Three-dimensionalizing the eight-vertex model

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Abstract

A simple ansatz is proposed for two-color R-matrix satisfying the tetrahedron equation. It generalizes, on one hand, a particular case of the eight-vertex model to three dimensions, and on another hand — Hietarinta's permutation-type operators to their linear combinations. Each separate R-matrix depends on one parameter, and the tetrahedron equation holds provided the quadruple of parameters belongs to an algebraic set containing five irreducible two-dimensional components.

1 Introductory remarks

1.1 Two-dimensional case: permutation-type R-matrices in the Baxter $\cap$ Felderhof model

Below we tend to use the notations from Hietarinta's paper [3]. Recall that Hietarinta proposed in that paper solutions to Yang–Baxter, tetrahedron, and higher simplex equations, acting as permutations on (tensor products of) basis vectors. For instance, the following simple permutation yields a solution to the constant Yang–Baxter equation (CYBE):

$$\mathcal{S}(e_i \otimes e_j) = e_j \otimes e_i$$

(compare [3, formula (9)]).

Convention 1. In this paper, we consider only *two-color* solutions, the colors being denoted as 0 and 1. So, our indices $i, j, \dots$ take only these two values.

Convention 2. The addition of indices mentioned in Convention 1 is understood modulo 2.

There is one more interesting operator, although it does not make by itself a solution to CYBE:

$$\mathcal{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \mathcal{S}, \qquad \mathcal{T}(e_i \otimes e_j) = e_{j+1} \otimes e_{i+1}.$$

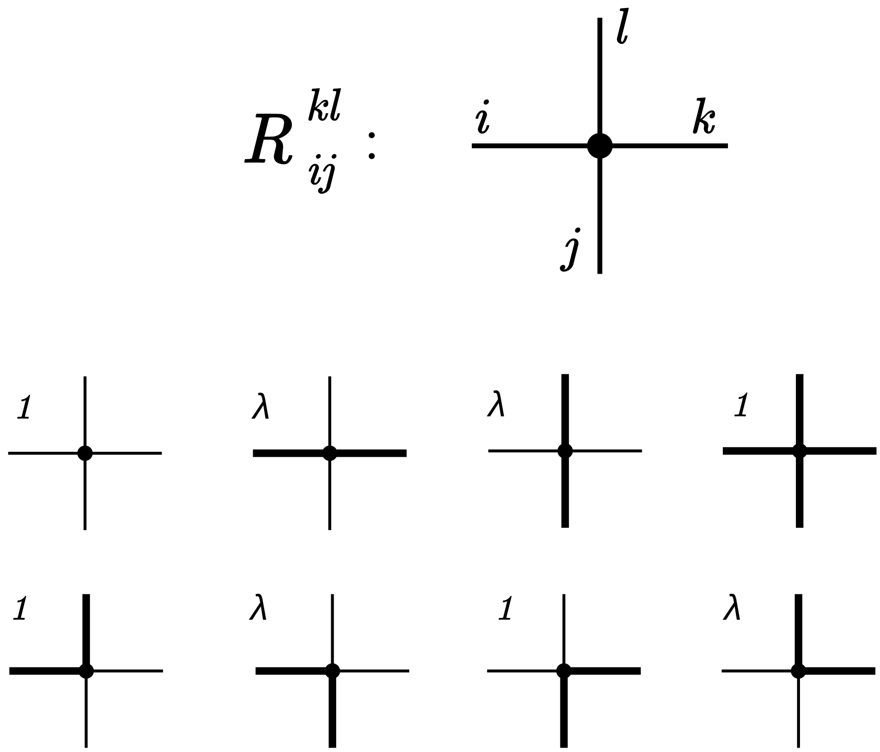

Figure 1: Matrix elements of operator (1). Thin lines correspond to color 0, thick lines — to color 1

Consider, however, the following *linear combination*:

$$\mathcal{R}(\lambda) = \mathcal{S} + \lambda \mathcal{T}, \tag{1}$$

and the following *non-constant* Yang–Baxter relation:

$$\mathcal{R}_{12}(\lambda)\mathcal{S}_{13}(\mu)\mathcal{T}_{23}(\nu) = \mathcal{T}_{23}(\nu)\mathcal{S}_{13}(\mu)\mathcal{R}_{12}(\lambda). \tag{2}$$

One can readily see that (2) holds provided

$$\lambda - \mu + \nu - \lambda\mu\nu = 0. \tag{3}$$

Our operator (1) is a particular case of both Baxter [1] and Felderhof [2] eight-vertex R-operators. Figure 1 depicts its matrix elements.

Remark 1. A reader with experience in various kinds of eight-vertex models will notice that our R-operator (1) can be represented, by taking different bases in our two-dimensional "color" spaces, by just a "four-vertex" R-matrix. Nevertheless, some simple further analysis shows that the corresponding lattice statistical model displays not completely trivial behavior, with a phase transition at $\lambda = 0$.

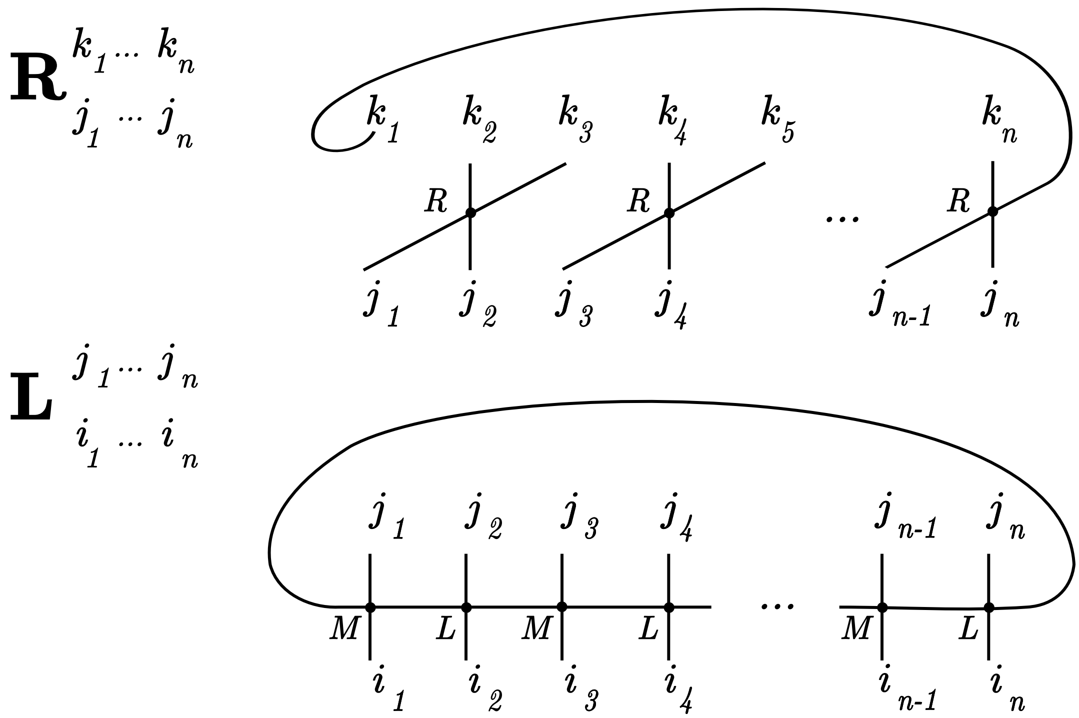

Figure 2: These transfer matrices commute if (4) holds.

Remark 2. Also, imagine that we knew only the model with Boltzmann weights as in Figure 1, satisfying equation (2) — then this could be already a good stimulus for (re)discovering Baxter's and Felderhof's models, by substituting different weights instead of ones and lambdas.

Remark 3. And finally, R-operator (1) will soon be generalized to much less trivial case of three dimensions, compare (3) with the formula in Case 3 in Section 2.

1.2 Commuting transfer matrices from not necessarily invertible R-operators

Sometimes we will encounter *non-invertible* R-operators satisfying Yang–Baxter or tetrahedron equation. A simple example is operator (1) with $\lambda = \pm 1$. We would like to remark here that such operators can still be relevant for constructing commuting transfer matrices.

To explain the idea, it is enough to take the Yang–Baxter case. So, let the relation

$$\mathcal{R}_{12}\mathcal{L}_{13}\mathcal{M}_{23} = \mathcal{M}_{23}\mathcal{L}_{13}\mathcal{R}_{12}, \tag{4}$$

with no requirements of invertibility for $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{M}$. Then, the two commuting transfer matrices $\mathbf{R}^{k_1 \dots k_n}_{j_1 \dots j_n}$ and $\mathbf{L}^{j_1 \dots j_n}_{i_1 \dots i_n}$ are most easily presented pictorially, see Figure 2.

Similar construction of commuting transfer matrices in the case of one more dimension, and tetrahedron equation instead of Yang–Baxter, can be found in paper [4]: the analogue of the upper transfer matrix in Figure 2 consists of "hedgehogs", while the other transfer matrix forms a *kagome* lattice, with three different types of vertices. We refer the reader to [4] for details.

1.3 Taking partial trace of Hietarinta's solution to the four-simplex equation

Recall that Hietarinta's permutation-type R-operators were defined, for an n-simplex equation, using an $(n \times n)$-matrix A and n-column B, usually written together as $[A|B]$, and acting *linearly on indices* in the following way [3, formula (10)]:

$$R_{i_1 \dots i_n}^{j_1 \dots j_n} = \delta_{A_1^\alpha i_\alpha + B_1}^{j_1} \dots \delta_{A_n^\alpha i_\alpha + B_n}^{j_n}.$$

In particular, Hietarinta discovered the following solution to the *four*-simplex equation [3, Subsection 6.21]:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]. \tag{5}$$

It is well-known that if there is a solution to n-simplex equation then a solution to $(n-1)$-simplex equation can be obtained by taking a *partial trace* in one pair of indices. For instance, if $\tilde{\mathcal{R}}_{i_1 i_2 i_3 i_4}^{j_1 j_2 j_3 j_4}$ enjoys the four-simplex equation, then

$$\mathcal{R}_{i_1 i_2 i_3}^{j_1 j_2 j_3} = \tilde{\mathcal{R}}_{i_1 i_2 i_3 k}^{j_1 j_2 j_3 k}$$

is a solution to tetrahedron. Partial trace of (5) in the fourth pair of indices gives the sum

$$\mathcal{R} = \mathcal{S} + \mathcal{T} \tag{6}$$

of two permutation-type operators, corresponding to the following two respective matrices $[A|B]$:

$$[A|B]_{\mathcal{S}} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right], \qquad [A|B]_{\mathcal{T}} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right]. \tag{7}$$

Note that operator $\mathcal{S}$ satisfies by itself the tetrahedron equation (and can be obtained by central reflection from the second solution in [3, Subsection 5.4]).

Operator (6) is highly degenerate: of rank 4, while acting in an eight-dimensional space. It can still be already interesting, as explained in our Subsection 1.2. Remarkably, it can also be generalized, according to formula (9) below, while still obeying the (now non-constant) tetrahedron equation.

2 Linear combinations of permutation-type solutions to the tetrahedron equation, and a 16-vertex model in three dimensions

We consider the following non-constant tetrahedron equation:

$$\mathcal{R}_{123}(a_{123})\mathcal{R}_{145}(a_{145})\mathcal{R}_{246}(a_{246})\mathcal{R}_{356}(a_{356}) \\ = \mathcal{R}_{356}(a_{356})\mathcal{R}_{246}(a_{246})\mathcal{R}_{145}(a_{145})\mathcal{R}_{123}(a_{123}), \quad (8)$$

where all $\mathcal{R}_{\alpha\beta\gamma}$ are defined according to the following ansatz:

$$\mathcal{R}_{\alpha\beta\gamma}(a_{\alpha\beta\gamma}) = \mathcal{S}_{\alpha\beta\gamma} + a_{\alpha\beta\gamma}\mathcal{T}_{\alpha\beta\gamma}, \quad (9)$$

and $\mathcal{S}_{\alpha\beta\gamma}$ and $\mathcal{T}_{\alpha\beta\gamma}$ are operators $\mathcal{S}$ and $\mathcal{T}$, defined according to (7) and acting in the tensor product of spaces numbered α, β and γ.

Matrix elements of operator (9) are depicted in Figure 3. Operator (9) can thus be said to correspond to a three-dimensional *sixteen-vertex* model.

Equation (8) imposes some restrictions on the four parameters a_{123}, a_{145}, a_{246} and a_{356}. Namely, a calculation using Singular[1] computer algebra system shows that (8) holds provided they belong to the algebraic variety containing five irreducible components written out below as "Cases 1–5".

Case 1.

$$a_{356} - 1 = 0, \qquad a_{246} - 1 = 0.$$

Case 2.

$$a_{356} + 1 = 0, \qquad a_{246} + 1 = 0.$$

Case 3.

$$a_{145}a_{246}a_{356} - a_{145} + a_{246} - a_{356} = 0, \qquad a_{123} - 1 = 0.$$

Case 4.

$$a_{145}a_{246}a_{356} - a_{145} - a_{246} + a_{356} = 0, \qquad a_{123} + 1 = 0.$$

Case 5.

$$a_{246} - a_{356} = 0, \qquad a_{145} = 0.$$

3 Discussion

This short note shows that there exists some interesting algebra related to the 16-vertex model in Figure 3. It is not yet known how nontrivial this model is or/and what interesting generalizations it admits. One feature stimulating further research in this direction is that Boltzmann weights in Figure 3 are clearly positive when $a > 0$.

[1]https://www.singular.uni-kl.de/

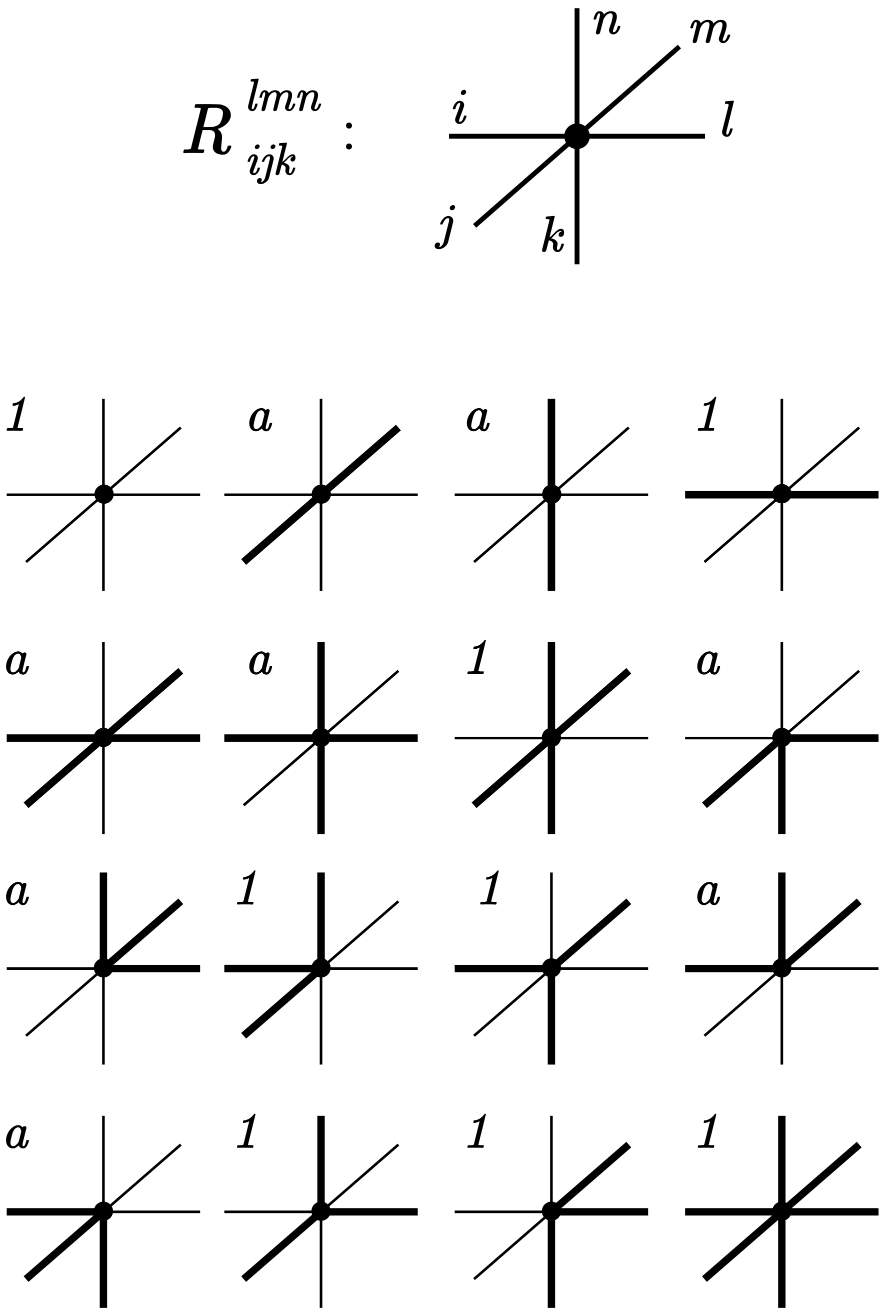

Figure 3: Matrix elements of operator (9), with subscripts α, β and γ left out. Thin lines correspond to color 0, thick lines — to color 1

Note that all R-operators are (generically) non-degenerate in our Case 5. Other cases are also of interest, for instance, the relation between parameters in Case 3, if written in the additive form

$$\operatorname{atanh} a_{246} = \operatorname{atanh} a_{145} + \operatorname{atanh} a_{356},$$

resembles the usual Yang–Baxter models very much — although more effort could be required to cope with the degeneracy of $\mathcal{R}_{123}$.

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