ERGODIC DECOMPOSITIONS OF STATIONARY MAX-STABLE PROCESSES IN TERMS OF THEIR SPECTRAL FUNCTIONS

CLÉMENT DOMBRY AND ZAKHAR KABLUCHKO

Abstract. We revisit conservative/dissipative and positive/null decompositions of stationary max-stable processes. Originally, both decompositions were defined in an abstract way based on the underlying non-singular flow representation. We provide simple criteria which allow to tell whether a given spectral function belongs to the conservative/dissipative or positive/null part of the de Haan spectral representation. Specifically, we prove that a spectral function is null-recurrent iff it converges to 0 in the Cesàro sense. For processes with locally bounded sample paths we show that a spectral function is dissipative iff it converges to 0. Surprisingly, for such processes a spectral function is integrable a.s. iff it converges to 0 a.s. Based on these results, we provide new criteria for ergodicity, mixing, and existence of a mixed moving maximum representation of a stationary max-stable process in terms of its spectral functions. In particular, we study a decomposition of max-stable processes which characterizes the mixing property.

1. Statement of main results

1.1. **Introduction.** A stochastic process $(\eta(x))_{x\in\mathcal{X}}$ on $\mathcal{X}=\mathbb{Z}^d$ or $\mathcal{X}=\mathbb{R}^d$ is called *max-stable* if

$$\frac{1}{n}\bigvee_{i=1}^{n}\eta_i \overset{f.d.d.}{=} \eta \quad \text{for all } n\geq 1,$$

where $\eta_1,\ldots,\eta_n$ are i.i.d. copies of η, $\bigvee$ is the pointwise maximum, and $\overset{f.d.d.}{=}$ denotes the equality of finite-dimensional distributions. Max-stable processes arise naturally when considering limits for normalized pointwise maxima of independent and identically distributed (i.i.d.) stochastic processes and hence play a major role in spatial extreme value theory; see, e.g., de Haan and Ferreira [4]. We restrict our attention to processes with non-degenerate (non-constant) margins. The above definition implies that the marginal distributions of η are 1–Fréchet, that is

$$\mathbb{P}[\eta(x)\leq z]=\mathrm{e}^{-c(x)/z} \quad \text{for all } z>0,$$

where $c(x)>0$ is a scale parameter.

A fundamental representation theorem by de Haan [3] states that any stochastically continuous max-stable process η can be represented (in distribution) as

$$\eta(x)=\bigvee_{i\geq 1}U_iY_i(x), \quad x\in\mathcal{X}, \tag{1}$$

2010 *Mathematics Subject Classification.* Primary: 60G70; Secondary: 60G52, 60G60, 60G55, 60G10, 37A10, 37A25.

Key words and phrases. max-stable random process, de Haan representation, non-singular flow, conservative/dissipative decomposition, positive/null decomposition, ergodic process, mixing process, mixed moving maximum process.

where

- $(U_i)_{i\geq 1}$ is a decreasing enumeration of the points of a Poisson point process on $(0, +\infty)$ with intensity measure $u^{-2}\mathrm{d}u$,
- $(Y_i)_{i\geq 1}$, which are called the *spectral functions*, are i.i.d. copies of a non-negative process $(Y(x))_{x\in\mathcal{X}}$ such that $\mathbb{E}[Y(x)] < +\infty$ for all $x \in \mathcal{X}$,
- the sequences $(U_i)_{i\geq 1}$ and $(Y_i)_{i\geq 1}$ are independent.

In this paper, we focus on *stationary* max-stable processes that play an important role for modelling purposes; see, e.g., Schlather [21]. The structure of stationary max-stable processes was first investigated by de Haan and Pickand [5] who related them to non-singular flows. Using the analogy between max-stable and sum-stable processes and the works of Rosiński [13, 14], Rosiński and Samorodnitsky [15] and Samorodnitsky [19, 20] on sum-stable processes, the representation theory of stationary max-stable processes via non-singular flows was developed by Kabluchko [7], Wang and Stoev [26, 25], Wang *et al.* [24]. In these papers, the conservative/dissipative (or Hopf) and positive/null (or Neveu) decompositions from non-singular ergodic theory were used to introduce the corresponding decompositions $\eta = \eta_C \vee \eta_D$ and $\eta = \eta_P \vee \eta_N$ of the max-stable process. These definitions were rather abstract (see Sections 3 and 4 where we will recall them) and did not allow to distinguish between conservative/dissipative or positive/null cases by looking just at the spectral functions Y_i from the de Haan representation (1). The purpose of this paper is to provide a *constructive* definition of these decompositions. Our main results in this direction can be summarized as follows. In Section 3 we will prove that in the case when the sample paths of η are a.s. locally bounded, a spectral function Y_i belongs to the dissipative (=mixed moving maximum) part of the process if and only if $\lim_{x\to\infty} Y_i(x) = 0$. In Section 4 we will prove that a spectral function Y_i belongs to the null (=ergodic) part if and only if it converges to 0 in the Cesàro sense. In Section 5, we will introduce one more decomposition which characterizes mixing.

1.2. **Ergodic properties of max-stable processes.** Our results can be used to give new criteria for ergodicity, mixing, and existence of mixed moving maximum representation of max-stable processes. These criteria extend and simplify the results of Stoev [22], Kabluchko and Schlather [8] and Wang *et al.* [24].

In the following, $(\eta(x))_{x\in\mathcal{X}}$ denotes a stationary, stochastically continuous max-stable process on $\mathcal{X} = \mathbb{Z}^d$ or $\mathbb{R}^d$ with de Haan representation (1). In the case when $\mathcal{X} = \mathbb{R}^d$, the process Y is continuous in L^1 by Lemma 2 in [3]. Since continuity in L^1 implies stochastic continuity and since every stochastically continuous process has a measurable and separable version, we will tacitly assume throughout the paper that both η and Y are measurable and separable processes. These assumptions (as well as the assumption of stochastic continuity) are empty (and can be ignored) in the discrete case $\mathcal{X} = \mathbb{Z}^d$.

Our first result is a characterization of ergodicity. Let $\lambda(\mathrm{d}x)$ be the counting measure on $\mathbb{Z}^d$ (in the discrete-time case) or the Lebesgue measure on $\mathbb{R}^d$ (in the continuous-time case), respectively. For $r > 0$, write $B_r = [-r, r]^d \cap \mathcal{X}$.

Theorem 1. *For a stationary, stochastically continuous max-stable process η the following conditions are equivalent:*

(a) *η is ergodic;*
(b) *η is weakly mixing;*

(c) *η has no positive recurrent component in its spectral representation, that is $\eta_P = 0$;*
(d) $\lim_{r\to\infty} \frac{1}{\lambda(B_r)} \int_{B_r} \mathbb{E}[Y(x) \wedge Y(0)]\lambda(\mathrm{d}x) = 0$*;*
(e) $\lim_{r\to\infty} \frac{1}{\lambda(B_r)} \int_{B_r} Y(x)\lambda(\mathrm{d}x) = 0$ *in probability;*
(f) $\liminf_{r\to\infty} \frac{1}{\lambda(B_r)} \int_{B_r} Y(x)\lambda(\mathrm{d}x) = 0$ *almost surely.*

The equivalence of (a), (b), (c), (d) in Theorem 1 was known before (see Theorem 3.2 in [8] for the equivalence of (a), (b), (d) in the case $d = 1$, Theorem 8 in [7] for the equivalence of (a) and (c) in the case $d = 1$, and Theorem 5.3 in [24] for an extension to the d-dimensional case). We will prove in Section 3 that (c), (e), (f) are equivalent by exploiting a new characterization of the positive/null decomposition.

The next theorem characterizes mixing (which is a stronger property than ergodicity).

Theorem 2. *For a stationary, stochastically continuous max-stable process η the following conditions are equivalent:*
(a) *η is mixing;*
(b) *η is mixing of all orders;*
(c) $\lim_{x\to\infty} \mathbb{E}[Y(x) \wedge Y(0)] = 0$*;*
(d) $\lim_{x\to\infty} Y(x) = 0$ *in probability.*

The equivalence of (a), (b), (c) in Theorem 3 was known before (see Theorem 3.4 in [22] for the equivalence of (a) and (c), and Theorem 1.1 in [8] for the equivalence of (a) and (b)). We will prove in Section 4 that (c) is equivalent to (d). Moreover, we will introduce a decomposition of the process η into a mixing part and and a part containing no mixing components.

Finally, we can characterize the mixed moving maximum property. The definition of this property will be recalled in Section 3.

Theorem 3. *For a stationary, stochastically continuous max-stable process η with locally bounded sample paths, the following conditions are equivalent:*
(a) *η has a mixed moving maximum representation;*
(b) *η has no conservative component in its spectral representation, that is $\eta_C = 0$;*
(c) $\int_{\mathcal{X}} Y(x)\lambda(\mathrm{d}x) < +\infty$ *almost surely;*
(d) $\lim_{x\to\infty} Y(x) = 0$ *almost surely.*

The equivalence of (a), (b), (c) in Theorem 3 was known before and holds even without the assumption of local boundedness (see Section 3.1 and the references therein). Our main contribution is an alternative characterization of the conservative/dissipative decomposition stated in Proposition 10 that implies the equivalence of (c) and (d). This equivalence may look strange at a first glance, but let us stress that the process Y is not arbitrary. For example, Y has the property that the corresponding process η is stationary (such Y's were called Brown–Resnick stationary in [9]). A special case of the implication (d) $\Rightarrow$ (c) when $\log Y$ is a Gaussian process with stationary increments and certain drift was obtained in [26, Theorem 7.1].

The rest of the paper is structured as follows. Section 2 is devoted to preliminaries on non-singular ergodic theory and cone decomposition for max-stable processes. Section 3 reviews known results on the conservative/dissipative decompositions and provides an alternative definition via a simple cone decomposition with an emphasis on the case of locally bounded max-stable processes. Section 4 introduces the positive/null decomposition and proposes an alternative construction via another simple cone decomposition. In Section 5 we study mixing.

2. Preliminaries

2.1. **Non-singular flow representations of max-stable processes.** We recall some information on non-singular flow representations of stationary max-stable processes. For more details on non-singular ergodic theory, the reader should refer to Krengel [10], Aaronson [1] or Danilenko and Silva [2].

Definition 4. *A measurable non-singular flow on a measure space $(S, \mathcal{B}, \mu)$ is a family of functions $\phi_x : S \to S$, $x \in \mathcal{X}$, satisfying*

(i) *(flow property) for all $s \in S$ and $x_1, x_2 \in \mathcal{X}$,*

$$\phi_0(s) = s \quad \textit{and} \quad \phi_{x_1+x_2}(s) = \phi_{x_2}(\phi_{x_1}(s));$$

(ii) *(measurability) the mapping $(x, s) \mapsto \phi_x(s)$ is measurable from $\mathcal{X} \times S$ to S;*

(iii) *(non-singularity) for all $x \in \mathcal{X}$, the measures $\mu \circ \phi_x^{-1}$ and μ are equivalent, i.e. for all $A \in \mathcal{B}$, $\mu(\phi_x^{-1}(A)) = 0$ if and only if $\mu(A) = 0$.*

The non-singularity property ensures that one can define the Radon–Nikodym derivative

$$\omega_x(s) = \frac{\mathrm{d}(\mu \circ \phi_x)}{\mathrm{d}\mu}(s). \tag{2}$$

By the measurability property, one may assume that the mapping $(x, s) \mapsto \omega_x(s)$ is jointly measurable on $\mathcal{X} \times S$.

According to de Haan and Pickands [5], see also [7] and [26], any stochastically continuous stationary max-stable process η admits a representation of the form

$$\eta(x) = \bigvee_{i \geq 1} U_i f_x(s_i), \quad x \in \mathcal{X}, \tag{3}$$

where $f_x(s) = \omega_x(s) f_0(\phi_x(s))$ and

- $(\phi_x)_{x \in \mathcal{X}}$ is a measurable non-singular flow on some σ-finite measure space $(S, \mathcal{B}, \mu)$, with $\omega_x(s)$ defined by (2),
- $f_0 \in L^1(S, \mathcal{B}, \mu)$ is non-negative such that the set $\{f_0 = 0\}$ contains no $(\phi_x)_{x \in \mathcal{X}}$–invariant set $B \in \mathcal{B}$ of positive measure,
- $\{(s_i, U_i)\}_{i \geq 1}$ is some enumeration of the points of the Poisson point process on $S \times (0, +\infty)$ with intensity $\mu(\mathrm{d}s) \times u^{-2}\mathrm{d}u$.

Starting with a non-singular flow representation (3) on a probability space, one easily gets a de Haan representation of the form (1) by considering the i.i.d. stochastic processes $Y_i(x) = f_x(s_i)$, $i \geq 1$. The flow representation (3) is comonly written as an extremal integral

$$\eta(x) = \int_S^{\mathrm{e}} f_x(s) M(\mathrm{d}s), \quad x \in \mathcal{X}, \tag{4}$$

where $M(\mathrm{d}s)$ denotes a 1-Fréchet random sup-measure on $(S, \mathcal{B})$ with control measure μ. The reader should refer to Stoev and Taqqu [23] for more details on extremal integrals. In the present paper, one can simply view the extremal integral (4) as a shorthand for the pointwise maximum over a Poisson point process (3).

2.2. **Cone based decompositions.** In the spirit of Wang and Stoev [26, Theorem 4.2] and Dombry and Kabluchko [6, Lemma 16], we will use decompositions of max-stable processes based on cones. We denote by $\mathcal{F}_0 = \mathcal{F}(\mathcal{X}, [0, +\infty)) \setminus \{0\}$ the set of non-negative measurable functions on $\mathcal{X}$ excluding the zero function. A subset $\mathcal{C} \subset \mathcal{F}_0$ is called a *cone* if for all $f \in \mathcal{C}$ and $u > 0$, $uf \in \mathcal{C}$. The cone $\mathcal{C}$ is said to be *shift-invariant* if for all $f \in \mathcal{C}$ and $x \in \mathcal{X}$ we have $f(\cdot + x) \in \mathcal{C}$.

Lemma 5 (Lemma 16 in [6])**.** *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two shift-invariant cones such that $\mathcal{F}_0 = \mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = \varnothing$. Let η be a stationary max-stable process given by representation* (1) *such that the events $\{Y_i \in \mathcal{C}_1\}$ and $\{Y_i \in \mathcal{C}_2\}$ are measurable. Consider the decomposition $\eta = \eta_1 \vee \eta_2$ with*

$$\eta_1(x) = \bigvee_{i \geq 1} U_i Y_i(x) \mathbb{1}_{\{Y_i \in \mathcal{C}_1\}} \quad \textit{and} \quad \eta_2(x) = \bigvee_{i \geq 1} U_i Y_i(x) \mathbb{1}_{\{Y_i \in \mathcal{C}_2\}}.$$

Then, η_1 and η_2 are stationary and independent max-stable processes whose distribution depends only on the distribution of η and not on the specific representation (1)*.*

3. Conservative/dissipative decomposition

3.1. **Definition of the conservative/dissipative decomposition.** We recall the Hopf (or conservative/dissipative) decomposition from non-singular ergodic theory; see Aaronson [1]. We start with the discrete case $\mathcal{X} = \mathbb{Z}^d$.

Definition 6. *Consider a measure space $(S, \mathcal{B}, \mu)$ and a non-singular flow $(\phi_x)_{x \in \mathbb{Z}^d}$. A measurable set $W \subset S$ is said to be wandering if the sets $\phi_x^{-1}(W)$, $x \in \mathbb{Z}^d$, are disjoint.*

The Hopf decomposition theorem states that there exists a partition of S into two disjoint measurable sets $S = C \cup D$, $C \cap D = \varnothing$, such that

(i) C and D are $(\phi_x)_{x \in \mathbb{Z}^d}$–invariant,
(ii) there exists no wandering set $W \subset C$ with positive measure,
(iii) there exists a wandering set $W_0 \subset D$ such that $D = \cup_{x \in \mathbb{Z}^d} \phi_x(W_0)$.

This decomposition is unique mod μ and is called the *Hopf decomposition* of S associated with the flow $(\phi_x)_{x \in \mathbb{Z}^d}$; the sets C and D are called the *conservative* and *dissipative* parts respectively. In the case when $\mathcal{X} = \mathbb{R}^d$, we follow Roy [17] by defining the Hopf decomposition of S associated with a measurable flow $(\phi_x)_{x \in \mathbb{R}^d}$ as the Hopf decomposition associated with the discrete skeleton flow $(\phi_x)_{x \in \mathbb{Z}^d}$.

One can then introduce the conservative/dissipative decomposition of the max-stable process η given by (3), (4): we have $\eta = \eta_C \vee \eta_D$ with

$$\eta_C(x) = \int_C^{\mathrm{e}} f_x(s) M(\mathrm{d}s) \quad \text{and} \quad \eta_D(x) = \int_D^{\mathrm{e}} f_x(s) M(\mathrm{d}s), \quad x \in \mathcal{X}. \tag{5}$$

The processes η_C and η_D are independent and their distribution depends only on the distribution of η and not on the particular choice of the representation (3).

The importance of the conservative/dissipative decomposition comes from the notion of mixed moving maximum representation.

Definition 7. *A stationary max-stable process $(\eta(x))_{x \in \mathcal{X}}$ is said to have a mixed moving maximum representation (shortly M3-representation) if*

$$\eta(x) \overset{f.d.d.}{=} \bigvee_{i \geq 1} V_i Z_i(x - X_i), \quad x \in \mathcal{X},$$

where

- $\{(X_i, V_i), i \geq 1\}$ *is a Poisson point process on* $\mathcal{X} \times (0, +\infty)$ *with intensity* $\lambda(\mathrm{d}x) \times u^{-2}\mathrm{d}u$,
- $(Z_i)_{i\geq 1}$ *are i.i.d. copies of a non-negative measurable stochastic process* Z *on* $\mathcal{X}$ *satisfying* $\mathbb{E}[\int_{\mathcal{X}} Z(x)\lambda(\mathrm{d}x)] < +\infty$,
- $\{(X_i, V_i), i \geq 1\}$ *and* $(Z_i)_{i\geq 1}$ *are independent.*

The following important theorem relates the dissipative/conservative decomposition and the existence of an M3-representation; see Wang and Stoev [26, Theorem 6.4] in the max-stable case with $d = 1$ or Roy [17, Theorem 3.4] in the sum-stable case with $d \geq 1$.

Theorem 8. *Let* η *be a stationary max-stable process given by the non-singular flow representation* (3)*. Then,* η *has an M3-representation if and only if* η *is generated by a dissipative flow.*

3.2. **Characterization using spectral functions.** The following simple integral test on the spectral functions allows us to retrieve the conservative/dissipative decomposition; see Roy and Samorodnitsky [18, Proposition], Roy [17, Proposition 3.2] and Wang and Stoev [26, Theorem 6.2].

Theorem 9. *We have*

(i) $\int_{\mathcal{X}} f_x(s)\lambda(\mathrm{d}x) = \infty$ $\mu(\mathrm{d}s)$*–a.e. on* C*;*
(ii) $\int_{\mathcal{X}} f_x(s)\lambda(\mathrm{d}x) < \infty$ $\mu(\mathrm{d}s)$*–a.e. on* D*.*

Consider a stationary max-stable process η given by de Haan's representation (1). In view of Theorem 9, we introduce the cones of functions

$$\mathcal{F}_C = \left\{ f \in \mathcal{F}_0;\ \int_{\mathcal{X}} f(x)\lambda(\mathrm{d}x) = \infty \right\}, \tag{6}$$

$$\mathcal{F}_D = \left\{ f \in \mathcal{F}_0;\ \int_{\mathcal{X}} f(x)\lambda(\mathrm{d}x) < \infty \right\}. \tag{7}$$

These cones are clearly shift-invariant and, assuming that Y is jointly measurable and separable, the events $\{Y \in \mathcal{F}_C\}$ and $\{Y \in \mathcal{F}_D\}$ are measurable. Using Lemma 5, we define

$$\eta_C(x) = \bigvee_{i\geq 1} U_i Y_i(x) \mathbb{1}_{\{Y_i \in \mathcal{F}_C\}} \quad \text{and} \quad \eta_D(x) = \bigvee_{i\geq 1} U_i Y_i(x) \mathbb{1}_{\{Y_i \in \mathcal{F}_D\}}. \tag{8}$$

One can easily prove thanks to Theorem 9 and Lemma 5 that we retrieve (in distribution) the conservative/dissipative decomposition (5) based on the flow representation (3).

The main contribution of this section concerns the case when the max-stable process η has locally bounded sample paths, which is usually the case in applications. Interestingly, one can then introduce another, more simple and convenient, cone decomposition equivalent to (8). Consider

$$\tilde{\mathcal{F}}_C = \left\{ f \in \mathcal{F}_0;\ \limsup_{x\to\infty} f(x) > 0 \right\},$$

$$\tilde{\mathcal{F}}_D = \left\{ f \in \mathcal{F}_0;\ \lim_{x\to\infty} f(x) = 0 \right\}.$$

Note that since the process Y is assumed to be separable, the events $\{Y \in \tilde{\mathcal{F}}_C\}$ and $\{Y \in \tilde{\mathcal{F}}_C\}$ are measurable.

Proposition 10. *Let η be a stationary max-stable process given by de Haan's representation* (1) *and assume that η has locally bounded sample paths. Then, modulo null sets,*

$$\{Y \in \mathcal{F}_C\} = \{Y \in \tilde{\mathcal{F}}_C\} \quad \textit{and} \quad \{Y \in \mathcal{F}_D\} = \{Y \in \tilde{\mathcal{F}}_D\}.$$

We deduce that the decomposition

$$\tilde{\eta}_C(x) = \bigvee_{i \geq 1} U_i Y_i(x) \mathbb{1}_{\{Y_i \in \tilde{\mathcal{F}}_C\}} \quad \textit{and} \quad \tilde{\eta}_D(x) = \bigvee_{i \geq 1} U_i Y_i(x) \mathbb{1}_{\{Y_i \in \tilde{\mathcal{F}}_D\}}.$$

is almost surely equal to the decomposition (8).

Proof. We consider first the discrete setting $\mathcal{X} = \mathbb{Z}^d$. The convergence of the series $\sum_{x \in \mathbb{Z}^d} f(x)$ implies the convergence $\lim_{x \to \infty} f(x) = 0$ so that the inclusion $\{Y \in \mathcal{F}_D\} \subset \{Y \in \tilde{\mathcal{F}}_D\}$ is trivial. We need only to prove the converse inclusion $\{Y \in \tilde{\mathcal{F}}_D\} \subset \{Y \in \mathcal{F}_D\}$. Then, the equality $\{Y \in \mathcal{F}_D\} = \{Y \in \tilde{\mathcal{F}}_D\}$ (modulo null sets) implies the equality of the complementary sets, i.e. $\{Y \in \mathcal{F}_C\} = \{Y \in \tilde{\mathcal{F}}_C\}$.

Proof of the inclusion $\{Y \in \tilde{\mathcal{F}}_D\} \subset \{Y \in \mathcal{F}_D\}$. Let $\tilde{Y}_D = Y \mathbb{1}_{\{Y \in \tilde{\mathcal{F}}_D\}}$ and $\tilde{\eta}_D = \vee_{i \geq 1} U_i Y_i \mathbb{1}_{\{Y_i \in \tilde{\mathcal{F}}_D\}}$. We will show that $\tilde{\eta}_D$ admits an M3-representation. By Theorem 8, this implies that $\tilde{Y}_D$ belongs a.s. to $\mathcal{F}_D$ and hence $\{Y \in \tilde{\mathcal{F}}_D\} \subset \{Y \in \mathcal{F}_D\}$ modulo null sets. For the sake of notational convenience, we assume that $Y \in \tilde{\mathcal{F}}_D$ a.s. so that $\tilde{Y}_D = Y$ and $\tilde{\eta}_D = \eta$. We prove that η has an M3-representation with a strategy similar to the proof of Theorem 14 in Kabluchko *et al.* [9] and we sketch only the main lines. We introduce the random variables

$$X_i = \operatorname*{argmax}_{x \in \mathcal{X}} Y_i(x), \quad Z_i(\cdot) = \frac{Y_i(X_i + \cdot)}{\max_{x \in \mathcal{X}} Y_i(x)}, \quad V_i = U_i \max_{x \in \mathcal{X}} Y_i(x). \tag{9}$$

If the argmax is not unique, we use the lexicographically smallest value. Clearly, we have $U_i Y_i(x) = V_i Z_i(x - X_i)$ for all $x \in \mathcal{X}$ so that

$$\eta(x) = \bigvee_{i \geq 1} V_i Z_i(x - X_i).$$

It remains to check that $(X_i, V_i, Z_i)_{i \geq 1}$ has the properties required in Definition 7, i.e. is a Poisson point process on $\mathcal{X} \times (0, \infty) \times \mathcal{F}_0$ with intensity measure $\lambda(\mathrm{d}x) \times u^{-2}\mathrm{d}u \times Q(\mathrm{d}f)$, where Q is a probability measure on $\mathcal{F}_0$. Clearly, $(X_i, V_i, Z_i)_{i \geq 1}$ is a Poisson point process as the image of the original point process $(U_i, Y_i)_{i \geq 1}$. Its intensity is the image of the intensity of the original point process. With a straightforward transposition of the arguments of [9, Theorem 14], one can check that it has the required form.

We now turn to the case $\mathcal{X} = \mathbb{R}^d$. The convergence of the integral $\int_{\mathcal{X}} f(x)\lambda(\mathrm{d}x)$ does not imply the convergence $\lim_{x \to \infty} f(x) = 0$. But it is easy to prove that for $K = [-1/2, 1/2]^d$, the convergence of the integral $\int_{\mathcal{X}} \sup_{u \in K} f(x+u)\lambda(\mathrm{d}x)$ implies the convergence $\lim_{x \to \infty} f(x) = 0$. We introduce the cone

$$\mathcal{F}'_D = \left\{ f \in \mathcal{F}_0;\ \int_{\mathcal{X}} \sup_{u \in K} f(x+u)\lambda(\mathrm{d}x) < \infty \right\}.$$

The inclusions of cones $\mathcal{F}'_D \subset \mathcal{F}_D$ and $\mathcal{F}'_D \subset \tilde{\mathcal{F}}_D$ imply the trivial inclusions of events

$$\{Y \in \mathcal{F}'_D\} \subset \{Y \in \mathcal{F}_D\} \quad \text{and} \quad \{Y \in \mathcal{F}'_D\} \subset \{Y \in \tilde{\mathcal{F}}_D\}.$$

We will prove below that, modulo null sets,

$$\{Y \in \mathcal{F}_D\} \subset \{Y \in \mathcal{F}'_D\} \quad \text{and} \quad \{Y \in \tilde{\mathcal{F}}_D\} \subset \{Y \in \mathcal{F}_D\}$$

whence we deduce the equalities, modulo null sets,

$$\{Y \in \mathcal{F}_D\} = \{Y \in \mathcal{F}'_D\} = \{Y \in \tilde{\mathcal{F}}_D\},$$

proving the proposition.

Proof of the inclusion $\{Y \in \mathcal{F}_D\} \subset \{Y \in \mathcal{F}'_D\}$. Let $Y_D = Y\mathbb{1}_{\{Y\in\mathcal{F}_D\}}$ and $\eta_D = \vee_{i\geq 1} U_i Y_i \mathbb{1}_{\{Y_i\in\mathcal{F}_D\}}$ be the dissipative part of η. Theorem 8 implies that η_D has an M3-representation of the form

$$\eta_D(x) \stackrel{f.d.d.}{=} \bigvee_{i\geq 1} V_i Z_{D,i}(x - X_i), \quad x \in \mathcal{X}.$$

The fact that η is locally bounded implies that η_D is a.s. finite on K and

$$\mathbb{P}\left[\sup_{x\in K} \eta_D(x) \leq z\right] = \exp\left(-\frac{\theta_D(K)}{z}\right) \tag{10}$$

with

$$\theta_D(K) = \mathbb{E}\left[\int_{\mathcal{X}} \sup_{x\in K} Z_D(x-y)\lambda(\mathrm{d}y)\right] < \infty.$$

We deduce that $\int_{\mathcal{X}} \sup_{x\in K} Z_D(x-y)\lambda(\mathrm{d}y)$ is a.s. finite and hence, Z_D belongs a.s. to the cone $\mathcal{F}'_D$. This implies that $Y\mathbb{1}_{\{Y\in\mathcal{F}_D\}} \in \mathcal{F}'_D$ almost surely, whence $\{Y \in \mathcal{F}_D\} \subset \{Y \in \mathcal{F}'_D\}$ modulo null sets.

Proof of the inclusion $\{Y \in \tilde{\mathcal{F}}_D\} \subset \{Y \in \mathcal{F}_D\}$. With the same notation as in the dicrete case, we show that $\tilde{\eta}_D$ is generated by a dissipative flow and hence has an M3-representation. By Theorem 8, this implies that $\tilde{Y}_D$ belongs a.s. to $\mathcal{F}_D$ and proves the inclusion $\{Y \in \tilde{\mathcal{F}}_D\} \subset \{Y \in \mathcal{F}_D\}$. Note that the discrete skeleton $\tilde{Y}_D^{skel} = (\tilde{Y}_D(x))_{x\in\mathbb{Z}^d}$ satisfies $\lim_{x\to\infty} \tilde{Y}_D^{skel} = 0$. We deduce $\tilde{Y}_D^{skel} \in \tilde{\mathcal{F}}_D$ a.s. which is equivalent to $\tilde{Y}_D^{skel} \in \mathcal{F}_D$ a.s. (proof above in the discrete case). Hence $(\tilde{\eta}_D(x))_{x\in\mathbb{Z}^d}$ is generated by a dissipative flow and this implies that $(\tilde{\eta}_D(x))_{x\in\mathbb{R}^d}$ is generated by a dissipative flow (see [17, Section 2]). □

Proof of Theorem 3. The equivalence of (a), (b), (c) in Theorem 3 was known before and holds even without the assumption of local boundedness (see Section 3.1 and the reference therein). The equivalence of (c) and (d) holds under the assumption of local boundedness and is a straightforward consequence of Proposition 10. □

Example 11. The assumption that the sample paths of η should be locally bounded cannot be removed from Proposition 10. To see this, consider the following (deterministic) process Z:

$$Z(x) = \sum_{n=1}^{\infty} f(n^2(x-n)), \quad x \in \mathbb{R},$$

where $f(t) = (1-t^2)\mathbb{1}_{|t|\leq 1}$. The process Z is non-zero only on the intervals of the form $(n - \frac{1}{n^2}, n + \frac{1}{n^2})$, $n \in \mathbb{N}$. The M3-process η corresponding to Z is well-defined because $\int_{\mathbb{R}} Z(x)\mathrm{d}x < \infty$. On the other hand, $\mathbb{P}[Z \in \tilde{\mathcal{F}}_D] = 0$ and hence, $\mathbb{P}[Y \in$

$\tilde{\mathcal{F}}_D] = 0$, where Y is the spectral function of η from the de Haan representation (1). It is easy to check that

$$\mathbb{P}\left[\sup_{x\in[0,1]} \eta(x) \leq z\right] = \exp\left(-\frac{\theta_{[0,1]}}{z}\right), \quad z > 0,$$

with

$$\theta_{[0,1]} = \int_{\mathbb{R}} \left(\sup_{x\in[0,1]} Z(x-y)\right) \mathrm{d}y = +\infty,$$

whence $\sup_{x\in[0,1]} \eta(x) = +\infty$ a.s. and the sample paths of η are not locally bounded.

4. Positive/null decomposition

4.1. **Definition of the positive/null decomposition.** We start by defining the Neveu decomposition of the non-singular flow $(\phi_x)_{x\in\mathcal{X}}$; see, e.g., Krengel [10, Theorem 3.9], Samorodnitsky [20] or Wang *et al.* [24, Theorem 2.4].

Definition 12. *Consider a measure space $(S, \mathcal{B}, \mu)$ and a measurable non-singular flow $(\phi_x)_{x\in\mathcal{X}}$ on S. A measurable set $W \subset S$ is said to be weakly wandering with respect to $(\phi_x)_{x\in\mathcal{X}}$ if there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ such that $\phi_{x_n}^{-1}(W) \cap \phi_{x_m}^{-1}(W) = \varnothing$ for all $n \neq m$.*

The Neveu decomposition theorem states that there exists a partition of S into two disjoint measurable sets $S = P \cup N$, $P \cap N = \varnothing$, such that

(i) P and N are $(\phi_x)_{x\in\mathcal{X}}$–invariant for all $x \in \mathcal{X}$,
(ii) P has no weakly wandering set of positive measure,
(iii) N is a union of countably many weakly wandering sets.

This decomposition is unique mod μ and is called the *Neveu decomposition* of S associated with $(\phi_x)_{x\in\mathcal{X}}$; P and N are called the *positive* and *null* components with respect to $(\phi_x)_{x\in\mathcal{X}}$, respectively. It can be shown that P is the largest subset of S supporting a finite measure which is equivalent to μ and invariant under the flow $(\phi_x)_{x\in\mathcal{X}}$ ([24, Lemma 2.2]). Hence, there exists a finite measure which is equivalent to μ and invariant under the flow if and only if $N = \varnothing$ mod μ.

The corresponding positive/null decomposition of the stationary max-stable process η represented as in (3), (4) is given by $\eta = \eta_P \vee \eta_N$ with

$$\eta_P(x) = \int_P^{\mathrm{e}} f_x(s) M(\mathrm{d}s) \quad \text{and} \quad \eta_N(x) = \int_N^{\mathrm{e}} f_x(s) M(\mathrm{d}s), \quad x \in \mathcal{X}. \tag{11}$$

The positive and null components η_P and η_N are independent, stationary max-stable processes, and their distribution does not depend on the particular choice of the representation (3).

4.2. **Characterization using spectral functions.** An integral test on the spectral functions which allows to retrieve the positive/null decomposition is known in the one-dimensional case (see Samorodnitsky [20] or Wang and Stoev [26, Theorem 5.3]).

Theorem 13. *Consider the case $d = 1$ and introduce the class $\mathcal{W}$ of positive weight functions $w : \mathcal{X} \to (0, +\infty)$ such that $\int_{\mathcal{X}} w(x)\lambda(\mathrm{d}x) < \infty$ and $w(x)$ and $w(-x)$ are non-decreasing on $\mathcal{X} \cap [0, +\infty)$. Then we have*

(i) *For all $w \in \mathcal{W}$, $\int_{\mathcal{X}} f_x(s)w(x)\lambda(\mathrm{d}x) = \infty$ $\mu(\mathrm{d}s)$–a.e. on P;*

(ii) *For some* $w \in \mathcal{W}$, $\int_{\mathcal{X}} f_x(s)w(x)\lambda(\mathrm{d}x) < \infty$ $\mu(\mathrm{d}s)$*–a.e. on* N.

The next theorem is a new integral test characterizing the positive/null decomposition. This test is simpler than Theorem 13 and is valid for all $d \geq 1$. Recall that we write $B_r = [-r, r]^d \cap \mathcal{X}$ for $r > 0$.

Theorem 14. *Let* η *be a stationary, stochastically continuous max-stable process given by the non-singular flow representation* (3). *We have*

(i) $\lim_{r\to\infty} \frac{1}{\lambda(B_r)} \int_{B_r} f_x(s)\lambda(\mathrm{d}x)$ *exists and is positive* $\mu(\mathrm{d}s)$*–a.e. on* P*;*

(ii) $\liminf_{r\to\infty} \frac{1}{\lambda(B_r)} \int_{B_r} f_x(s)\lambda(\mathrm{d}x) = 0$ $\mu(\mathrm{d}s)$*–a.e. on* N.

Proof. We consider the positive case and the null case separately.

Case 1. Assume first that η is generated by a positive flow. Then, there is a probability measure μ^* on $(S, \mathcal{B})$ which is equivalent to μ and which is invariant under the flow. Note that any property holds μ–a.e. if and only if it holds μ^*–a.e. We denote by $D(s) = \frac{\mathrm{d}\mu}{\mathrm{d}\mu^*}(s) \in (0, \infty)$ the Radon–Nikodym derivative and observe that for every $x \in \mathcal{X}$, the function $f_x^*(s) := f_x(s)D(s)$ satisfies

$$f_x^*(s) = f_0^*(\phi_x(s)) \quad \text{for } \lambda \times \mu\text{–a.e. } (x, s) \in \mathcal{X} \times S. \tag{12}$$

Indeed, by definition of f_x^* and ω_x, we have

$$f_x^*(s) = D(s)f_x(s) = D(s)\omega_x(s)f_0(\phi_x(s)) = \frac{D(s)\omega_x(s)}{D(\phi_x(s))} f_0^*(\phi_x(s)).$$

However, recalling the definition (2) of $\omega_x(s)$ and that $D(s) = \frac{\mathrm{d}\mu}{\mathrm{d}\mu^*}(s) \in (0, \infty)$, we obtain

$$\frac{D(s)\omega_x(s)}{D(\phi_x(s))} = \frac{\mathrm{d}\mu}{\mathrm{d}\mu^*}(s)\frac{\mathrm{d}(\mu \circ \phi_x)}{\mathrm{d}\mu}(s)\frac{\mathrm{d}(\mu^* \circ \phi_x)}{\mathrm{d}(\mu \circ \phi_x)}(s) = \frac{\mathrm{d}(\mu^* \circ \phi_x)}{\mathrm{d}\mu^*}(s) = 1$$

μ–a.e. for every $x \in \mathcal{X}$ because the measure μ^* is invariant. This yields (12). By the multiparameter Birkhoff Theorem (see [24, Theorem 2.8]), we have

$$\lim_{r\to\infty} \frac{1}{\lambda(B_r)} \int_{B_r} f_x^*(s)\lambda(\mathrm{d}x) = \mathbb{E}[f_0^*|\mathcal{I}] \quad \mu^*\text{–a.e.}, \tag{13}$$

where $\mathcal{I}$ is the σ-algebra of $(\phi_x)_{x\in\mathcal{X}}$–invariant measurable sets and $\mathbb{E}$ denotes the expectation w.r.t. μ^*. We prove that the conditional expectation on the right-hand side is a.e. strictly positive. The set $B = \{\mathbb{E}[f_0^*|\mathcal{I}] = 0\}$ is measurable and $(\phi_x)_{x\in\mathcal{X}}$–invariant. Moreover, f_0^* (and hence, f_0) vanishes a.e. on B since f_0^* is non-negative. This implies that $\mu(B) = 0$ by the second condition in the definition of the flow representation (3). Thus, $\mathbb{E}[f_0^*|\mathcal{I}] > 0$ a.e. It follows from (13) and the above considerations that

$$\lim_{r\to\infty} \frac{1}{\lambda(B_r)} \int_{B_r} f_x(s)\lambda(\mathrm{d}x) = \frac{\mathbb{E}[f_0^*|\mathcal{I}]}{D(s)} > 0 \quad \mu\text{–a.e.}, \tag{14}$$

which proves part (i) of the theorem.

Case 2. We consider now the case when η is generated by a null flow. Let μ^* be any probability measure on $(S, \mathcal{B})$ which is equivalent to μ. Write $D(s) = \frac{\mathrm{d}\mu}{\mathrm{d}\mu^*}(s) \in (0, \infty)$ for the Radon–Nikodym derivative. The functions $f_x^*(s) := f_x(s)D(s)$ satisfy

$$f_x^*(s) = \omega_x^*(s)f_0^*(\phi_x(s)), \quad \text{where } \omega_x^*(s) := \frac{\mathrm{d}(\mu^* \circ \phi_x)}{\mathrm{d}\mu^*}(s),$$

by the same considerations as in the positive case. Birkhoff's ergodic theorem is valid for measure preserving flows only, but we can use Krengel's stochastic ergodic theorem for non-singular actions (see [24, Theorem 2.7]) which yields

$$\frac{1}{\lambda(B_r)}\int_{B_r} f_x^*(\cdot)\lambda(\mathrm{d}x) \stackrel{\mu^*}{\to} F(\cdot) \quad \text{as } r\to\infty$$

where $\stackrel{\mu^*}{\to}$ denotes convergence in μ^*-probability and the limit function $F\in L^1(S,\mu^*)$ is such that for all $x\in\mathcal{X}$,

$$\omega_x^*(s)F(\phi_x(s)) = F(s) \quad \text{a.e.}$$

This relation implies that the measure $F(s)\mu^*(\mathrm{d}s)$ is a finite measure which is absolutely continuous with respect to μ and invariant under the flow $(\phi_x)_{x\in\mathcal{X}}$. Since the flow has no positive component, this means that $F=0$ a.e. We deduce that $\frac{1}{\lambda(B_r)}\int_{B_r} f_x^*(\cdot)\lambda(\mathrm{d}x)$ converges in μ^*-probability to 0. Convergence in probability implies a.s. convergence along a subsequence, whence

$$\liminf_{r\to\infty}\frac{1}{\lambda(B_r)}\int_{B_r} f_x^*(s)\lambda(\mathrm{d}x) = 0 \quad \mu^*\text{–a.e.}$$

Since f_x differs from f_x^* by a positive factor and the measures μ and μ^* are equivalent, we have

$$\liminf_{r\to\infty}\frac{1}{\lambda(B_r)}\int_{B_r} f_x(s)\lambda(\mathrm{d}x) = 0 \quad \mu\text{–a.e.,}$$

which proves part (ii) of the theorem. □

As a consequence of Theorem 14, we can provide a new construction for the positive/null decomposition (11). Consider the following shift-invariant cones

$$\mathcal{F}_P = \left\{ f\in\mathcal{F}_0;\ \lim_{r\to\infty}\frac{1}{\lambda(B_r)}\int_{B_r} f(x)\lambda(\mathrm{d}x) > 0\right\}, \tag{15}$$

$$\mathcal{F}_N = \left\{ f\in\mathcal{F}_0;\ \liminf_{r\to\infty}\frac{1}{\lambda(B_r)}\int_{B_r} f(x)\lambda(\mathrm{d}x) = 0\right\}. \tag{16}$$

In the definition of $\mathcal{F}_P$ the limit is required to exist and to be positive.

Corollary 15. *Let η be a stationary, stochastically continuous max-stable process given by de Haan's representation* (1). *Then the decomposition $\eta=\eta_P\vee\eta_N$ with*

$$\eta_P(x) = \bigvee_{i\geq 1} U_iY_i(x)\mathbb{1}_{\{Y_i\in\mathcal{F}_P\}} \quad \textit{and} \quad \eta_N(x) = \bigvee_{i\geq 1} U_iY_i(x)\mathbb{1}_{\{Y_i\in\mathcal{F}_N\}}$$

is equal (in distribution) to the positive/null decomposition (11).

Proof. Corollary 15 is a direct consequence of Theorem 14 and Lemma 5. Note that although instead of $\mathcal{F}_P\cup\mathcal{F}_N=\mathcal{F}_0$ it holds only that $\mathbb{P}[Y\in\mathcal{F}_P\cup\mathcal{F}_N]=1$, Lemma 5 still applies. □

Proof of Theorem 1. We need to prove the equivalence of (c), (e), (f) only; see Section 1.2 for references to the other equivalences. We recall that (c) states that η has no positive recurrent component, and

(e) $\lim_{r\to\infty}\frac{1}{\lambda(B_r)}\int_{B_r} Y(x)\lambda(\mathrm{d}x) = 0$ in probability;
(f) $\liminf_{r\to\infty}\frac{1}{\lambda(B_r)}\int_{B_r} Y(x)\lambda(\mathrm{d}x) = 0$ a.s.

The equivalence of (c) and (f) follows from Corollary 15. Clearly, (e) implies (f) because any sequence converging to 0 in probability has a subsequence converging to 0 a.s.

It remains to show that (c) implies (e). Since the positive/null decomposition of η does not depend on the choice of the flow representation, we can consider a *minimal* representation $(f_x)_{x\in\mathcal{X}}$ of η by a null-recurrent flow $(\phi_x)_{x\in\mathcal{X}}$ on a probability space $(S^*,\mathcal{B}^*,\mu^*)$; see [26, Section 3] for definition and existence of the minimal representation. In the proof of Theorem 14, Case 2, we have shown that

$$M_r := \frac{1}{\lambda(B_r)}\int_{B_r} f_x\lambda(\mathrm{d}x) \underset{r\to\infty}{\longrightarrow} 0 \quad \text{in probability on } (S^*,\mathcal{B}^*,\mu^*).$$

However, we are interested in an arbitrary de Haan representation $(Y(x))_{x\in\mathcal{X}}$ of η on a probability space $(S,\mathcal{B},\mu)$. This representation need not be generated by a flow, but it can be mapped to the minimal one (see [26, Theorem 3.2]). More concretely, there is a measurable map $\Phi : S \to S^*$ and a measurable function $h : S \to (0,\infty)$ such that for every $x \in \mathcal{X}$,

$$Y(x;s) = h(s)f_x(\Phi(s)) \quad \text{for } \mu\text{-a.e. } s\in S,$$

and μ^* is the push-forward of the (probability) measure $\mu_h(\mathrm{d}s) := h(s)\mu(\mathrm{d}s)$ by the map Φ. We have

$$\frac{1}{\lambda(B_r)}\int_{B_r} Y(x;s)\lambda(\mathrm{d}x) = h(s)\cdot M_r(\Phi(s)) \quad \text{for } \mu\text{-a.e. } s\in S.$$

Since $M_r \to 0$ in μ^*-probability as $r\to\infty$, we obtain that for every $\varepsilon>0$,

$$\mu_h\{M_r\circ\Phi>\varepsilon\} = (\mu_h\circ\Phi^{-1})\{M_r>\varepsilon\} = \mu^*\{M_r>\varepsilon\} \underset{r\to\infty}{\longrightarrow} 0.$$

Since h is strictly positive, this implies that $\mu\{M_r\circ\Phi>\varepsilon\}\to 0$ and hence, $h\cdot(M_r\circ\Phi)\to 0$ in μ-probability, thus proving (e). □

5. Mixing

5.1. Proof of Theorem 2. We need to prove the equivalence of (c) and (d) only, that is

$$\text{(c): } \lim_{x\to\infty}\mathbb{E}[Y(x)\wedge Y(0)] = 0 \quad \Leftrightarrow \quad \text{(d): } \lim_{x\to\infty} Y(x) = 0 \text{ in probability.}$$

See Section 1.2 for references to the other equivalences.

Assume that (d) holds, i.e. $\lim_{x\to\infty} Y(x)=0$ in probability. The upper bound $Y(x)\wedge Y(0) \le Y(0)$ with $Y(0)$ integrable implies that the collection $(Y(x)\wedge Y(0))_{x\in\mathcal{X}}$ is uniformly integrable. Assumption (d) implies that $Y(x)\wedge Y(0)$ converges in probability to 0 as $x\to\infty$, whence we deduce that $\mathbb{E}[Y(x)\wedge Y(0)]\to 0$ as $x\to\infty$, i.e. (c) is satisfied.

Conversely, we prove the implication (c) ⇒ (d). The relation

$$\mathbb{E}[Y(x)\wedge Y(0)] = 2+\log\mathbb{P}[\eta(x)\le 1, \eta(0)\le 1]$$

together with the stationarity of η implies that for all $x_0\in\mathcal{X}$,

$$\lim_{x\to\infty}\mathbb{E}[Y(x)\wedge Y(x_0)] = 0. \tag{17}$$

Without restriction of generality we can assume that $\mathbb{P}[Y\equiv 0]=0$ (where, by separability, the event $\{Y\equiv 0\}$ is interpreted as $\cap_{x\in T}\{Y(x)=0\}$ with countable

$T \subset \mathcal{X}$). Then, for arbitrary $\varepsilon > 0$, there exists $\alpha > 0$ and $x_1, \dots, x_k \in \mathcal{X}$ such that $\mathbb{P}[\cup_{1\le i\le k}\{Y(x_i) > \alpha\}] \ge 1 - \varepsilon/2$, whence

$$\mathbb{P}[Y(x_1) + \dots + Y(x_k) > \alpha] \ge 1 - \varepsilon/2.$$

With the inequality $(a_1 + \dots + a_k) \wedge b \le a_1 \wedge b + \dots + a_k \wedge b$, we obtain from (17) that

$$\lim_{x\to\infty} \mathbb{E}[Y(x) \wedge (Y(x_1) + \dots + Y(x_k))] = 0.$$

These two equations imply, for all $\delta > 0$,

$$\begin{aligned}\mathbb{P}[Y(x) > \delta] &\le \mathbb{P}[Y(x) > \delta,\ Y(x_1) + \dots + Y(x_k) > \alpha] + \varepsilon/2\\ &\le \mathbb{P}[Y(x) \wedge (Y(x_1) + \dots + Y(x_k)) > \delta \wedge \alpha] + \varepsilon/2\\ &\le \mathbb{E}[Y(x) \wedge (Y(x_1) + \dots + Y(x_k))]/(\delta \wedge \alpha) + \varepsilon/2\\ &\le \varepsilon\end{aligned}$$

for large $|x|$. This proves that $Y(x) \to 0$ in probability as $x \to \infty$.

5.2. **Criterium for mixing in terms of flows.** Given a measurable non-singular flow $(\phi_x)_{x\in\mathcal{X}}$ on a σ-finite measure space $(S, \mathcal{B}, \mu)$ define the corresponding group of L^1–isometries $(U_x)_{x\in\mathcal{X}}$ by

$$(U_x g)(s) = \omega_x(s) g(\phi_x(s)), \quad g \in L^1(S,\mu), \quad x \in \mathcal{X},$$

where ω_x is the Radon–Nikodym derivative; see (2).

Theorem 16. *Let η be a stationary, stochastically continuous max-stable process with a flow representation* (3). *Then, the following conditions are equivalent:*

(a) *η is mixing.*
(b) $\lim_{x\to\infty} \int_S (f_x \wedge f_0) \mathrm{d}\mu = 0$.
(c) *$f_x \to 0$ locally in measure as $x \to \infty$. That is, for every measurable set $B \subset S$ with $\mu(B) < \infty$ and every $\varepsilon > 0$ we have*

$$\lim_{x\to\infty} \mu(B \cap \{f_x > \varepsilon\}) = 0.$$

(d) *For every non-negative function $g \in L^1(S,\mu)$ we have*

$$\lim_{x\to\infty} \int_S ((U_x g) \wedge g) \mathrm{d}\mu = 0.$$

(e) *For every non-negative function $g \in L^1(S,\mu)$, $U_x g \to 0$ locally in measure.*

Proof. The equivalence of (a) and (b) is due to Stoev; see Theorem 3.4 in [22]. We prove that (b) is equivalent to (c), (d), (e).

Take a non-negative function $g \in L^1(S,\mu)$. We prove that the following conditions are equivalent:

(b') $\lim_{x\to\infty} \int_S ((U_x g) \wedge g) \mathrm{d}\mu = 0$.
(c') $U_x g \to 0$ locally in measure, as $x \to \infty$.

Once the equivalence of (b') and (c') has been established, we immediately obtain the equivalence of (b) and (c) (by taking $g = f_0$) and the equivalence of (d) and (e).

Proof of (c') $\Rightarrow$ (b'). Let $U_x g \to 0$ locally in measure, as $x \to \infty$. We prove that (b') holds. Fix some $\varepsilon > 0$. The sets $B_n := \{g > \frac{1}{n}\}$, $n \in \mathbb{N}$, are measurable, have finite measure (since $g \in L^1(S,\mu)$), and

$$\lim_{n\to\infty} \int_S g \mathbb{1}_{S\backslash B_n} \mathrm{d}\mu = 0$$

by the dominated convergence theorem. Hence, by taking n sufficiently large we can achieve that the set $B = B_n$ satisfies $\mu(B) < \infty$ and

$$\int_{S\backslash B} g \mathrm{d}\mu \le \varepsilon.$$

The collection $(U_x g \wedge g)_{x\in\mathcal{X}}$ is uniformly integrable on B since $U_x g \wedge g \le g$. Also, we know that $U_x g \wedge g \to 0$ (as $x \to \infty$) in measure on B. It follows that

$$\lim_{x\to\infty} \int_B U_x g \wedge g \mathrm{d}x = 0.$$

Thus, condition (b') holds.

Proof of (b') ⇒ (c'). We argue by contradiction. Assume that $U_x g \nrightarrow 0$ locally in measure as $x \to \infty$. Our aim is to prove that (b') is violated. By our assumption, there is a measurable set $B \subset S$ and $\varepsilon > 0$ such that $0 < \mu(B) < \infty$ and

$$\mu(\{U_{x_i} g > \varepsilon\} \cap B) > \varepsilon, \quad i \in \mathbb{N}, \tag{18}$$

where $x_1, x_2, \ldots \to \infty$ is some sequence in $\mathcal{X}$. Denote by $\mathcal{H}$ the family consisting of the sets $\operatorname{supp} U_x g$, $x \in \mathcal{X}$, together with all measurable subsets of these sets. Let S^* be the measurable union of this family; see [1, pp. 7–8] for the proof of its existence. By the exhaustion lemma [1, pp. 7–8], we can find countably many sets $A_1, A_2, \ldots \in \mathcal{H}$ such that $S^* = A_1 \cup A_2 \cup \ldots$. It follows that we can find finitely many $z_1, \ldots, z_m \in \mathcal{X}$ such that

$$\mu\left((B \cap S^*)\backslash \bigcup_{j=1}^m \operatorname{supp} U_{z_j} g\right) < \frac{\varepsilon}{2}.$$

Together with (18) (where B can be replaced by $B \cap S^*$ because $\{U_{x_i} g > \varepsilon\} \subset S^*$ mod μ), this implies that for all $i \in \mathbb{N}$,

$$\mu\left(\{U_{x_i} g > \varepsilon\} \cap \bigcup_{j=1}^m \operatorname{supp} U_{z_j} g\right) > \frac{\varepsilon}{2}.$$

It follows that there is $j \in \{1, \ldots, m\}$ and a subsequence $y_1, y_2, \ldots \to \infty$ of $x_1, x_2, \ldots$ such that for all $i \in \mathbb{N}$,

$$\mu\left(\{U_{y_i} g > \varepsilon\} \cap \operatorname{supp} U_{z_j} g\right) > \frac{\varepsilon}{2m}.$$

Put $z = z_j$. For a sufficiently small $\delta \in (0, \varepsilon)$ we have

$$\mu\left(\{U_{y_i} g > \delta\} \cap \{U_z g > \delta\}\right) > \frac{\varepsilon}{4m}. \tag{19}$$

By the flow property and (19) it follows that for all $i \in \mathbb{N}$,

$$\int_S ((U_{y_i - z} g) \wedge g) \mathrm{d}\mu = \int_S ((U_{y_i} g) \wedge (U_z g)) \mathrm{d}\mu > \frac{\varepsilon}{4m}\delta > 0.$$

But this contradicts (b').

Proof of (d) $\Rightarrow$ (b). Trivial, because $f_x = U_x f_0$.

Proof of (b) $\Rightarrow$ (d). For every non-negative function $g \in L^1(S,\mu)$ we have to show that

$$\lim_{x\to\infty}\int_S (U_x g \wedge g)\mathrm{d}\mu = 0.$$

Fix some $\varepsilon > 0$. By the same argument relying on the dominated convergence theorem as above, we can find a sufficiently large $K > 0$ such that the set $B := \{1/K \le g \le K\}$ satisfies

$$\int_{S\backslash B} g\mathrm{d}\mu < \varepsilon. \tag{20}$$

The set B has finite measure because g is integrable. By the uniform integrability of a single function g, there is $\delta > 0$ such that every for every measurable set $A \subset B$ with $\mu(A) < \delta$ we have $\int_A g\mathrm{d}\mu < \varepsilon$.

We argue that it is possible to find finitely many $z_1,\dots,z_m \in \mathcal{X}$ such that the sets $\operatorname{supp} f_{z_1},\dots,\operatorname{supp} f_{z_m}$ cover B up to a set of measure at most $\delta/2$. Indeed, let $\mathcal{H}$ be the family consisting of the sets $\operatorname{supp} f_x$, $x \in \mathcal{X}$, together with all measurable subsets of these sets. In the definition of the flow representation (3) we made a "full support" assumption which assures that the measurable union of $\mathcal{H}$ is the whole of S. By the exhaustion lemma [1, pp. 7–8], we can represent S as a disjoint union of countably many sets $A_1, A_2, \dots \in \mathcal{H}$. It follows that we can find finitely many $z_1,\dots,z_m \in \mathcal{X}$ such that

$$\mu\left(B\backslash \bigcup_{j=1}^m \operatorname{supp} f_{z_j}\right) < \frac{\delta}{2}.$$

By taking $c > 0$ sufficiently small, we can even achieve that the sets $\{f_{z_1} > c\},\dots,\{f_{z_m} > c\}$ cover B up to a set of measure at most δ, that is for

$$D := B\backslash \bigcup_{j=1}^m \{f_{z_j} > c\}$$

we have $\mu(D) < \delta$. By construction of δ it follows that

$$\int_D g\mathrm{d}\mu < \varepsilon. \tag{21}$$

For every $j \in \{1,\dots,m\}$, on the set $A_j := B \cap \{f_{z_j} > c\}$ we have the estimates $g \le K$ and $f_{z_j} > c$. Hence, $g\mathbb{1}_{A_j} \le \frac{K}{c} f_{z_j}$ and, by non-negativity of U_x,

$$\int_B U_x(g\mathbb{1}_{A_j}) \wedge g\mathrm{d}\mu \le \int_B \left(\frac{K}{c} f_{x+z_j}\right) \wedge K\mathrm{d}\mu \underset{x\to\infty}{\longrightarrow} 0 \tag{22}$$

because $\frac{K}{c} f_{x+z_j} \to 0$ locally in measure by assumption (b) which, as we already know, is equivalent to (c). Writing $g = g\mathbb{1}_B + g\mathbb{1}_{S\backslash B}$, we obtain

$$\int_S (U_x g) \wedge g\mathrm{d}\mu \le \int_S U_x(g\mathbb{1}_{S\backslash B})\mathrm{d}\mu + \int_S U_x(g\mathbb{1}_B) \wedge g\mathrm{d}\mu.$$

We have $\int_S U_x(g\mathbb{1}_{S\setminus B})\mathrm{d}\mu \le \varepsilon$ using (20) and because U_x is L^1-isometry. The second integral can be estimated as follows:

$$\int_S U_x(g\mathbb{1}_B)\wedge g\mathrm{d}\mu \le \int_{S\setminus B} g\mathrm{d}\mu + \int_B U_x(g\mathbb{1}_B)\wedge g\mathrm{d}\mu \le \varepsilon + \int_B U_x\left(g\mathbb{1}_D + \sum_{j=1}^m g\mathbb{1}_{A_j}\right)\wedge g\mathrm{d}\mu.$$

Using the inequality $(a_1+\ldots+a_k)\wedge b \le a_1\wedge b+\ldots+a_k\wedge b$, we obtain

$$\int_S U_x(g\mathbb{1}_B)\wedge g\mathrm{d}\mu \le \varepsilon + \int_B U_x(g\mathbb{1}_D)\mathrm{d}\mu + \sum_{j=1}^m \int_B U_x(g\mathbb{1}_{A_j})\wedge g\mathrm{d}\mu.$$

Since U_x is L^1-isometry, we have $\int_B U_x(g\mathbb{1}_D)\mathrm{d}\mu \le \varepsilon$ by (21). Recalling (22) we obtain that

$$\limsup_{x\to\infty}\int_S ((U_x g)\wedge g)\mathrm{d}\mu \le 3\varepsilon.$$

Since this is true for every $\varepsilon > 0$, the limit is in fact 0 and we obtain (d). □

Remark 17. Condition (d) in Theorem 16 can be replaced by the following seemingly stronger one: For every non-negative functions $g, h \in L^1(S,\mu)$ we have

$$\lim_{x\to\infty}\int_S ((U_x g)\wedge h)\mathrm{d}\mu = 0.$$

It is clear that this condition implies (d). To see the converse, note that by the non-negativity property of U_x,

$$\int_S (U_x g\wedge h)\mathrm{d}\mu \le \int_S (U_x(g\vee h)\wedge(g\vee h))\mathrm{d}\mu.$$

5.3. **Mixing/non-mixing decomposition.** It is known that the Hopf decomposition can be used to characterize the mixed moving maximum property, whereas Neveu decomposition characterizes ergodicity. In the next proposition we construct a decomposition which characterizes mixing. For measure-preserving maps, this decomposition was introduced by Krengel and Sucheston [12, 11]. E. Roy [16] used it to characterize mixing of sum-infinitely divisible processes. Note that we consider non-singular flows (which is a broader class than measure preserving flows).

Theorem 18. *Consider a non-singular, measurable flow $(\phi_x)_{x\in\mathcal{X}}$ acting on a σ-finite measure space $(S,\mathcal{B},\mu)$. There is a decomposition of S into two disjoint measurable sets $S = N_0\cup N_+$, $N_0\cap N_+ = \varnothing$, such that*

(i) *N_0 and N_+ are $(\phi_x)_{x\in\mathcal{X}}$-invariant, modulo null sets.*
(ii) *For every non-negative function $g\in L^1(S,\mu)$ supported on N_0,*

$$\lim_{x\to\infty}\int_S (U_x g\wedge g)\mathrm{d}\mu = 0.$$

(iii) *For every nonnegative function $h \in L^1(S,\mu)$ supported on N_+ and not vanishing identically,*

$$\limsup_{x\to\infty}\int_S (U_x h\wedge h)\mathrm{d}\mu > 0.$$

Properties (ii) and (iii) define the components N_+ and N_0 uniquely, modulo null sets.

Proof. Let $\mathcal{H}$ be the family of all measurable sets $A \subset S$ such that $\mu(A) < \infty$ and $U_x \mathbb{1}_A \to 0$ locally in measure, as $x \to \infty$. By the positivity of U_x, the family $\mathcal{H}$ is hereditary, that is it contains with every set A all its measurable subsets. Denote by N_0 the measurable union of $\mathcal{H}$; see [1, pp. 7–8] for its existence.

Proof of (ii). Take any non-negative function $g \in L^1(S, \mu)$ supported on N_0. Fix $\varepsilon > 0$. Let K be sufficiently large so that the set $B := \{g \le K\}$ satisfies

$$\int_{S \backslash B} g \mathrm{d}\mu < \varepsilon. \tag{23}$$

Let $\delta > 0$ be such that for every measurable set $D \subset B$ with $\mu(D) < \delta$ we have $\int_D g \mathrm{d}\mu < \varepsilon$. By the exhaustion lemma [1, pp. 7–8] we can find finitely many sets $A_1, \ldots, A_m \in \mathcal{H}$ such that $\mu(B \backslash \cup_{j=1}^m A_j) < \delta$ and hence,

$$\int_{B \backslash A} g \mathrm{d}\mu < \varepsilon, \tag{24}$$

where we introduced the set $A := A_1 \cup \ldots \cup A_m$. For every $j \in \{1, \ldots, m\}$ we have, by the positivity of U_x,

$$\int_B (U_x(g \mathbb{1}_{A_j \cap B})) \wedge g \mathrm{d}\mu \le \int_B (K U_x(\mathbb{1}_{A_j \cap B})) \wedge K \mathrm{d}\mu \underset{x \to \infty}{\longrightarrow} 0 \tag{25}$$

because $U_x \mathbb{1}_{A_j \cap B} \to 0$ locally in measure. We have the estimate

$$\int_S U_x g \wedge g \mathrm{d}\mu \le \int_{S \backslash B} g \mathrm{d}\mu + \int_B (U_x g \wedge g) \mathrm{d}\mu \le \varepsilon + \int_B U_x \left(g \mathbb{1}_{S \backslash (A \cap B)} + \sum_{j=1}^m g \mathbb{1}_{A_j \cap B} \right) \wedge g \mathrm{d}\mu.$$

Using the inequality $(a_1 + \ldots + a_k) \wedge b \le a_1 \wedge b + \ldots + a_k \wedge b$, we obtain

$$\int_S U_x g \wedge g \mathrm{d}\mu \le \varepsilon + \int_B U_x(g \mathbb{1}_{S \backslash (A \cap B)}) \mathrm{d}\mu + \sum_{j=1}^m \int_B U_x(g \mathbb{1}_{A_j \cap B}) \wedge g \mathrm{d}\mu.$$

Since U_x is an L^1-isometry, we have $\int_B U_x(g \mathbb{1}_{S \backslash (A \cap B)}) \mathrm{d}\mu \le 2\varepsilon$ by (23) and (24). By (22) we obtain that

$$\limsup_{x \to \infty} \int_S U_x g \wedge g \mathrm{d}\mu \le 3\varepsilon,$$

which proves (ii) since $\varepsilon > 0$ is arbitrary.

Proof of (iii). We argue by contraposition. Assume that a non-negative function $h \in L^1(S, \mu)$ supported on $N_+ := S \backslash N_0$ and not vanishing identically satisfies $\lim_{x \to \infty} \int_S (U_x h \wedge h) \mathrm{d}\mu = 0$. For a sufficiently small $b > 0$, the set $A := \{h > b\}$ has positive, finite measure, and (by the positivity of U_x) satisfies

$$\lim_{x \to \infty} \int_S U_x \mathbb{1}_A \wedge \mathbb{1}_A \mathrm{d}\mu = 0.$$

Since U_x preserves pointwise minima and is an L^1-isometry, we obtain that for every $x_0 \in \mathcal{X}$,

$$\lim_{x \to \infty} \int_S (U_x \mathbb{1}_A) \wedge (U_{x_0} \mathbb{1}_A) \mathrm{d}\mu = 0. \tag{26}$$

Since $A \subset N_+$ and $\mu(A) > 0$, the definition of N_0 implies that the sequence $U_x \mathbb{1}_A$ does not converge locally in μ-measure, as $x \to \infty$. Hence, we can find a measurable set $B \subset S$ with $\mu(B) < \infty$ and $a > 0$ such that

$$\limsup_{x\to\infty} \mu(B \cap \{U_x \mathbb{1}_A > a\}) > a. \tag{27}$$

Let B_0 be the measurable union of $\operatorname{supp} U_x \mathbb{1}_A$, $x \in \mathcal{X}$. Since replacing B by $B \cap B_0$ does not change the validity of (27), we can assume that $B \subset B_0$. By the exhaustion lemma, see [1, pp. 7–8], we can find finitely many $x_1, \ldots, x_m \in \mathcal{X}$ and $c > 0$ such that the set B is covered, up to a subset of measure at most $a/2$, by the sets $\{U_{x_1}\mathbb{1}_A > c\}, \ldots, \{U_{x_m}\mathbb{1}_A > c\}$. It follows that for every $x \in \mathcal{X}$ satisfying $\mu(B \cap \{U_x \mathbb{1}_A > a\}) \geq a$ we also have

$$\mu(\{U_x \mathbb{1}_A > a\} \cap \{U_{x_i}\mathbb{1}_A > c\}) > a/(4m)$$

for at least one $i \in \{1, \ldots, m\}$. But this contradicts (26), thus proving (iii).

Proof of the uniqueness. Let $S = \tilde{N}_0 \cup \tilde{N}_+$ be another disjoint decomposition enjoying properties (ii) and (iii). If $\mu(N_0 \cap \tilde{N}_+) > 0$, then we can find a set $A \subset N_0 \cap \tilde{N}_+$ with $\mu(A) \neq 0, \infty$ (recall that μ is σ-finite). The indicator function of this set must satisfy both $\lim_{x\to\infty} \int_S (U_x \mathbb{1}_A \wedge \mathbb{1}_A) \mathrm{d}\mu = 0$ (because $A \subset N_0$) and $\limsup_{x\to\infty} \int_S (U_x \mathbb{1}_A \wedge \mathbb{1}_A) \mathrm{d}\mu > 0$ (because $A \subset \tilde{N}_+$), which is a contradiction. Similarly, the assumption $\mu(\tilde{N}_0 \cap N_+) > 0$ leads to a contradiction. Hence, the decompositions $S = N_0 \cup N_+$ and $S = \tilde{N}_0 \cup \tilde{N}_+$ coincide modulo μ.

Proof of (i). We show that the decomposition $S = N_0 \cup N_+$ is $(\phi_x)_{x\in\mathcal{X}}$-invariant, modulo null sets. It is easy to check that for every $y \in \mathcal{X}$ the decomposition $S = \phi_y(N_0) \cup \phi_y(N_+)$ enjoys properties (ii) and (iii). Indeed, if g is a function supported on $\phi_y(N_0)$, then $U_y g$ is supported on N_0 and hence,

$$\lim_{x\to\infty} \int_S (U_x g \wedge g) \mathrm{d}\mu = \lim_{x\to\infty} \int_S U_y(U_x g \wedge g) \mathrm{d}\mu = \lim_{x\to\infty} \int_S (U_x U_y g \wedge U_y g) \mathrm{d}\mu = 0$$

by (ii). Similarly, one verifies that $\phi_y(N_+)$ satisfies (iii). The uniqueness of the decomposition implies that $N_0 = \phi_y(N_0)$ and $N_+ = \phi_y(N_+)$ modulo null sets. $\square$

Remark 19. Krengel and Sucheston [12] called a measure-preserving flow $(\phi_x)_{x\in\mathbb{Z}}$ mixing if

$$\lim_{x\to\infty} \mu(\phi_x A \cap A) = 0$$

for every set $A \in \mathcal{B}$ with $\mu(A) < \infty$. Thus, in the measure-preserving case, the decomposition from Theorem 18 coincides with the decomposition of Krengel and Sucheston [12, 11].

The decomposition introduced in Theorem 18 characterizes mixing of max-stable processes.

Theorem 20. *Let η be a stationary, stochastically continuous max-stable processes with a flow representation* (3)*. Then η is mixing if and only if $N_+ = \varnothing \mod \mu$.*

Proof. Follows immediately from Theorem 16. $\square$

We can introduce a decomposition of a stationary max-stable process η into mixing and non-mixing components as follows: $\eta = \eta_0 \vee \eta_+$ with

$$\eta_0(x) = \int_{N_0}^{e} f_x(s) M(\mathrm{d}s) \quad \text{and} \quad \eta_+ = \int_{N_+}^{e} f_x(s) M(\mathrm{d}s), \quad x \in \mathcal{X}.$$

Clearly, η_0 and η_+ are independent stationary max-stable processes. Using argumentation as in the proof of Theorem 2.4 in [20] (mapping to the minimal representation), it can be shown that the laws of η_0 and η_+ do not depend on the choice of the flow representation.

5.4. **An open question.** We have provided characterizations of the null recurrent and the dissipative components of a max-stable process in terms of its spectral functions, see condition (f) in Theorem 1 and conditions (c)-(d) in Theorem 3. This allows us to obtain the positive/null and conservative/dissipative decompositions of a max-stable process given by de Haan representation (1) directly via cone decompositions (see Proposition 10 and Corollary 15). We have also provided a new decomposition into mixing/non mixing components. It is therefore natural to ask whether a pathwise characterization of this decomposition is available. In view of the equivalence (e)-(f) in Theorem 1, we can wonder whether mixing can be characterized by the condition

$$\liminf_{x\to\infty} Y(x) = 0 \quad \text{a.s.} \tag{28}$$

The answer is negative. Although mixing implies (28) (because mixing is equivalent to $Y(x) \to 0$ in probability which implies a.s. convergence to 0 along a subsequence), the converse is not true. We will show that a counterexample is provided by a process constructed in [8].

Consider a max-stable process $\eta(t) = \vee_{i=1}^{\infty} U_i Y_i(t)$ as in (1), where the spectral functions $(Y_i)_{i\in\mathbb{N}}$ are i.i.d. copies of the log-normal process

$$Y(t) = \exp\left\{ Z(t) - \frac{1}{2}\sigma^2(t) \right\}, \quad t \in \mathbb{R}, \tag{29}$$

with $(Z(t))_{t\in\mathbb{R}}$ a zero-mean Gaussian process with stationary increments, $Z(0) = 0$, and incremental variance

$$\sigma^2(t) := \mathrm{Var}(Z(s+t) - Z(s)) = \sum_{k=1}^{\infty} \left(1 - \cos\left(\frac{2\pi t}{2^k} \right) \right).$$

An explicit series representation of $(Z(t))_{t\in\mathbb{R}}$ is given by

$$Z(t) = \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \left(N_k' \left(1 - \cos \frac{2\pi t}{2^k} \right) + N_k'' \sin \frac{2\pi t}{2^k} \right),$$

where N_k', N_k'', $k \in \mathbb{N}$, are independent standard normal random variables. The max-stable process η belongs to the family of the so-called Brown–Resnick processes and is stationary; see [9].

Proposition 21. *The max-stable process η is ergodic but non-mixing although it satisfies* (28).

Proof. The fact that η is ergodic but non-mixing was proven in [8]. We show here that Equation (28) is satisfied. It was shown in [8] that there is a sequence $x_1 < x_2 < \ldots \to +\infty$ such that $\lim_{n\to\infty} \sigma^2(x_n) = +\infty$. Passing, if necessary, to a subsequence, we can assume that $\sigma^2(x_n) > n^2$. For every $\varepsilon \in (0,1)$ we have

$$\mathbb{P}[Y(x_n) > \varepsilon] = \mathbb{P}\left[Z(x_n) > \log\varepsilon + \frac{1}{2}\sigma^2(x_n) \right] = \mathbb{P}\left[N > \frac{\log \varepsilon}{\sigma(x_n)} + \frac{1}{2}\sigma(x_n) \right],$$

where N is a standard normal random variable. It follows that

$$\sum_{n=1}^{\infty} \mathbb{P}[Y(x_n) > \varepsilon] \leq \sum_{n=1}^{\infty} \mathbb{P}\left[N > \frac{n}{2} + \log \varepsilon\right] < \infty.$$

By the Borel–Cantelli lemma, the probability that only finitely many events $\{Y(x_n) > \varepsilon\}$ occur equals 1. Since this holds for every $\varepsilon \in (0,1)$, we obtain that $\lim_{n\to\infty} Y(x_n) = 0$ a.s. and this implies (28). $\square$

References

[1] J. Aaronson. *An introduction to infinite ergodic theory*, volume 50 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[2] A. I. Danilenko and C. E. Silva. Ergodic theory: non-singular transformations. In *Mathematics of complexity and dynamical systems. Vols. 1–3*, pages 329–356. Springer, New York, 2012.

[3] L. de Haan. A spectral representation for max-stable processes. *Ann. Probab.*, 12(4):1194–1204, 1984.

[4] L. de Haan and A. Ferreira. *Extreme value theory. An introduction*. Springer Series in Operations Research and Financial Engineering. Springer, New York, 2006.

[5] L. de Haan and J. Pickands, III. Stationary min-stable stochastic processes. *Probab. Theory Relat. Fields*, 72(4):477–492, 1986.

[6] C. Dombry and Z. Kabluchko. Random tessellations associated with max-stable random fields. Preprint arXiv:1410.2584, 2015.

[7] Z. Kabluchko. Spectral representations of sum- and max-stable processes. *Extremes*, 12(4):401–424, 2009.

[8] Z. Kabluchko and M. Schlather. Ergodic properties of max-infinitely divisible processes. *Stochastic Process. Appl.*, 120(3):281–295, 2010.

[9] Z. Kabluchko, M. Schlather, and L. de Haan. Stationary max-stable fields associated to negative definite functions. *Ann. Probab.*, 37(5):2042–2065, 2009.

[10] U. Krengel. *Ergodic theorems*, volume 6 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1985.

[11] U. Krengel and L. Sucheston. On mixing in infinite measure spaces. *Bull. Amer. Math. Soc.*, 74:1150–1155, 1968.

[12] U. Krengel and L. Sucheston. On mixing in infinite measure spaces. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 13:150–164, 1969.

[13] J. Rosiński. On the structure of stationary stable processes. *Ann. Probab.*, 23(3):1163–1187, 1995.

[14] J. Rosiński. Decomposition of stationary α-stable random fields. *Ann. Probab.*, 28(4):1797–1813, 2000.

[15] J. Rosiński and G. Samorodnitsky. Classes of mixing stable processes. *Bernoulli*, 2(4):365–377, 1996.

[16] E. Roy. Ergodic properties of Poissonian ID processes. *Ann. Probab.*, 35(2):551–576, 2007.

[17] P. Roy. Nonsingular group actions and stationary $S\alpha S$ random fields. *Proc. Amer. Math. Soc.*, 138(6):2195–2202, 2010.

[18] P. Roy and G. Samorodnitsky. Stationary symmetric α-stable discrete parameter random fields. *J. Theoret. Probab.*, 21(1):212–233, 2008.

[19] G. Samorodnitsky. Maxima of continuous-time stationary stable processes. *Adv. in Appl. Probab.*, 36(3):805–823, 2004.

[20] G. Samorodnitsky. Null flows, positive flows and the structure of stationary symmetric stable processes. *Ann. Probab.*, 33(5):1782–1803, 2005.

[21] M. Schlather. Models for stationary max-stable random fields. *Extremes*, 5(1):33–44, 2002.

[22] S. A. Stoev. On the ergodicity and mixing of max-stable processes. *Stochastic Process. Appl.*, 118(9):1679–1705, 2008.

[23] S. A. Stoev and M. S. Taqqu. Extremal stochastic integrals: a parallel between max-stable processes and α-stable processes. *Extremes*, 8(4):237–266 (2006), 2005.

[24] Y. Wang, P. Roy, and S.A. Stoev. Ergodic properties of sum- and max-stable stationary random fields via null and positive group actions. *Ann. Probab.*, 41(1):206–228, 2013.

[25] Y. Wang and S.A. Stoev. On the association of sum- and max-stable processes. *Statist. Probab. Lett.*, 80(5-6):480–488, 2010.

[26] Y. Wang and S.A. Stoev. On the structure and representations of max-stable processes. *Adv. in Appl. Probab.*, 42(3):855–877, 2010.

Université de Franche-Comté, Laboratoire de Mathématiques de Besançon, UMR CNRS 6623, 16 route de Gray, 25030 Besançon cedex, France

E-mail address: `Email: clement.dombry@univ-fcomte.fr`

Universität Münster, Institut für Mathematische Statistik, Orléans-Ring 10, 48149 Münster, Germany

E-mail address: `zakhar.kabluchko@uni-muenster.de`