi_{2^\sigma N}(x),\;\; x\in K,\xi\in\Gamma,|\xi|>\lfloor N^{1/\sigma}\rfloor,$$

for every $k\in\{1,\dots,N\}$, so that after summing up those N equalities we obtain

$$w(x,\xi)-R^N(\xi)w(x,\xi)=\sum_{k=0}^{N-1}R^k(\xi)\chi_{2^\sigma N}(x),$$

which gives formal approximate solution

$$\begin{aligned}
w(x,\xi)&=\sum_{k=0}^\infty R^k\chi_{2^\sigma N}(x,\xi)\\
&\quad=\sum_{|a|=0}^\infty\binom{|a|}{a_1,a_2,\dots,a_m}R_1^{a_1}R_2^{a_2}\dots R_m^{a_m}\chi_{2^\sigma N}(x,\xi). \qquad (4.23)
\end{aligned}$$

The operators $R_k^{a_k}(\xi)$, $1\le k\le m$, are of order less then or equal to ka_k and homogeneous of order $-ka_k$ with respect to ξ. Since $P(D)$ have constant coefficients, the operators R_j commute, and we used the generalized Newton formula, cf. [28].

We proceed with the following approximation procedure. We consider partial sums

$$w_N(x,\xi)=\sum_{\mathfrak{S}=0}^{\lfloor(\frac{N}{T})^{\frac{1}{\sigma}}\rfloor-m}\binom{|a|}{a_1,a_2\dots a_m}(R_1^{a_1}R_2^{a_2}\dots R_m^{a_m}\chi_{2^\sigma N})(x,\xi),$$

$\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma} \rfloor$, and $N \in \mathbf{N}$ is large enough, so that (4.22) takes the form (4.4) and the error term e_N is given by:

$$e_N(x,\xi) = \sum_{k=1}^{m} \sum_{\mathfrak{S}=\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m+1}^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m+k} \binom{|a|}{a_1, ..., a_m} (R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N})(x,\xi).$$

The precise calculation which leads to (4.4) is given in Subsection 4.2. Note that the number of terms in (4.7) is bounded by $4 \cdot 2^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor}$, since from $\binom{n}{k} \leq 2^n$, $k \leq n$, $n \in \mathbf{N}$, we obtain

$$\binom{|a|}{a_1, a_2, \ldots a_m} \leq 2^{|a|} 2^{|a|-a_1} \ldots 2^{|a|-a_1-\cdots-a_{m-2}} \leq 2^{a_1+2a_2+\cdots+ma_m},$$

and therefore

$$\sum_{k=1}^{m} \sum_{\mathfrak{S}=\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m+1}^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m+k} \binom{|a|}{a_1, \ldots, a_m} \leq \sum_{k=1}^{m} \sum_{\mathfrak{S}=\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m+1}^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m+k} 2^{a_1+2a_2\cdots+ma_m}$$

$$\leq 2^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m+1} \sum_{k=1}^{m} 2^k \leq 4 \cdot 2^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor},$$

where we put $\mathfrak{S} = a_1 + 2a_2 + \cdots + ma_m$.

4.2. **The calculation of the error term.** For multinomial coefficients

$$\binom{|a|}{a_1, a_2, \ldots a_m} := \binom{|a|}{a_1} \binom{|a| - a_1}{a_2} \cdots \binom{|a| - a_1 - \cdots - a_{m-2}}{a_{m-1}}$$

$$= \frac{|a|!}{a_1! a_2! \ldots a_m!}, \quad |a| = a_1 + a_2 + \cdots + a_m,\ a_k \in \mathbf{N},\ k \leq m, \quad (4.24)$$

a generalization of Pascal's triangle equality for the binomial formula gives

$$\binom{|a|}{a_1, ..., a_m} = \sum_{k=1}^{m} \binom{|a| - 1}{a_1, ..., a_k - 1, ...a_m}, \quad |a| \geq 1, \quad (4.25)$$

wherefrom for $|a| \geq 1$, and putting $\mathfrak{S} = a_1 + 2a_2 + \cdots + ma_m$ we obtain

$$\sum_{\mathfrak{S}=0}^{(\lfloor \frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m} \binom{|a|}{a_1, ..., a_m} R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N}$$

$$= \sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m} \Big(\sum_{k=1}^{m} \binom{|a| - 1}{a_1, ..., a_k - 1, ...a_m} \Big) R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N}$$

$$= \sum_{k=1}^{m} \sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m - k} \binom{|a|}{a_1, ..., a_k, ...a_m} R_1^{a_1} ... R_k^{a_k+1} ... R_m^{a_m} \chi_{2^\sigma N}$$

$$= \sum_{k=1}^{m} R_k \Big(\sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m - k} \binom{|a|}{a_1, ..., a_m} R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N} \Big), \tag{4.26}$$

where for the second equality we interchange the summation and substitute a_k with $a_k + 1$.

Hence, for $|a| \geq 0$ we have

$$(I - R) w_N = \sum_{\mathfrak{S}=0}^{(\lfloor \frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m} \binom{|a|}{a_1, ..., a_m} R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N}$$

$$- \sum_{k=1}^{m} R_k \Big(\sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m - k} \binom{|a|}{a_1, ..., a_m} R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N}$$

$$+ \sum_{\mathfrak{S}=\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m - k + 1}^{\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m} \binom{|a|}{a_1, ..., a_m} R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N} \Big)$$

$$= \chi_{2^\sigma N} - \sum_{k=1}^{m} \sum_{\mathfrak{S}=\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m - k + 1}^{\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m} \binom{|a|}{a_1, ..., a_m} R_1^{a_1} ... R_k^{a_k+1} ... R_m^{a_m} \chi_{2^\sigma N}$$

$$= \chi_{2^\sigma N} - \sum_{k=1}^{m} \sum_{\mathfrak{S}=\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m + 1}^{\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m + k} \binom{|a|}{a_1, ..., a_m} R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N}, \tag{4.27}$$

where for the second equality we used (4.26) and for the last one we substitute a_k with $a_k - 1$.

Therefore, if we set

$$e_N(x, \xi) = \sum_{k=1}^{m} \sum_{\mathfrak{S}=\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m + 1}^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m + k} \binom{|a|}{a_1, ..., a_m} (R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N})(x, \xi),$$

then the computation of this subsection gives the equality (4.4), which in turn implies the fundamental representation (4.5).

4.3. **Estimates for $D^\beta (R_1^{a_1} ... R_m^{a_m} \chi_{2^\sigma N})$.** Note that for N large enough we have

$$(\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor + M)^\sigma \leq 2^{\sigma - 1} (N/\tilde{\tau} + M^\sigma) < 2^\sigma N/\tilde{\tau}$$

so that for $|\beta| \leq M$ the following estimate holds:

$$\mathfrak{S} + |\beta| \leq \lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor + M = \lfloor (N/\tilde{\tau})^{1/\sigma} + M \rfloor < \lfloor 2 (N/\tilde{\tau})^{1/\sigma} \rfloor.$$

Thus, for $x \in K$, $\xi \in \Gamma$, and $\mathfrak{S} \geq \lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor - m$, by using (4.3) we obtain

$$
\begin{aligned}
|D^\beta(R_1^{a_1}...R_m^{a_m}\chi_{2^\sigma N})(x,\xi)| &\leq |\xi|^{-\mathfrak{S}} A^{\mathfrak{S}+|\beta|+1} \lfloor N^{1/\sigma} \rfloor^{\mathfrak{S}+|\beta|} \\
&\leq |\xi|^{m-\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor} A^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor + M+1} \lfloor N^{1/\sigma} \rfloor^{\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor + M} \\
&\leq |\xi|^{m-\lfloor (N/\tilde{\tau})^{1/\sigma} \rfloor} C^{N+1} N^{\frac{\tilde{\tau}^{-1/\sigma}}{\sigma} N}, \qquad (4.28)
\end{aligned}
$$

for some $C > 0$, which is, after enumeration $N \to N + 2^{\sigma-1}\tilde{\tau}(m+M)^\sigma$ bounded by

$$
|\xi|^{m-\lfloor ((N+2^{\sigma-1}\tilde{\tau}(m+M)^\sigma)/\tilde{\tau})^{1/\sigma} \rfloor} A^{N+2^{\sigma-1}\tilde{\tau}(m+M)^\sigma+1} \times (N+2^{\sigma-1}\tilde{\tau}(m+M)^\sigma)^{\frac{\tilde{\tau}^{-1/\sigma}}{\sigma}(N+2^{\sigma-1}\tilde{\tau}(m+M)^\sigma)},
$$

for some $A > 0$. Moreover,

$$
\begin{aligned}
\Big(\frac{N+2^{\sigma-1}\tilde{\tau}(m+M)^\sigma}{\tilde{\tau}}\Big)^{1/\sigma} &\geq 2^{\frac{1-\sigma}{\sigma}}((N/\tilde{\tau})^{1/\sigma} + 2^{\frac{\sigma-1}{\sigma}}(m+M)) \\
&= 2^{\frac{1-\sigma}{\sigma}}(N/\tilde{\tau})^{1/\sigma} + m + M. \qquad (4.29)
\end{aligned}
$$

Finally, (4.29), $(M.2)'$ property of $N^{\frac{\tilde{\tau}^{-1/\sigma}}{\sigma}N}$ and Stirling's formula give the estimate

$$
|D^\beta R_1^{a_1}...R_m^{a_m}\chi_{2^\sigma N}(x)| \leq |\xi|^{-\lfloor 2^{\frac{1-\sigma}{\sigma}}(N/\tilde{\tau})^{1/\sigma} \rfloor - M} C^{N+1} N!^{\frac{\tilde{\tau}^{-1/\sigma}}{\sigma}} \qquad (4.30)
$$

for some $C > 0$.

4.4. **The computation of $\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)$.** From

$$
(R_1^{a_1}...R_m^{a_m}\chi_{2^\sigma N})(x,\xi) = \prod_{j=1}^m p_j^{a_j}(\xi) \sum_{|\alpha|\leq \mathfrak{S}} c_\alpha D^\alpha \chi_{2^\sigma N}(x)
$$

for suitable constants c_α, it follows that

$$
\mathcal{F}_{x\to\eta}(R_1^{a_1}...R_m^{a_m}\chi_{2^\sigma N})(\eta,\xi) = \prod_{j=1}^m p_j^{a_j}(\xi) \sum_{|\alpha|\leq \mathfrak{S}} c''_\alpha \eta^\alpha \widehat{\chi}_{2^\sigma N}(\eta),
$$

so that

$$
\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi) = \sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tilde{\tau}})^{\frac{1}{\sigma}} \rfloor - m} \binom{|a|}{a_1, a_2 \ldots a_m} \Big(\prod_{j=1}^m p_j^{a_j}(\xi)\Big) \sum_{|\alpha|\leq \mathfrak{S}} c''_\alpha \eta^\alpha \widehat{\chi}_{2^\sigma N}(\eta).
$$

Note that the number of terms in $\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)$ is bounded by $C2^{\lfloor (N/\tau)^{1/\sigma} \rfloor}$ for some $C > 0$ which does not depend on N.

When $|\eta| \leq \varepsilon|\xi|$, $\xi \in \Gamma$, $|\xi| > \lfloor N^{1/\sigma} \rfloor$, and N sufficiently large we have

$$|\mathcal{F}_{x\to\eta}(w_N)(\eta,\xi)| \leq$$

$$\sum_{\mathfrak{S}=0}^{\lfloor (\frac{N}{\tau})^{\frac{1}{\sigma}} \rfloor - m} \binom{|a|}{a_1, a_2 \dots a_m} \Big(\prod_{j=1}^{m} (|p_j(\xi)||\varepsilon\xi|^j)^{a_j} \Big) \sum_{|\alpha| \leq \mathfrak{S}} c''_\alpha |\widehat{\chi}_{2^\sigma N}(\eta)|$$

$$\leq AC^{\lfloor (N/\tau)^{1/\sigma} \rfloor} |\widehat{\chi}_{2^\sigma N}(\eta)|,$$

for some $A, C > 0$, and we used

$$\prod_{j=1}^{m} (|p_j(\xi)||\varepsilon\xi|^j)^{a_j} \leq A\varepsilon^{\mathfrak{S}} \leq A, \quad \xi \in \Gamma, |\xi| > \lfloor N^{1/\sigma} \rfloor,$$

which follows from $\varepsilon < 1$ and the fact that $\prod_{j=1}^{m} (|p_j(\xi)||\xi|^j)^{a_j}$ is homogeneous of order zero.

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