ON CLIFFORD DOUBLE MIRRORS OF TORIC COMPLETE INTERSECTIONS

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Abstract. We present a construction of noncommutative double mirrors to complete intersections in toric varieties. This construction unifies existing sporadic examples and explains the underlying combinatorial and physical reasons for their existence.

Contents

1. Introduction.

Two Calabi-Yau varieties X and Y are called a mirror symmetric pair if, together with some Kähler data, they give rise to two superconformal field theories that differ by a twist, see [CK99]. While this string-theoretic statement can not be at present rigorously understood, some of its consequences can be stated and even proved mathematically. For example, the (stringy) Hodge numbers of the mirror pair X and Y are expected to obey the relation $h^{p,q}(X) = h^{\dim Y - p,q}(Y)$. This so-called mirror duality test of an alleged mirror pair connects rather accessible invariants of X and Y and is the easiest to verify. A significantly more complicated test of mirror symmetry connects quantum cohomology of X with period integrals of Y. Even in the simple case when X is a smooth quintic hypersurface in $\mathbb{P}^4$, this is a highly nontrivial result due to Givental [Gi96], which was later clarified by Lian, Liu and Yau in [LLY97]. When one considers surfaces with boundary (open strings), then homological mirror symmetry [Kon94] predicts that the bounded derived category of coherent sheaves on X is equivalent to the Fukaya category of Y with the appropriate symplectic structure.

An arbitrary Calabi-Yau variety may not always possess a mirror, moreover, even if a mirror exists, it may not be unique. In fact, it is common for a Calabi-Yau variety Y to possess multiple mirror partners X_i. In physics terms the expectation is that the superconformal field theories associated to X_i are obtained from each other by some kind of analytic continuation along the parameter space of such theories. In this case, it is reasonable to refer to X_1 and X_2 as double mirrors of each other in the sense that X_2 is a mirror of a mirror of X_1 and vice versa. Even more generally, we will say that X_1 and X_2 are double mirror to each other if they pass some basic compatibility tests below.[1]

- Hodge numbers of X_1 and X_2 coincide.
- Complex moduli spaces of X_1 and X_2 coincide.
- Quantum cohomology of X_1 and X_2 are obtained from each other by analytic continuation.
- Bounded derived categories of coherent sheaves on X_1 and X_2 coincide (under some identification of the aforementioned complex moduli spaces).

Common examples of such X_1 and X_2 differ from each other by flops or more generally by K-equivalences. In this case the last two statements are known as Ruan's and Kawamata's conjectures respectively.

There are other prominent examples of double mirror Calabi-Yau varieties, such as the Pfaffian-Grassmannian example, where X_1 and X_2 are not birational. In addition, it is natural to move a little bit beyond the category of algebraic varieties to allow Deligne-Mumford stacks, as well as mildly

[1] As such, we don't require that their mirror exists in any sense.

noncommutative "varieties". [2] It is the latter kind of varieties that are the subject of this paper.

There is a number of results in the literature where some usual or noncommutative varieties X_1 and X_2 satisfy $D^b(Coh - X_1) = D^b(Coh - X_2)$ in the sense of equivalence of categories. It is reasonable to postulate that most if not all such examples should be viewed as particular cases of the double mirror phenomenon.[3] One such example is a construction of Kuznetsov, who shows that the derived category of a Calabi-Yau complete intersection of k quadrics in $\mathbb{CP}^{2k-1}$ is derived equivalent to a certain noncommutative crepant resolution of the double cover of $\mathbb{CP}^{k-1}$ ramified over a determinant of a symmetric $2k \times 2k$ matrix of linear forms, twisted by a Brauer class. More precisely, this variety can be viewed as a certain sheaf of even Clifford algebras over $\mathbb{CP}^{k-1}$. A related example has been also considered by Calabrese and Thomas [CT14].

The goal of this paper is to uncover the toric geometry that underlies Kuznetsov's and Calabrese-Thomas' constructions. This more general framework allows us to construct additional examples and leads to the more conceptual understanding of these derived equivalences.

Remark 1.1. *We should also point out that there are examples of derived equivalences between noncommutative varieties which are not covered by our construction. For instance, Căldăraru studied the derived category of elliptic fibration and the twisted derived category of its relative Jacobian [Cal00a, Cal00b]. This is a family version of classical derived equivalence between abelian varieties. The twists are used to glue the universal object in the Fourier-Mukai transform which may not exist in the ordinary sense. There are also Hosono-Takagi examples [HT14], which are closer in spirit to this paper, since they involve quadric fibrations. However, they appear to be non-toric in nature and thus not covered by our construction.*

Our construction starts off with a pair of dual reflexive Gorenstein cones $(K, K^\vee)$ in dual lattices M and N. These are dual cones in the usual sense, with the property that lattice generators of rays of K and $K^\vee$ lie in the hyperplanes $\langle -, \deg^\vee \rangle = 1$ and $\langle \deg, - \rangle = 1$ respectively, where deg and $\deg^\vee$ are (uniquely defined) lattice elements of K and $K^\vee$. We consider decompositions of the degree element $\deg^\vee$ in $K^\vee$ under certain appropriate conditions. We associate noncommutative varieties to such decompositions, and the change of the decomposition results in conjectural double mirrors.

[2] While we do not wish to be explicit in this definition, by a mildly noncommutative variety we mean a sheaf of finite rank algebras over a usual variety or stack.

[3] As far as we know, there is no systematic study of nonlinear sigma models with noncommutative targets in the physics literature. Perhaps this paper may provide a motivation for it.

To make this a bit more precise, suppose we have

$$\deg^\vee = \frac{1}{2}(s_1 + \cdots + s_{2r}) + t_1 + \cdots + t_{k-r} \tag{1.1}$$

where s_i and t_j are linearly independent lattice elements of $K^\vee$ which satisfy $\langle \deg, s_i \rangle = \langle \deg, t_j \rangle = 1$. The index $k = \langle \deg, \deg^\vee \rangle$ of reflexive Gorenstein pair $(K, K^\vee)$ is fixed, but r could vary. We construct a stack $\mathcal{S}$ which is a complete intersection in a toric stack by equations associated to $\{t_i\}$. Then $\{s_i\}$ give a sheaf of even Clifford algebras $\mathcal{B}_0$, and they combine to give us a noncommutative variety $(\mathcal{S}, \mathcal{B}_0)$. [4]

Two extreme cases are of particular importance: when $r = 0$, there are no $\{s_i\}$ in the expression of $\deg^\vee$ and $\mathcal{B}_0 = \mathcal{O}$. The noncommutative stacks $(\mathcal{S}, \mathcal{B}_0)$ are just usual DM stacks $\mathcal{S}$ which are crepant resolutions of Calabi-Yau complete intersections in toric Gorenstein Fano varieties. If there is a mirror, then one gets the classical Batyrev-Borisov construction. On the other end of the spectrum, when $r = k$, there are no $\{t_i\}$ in the expression of $\deg^\vee$. In this case $\mathcal{S}$ is a toric DM stack.

The main result of the paper is the following theorem.

Theorem 6.2. Suppose that a complete intersection $\mathcal{X}$ and a Clifford noncommutative variety $\mathcal{Y}$ are given by different decompositions of the degree element $\deg^\vee$ of a reflexive Gorenstein cone $K^\vee$ and the appropriate regular simplicial fans in $K^\vee$. Then the bounded derived categories of $\mathcal{X}$ and $\mathcal{Y}$ are equivalent, provided the centrality and the flatness assumptions on $\mathcal{Y}$ hold.

For the benefit of the reader, we try to keep the paper as self-contained as possible. It is structured as follows.

In Section 2 we review the definition of reflexive Gorenstein cones and set up some of the notations that recur throughout the paper. We give a quick introduction to Batyrev-Borisov construction from the viewpoint of the pairs of reflexive Gorenstein cones $(K, K^\vee)$. This is a slightly different approach to the subject than the more traditional study of nef-partitions as in [Bor93]. We find it more natural both in its own right and for the purposes of this paper. In fact, we only mention nef-partitions in passing remarks.

In Section 3 we outline the physical intuition that guides this paper and is key to the proper understanding of the construction and its possible generalizations. We hope that even readers not interested in the more technical details of the rest of the paper will read this section. We argue that the principal category of interest is the graded equivariant derived category of singularities of the potential in the homogeneous coordinates, considered in [BFK12]. It is defined for every regular simplicial fan in $K^\vee$ but is unique up to equivalence.

[4] For our construction to work best, we need some additional technical centrality condition on a fan Σ in $K^\vee$ and a certain flatness assumption.

In Section 4 we recall the construction of Kuznetsov of Clifford double mirrors of complete intersections of quadrics in $\mathbb{CP}^n$. This section serves a dual purpose. On the one hand, we introduce the key example that served as the original motivation behind this paper. On the other hand, we introduce (even) Clifford algebras which will be used further in the paper.

In Section 5 we consider the decompositions (1.1) with $r = k$. We construct sheaves of Clifford algebras over toric DM stacks which generalize Kuznetsov's construction and open the door to many more examples of the phenomenon. In particular, we describe a double mirror to the quotient of complete intersection of four quadrics in $\mathbb{CP}^7$ by a fixed point free involution.

In Section 6 we prove the first case of our main result, namely the equivalence of derived categories of Clifford double mirrors to the graded equivariant derived categories of singularities from Section 3. Our argument is based heavily on the work of [BFK12].

In Section 7 we generalize the construction of Section 5 to the $0 < r < k$ case of (1.1). We also conjecture that the derived equivalence statement of Section 6 extends to this more general case.

In Section 8 we discuss in detail the combinatorics of Clifford double mirrors. It is likely to be useful in further study of the phenomenon.

In Section 9 we describe several additional examples of the construction. Some of them such as Calabrese-Thomas' example [CT14] already appear in the literature, and the others are new. We specifically look at what happens if some of the assumptions of the main theorem are relaxed.

Finally, in Section 10 we make some concluding remarks and pose open questions that we hope the readers or the authors will address in future research.

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2. Review of reflexive Gorenstein cones, Batyrev-Borisov mirror construction and double mirror phenomenon.

In this section we give an overview of Batyrev-Borisov mirror construction with the emphasis on reflexive Gorenstein cones. This viewpoint is essential for understanding the rest of the paper. So even a reader familiar with Batyrev-Borisov construction in terms of nef-partitions will find it necessary to at least briefly look through this section. We describe the crucial work of Batyrev and Nill [BatN08] which forms the basis for the double mirror phenomenon in the Batyrev-Borisov setting. In the process we fix the notations and explain the way they are used throughout the paper.

2.1. **Reflexive Gorenstein cones.** Let $M \cong \mathbb{Z}^{\operatorname{rank} M}$ be a lattice and let $N := \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ be its dual. The natural pairing is given by

$$\langle , \rangle : M \times N \to \mathbb{Z}.$$

Let $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ be the $\mathbb{R}$-linear extensions of M, N. The pairing can be $\mathbb{R}$-linearly extended as well, and we still use $\langle , \rangle$ to denote this extension.

Definition 2.1. *A rational polyhedral cone $K \subset M_{\mathbb{R}}$ is a convex cone generated by a finite set of lattice points. We assume that $K \cap (-K) = \{0\}$. We call the first lattice point of a ray ρ of K a primitive element or a lattice generator of ρ.*

Definition 2.2. *([BatBor94]) A full-dimensional rational polyhedral cone $K \subset M_{\mathbb{R}}$ is called a* Gorenstein *cone if all the primitive elements of its rays lie on some hyperplane $\langle -, n \rangle = 1$ for some* degree element *n in N. A Gorenstein cone $K \subset M_{\mathbb{R}}$ is called a* reflexive *Gorenstein cone iff its dual cone $K^{\vee} := \{y \mid \langle x, y \rangle \geq 0 \ \forall \ x \in K\}$ is also a Gorenstein cone with respect to the dual lattice N.*

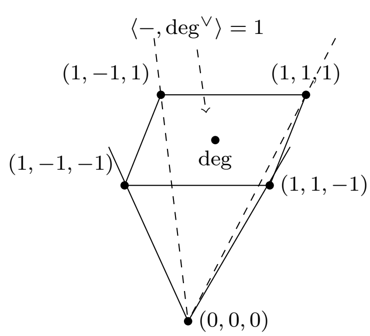
Remark 2.3. *Note that $(K^{\vee})^{\vee} = K$, which is why we typically talk about a pair of dual reflexive Gorenstein cones K and $K^{\vee}$. For any such pair the degree elements are unique and are denoted by $\deg^{\vee}$ and $\deg$ respectively. Observe that $\deg^{\vee}$ is a lattice point in the interior $(K^{\vee})^{\circ}$ of $K^{\vee}$ and there holds*

$$(K^{\vee})^{\circ} \cap N = \deg^{\vee} + (K^{\vee} \cap N).$$

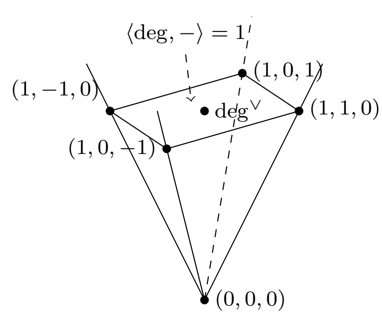
Similarly $K^{\circ} \cap M = \deg + (K \cap M)$.

Definition 2.4. *For a pair of dual reflexive Gorenstein cones $(K, K^{\vee})$, the pairing of their two degree elements $\langle \deg, \deg^{\vee} \rangle = k$ is called the* index *of the pair. The index is always a positive integer.*

The following is an example of a 3-dimensional reflexive Gorenstein cone and its dual cone. Notice that two degree elements $\deg, \deg^{\vee}$ happen to lie on the hyperplanes $\langle -, \deg^{\vee} \rangle = 1$ and $\langle \deg, - \rangle = 1$ which may not be the case in general. Indeed, this will never happen as soon as the index of the pair of Gorenstein cones is larger than 1.



Gorenstein cone K.



The dual Gorenstein cone $K^{\vee}$.

We will revisit this example in Section 9.4.

Remark 2.5. *If a pair of Gorenstein cones leads to a complete intersection in a toric variety, the index k of the pair is the codimension of the complete intersection. The dimension of the complete intersection is $d - 2k$ where $d = \operatorname{rank} M = \operatorname{rank} N$. In particular, the original Batyrev's hypersurface construction corresponds to $k = 1$, as is the case in the above figure. In this paper we are primarily interested in the case $k > 1$, so the reader should not rely too much on their knowledge of the original Batyrev's hypersurface construction.*

Given a pair of reflexive Gorenstein cones $(K, K^\vee)$ we define two lattice polytopes

$$\Delta = \{x \in K, \langle x, \deg^\vee \rangle = 1\},\ \Delta^\vee = \{y \in K^\vee, \langle \deg, y \rangle = 1\}.$$

Their sets of lattice points are denoted by

$$K_{(1)} := \Delta \cap M,\ K^\vee_{(1)} := \Delta^\vee \cap N$$

respectively. When the index k is one, these are the original reflexive polytopes of Batyrev [Bat94]. For $k > 1$, these polytopes have no interior lattice points, although they can and often do have non-vertex lattice points on the boundary.

A crucial part of the data necessary to define Calabi-Yau varieties (or more generally triangulated categories of Calabi-Yau type) in the setting of Gorenstein reflexive cones, is a family of *coefficient functions*

$$c : K_{(1)} \to \mathbb{C}.$$

We typically fix an element of this family in general position.

2.2. **Reflexive cones to complete intersections.** The original Borisov's extension of Batyrev's construction was accomplished by the use of nef-partitions. However, a more flexible and conceptually superior way of looking at the construction has been later provided by the work of Batyrev and Nill [BatN08]. This new approach allows for a very clear way of constructing double mirrors in the Batyrev-Borisov setting. While we follow the idea of Batyrev and Nill, the exposition below is different from their paper. Indeed, we are trying to set up the viewpoint that will naturally extend to our noncommutative double mirror setting.

The main idea of Batyrev-Nill's paper is the following concept of the decomposition of the dual degree element.

Definition 2.6. *Let $(K \subset M_\mathbb{R}, K^\vee \subset N_\mathbb{R})$ be a pair of dual reflexive Gorenstein cones of index k. We call*

$$\deg^\vee = t_1 + \cdots + t_k$$

a decomposition of $\deg^\vee$ *if all* t_i *are elements of* $K^\vee_{(1)}$. [5]

Given a decomposition of $\deg^\vee$, one can construct a singular toric variety $\mathbb{P}_{sing}$ and a family of Calabi-Yau complete intersections in it. We will later describe how one can construct a Deligne-Mumford stack crepant resolution of this family.

Definition 2.7. *Since* $t_i \in K^\vee \cap N$, *the pairing with* t_i *provides the semi-group ring* $\mathbb{C}[K\cap M]$ *with a* $(\mathbb{Z}_{\geq 0})^k$ *grading. Indeed, a monomial associated to* m *will have grading* $(\langle m,t_1\rangle,\ldots,\langle m,t_k\rangle)$. *One can then define the multi-Proj of this ring the same way one defines a usual* Proj *to get*

$$\mathbb{P}_{sing} := \mathrm{multiProj}(\mathbb{C}[K\cap M]).$$

Remark 2.8. *We will later see a more conventional definition of this toric variety* $\mathbb{P}_{sing}$, *which does not use the multi-Proj construction but is a bit less intuitive.*

The decomposition $\deg^\vee = \sum_{i=1}^k t_i$ provides a decomposition of the set of lattice points of Δ into a disjoint union of the sets

$$K_{(1)} = \bigsqcup_{i=1}^k K_{(1),i},\ K_{(1),i} = \{x\in K_{(1),i}, \langle x,t_j\rangle = \delta_i^j\}$$

where δ_i^j is the Kronecker delta. Importantly, these sets $K_{(1),i}$ are the sets of lattice points of lattice polytopes Δ_i in M, which are faces of Δ. A generic coefficient function $c: K_{(1)}\to\mathbb{C}$ now allows one to define k homogeneous elements of $\mathbb{C}[K\cap M]$

$$\sum_{m\in K_{(1),1}} c(m)[m],\ldots,\sum_{m\in K_{(1),k}} c(m)[m],$$

where $[m]$ is the monomial element of $\mathbb{C}[K\cap M]$ that corresponds to m.

Definition 2.9. *For a generic choice of* c, *we define the complete intersection* $X_{c,sing}\subset\mathbb{P}_{sing}$ *by*

$$X_{c,sing} := \mathrm{multiProj}(\mathbb{C}[K\cap M]/\langle \sum_{m\in K_{(1),1}} c(m)[m],\ldots,\sum_{m\in K_{(1),k}} c(m)[m]\rangle).$$

Remark 2.10. *In the absence of a decomposition of* $\deg^\vee$, *one may only consider the usual*

$$H = \mathrm{Proj}(\mathbb{C}[K\cap M]/\langle\sum_{m\in\Delta} c(m)[m]\rangle)\subset \mathrm{Proj}(\mathbb{C}[K\cap M])$$

which is a hypersurface in a toric variety. A choice of a decomposition allows one to realize the above hypersurface as a so-called Cayley hypersurface of a complete intersection. There are also situations where a decomposition of

[5] It is easy to show that these t_i are linearly independent, so it does not need to be a part of the definition.

$\deg^\vee$ *does not exist, as in the example of [BatBor94]. In these cases one can try to work with the hypersurface H as if it were a Cayley hypersurface of a complete intersection, even though no such complete intersection is available.*

Let us now describe $\mathbb{P}_{sing}$ in a more traditional way, which will also allow us to talk about its desingularizations.

Observe that the sublattice of N given by $\mathbb{Z}t_1+\ldots+\mathbb{Z}t_k$ is of rank k and is saturated. Indeed, if a set $K_{(1),i}$ were empty for some i, it would mean that all elements of $K_{(1)}$ had zero pairing with t_i, in contradiction to K being full-dimensional. Thus, there are elements in each $K_{(1),i}$ and a pairing with them provides a splitting to the natural map $\mathbb{Z}^k \to N$ given by t_i. We define the quotient lattice $\overline{N}$ of N and a sublattice $\overline{M}$ of M by

$$\overline{N} := N/(\mathbb{Z}t_1+\ldots+\mathbb{Z}t_k),\ \overline{M} := \mathrm{Ann}(\mathbb{Z}t_1+\ldots+\mathbb{Z}t_k).$$

Note that $\overline{M}$ and $\overline{N}$ are naturally dual to each other. The image of the polytope $\Delta^\vee$ under the quotient map $\phi : N_\mathbb{R} \to \overline{N}_\mathbb{R}$ is a polytope $\phi(\Delta^\vee)$ which is reflexive (see [Li13]). Consider the minimum fan Σ_1 in $\overline{N}_\mathbb{R}$ associated to $\phi(\Delta^\vee)$, i.e. the fan whose maximum cones correspond to facets of $\phi(\Delta^\vee)$. Then $\mathbb{P}_{sing}$ is the toric Fano variety associated to Σ_1.

Note that the polytopes Δ_i lie in parallel translates of $\overline{M}$. We also observe [Li13] that

$$\sum_{i=1}^k \Delta_i - \deg$$

is the reflexive polytope dual to $\phi(\Delta^\vee)$. This allows us to view polytopes Δ_i as support polytopes for global sections of k globally generated line bundles on $\mathbb{P}_{sing}$. The Minkowski sum of the polytopes is the anti-canonical polytope, which means that the tensor product of the line bundles is the anti-canonical bundle of $\mathbb{P}_{sing}$. Thus, provided that the intersection $X_{c,sing}$ is of expected dimension, it will be a Calabi-Yau variety by the adjunction formula. We will see that for a generic choice of the coefficient function c the resulting $X_{c,sing}$ is of correct dimension and has a DM stack resolution induced by a resolution of the ambient space $\mathbb{P}_{sing}$.

Before we prove the main result of this section, let us state and prove a simple lemma.

Lemma 2.11. *We have*

$$K = \sum_i \mathbb{R}_{\geq 0}\Delta_i.$$

Moreover, if a point $v \in K$ has $\langle v, t_i\rangle = \alpha_i$, then $v \in \sum_i \alpha_i\Delta_i$.

Proof. Since $\Delta_i \subset K$, we have $\sum_i \mathbb{R}_{\geq 0}\Delta_i \subseteq K$. In the other direction, observe that for every ray of K its generator lies in one of the Δ_i. Finally, the last statement follows by considering the pairing with $t_1,\ldots,t_k$. □

Let us now formulate an important result that connects triangulations of the boundary of $\phi(\Delta^\vee)$ with fans on $K^\vee$.

Proposition 2.12. *Let $\overline{\Sigma}$ be a simplicial fan in $\overline{N}$ which comes from a regular triangulation of the boundary of $\phi(\Delta^\vee)$. Consider the fan Σ in N which is the preimage of $\overline{\Sigma}$ intersected with $K^\vee$. Then preimages of the maximum cones of $\overline{\Sigma}$ are themselves simplicial cones in $N_{\mathbb{R}}$, and the fan Σ comes from a regular triangulation of $\Delta^\vee$.*

Proof. It is clear that the preimages of the cones of $\overline{\Sigma}$ form a fan, when intersected with $K^\vee$. To make sure that the generators of the rays of Σ are lattice points of $\Delta^\vee$, we need to show that all lattice points of $\phi(\Delta^\vee)$ are images of lattice points of $\Delta^\vee$. Suppose that there is $p \in \phi(\Delta^\vee) \cap \overline{N}$. We have $p = \phi(q)$ for some $q \in \Delta^\vee$. While we can not assume that $q \in N$, we know that there exist $\alpha_i \in \mathbb{R}$ such that

$$\hat{q} = \sum_{i=1}^{k} \alpha_i t_i + q \in N.$$

We may assume that $\hat{q} \in K^\vee$ and by picking such $\hat{q}$ of minimum degree $\langle \deg, \hat{q}\rangle$ we can assure that for all i we have $\hat{q} - t_i \notin K^\vee$. This means that there are generators v_i of K such that $\langle v_i, \hat{q} - t_i\rangle < 0$. Since $\langle v_i, \hat{q}\rangle \geq 0$, this shows that $\langle v_i, t_i\rangle > 0$, thus $v_i \in \Delta_i \cap M$. Since $\langle v_i, t_i\rangle = 1$ and $\langle v_i, \hat{q}\rangle$ is integer, we see that $\langle v_i, \hat{q}\rangle = 0$. Since $\langle v_i, \hat{q}\rangle = \alpha_i + \langle v_i, q\rangle$ and $\langle v_i, q\rangle$ are nonnegative, we see that $\alpha_i \leq 0$. Unless all α_i are zero, we see that $\hat{q}$ is of degree zero, which implies that $p = 0$, in which case the above argument shows that the only lattice preimages q of $0 \in \overline{N}$ inside $\Delta^\vee$ are t_i. If all α_i are zero, it means that $q = \hat{q}$ is a lattice point. In fact, the above argument show the uniqueness of such q.

Let us now show that the intersection of the preimage of a maximum-dimensional cone

$$\sigma_1 = \mathbb{R}_{\geq 0}\phi(w_1) + \cdots + \mathbb{R}_{\geq 0}\phi(w_{d-k})$$

of $\overline{\Sigma}$ with $K^\vee$ is the cone

$$\sigma = \mathbb{R}_{\geq 0} w_1 + \cdots + \mathbb{R}_{\geq 0} w_{d-k} + \sum_{i=1}^{k} \mathbb{R}_{\geq 0} t_i.$$

The inclusion

$$\sigma \subseteq \phi^{-1}(\sigma_1) \cap K^\vee$$

is clear. In the other direction, observe that there is a facet of $\phi(\Delta^\vee)$ that contains all $\phi(w_j)$. This means that there is an element of $v \in \overline{M}_{\mathbb{R}} \subset M_{\mathbb{R}}$ (we can pick it in M, but we will not need it) such that $\langle v, t_i\rangle = 0$ for all i, $\langle v, \Delta^\vee\rangle \geq -1$ and $\langle v, w_j\rangle = -1$ for all j. We see that $(v+\deg)$ is nonnegative on $\Delta^\vee$, therefore $v + \deg \in K$. By Lemma 2.11 we see that

$$v + \deg = \sum_{i=1}^{k} v_k$$

with $v_i \in \Delta_i$. Observe that $\langle v + \deg, w_j \rangle = 0$ and $v + \deg \in K$ implies that $\langle v_i, w_j \rangle = 0$ for all i and j.

Now suppose that

$$w = \sum_i \alpha_i t_i + \sum_j \beta_j w_j \in \phi^{-1}(\sigma_1) \cap K^\vee$$

Then for all i we have $\langle v_i, w \rangle \geq 0$ which implies that $\alpha_i \geq 0$, so $w \in \sigma$.

Last, we observe that the fan Σ is regular, since we may simply use the pullback of the convex piecewise-linear function on $\overline{\Sigma}$ to give one for Σ. □

As the result of the above Proposition, we can construct a regular simplicial fan Σ on $K^\vee$ with the following property.

(2.1) *All maximum dimensional cones of* Σ *contain* $t_1, \ldots, t_k$.

The toric variety $\mathbb{P}_\Sigma$ is a vector bundle over the toric variety $\mathbb{P}_{\overline{\Sigma}}$. However, we are interested in the corresponding Deligne-Mumford stacks. We will review the construction of these stacks briefly. It is much easier in this setting than in general [BCS05] since the lattices in question do not have torsion.

Consider the open torus-invariant subset U_Σ of the variety $\mathbb{C}^{K^\vee_{(1)}}$ of complex-valued functions on the finite set $K^\vee_{(1)}$ given by the points $\mathbf{z} : K^\vee_{(1)} \to \mathbb{C}$ with the zero locus a subset of some cone $\sigma \in \Sigma$. [6] There is a group G which acts on U_Σ. It is the subgroup of the torus $(\mathbb{C}^*)^{K^\vee_{(1)}}$ which consists of $\lambda : K^\vee_{(1)} \to \mathbb{C}^*$ with the property that

$$\prod_{n \in K^\vee_{(1)}} \lambda(n)^{\langle m,n \rangle} = 1$$

for all $m \in M$. The smooth Deligne-Mumford stack $[U_\Sigma/G]$ is a resolution of singularities of $\mathrm{Spec}(\mathbb{C}[K \cap M])$.

Similarly, we consider the open subset $U_{\overline{\Sigma}}$ of the variety $\mathbb{C}^{K^\vee_{(1)} - \{t_1,\ldots,t_k\}}$ given by the property that the zero locus of $\mathbf{z}$ lies in a cone of $\overline{\Sigma}$. We see that

$$U_\Sigma = U_{\overline{\Sigma}} \times \mathbb{C}^k$$

where the second factor corresponds to the values of $\mathbf{z}$ on $t_1, \ldots, t_k$. Moreover, our description of the set of lattice points of $\phi(\Delta^\vee)$ implies that the

[6] Here we mean that the indices of the zero locus form a set in the simplicial complex that corresponds to Σ. In particular, if a point in $K^\vee_{(1)}$ is not used in Σ, the corresponding variable is always nonzero.

group $\overline{G}$ used to construct the Deligne-Mumford stack that corresponds to $\overline{\Sigma}$ coincides with the group G above. Specifically, every function

$$\overline{\lambda} : K^\vee_{(1)} - \{t_1, \ldots, t_k\} \to \mathbb{C}^*$$

which satisfies

$$\prod_{n \in K^\vee_{(1)} - \{t_1,\ldots,t_k\}} \overline{\lambda}(n)^{\langle m,n\rangle} = 1$$

for all $m \in \overline{M} = \mathrm{Ann}(\mathbb{Z}t_1 + \cdots \mathbb{Z}t_k)$ can be uniquely extended to

$$\lambda : K^\vee_{(1)} \to \mathbb{C}^*$$

which satisfies the above condition for all $m \in M$. This gives U_Σ a structure of the G-equivariant vector bundle over $U_{\overline{\Sigma}}$ and gives $\mathbb{P}_\Sigma = [U_\Sigma/G]$ a structure of a vector bundle over $\mathbb{P}_{\overline{\Sigma}} = [U_{\overline{\Sigma}}/G]$. Moreover, this vector bundle is naturally the direct sum of k line bundles that correspond to individual coordinates $z(t_i)$.

As before, consider a generic coefficient function c. It allows us to define polynomials $\overline{h}_i$ in coordinates of $\mathbb{C}^{K^\vee_{(1)} - \{t_1,\ldots,t_k\}}$

$$\overline{h}_i := \sum_{m \in K_{(1),i}} c(m) \prod_{n \in K^\vee_{(1)} - \{t_1,\ldots,t_k\}} z(n)^{\langle m,n\rangle},$$

and the corresponding complete intersection on $U_{\overline{\Sigma}}$

$$\overline{X}_c := \{\overline{h}_1 = \cdots = \overline{h}_k = 0\} \subset U_{\overline{\Sigma}}.$$

Observe that we have

$$C(\mathbf{z}) = \sum_{i=1}^k z(t_i)\overline{h}_i = \sum_{m \in K_{(1)}} c(m) \prod_{n \in K^\vee_{(1)}} z(n)^{\langle m,n\rangle},$$

and therefore $\{C = 0\}$ is the Cayley hypersurface associated to the complete intersection of $\{\overline{h}_i = 0\}$. Since the above construction is G-equivariant, we see that the same is true for the stacks

$$\mathcal{H}_c := [\{C(\mathbf{z}) = 0\}/G] \subset \mathbb{P}_\Sigma$$

and

$$\mathcal{X}_c := [\cap_i \{\overline{h}_i = 0\}/G] \subset \mathbb{P}_{\overline{\Sigma}}.$$

We observe that if c is generic (for example Δ-nondegenerate in the sense of Batyrev) then $\mathcal{X}_c$ is a smooth Deligne-Mumford stack. The stack $\mathcal{H}_c$ is singular along $\mathcal{X}_c$ which is naturally included as a substack in the zero section of the vector bundle $\mathbb{P}_\Sigma \to \mathbb{P}_{\overline{\Sigma}}$.

We have thus described how to associate to a decomposition

$$\deg^\vee = t_1 + \cdots + t_k$$

and a fan Σ with the property (2.1) a family of smooth Calabi-Yau DM stacks.

2.3. **Mirrors and double mirrors.** To construct the mirror family, one should consider a decomposition

$$\deg = u_1 + \ldots + u_k$$

where $u_i \in K_{(1)}$. One can reindex u_i to ensure that $\langle u_i, t_j \rangle = \delta_{i,j}$. One can show that the combinatorial data of $(K, K^\vee)$ together with the pair of decompositions of deg and $\deg^\vee$ are precisely equivalent to the data of the nef-partition considered originally in [BatBor94]. It allows one to construct a family

$$\{\mathcal{Y}_{c^\vee}\}$$

for generic mirror coefficient functions $c^\vee : K^\vee_{(1)} \to \mathbb{C}$, provided one picks a simplicial subdivision of K.

The double mirrors of $\{\mathcal{X}_c\}$ are then defined as *mirrors of mirrors* of $\{\mathcal{X}_c\}$. As the above discussion shows, they can be obtained from the same pair of cones $(K, K^\vee)$ and the coordinate functions c by changing the fan Σ and more interestingly *by changing a decomposition of* $\deg^\vee$ to

$$\deg^\vee = t'_1 + \cdots + t'_k.$$

A striking observation regarding this double mirror construction is that the coefficients $c(m)$ are generally sorted into different subsets to define the complete intersection! Nonetheless the resulting stacks are expected to share many properties.

Remark 2.13. *In what follows we will call* $\{\mathcal{X}'_c\}$ *a double mirror of* $\{\mathcal{X}_c\}$ *even in the absence of a choice of a decomposition of* deg. [7] *We are less interested in the choice of* Σ *and* Σ'*, although they of course are needed. The reason is that different choices of the fans amount to a combination of toric flops.*

We collect existing results related to Batyrev-Borisov double mirrors $\mathcal{X}, \mathcal{X}'$.

Theorem 2.14. *For any* p, q*, the stringy Hodge numbers* $h^{p,q}_{\rm st}(X_{c,sing})$ *and* $h^{p,q}_{\rm st}(X'_{c,sing})$ *coinside.*

This is a direct consequence of the main theorem of [BatBor96] which provides a combinatorial formula for $h^{p,q}_{\rm st}(X_{c,sing})$ in terms of the combinatorics of the cones K and $K^\vee$. Note that these Hodge numbers are the orbifold Hodge numbers of the crepant stacky resolutions $\mathcal{X}_c$ and $\mathcal{X}'_c$ considered above.

In [BatN08], Batyrev and Nill proposed conjectures on the birationality and derived equivalence of double mirrors. These conjectures have been answered in [Li13, FK14].

[7] Such decomposition may not exist, in which case the families in questions are in some sense generalized double-mirrors of each other. We do not expect this to be essential.

Theorem 2.15. *([Li13]) Under some mild technical assumptions, the double mirrors $\mathcal{X}_c$ and $\mathcal{X}'_c$ are birational.*

Theorem 2.16. *([FK14]) The double mirrors $\mathcal{X}_c$ and $\mathcal{X}'_c$ are derived equivalent.*

There are other types of (commutative) mirror constructions in mirror symmetry which also exhibit double mirror phenomenon. Among them, the so called Berglund-Hübsch-Krawitz mirror construction [BerH93, Kra09] is particularly well understood. We briefly mention the parallel results in BHK setting.

Let $Z_{A,G}, Z_{A',G}$ be a pair of BHK double mirrors. The equivalence between orbifold Chen-Ruan cohomology of corresponding DM-stacks $[Z_{A,G}]$ and $[Z_{A',G}]$ is a consequence of the main result of [CR11]. The birationality of double mirrors is established in various generality in [Sho12, Kel13, Cla13, Bor13]. In [FK14], the derived categories of double mirrors are shown to be equivalent.

3. The underlying philosophy: triangulated categories associated to reflexive Gorenstein cones.

In this section we explain the underlying philosophy that guides our construction. We construct triangulated categories of type IIB boundary conditions for the data of reflexive Gorenstein cones and corresponding potentials. We argue that these categories provide the correct definition and should be the primary object of study.

Let $K \subset M_{\mathbb{R}}$ and $K^\vee \subset N_{\mathbb{R}}$ be dual reflexive Gorenstein cones. Let

$$c : K_{(1)} \to \mathbb{C},\ c^\vee : K^\vee_{(1)} \to \mathbb{C}$$

be generic coefficient functions. To this data one should be able to associate (in some vague physical sense) $N = (2,2)$ superconformal field theories of type IIA and IIB. The switch between the IIA and IIB should correspond to the switch of the data and the dual data. For the purposes of the following discussion we will focus on the IIB theory. We should view this theory as some kind of generalized Landau-Ginzburg theory with the potential c and the generalized Kähler data (or mirror potential) given by $c^\vee$.

A reasonably well understood feature of type IIB superconformal field theory is the triangulated category of the boundary conditions on the open strings. If the target is a Calabi-Yau manifold, then this is simply the derived category of coherent sheaves on it. In what follows we propose a definition of such triangulated category in our setting.

Consider the singular affine toric variety $\mathrm{Spec}(\mathbb{C}[K \cap M])$ and the hypersurface in it

$$X_c := \mathrm{Spec}(\mathbb{C}[K \cap M]/\langle \sum_{m \in K_{(1)}} c(m)[m]\rangle).$$

We want to define the triangulated category in question as some resolution of the graded singularity category of X_c. The grading is given by $\deg^\vee \in K^\vee$. The actual definition is given below in terms of the Cox construction.

Consider a regular triangulation Σ of $K^\vee$ and the corresponding Cox open subset

$$U_\Sigma \subset \mathbb{C}^{K^\vee_{(1)}}$$

given by the points $\mathbf{z} : K^\vee_{(1)} \to \mathbb{C}$ with the zero locus a subset of some cone $\sigma \in \Sigma$. There is a group G which acts on U_Σ. It is a subgroup of the torus $(\mathbb{C}^*)^{K^\vee_{(1)}}$ which consists of $\lambda : K^\vee_{(1)} \to \mathbb{C}^*$ with the property that

$$\prod_{n \in K^\vee_{(1)}} \lambda(n)^{\langle m,n\rangle} = 1$$

for all $m \in M$. The smooth Deligne-Mumford stack $[U_\Sigma/G]$ is a resolution of singularities of $\mathrm{Spec}(\mathbb{C}[K \cap M])$.

The coefficient function c defines a hypersurface $C = 0$ in U_Σ where

$$C(\mathbf{z}) = \sum_{m \in K_{(1)}} c(m) \prod_{n \in K^\vee_{(1)}} z(n)^{\langle m,n\rangle}$$

is the G-invariant polynomial that corresponds to $\sum_m c(m)[m] \in \mathbb{C}[K \cap M]$.

The action of $\mathbb{C}^*$ on $[U_\Sigma/G]$ that we alluded to before manifests itself in a natural supgroup $\hat{G} \supset G$ defined as

$$\hat{G} := \{\lambda : K^\vee_{(1)} \to \mathbb{C}^* \Big| \prod_{n \in K^\vee_{(1)}} \lambda(n)^{\langle m,n\rangle} = 1, \text{ for all } m \in \mathrm{Ann}(\deg^\vee)\}. \tag{3.1}$$

We have a natural inclusion $G \subset \hat{G}$ and the quotient $\hat{G}/G$ can be identified with $\mathbb{C}^*$ by choosing $m \in M$ with $\langle m, \deg^\vee\rangle = 1$. Then the map $\hat{G} \to \mathbb{C}^*$ given by

$$\lambda \mapsto \prod_{n \in K^\vee_{(1)}} \lambda(n)^{\langle m,n\rangle}$$

is surjective, has kernel G and is independent from the choice of m above.

Definition 3.1. *We define the triangulated category*

$$D_B(K, c; \Sigma)$$

as the graded category of singularities of $\{C = 0\} \subset [U_\Sigma/G]$. Specifically, it is obtained from the bounded derived category of $\hat{G}$-equivariant coherent sheaves on $\{C = 0\}$ by taking quotient by the full subcategory generated by locally free $\hat{G}$-equivariant sheaves.

We would like to argue that thus defined category should be viewed as the correct mathematical definition of the triangulated category of type IIB branes on the theory that corresponds to the above combinatorial data, even if the said theory itself is not defined mathematically. Our first observation is the result of Ballard, Favero and Katzarkov [BFK12], which clarifies the earlier work of Herbst-Walcher [HW12].

Theorem 3.2. *The category $D_B(K,c;\Sigma)$ does not depend on Σ in the sense that for any two regular simplicial fans Σ_+ and Σ_- as above there is an equivalence of triangulated categories*

$$D_B(K,c;\Sigma_+) = D_B(K,c;\Sigma_-).$$

Proof. This statement follows from [BFK12, Theorem 3.5.2]. Since all of the rays of the fans lie in a hyperplane, the parameter μ of [BFK12] is 0. □

Remark 3.3. *As expected, the triangulated category of the data $(K,K^\vee;c,c^\vee)$ is independent of $c^\vee$. However, there should be some, yet unknown, construction of the family of such categories as $c^\vee$ varies. This family of categories should have a flatness property. Then the above equivalences correspond to the path in the Kähler moduli space of $c^\vee$ between the large Kähler limit points that correspond to Σ_+ and Σ_-.*

Importantly, we can relate the above defined category $D_B(K,c)$ to the derived category of a Calabi-Yau complete intersection for any choice of a decomposition

$$\deg^\vee = t_1 + \cdots + t_k.$$

Specifically, there is the following result, due to multiple authors, see [FK14, Isik13, Shi12].

Theorem 3.4. *Let $\deg^\vee = t_1 + \cdots + t_k$ be a decomposition of $\deg^\vee$ as in Section 2. Let Σ be a regular simplicial fan in $K^\vee$ considered in that section. Let $\mathcal{X}_{c;\Sigma}$ be the complete intersection considered in that section. Then*

$$D_B(K,c;\Sigma) = D^b(Coh - \mathcal{X}_{c;\Sigma})$$

in the sense of equivalence of triangulated categories.

Remark 3.5. *We can thus view the category $D_B(K,c) = D_B(K,c;\Sigma)$ as a primary object of interest. In a somewhat vague sense we view the above Theorem 3.4 as* large Kähler limit description of *$D_B(K,c)$ as the size of the coefficients $c^\vee(t_i)$ for the mirror coefficient function*

$$c^\vee : K^\vee_{(1)} \to \mathbb{C}$$

is large compared to the other values. In what follows we will describe another explicit geometric realization of $D_B(K,c)$, this time coming from a more complicated decomposition of $\deg^\vee$ which corresponds to a more complicated large Kähler limit.

Remark 3.6. *If the centrality assumption on the fan* (2.1) *does not hold, then we expect some interesting structures along the lines of exoflops of Aspinwall [Asp09, Asp15]. However, these are not the focus of the current paper.*

Remark 3.7. *There must be a relation between the Hochschild cohomology of the category* $D_B(K,c)$ *and the stringy cohomology of* $(K, K^\vee; c, c^\vee)$ *in [Bor14]. However, we can not presently formulate a precise conjecture, beyond the basic equality of dimensions.*

4. Review of Kuznetsov's Clifford double mirrors of complete intersections of quadrics in $\mathbb{CP}^n$.

The goal of this section is to review the construction due to Kuznetsov of noncommutative (Clifford) double mirrors of complete intersections of quadrics in projective spaces. One can view this paper as the generalization of Kuznetsov's construction to more general toric varieties, as well as an explanation of the combinatorics behind it. We are interested in the Calabi-Yau case of the construction, so we will be working with the intersection of k quadrics in $\mathbb{CP}^{2k-1}$.

Let $f_1, \ldots, f_k$ be generic homogeneous degree two polynomials in the variables $\mathbf{x} := (x_1 : \ldots : x_{2k})$. Consider the complete intersection $Y \subset \mathbb{CP}^{2k-1}$

$$Y = \{f_1(\mathbf{x}) = f_2(\mathbf{x}) = \ldots = f_k(\mathbf{x}) = 0\}.$$

This will be a smooth Calabi-Yau variety by Bertini's theorem and the adjunction formula. Note that the Cayley hypersurface of this complete intersection Y may be thought of as the generic bi-degree $(1,2)$ divisor in $\mathbb{CP}^{k-1} \times \mathbb{CP}^{2k-1}$ given by

$$u_1 f_1(\mathbf{x}) + \cdots + u_k f_k(\mathbf{x}) = 0.$$

We denote this hypersurface by X.

The double mirror noncommutative variety can be described as follows. The polynomial

$$C(\mathbf{u}, \mathbf{x}) = u_1 f_1(\mathbf{x}) + \cdots + u_k f_k(\mathbf{x})$$

allows one to define a graded noncommutative ring which is the quotient of the free ring in k commuting central variables u_i and $2k$ noncommuting variables y_j

$$\mathcal{A} = \mathbb{C}[u_1, \ldots, u_k]\{y_1, \ldots, y_{2k}\}/\langle (\sum_{i=1}^{2k} x_i y_i)^2 + C(\mathbf{u}, \mathbf{x}),\ \mathbf{x} \in \mathbb{CP}^{2k-1}\rangle.$$

This is a finitely generated algebra over the homogeneous coordinate ring $\mathbb{C}[u_1, \ldots, u_k]$ of $\mathbb{CP}^{k-1}$. It is equipped with a half-integer grading such that the degree of u_i is 1 and the degree of y_j is $\frac{1}{2}$. By localizing for the central variables u_i and taking the degree zero component we get a sheaf of even

Clifford algebras $\mathcal{B}_0$ over $\mathbb{CP}^{k-1}$. Specifically, if we introduce a vector bundle $\mathcal{E} = \mathcal{O}^{n+1}$ over $\mathbb{CP}^{k-1}$ then $\mathcal{B}_0$ is the direct sum of vector bundles

$$\mathcal{B}_0 = \mathcal{O} \oplus (\wedge^2 \mathcal{E})(1) \oplus \ldots \oplus (\wedge^{2k}\mathcal{E})(k)$$

with a certain Clifford multiplication structure. [8]

The following key result is due to Kuznetsov. It generalizes the work of Kapranov [Kap89] and relates the bounded derived category of the Cayley hypersurface X in $\mathbb{CP}^{k-1}\times\mathbb{CP}^n$ with the bounded derived category of sheaves of $\mathcal{B}_0$-modules on $\mathbb{CP}^{k-1}$. It is the consequence of Theorem 4.2 in [Kuz08].

Theorem 4.1. *Denote by $p : X \to \mathbb{P}^{k-1}$ the natural projection with quadric fibers of dimension $n-1$. The derived category $D^b(X)$ admits a semiorthogonal decomposition*

$$D^b(X) = \langle D^b(\mathbb{P}^{k-1}, \mathcal{B}_0), p^*D(\mathbb{P}^{k-1})\otimes\mathcal{O}_{X/\mathbb{P}^{k-1}}(1), \ldots, p^*D(\mathbb{P}^{k-1})\otimes\mathcal{O}_{X/\mathbb{P}^{k-1}}(2k-2)\rangle$$

in the sense that the orthogonal complement of $\langle p^*D(\mathbb{P}^{k-1})(1), \ldots, p^*D(\mathbb{P}^{k-1})(2k-2)\rangle$ is naturally equivalent to $D^b(\mathbb{P}^{k-1}, \mathcal{B}_0)$.

It was shown by Kuznetsov that $D^b(\mathbb{P}^{k-1}, \mathcal{B}_0)$ is equivalent to $D^b(Y)$. Instead of Kuznetsov's original proof that uses Lefschetz decompositions and homological projective duality, we will sketch a proof of this statement that will generalize later to more sophisticated examples.

Theorem 4.2. *There is an equivalence of categories*

$$D^b(\mathbb{P}^{k-1}, \mathcal{B}_0) = D^b(Y).$$

Proof. We can relate the derived category of X to the category of singularities of the corresponding affine bundle over $\mathbb{P}^{k-1}$. Specifically, let X_+ be the singular quadric in

$$\mathbb{CP}^{k-1} \times \mathbb{C}^{2k}$$

given by $C(\mathbf{u}, \mathbf{x}) = 0$. This variety X_+ admits a $\mathbb{C}^*$ action which scales $\mathbf{x}$. We consider the graded derived category of its singularities

$$D^b_{sg}(\hat{X}, \mathbb{C}^*).$$

The relative version of the famous theorem of Orlov [Orl09] gives a semiorthogonal decomposition of $D^b(X)$

$$D^b(X) = \langle p^*D(\mathbb{P}^{k-1})(1), \ldots, p^*D(\mathbb{P}^{k-1})(2k-2), D^b_{sg}(X_+, \mathbb{C}^*)\rangle$$

because the Gorenstein parameter a is given by $(2k-2)$ by adjunction formula. This implies that

$$D^b_{sg}(X_+, \mathbb{C}^*) = D^b(\mathbb{P}^{k-1}, \mathcal{B}_0).$$

[8] We identify the exterior algebra with a Clifford algebra via the composition of the embedding into tensor algebra and projection.

For the second step, the work of Ballard, Favero and Katzarkov [BFK12] allows one to pass from X_+ to the hypersurface $X_- \subset \mathbb{C}^k \times \mathbb{CP}^{2k-1}$ given by

$$C(\mathbf{u}, \mathbf{x}) = 0,$$

together with the action of $\mathbb{C}^*$ on $\mathbf{u}$. Indeed, we may consider two open subsets $U_\pm \subset \mathbb{C}^k \times \mathbb{C}^{2k}$ defined by $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{u} \neq \mathbf{0}$ respectively. We define the group $\hat{G} = \mathbb{C}^* \times \mathbb{C}^*$ that scales both sets of coordinates. Then the categories $D^b_{sg}(X_+, \mathbb{C}^*)$ and $D^b_{sg}(X_-, \mathbb{C}^*)$ are the quotients of the derived categories of $\hat{G}$-equivariant sheaves on $U_\pm$ by the subcategory of free complexes. We have by [BFK12]

$$D^b_{sg}(X_+, \mathbb{C}^*) = D^b_{sg}(U_+, \hat{G}) = D^b_{sg}(U_-, \hat{G}) = D^b_{sg}(X_-, \mathbb{C}^*).$$

The middle equivalence is due to a certain "conservation of canonical class". Specifically, the subgroup G of $\hat{G}$ that preserves $C(\mathbf{u}, \mathbf{x})$ acts by

$$\lambda_t \mathbf{u} = t^{-2}\mathbf{u}, \ \ \lambda_t \mathbf{x} = t\mathbf{x}.$$

Therefore, the weight of it on the anticanonical bundle of $\mathbb{C}^k \times \mathbb{C}^{2k}$ restricted to the fixed point $(\mathbf{0}, \mathbf{0})$ is $\sum_{i=1}^k (-2) + \sum_{j=1}^{2k} 1 = 0$. This is the condition needed for the equivalence $D^b_{sg}(U_+, \hat{G}) = D^b_{sg}(U_-, \hat{G})$, see [BFK12].

Thus we get

$$D^b(\mathbb{P}^{k-1}, \mathcal{B}_0) = D^b_{sg}(X_-, \mathbb{C}^*).$$

Finally, we observe that $D^b_{sg}(X_-, \mathbb{C}^*)$ is equivalent to the derived category of the complete intersection $D^b(Y)$ by the work of Isik [Isik13] and Shipman [Shi12]. □

5. Clifford double mirror construction.

In this section we generalize the Kuznetsov's example by uncovering the underlying toric geometry. Specifically, we construct noncommutative double mirrors of Calabi-Yau complete intersections in Gorenstein toric varieties, given certain natural combinatorial data. These noncommutative mirrors consist of a pair $(\mathcal{S}, \mathcal{B}_0)$ where $\mathcal{S}$ is a smooth toric DM stack and $\mathcal{B}_0$ is a sheaf of algebras which serves as the structure sheaf of $(\mathcal{S}, \mathcal{B}_0)$. The coherent sheaves on $(\mathcal{S}, \mathcal{B}_0)$ are coherent sheaves on $\mathcal{S}$ which are also $\mathcal{B}_0$-modules.

We work in the notations of Sections 2 and 3. Namely, we have dual Gorenstein cones K and $K^\vee$ in lattices M and N, with degree elements deg and $\deg^\vee$. We denote by k the index $\langle \deg, \deg^\vee \rangle$. There is also given a generic coefficient function

$$c : K_{(1)} \to \mathbb{C}.$$

5.1. **Definition of $(\mathcal{S}, \mathcal{B}_0)$.** The key idea of this paper is that Kuznetsov's and related examples correspond to the decomposition of $\deg^\vee$

$$\deg^\vee = \frac{1}{2}(s_1 + \cdots + s_{2k}) \tag{5.1}$$

with $s_i \in K^\vee_{(1)}$. We assume the elements $s_1, \ldots, s_{2k}$ are $\mathbb{R}$-linearly independent. Moreover, we assume that there exists (and is chosen) a regular simplicial fan Σ with support $K^\vee$ and rays based on $K^\vee_{(1)}$ such that

(5.2) *All maximum dimensional cones of* Σ *contain* $s_1, \ldots, s_{2k}$.

We call this *the centrality assumption* on Σ. As usual, we will also denote by Σ the corresponding simplicial complex on the set $K^\vee_{(1)}$. Note that in the case of decompositions that correspond to complete intersections considered in Section 2 the analogous centrality condition (2.1) can always be assured. It may no longer be the case in this setting, even for $k = 1$.

To orient the reader, the idea of our construction is the following. The centrality assumption (5.2) allows us to view the resolution $\mathbb{P}_\Sigma$ of $\mathrm{Spec}(\mathbb{C}[K \cap M])$ as vector bundle of rank $2k$ over a toric base. Then the coefficient function c gives a quadric section of this vector bundle in the sense of Kuznetsov [Kuz08]. This defines a sheaf of even Clifford algebras over the base of the fibration.

Unfortunately, there are some inevitable technical difficulties that need to be overcome to make the above picture precise. First of all, we need to work with smooth toric DM stacks, rather than schemes. Second, we need to be careful in our choice of the lattice for the base of the fibration as described below.

Let U_Σ be the Cox open subset of $\mathbb{C}^{K^\vee_{(1)}}$ which consists of maps $\mathbf{z} : K^\vee_{(1)} \to \mathbb{C}$ such that the preimage of 0 is an element of the simplicial complex Σ.

We define an abelian group

$$\overline{N} := N/\mathbb{Z}s_1 + \cdots + \mathbb{Z}s_{2k} + \mathbb{Z}\deg^\vee.$$

Notice that we also quotient $\deg^\vee$ in the last component. Otherwise, $\overline{N}$ would always have an order two torsion element which is the image of $\deg^\vee$. The abelian group $\overline{N}$ may still have torsion elements as explained in Remark 5.10. Next, we consider the stacky fan $\overline{\Sigma}$ in $\overline{N}$ which corresponds to the simplicial complex in $K^\vee_{(1)} - \{s_1, \ldots, s_{2k}\}$ whose maximum sets are obtained from those of Σ by removing all of s_i. We immediately observe that the centrality condition (5.2) implies that the natural identification

$$\mathbb{C}^{K^\vee_{(1)}} = \mathbb{C}^{K^\vee_{(1)} - \{s_1, \ldots, s_{2k}\}} \times \mathbb{C}^{2k}$$

induces the natural identification

$$U_\Sigma = U_{\overline{\Sigma}} \times \mathbb{C}^{2k}.$$

We will now discuss the various groups associated to the construction. Recall that in Section 3 we considered two subgroups G and $\hat{G}$ of $(\mathbb{C}^*)^{K^\vee_{(1)}}$ defined by

$$G := \{\lambda : K^\vee_{(1)} \to \mathbb{C}^* \Big| \prod_{n \in K^\vee_{(1)}} \lambda(n)^{\langle m,n\rangle} = 1, \text{ for all } m \in M\}$$

$$\hat{G} := \{\lambda : K^\vee_{(1)} \to \mathbb{C}^* \Big| \prod_{n \in K^\vee_{(1)}} \lambda(n)^{\langle m,n\rangle} = 1, \text{ for all } m \in \mathrm{Ann}(\deg^\vee)\}.$$

The group G is the group that corresponds to the smooth toric DM stack $\mathbb{P}_\Sigma = [U_\Sigma/G_\Sigma]$, and the group $\hat{G}$ defines a $\mathbb{C}^*$ action on $\mathbb{P}_\Sigma$.

There is an additional group of interest H isomorphic to $\mathbb{C}^*$ given by

$$\lambda(s_i) = t, \lambda(v) = 1, \quad \text{for all } v \in K^\vee_{(1)} - \{s_1, \ldots, s_{2k}\}$$

with $t \in \mathbb{C}^*$. Notice that $H \subseteq \hat{G}$, because for any $m \in \mathrm{Ann}(\deg^\vee)$ there holds

$$\prod_{n \in K^\vee_{(1)}} \lambda(n)^{\langle m,n\rangle} = t^{\sum_i \langle m,s_i\rangle} = t^{\langle m,2\deg^\vee\rangle} = 1.$$

Analogous calculation shows that $H \cap G = \{\pm 1\}$.

Remark 5.1. *The group H acts by scaling the fibers of the $\mathbb{C}^{2k}$ fibration $U_\Sigma \to U_{\overline{\Sigma}}$. Note also that the inclusion $H \subseteq \hat{G}$ is split, since one can consider the evaluation at s_1 as the splitting map $\hat{G} \to H$. While such splitting is not completely natural, as it requires a choice of one of s_i, it will suffice for our purposes.*

Notice that there is a natural map $\hat{G}/H \to (\mathbb{C}^*)^{K^\vee_{(1)} - \{s_1,\ldots,s_{2k}\}}$ since the coordinates of H that correspond to $K^\vee_{(1)} - \{s_1, \ldots, s_{2k}\}$ are equal to 1. This gives an action of $\hat{G}/H$ on $U_{\overline{\Sigma}}$. We observe that this is precisely the action used in the definition of the toric DM stack $\mathbb{P}_{\overline{\Sigma}}$.

Lemma 5.2. *We denote by $\overline{G}$ the quotient group $\hat{G}/H$. Then the toric DM stack associated to $\overline{\Sigma}$ in $\overline{N}$ is given by $[U_{\overline{\Sigma}}/\overline{G}]$.*

Proof. According to [BCS05] we need to identify $\hat{G}/H$ with the character group of the derived Gale dual of the complex

$$0 \to \mathbb{Z}^{K^\vee_{(1)} - \{s_1,\ldots,s_{2k}\}} \to \overline{N} \to 0 \tag{5.3}$$

defined by looking at linear combinations of images in $\overline{N}$ of degree one elements of $K^\vee$.

To define the derived Gale dual one needs to resolve $\overline{N}$ by free groups. Even in the case when it is already free, it will be convenient to consider the short exact sequence of free abelian groups

$$0 \to L \to N/\mathbb{Z}\deg^\vee \to \overline{N} \to 0.$$

Here L is the subgroup of $N/\mathbb{Z}\deg^\vee$ generated by the images of $s_1,\ldots,s_{2k}$. It is naturally isomorphic to the quotient of $\mathbb{Z}^{2k}$ by $(1,\ldots,1)$. Note that $\mathbb{Z}\deg^\vee$ is saturated, since $\deg^\vee$ is the smallest degree lattice element in the interior of $K^\vee$.

Following the definitions of [BCS05] we now replace the complex (5.3) above by a quasiisomorphic complex of free groups

$$0\to\mathbb{Z}^{K^\vee_{(1)}-\{s_1,\ldots,s_{2k}\}}\oplus L\to N/\mathbb{Z}\deg^\vee\to 0.$$

The group that corresponds to the stacky fan $\overline{\Sigma}$ in $\overline{N}$ is given as the character group of the cokernel L_1 of the (injective) dual map

$$\mathrm{Ann}(\deg^\vee)\to\mathbb{Z}^{K^\vee_{(1)}-\{s_1,\ldots,s_{2k}\}}\oplus L^\vee.$$

We have the exact sequence (in the multiplicative notation)

$$1\to\mathrm{Hom}(L_1,\mathbb{C}^*)\to(\mathbb{C}^*)^{K^\vee_{(1)}-\{s_1,\ldots,s_{2k}\}}\times(\mathbb{C}^*)^{2k}/\{t,\ldots,t\}\to\mathrm{Hom}(\mathrm{Ann}(\deg^\vee),\mathbb{C}^*).$$

When compared with the definition of $\hat{G}$ via

$$1\to\hat{G}\to(\mathbb{C}^*)^{K^\vee_{(1)}-\{s_1,\ldots,s_{2k}\}}\times(\mathbb{C}^*)^{2k}\to\mathrm{Hom}(\mathrm{Ann}(\deg^\vee),\mathbb{C}^*)$$

we recover the needed natural isomorphism between $\mathrm{Hom}(L_1,\mathbb{C}^*)$ and $\overline{G}=\hat{G}/H$. It is also easy to see that the action on $U_{\overline{\Sigma}}$ is induced by the map $\overline{G}\to(\mathbb{C}^*)^{K^\vee_{(1)}-\{s_1,\ldots,s_{2k}\}}$. □

Remark 5.3. *We recall that a quadric fibration in the sense of [Kuz08, Section 3] is given by the following data:*

- *a smooth algebraic variety S;*
- *a vector bundle $E\to S$;*
- *a line bundle $\mathcal{L}$ on S;*
- *an embedding of vector bundles $\sigma:\mathcal{L}\to Sym^2(E^\vee)$.*

This data defines $\pi:\mathbb{P}_S(E)\to E$ the projectivization of $E\to S$. The data of σ gives a section of $H^0(\mathbb{P}_S(E),\mathcal{O}(2)\otimes\pi^*\mathcal{L}^\vee)$ where $\mathcal{O}(1)$ be the dual of the tautological line bundle on $\mathbb{P}_S(E)$. Kuznetsov denotes by $\mathcal{X}\subset\mathbb{P}_S(E)$ the zero locus of σ in $\mathbb{P}_S(E)$. Then the restriction of π to $\mathcal{X}$ denoted by $p:\mathcal{X}\to S$ is a flat fibration with (possibly singular) quadric fibers. The embedding assumption above is crucial. It is equivalent to the flatness of the fibration $p:\mathcal{X}\to S$.

The coefficient function c gives a global function on U_Σ given by

$$C(\mathbf{z})=\sum_{m\in K_{(1)}}c(m)\prod_{n\in K^\vee_{(1)}}z(n)^{\langle m,n\rangle}.$$

We observe that C naturally fits into the definition of quadric fibration as above which is $\overline{G}$-equivariant.

Proposition 5.4. *The zero set $\{C=0\}$ defines a $\overline{G}$-equivariant quadric fibration over $U_{\overline{\Sigma}}$.*

Proof. It follows immediately from the definition of C that it is invariant under G and is semi-invariant under $\hat{G}$. Specifically, the action of λ on C scales each summand by

$$\prod_{n\in K_{(1)}^{\vee}} \lambda(n)^{\langle m,n\rangle}.$$

Since different $m \in K_{(1)}$ differ by an element of $\mathrm{Ann}(\deg^{\vee})$, the above term is independent of m. Moreover, for $\lambda \in H$ the above term is $t^{\langle m,\sum_i s_i\rangle} = t^{\langle m,2\deg^{\vee}\rangle} = t^2$. In other words, $C(\mathbf{z})$ has total degree 2 in the variables $z(s_1),\ldots,z(s_{2k})$. □

Remark 5.5. *It is important to point out that flatness of the quadric fibration can not be taken for granted. We will see later in Section 9.6 that it is not always the case. Since the fibration is defined by a hypersurface $C = 0$, the geometric criterion for flatness is that all of the fibers are hypersurfaces, i.e. for all points $\overline{\mathbf{z}} \in U_{\overline{\Sigma}}$ the restriction of C to the fiber of $U_\Sigma \to U_{\overline{\Sigma}}$ is not identically zero. We will refer to this condition on our combinatorial data as the flatness assumption. If true, it can be typically established by Bertini theorem arguments, so we will tacitly assume it, unless stated otherwise.*

We are now ready to define the noncommutative variety $(\mathcal{S},\mathcal{B}_0)$ that corresponds to the decomposition 5.1, fan Σ and the choice of the coefficient function c.

Definition 5.6. *The noncommutative mirror $(\mathcal{S},\mathcal{B}_0)$ is the sheaf of even Clifford algebras $\mathcal{B}_0$ over the smooth DM stack $\mathcal{S} = [U_{\overline{\Sigma}}/\overline{G}]$ associated to the quadratic function $C(\mathbf{z})$ of the stacky bundle*

$$[U_\Sigma/\overline{G}] \to [U_{\overline{\Sigma}}/\overline{G}]$$

where we use the splitting of $\hat{G} \to \overline{G}$ to define the action of $\overline{G}$ on U_Σ.

Remark 5.7. *More explicitly, the category of coherent sheaves on $(\mathcal{S},\mathcal{B}_0)$ is defined as the category of $\overline{G}$-equivariant sheaves over the even part of the (locally constant) sheaf of Clifford algebras over $U_{\overline{\Sigma}}$ given by*

$$\Big(\mathcal{O}_{U_{\overline{\Sigma}}}\{y_1,\ldots,y_{2k}\}/\langle(\sum_{i=1}^{2k} z_i y_i)^2 + C(\mathbf{z}),\ \text{for all } z_1,\ldots,z_{2k}\rangle\Big)_{even} \tag{5.4}$$

where $y_1,\ldots,y_{2k}$ are free noncommuting variables and z_i is a shorthand notation for $z(s_i)$. The subscript even *refers to the parity of the number of y_i. Here the action of $\overline{G}$ on $y_1,\ldots,y_{2k}$ is defined as follows. For an element $\overline{\lambda} \in \overline{G}$ consider the lift λ to $\hat{G}$ from the splitting $\hat{G} = \overline{G} \times H$. Denote by $\varphi(\lambda)$ the image of λ in $\hat{G}/G = \mathbb{C}^*$. Then we define*

$$\overline{\lambda}(y_i y_j) = \lambda_i^{-1}\lambda_j^{-1}\varphi(\lambda) y_i y_j$$

where λ_i is the coordinate of λ that corresponds to s_i. This definition is ensures that $(\sum_i z_i y_i)^2 + C(\mathbf{z})$ is semi-invariant with respect to $\overline{G}$ with character φ.

Remark 5.8. *One could in principle define the action of $\overline{G}$ on the sheaf of Clifford algebras without a choice of the splitting $\hat{G} = \overline{G} \times H$. For any lift $\lambda \in \hat{G}$ of $\overline{\lambda} \in \overline{G}$ the formula*

$$\overline{\lambda}(y_i y_j) = \lambda_i^{-1} \lambda_j^{-1} \varphi(\lambda) y_i y_j$$

gives the same result. Indeed, for an element $t \in H = \mathbb{C}^*$ we have $\varphi(h) = t^2$, and $\lambda_i = \lambda_j = t$, so the right hand side is 1. However, we find it convenient to pick a splitting.

We will now describe some simple examples of our construction. In particular, we make a connection to the Kuznetsov's example that we studied in Section 4 to show how it fits into the above toric formalism. For the sake of simplicity, we will start with the classical example of $(2,2,2,2)$-complete intersections in $\mathbb{CP}^7$ which has been thoroughly studied by multiple authors (see [Add09, CDHPS10]). It corresponds to the $k = 4$ case of Section 4.

Afterwards, we will consider a free $\mathbb{Z}_2$ quotient of the above example which is a family of Calabi-Yau threefolds with Hodge numbers $(1, 33)$. Its Clifford double mirror could be obtained completely analogously but appears to be new.

5.2. **Example: $(2,2,2,2)$-complete intersections in $\mathbb{CP}^7$.** We start with the lattice $N_1 \cong \mathbb{Z}^7$ of the fan of $\mathbb{CP}^7$. We try to keep the construction as natural as possible, in particular, we try to keep it symmetric with respect to the permutations of coordinates of $\mathbb{CP}^7$. Thus we view this lattice as the quotient of the lattice $\bigoplus_{i=1}^8 \mathbb{Z}e_i$ by the sublattice $\mathbb{Z}(\sum_{i=1}^8 e_i)$. The fan of $\mathbb{CP}^7$ has maximum cones that are generated by subsets of 7 out of 8 elements of $\{e_i\}$. The rays of the fan are generated by e_i and correspond to coordinate hyperplanes $D_i \subset \mathbb{CP}^7$.

The dual lattice M_1 is naturally identified with the rank 7 sublattice of $\bigoplus \mathbb{Z}e_i^\vee$ with the property that the sum of the entries is 0. Here we use $\{e_i^\vee\}$ to denote the dual basis of $\{e_i\}$.

To consider the complete intersection of four general quadrics in $\mathbb{CP}^7$, we, as usual, subdivide the standard anitcanonical divisor of $\mathbb{CP}^7$ as

$$(D_1 + D_2) + (D_3 + D_4) + (D_5 + D_6) + (D_7 + D_8) = -K_{\mathbb{CP}^7}.$$

We introduce the extended lattices $M = M_1 \oplus \mathbb{Z}^4$ and $N = N_1 \oplus \mathbb{Z}^4$ and consider the reflexive Gorenstein cone $K^\vee$ in N generated by the lattice elements

$$\begin{aligned}
s_1 &= (e_1; 1,0,0,0), s_2 = (e_2; 1,0,0,0), t_1 = (\mathbf{0}; 1,0,0,0),\\
s_3 &= (e_3; 0,1,0,0), s_4 = (e_4; 0,1,0,0), t_2 = (\mathbf{0}; 0,1,0,0),\\
s_5 &= (e_5; 0,0,1,0), s_6 = (e_6; 0,0,1,0), t_3 = (\mathbf{0}; 0,0,1,0),\\
s_7 &= (e_7; 0,0,0,1), s_8 = (e_8; 0,0,0,1), t_4 = (\mathbf{0}; 0,0,0,1).
\end{aligned}$$

The above s_i and t_j form the set of degree one elements $K^\vee_{(1)}$.

The dual cone K in M is generated by 32 elements

$$(-e_1^\vee - e_2^\vee + 2e_i^\vee; 1,0,0,0), (-e_3^\vee - e_4^\vee + 2e_i^\vee; 0,1,0,0),$$
$$(-e_5^\vee - e_6^\vee + 2e_i^\vee; 0,0,1,0), (-e_7^\vee - e_8^\vee + 2e_i^\vee; 0,0,0,1)$$

for all $i = 1, \ldots, 8$. The degree 1 lattice elements that form $K_{(1)}$ are given by

$$(-e_1^\vee - e_2^\vee + e_i^\vee + e_j^\vee; 1,0,0,0), (-e_3^\vee - e_4^\vee + e_i^\vee + e_j^\vee; 0,1,0,0),$$
$$(-e_5^\vee - e_6^\vee + e_i^\vee + e_j^\vee; 0,0,1,0), (-e_7^\vee - e_8^\vee + e_i^\vee + e_j^\vee; 0,0,0,1)$$

for all 36 unordered pairs (i, j). The corresponding coefficient function

$$c : K_{(1)} \to \mathbb{C}$$

encodes the coefficients of 4 quadrics at the standard monomials $x_i x_j$.

In what follows, we will adapt a somewhat different way of looking at K and $K^\vee$. We can think of the lattice N as the rank 11 quotient of $\mathbb{Z}^{12}$ by the sublattice $\mathbb{Z}(\sum_{i=1}^8 s_i - 2\sum_{j=1}^4 t_j)$. The cone $K^\vee$ is then just the image of the positive orthant $\mathbb{Z}^{12}_{\geq 0}$. The dual lattice M can be viewed as a corank one sublattice of $\bigoplus_{i=1}^8 \mathbb{Z}s_i^\vee \oplus \bigoplus_{j=1}^4 \mathbb{Z}t_j^\vee$, namely

$$M = \{\sum_{i=1}^8 a_i s_i^\vee + \sum_{j=1}^4 b_j t_j^\vee \mid \sum_i a_i = 2\sum_j b_j\}.$$

The cone K is then the intersection of M with the positive orthant.

The degree element $\deg^\vee$ in $K^\vee$ is given by

$$\deg^\vee = \frac{1}{2}\sum_{i=1}^8 s_i = \sum_{j=1}^4 t_j. \tag{5.5}$$

The degree element $\deg \in K$ is $\sum_{i=1}^8 s_i^\vee + \sum_{j=1}^4 t_j^\vee$. The elements of $K^\vee_{(1)}$ are the aforementioned s_i and t_j. The elements of $K_{(1)}$ are of the form $s_i^\vee + s_j^\vee + t_k^\vee$ where i may or may not equal j. When $i = j$ we get the above 32 generators of the rays of K.

The equation (5.5) is a prototypical situation where for two different large Kähler limits of the $N = (2,2)$ theories one gets the description of the triangulated category in two ways. On the one hand, the decomposition

$$\deg^\vee = \sum_{j=1}^4 t_j$$

allows one to see this category as the derived category of the complete intersection of four quadrics in $\mathbb{CP}^7$. On the other hand, the decomposition

$$\deg^\vee = \frac{1}{2}\sum_{i=1}^8 s_i$$

leads to its description as the derived category of coherent sheaves for the Clifford algebra over $\mathbb{CP}^3$.

For the first construction, we consider the regular simplicial fan on $K^\vee$ with eight maximum cones obtained by removing one of the eight elements s_i. It is easy to see that the resulting complete intersection is that of four quadrics in $\mathbb{CP}^7$. The coefficient of the x_ix_j monomial of the k-th quadric is the coefficient $c(s_i^\vee + s_j^\vee + t_k^\vee)$.

For the second construction, we use the fan Σ in $K^\vee$ whose maximal cones are given by eight elements s_i and three out of four t_i. We observe that the group $\hat{G}$ described in (3.1) is given by $\mathbb{C}^* \times \mathbb{C}^*$ with the action

$$(\lambda_1, \lambda_2)(t_j) = \lambda_1;\ (\lambda_1, \lambda_2)(s_i) = \lambda_2.$$

The map to $\mathbb{C}^*$ whose kernel is G is given by

$$(\lambda_1, \lambda_2) \mapsto \lambda_1 \lambda_2^2.$$

The group H is given by $\{(1, \lambda_2), \lambda_2 \in \mathbb{C}^*\} \subset \hat{G}$, and the group $\overline{G} = \hat{G}/H$ is naturally isomorphic to $\mathbb{C}^*$ with the diagonal map to $(\mathbb{C}^*)^4$ in view of $(\lambda_1, -)(t_j) = \lambda_1$. The lattice $\overline{N}$ is the quotient of N by the lattice generated by s_i and $\deg^\vee$. Therefore, it can be naturally identified with the quotient of $\oplus_{j=1}^4 \mathbb{Z} t_j$ by the span of $\sum_j t_j$. The fan $\overline{\Sigma}$ is the standard fan of $\mathbb{CP}^3$.

Then it is easy to see that the sheaf of Clifford algebras constructed in Definition 5.6 is the same sheaf on $\mathbb{CP}^3$ as the one constructed in Section 4.

Remark 5.9. *The Clifford noncommutative variety $(\mathcal{B}, \mathcal{S}_0)$ should be viewed as a crepant resolution of the singular double cover of $\mathbb{CP}^3$ ramified over the determinantal octic which is the determinant of the symmetric 8×8 matrix of degree 1 forms encoded by c. See [Kuz15] for details.*

5.3. **Free involution quotients of $(2,2,2,2)$-complete intersections.** In this example, we consider complete intersections of four quadrics in $\mathbb{CP}^7$ which admit a free $\mathbb{Z}_2$ action. This is a particular case of more general construction of such complete intersections with more sophisticated free group actions, see [Bea98, Hua11]. While the more interesting non-abelian actions can not be easily realized in the toric setting of this paper, [9] the simple case of an involution fits nicely into our construction.

An involution τ on a smooth complete intersection X of four quadrics in $\mathbb{CP}^7$ always comes from a linear action on $\mathbb{C}^8$. In view of holomorphic Lefschetz formula (or by direct inspection of fixed point sets), if this involution is fixed point free on X, then it must act with trace 0 on $H^0(X, \mathcal{O}(1))$. Without loss of generality we may assume that this involution τ acts by

$$\tau(x_1 : x_2 : \ldots : x_8) = (-x_1 : -x_2 : -x_3 : -x_4 : x_5 : x_6 : x_7 : x_8).$$

[9] One can try to extend to such actions by looking at automorphisms of the fan, similar to [St12].

Then the action of τ on the space of degree two polynomials in x_i has 1 and (-1)-eigenspaces of dimension 20 and 16 respectively. Holomorphic Lefschetz formula then forces us to consider 4 τ-invariant quadrics in $\mathbb{CP}^7$, so that the action of τ on $H^0(X, \mathcal{O}(2))$ has eigenspaces of dimension 16 each, see [Hua11].

In terms of toric geometry, the situation is extremely similar to the setting of the previous example, except that the lattice $\mathbb{Z}^8/\mathbb{Z}\sum_{i=1}^8 e_i$ is now replaced by a suplattice of index two to include $\frac{1}{2}(e_1+e_2+e_3+e_4)$. On the dual side, we must consider the corresponding sublattice of index two. When extended to N, we now have

$$N = \Big((\bigoplus_{i=1}^8 \mathbb{Z}s_i + \frac{1}{2}\mathbb{Z}(s_1+s_2+s_3+s_4)) \oplus \bigoplus_{j=1}^4 \mathbb{Z}t_j\Big)/\mathbb{Z}(\sum_{i=1}^8 s_i - 2\sum_{j=1}^4 t_j)$$

with the cone $K^\vee$ still given as the image of the nonnegative orthant. The dual lattice M is now given by

$$M = \{\sum_{i=1}^8 a_i s_i^\vee + \sum_{j=1}^4 b_j t_j^\vee \mid \sum_{i=1}^8 a_i = 2\sum_{j=1}^4 b_j \text{ and } \sum_{i=1}^4 a_i \text{ is even}\}.$$

The cone K is the intersection of M with the positive orthant.

The degree elements $\deg^\vee$ and $\deg$ are given by the same formulas as before. The set $K^\vee_{(1)}$ is unchanged, however, the set $K_{(1)}$ is now smaller. It consists of elements of the form

$$s_i^\vee + s_j^\vee + t_k^\vee$$

where i and j are either both in the range of $1,\ldots,4$ or are both in the range $5,\ldots,8$. Observe that this is consistent with the choice of quadrics from the invariant eigenspace of τ.

Again, we now look at the groups $\hat{G}$, G and H. The group $\hat{G}$ is now equal to $\mathbb{C}^*\times\mathbb{C}^*\times\{\pm1\}$ which is written in coordinates as

$$(\lambda_1,\lambda_2,\pm1)(t_j) = \lambda_1,\ (\lambda_1,\lambda_2,\pm1)(s_i) = \lambda_2, \text{ if } 1\le i\le 4,$$
$$(\lambda_1,\lambda_2,\pm1)(s_i) = \pm\lambda_2, \text{ if } 5\le i\le 8.$$

The map $\hat{G}\to\mathbb{C}^*$ still sends

$$(\lambda_1,\lambda_2,\pm1)\mapsto \lambda_1\lambda_2^2.$$

The subgroup H of $\hat{G}$ that scales the variables that correspond to s_i is $(1,\mathbb{C}^*,1)$. The quotient group $\overline{G} = \hat{G}/H$ is naturally identified with $\mathbb{C}^*\times\{\pm1\}$.

Remark 5.10. *In this example, the lattice $\mathbb{Z}s_1+\mathbb{Z}s_2+\ldots+\mathbb{Z}s_8$ is not saturated in M, even after adding $\deg^\vee = \frac{1}{2}\sum_{i=1}^8 s_i$. Indeed, there is also an element $\frac{1}{2}(s_1+s_2+s_3+s_4)$ in the real span of s_i, which is not an integer linear combination of s_i and $\deg^\vee$.*

Let us now discuss the noncommutative variety $(\mathcal{S}, \mathcal{B}_0)$ in view of the above remark. The images of $t_1, \ldots, t_4$ still add up to 0, so the base $\mathcal{S}$ is a $\mathbb{Z}_2$ gerbe over $\mathbb{CP}^3$ given by the quotient $[\mathbb{CP}^3/\mathbb{Z}_2]$ by the trivial $\mathbb{Z}_2$ action. Alternatively, it is a quotient of $\mathbb{C}^4 - \{\mathbf{0}\}$ by $\mathbb{C}^* \times \{\pm 1\}$ where the first factor acts in the usual way and the second factor acts trivially. The vector bundle used to construct the Clifford algebra is now a bit different. Namely, it is a direct sum of two rank four bundles on which the extra involution acts as (-1) and 1 respectively.

It is interesting to investigate this case further along the lines of Remark 5.9. On the coarse moduli space $\mathbb{CP}^3$ of $\mathcal{S}$, the symmetric determinantal octic is now a union of two symmetric determinantal quartics Q_+ and Q_-, since the corresponding matrix consists of two 4×4 blocks.

The fact that we consider a gerbe $[\mathbb{CP}^3/\mathbb{Z}_2]$ can be encoded by looking at the semidirect product of the even Clifford algebra of Kuznetsov's original construction with the group algebra $\mathbb{C}[h]/\langle h^2 - 1\rangle$ of $\mathbb{Z}_2$. Let us investigate the center of this algebra in more detail. Let us first localize over the generic point of $\mathbb{CP}^3$, i.e. we will work over the field F of rational functions on $\mathbb{CP}^3$. If we diagonalize the quadratic forms on V_+ and V_-, then the semidirect product of the even Clifford algebra with the group ring of $\mathbb{Z}_2$ gives the even part (in y) of the quotient of the free algebra

$$F\{y_1^+, \ldots, y_4^+, y_1^-, \ldots, y_4^-, h\}$$

by the two-sided ideal generated by the relations that h commutes with y_i^+ and anti-commutes with y_i^-, the relation $h^2 - 1$, and the usual Clifford relations

$$(y_i^+)^2 + c_i^+, (y_i^-)^2 + c_i^-, y_i^+ y_j^- + y_j^- y_i^+$$

for all i and j and

$$y_i^+ y_j^+ + y_j^+ y_i^+, y_i^- y_j^- + y_j^- y_i^-$$

for $i \neq j$. Note that c_i may not assumed to be 1, since F is not algebraically closed.

There is a grading by $\mathbb{Z}_2^9$ that looks at parity of monomials in y_i^+, y_i^- and h. In fact, the algebra is easily seen to be of dimension 2^8 over F with the basis given by

$$h^l \prod_{i \in I} y_i^+ \prod_{j \in J} y_j^- \tag{5.6}$$

for sets I, J with $|I| + |J|$ even and $l \in \{0, 1\}$.

The same grading descends to the center, so to calculate the center we just need to determine which monomials in (5.6) are central. If one has I which is neither empty nor the whole set $\{1, \ldots, 4\}$, then by taking a commutator with $y_i^+ y_j^+$ with i in I and j not in I we get a factor of (-1). The same happens for J. So there are 8 monomials to consider as possible elements of the center.

$$1, h, y_1^+ \cdots y_4^+, hy_1^+ \cdots y_4^+, y_1^- \cdots y_4^-, hy_1^- \cdots y_4^-,$$
$$y_1^+ \cdots y_4^+ y_1^- \cdots y_4^-, hy_1^+ \cdots y_4^+ y_1^- \cdots y_4^-.$$

Of these, commutator with $y_1^+ y_1^-$ excludes all but

$$1, hy_1^+ \cdots y_4^+, hy_1^- \cdots y_4^-, y_1^+ \cdots y_4^+ y_1^- \cdots y_4^-$$

which are indeed central. It remains to observe that the relations of the algebra imply that

$$(hy_1^+ \cdots y_4^+)^2 = c_{1,+} \cdots c_{4,+} = \det(C_+)$$

and

$$(hy_1^- \cdots y_4^-)^2 = c_{1,-} \cdots c_{4,+} = \det(C_-)$$

so at least generically the center of the algebra is the $(\mathbb{Z}_2)^2$ Galois cover of $\mathbb{CP}^3$ obtained by attaching the square roots of the two quartics Q_+ and Q_-.

This motivates the following more precise conjecture.

Conjecture 5.11. *The Clifford double mirror of Definition 5.6 is a crepant categorical resolution of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ Galois cover of $\mathbb{CP}^3$ ramified at the two quartics $\mathbb{Q}_+$ and $\mathbb{Q}_-$.*

In support of this conjecture, let us calculate the stringy Euler numbers of this Galois cover. Each quartic $Q_\pm$ has 10 nodes and Euler characteristics 14. They intersect in a smooth curve Y with $2g - 2 = 64$, so $\chi(Y) = -64$. We have the following Euler characteristics

$$\chi(Y) = -64, \chi(Q_\pm - Y) = 78, \chi(\mathbb{CP}^3 - (Q_+ \cup Q_-)) = -88.$$

On the Galois cover, the first kind of points gives preimage of 1 point, the second gives 2 points, and the last gives 4 points. Therefore, we get the usual Euler characteristics of the Galois cover

$$(-64) + 2(78 + 78) + 4(-88) = -104.$$

The singularities on the Galois cover are the preimages of the 10 nodes of Q_+ and 10 nodes of Q_-, which gives 40 three-dimensional ODPs. Each of them contributes an extra 1 to stringy Euler characteristics since their crepant resolution locally involves replacing a point with a 2-sphere. So we get the stringy Euler characteristics of $-104 + 40 = -64$. As expected, this matches the Euler characteristics of the quotient of the complete intersection of type $(2, 2, 2, 2)$ in $\mathbb{CP}^7$ by a free involution.

Remark 5.12. *We find it fascinating that the free quotient on the complete intersection side leads to a double cover on the Clifford double mirror side by means of enlarging the center of the corresponding sheaf of algebras. This phenomenon depends crucially on the parity of the number of quadrics. We will see later in Section 9.2 that in the case of Enriques surfaces realized as* *quotients of* $(2,2,2)$ *complete intersections in* $\mathbb{CP}^5$ *by a free involution, the center will be generically just the field of functions on* $\mathbb{CP}^2$*.*

6. Derived categories of Clifford double mirrors.

The goal of this section is to provide a proof that the derived category of the sheaf of Clifford algebras over a toric DM stack constructed in the previous section is equivalent to the $\hat{G}$-equivariant derived category of singularities considered in Section 3. This immediately implies that in the case when different large Kähler limits give complete intersection and a sheaf of Clifford algebras, there is an equivalence of the corresponding derived categories. This is our main result Theorem 6.2.

Recall that we have a reflexive Gorenstein cone $K \subset M_{\mathbb{R}}$ with the dual cone $K^\vee \subset N_{\mathbb{R}}$. Suppose there exists a decomposition

$$\deg^\vee = \frac{1}{2}(s_1 + \cdots + s_{2k}), \quad s_j \in K^\vee_{(1)},$$

and a regular triangulation Σ of $K^\vee_{(1)}$ such that every maximum dimensional cone of Σ contains all $\{s_j\}$. For a fixed generic coefficient function

$$c : K_{(1)} \to \mathbb{C}$$

we define a noncommutative variety/stack $(\mathcal{S}, \mathcal{B}_0)$, as in Section 5. Recall that $\mathcal{S} = [U_{\overline{\Sigma}}/\overline{G}]$ is the toric DM stack associated to the action of the group

$$\overline{G} = \hat{G}/H$$

on an open set U_Σ of $\mathbb{C}^{K^\vee_{(1)} - \{s_1,\ldots,s_{2k}\}}$.

Let $D(\mathcal{S}, \mathcal{B}_0)$ be the bounded derived category of coherent sheaves on $\mathcal{S}$ which are also $\mathcal{B}_0$-modules. Let us also consider the derived category

$$D_B(K, c; \Sigma)$$

as the graded category of singularities of the hypersurface in $[U_\Sigma/G]$ defined by c. Specifically, it is obtained from the bounded derived category of $\hat{G}$-equivariant coherent sheaves on $\{C(\mathbf{z}) = 0\} \subset U_\Sigma$ by factoring out the complexes of locally free $\hat{G}$-equivariant sheaves.

The main result of this section is the following.

Theorem 6.1. *Under the flatness assumption of Remark 5.5 there exists a derived equivalence*

$$D(\mathcal{S}, \mathcal{B}_0) \cong D_B(K, c; \Sigma).$$

Proof. The argument uses the intermediate category $D^b(\mathcal{X})$, which contains both of the triangulated categories in question as left and right orthogonal complements to an admissible subcategory. We will use the results of [Kuz08, BDFIK14].

In order to construct $\mathcal{X}$ and $D^b(\mathcal{X})$, recall that we have

$$U_\Sigma = U_{\overline{\Sigma}} \times \mathbb{C}^{2k}.$$

There is a group $\hat{G}$ acting on U_Σ. There is a subgroup H of $\hat{G}$ which acts by scaling the coordinates of $\mathbb{C}^{2k}$. Moreover, the inclusion $H \subseteq \hat{G}$ is split by Remark 5.1. Thus we will now consider

$$\hat{G} = H \times \overline{G}$$

where $\overline{G} = \hat{G}/H$ and have $\overline{G}$ act U_Σ as well.

Consider $\mathbb{CP}^{2k-1}$ bundle $U_{\overline{\Sigma}} \times \mathbb{CP}^{2k-1}$ over $U_{\overline{\Sigma}}$ given by

$$U_{\overline{\Sigma}} \times (\mathbb{C}^{2k} - \{\mathbf{0}\})/H.$$

The coefficient function $c : K_{(1)} \to \mathbb{C}$ gives rise to a quadric fibration over $U_{\overline{\Sigma}}$ in the sense of [Kuz08, Section 3]. Specifically, the polynomial

$$C(\mathbf{z}) = \sum_{m \in K_{(1)}} c(m) \prod_{n \in K_{(1)}^\vee} z(n)^{\langle m,n\rangle}$$

has total degree 2 in the variables $z(s_1), \ldots, z(s_{2k})$, because

$$\sum_{n=s_i, i=1,\ldots,2k} \langle m, n\rangle = \langle m, \sum_{i=1}^{2k} s_i\rangle = \langle m, 2\deg^\vee\rangle = 2.$$

While our situation is slightly more general, since we are interested in working equivariantly with respect to the group $\overline{G}$ (or alternatively work over a DM stack base rather than a scheme base), we can still use the framework of Kuznetsov. However, we do need to assume that the fibration is flat, see Remark 5.5 and Section 9.6.

We denote by $\mathcal{X}$ the DM quotient substack $[\{C = 0\}/H \times \overline{G}]$ of the DM stack $[U_{\overline{\Sigma}} \times (\mathbb{C}^{2k} - \{\mathbf{0}\})/H \times \overline{G}]$. Note that the action of $\overline{G}$ descends to the action on $U_{\overline{\Sigma}} \times \mathbb{CP}^{2k-1}$ and is compatible with the action on the base $U_{\overline{\Sigma}}$. Thus we have a quadric fibration

$$\pi : \mathcal{X} \to \mathcal{S}.$$

Our definition of the sheaf of even Clifford algebras $(\mathcal{S}, \mathcal{B}_0)$ matches that of [Kuz08, Section 3]. We now wish to apply a slight generalization of [Kuz08, Theorem 4.2] to the equivariant setting which states that $D^b(\mathcal{X})$ admits a semiorthogonal decomposition

(6.1)

$$D^b(\mathcal{X}) = \left\langle D^b(\mathcal{S}, \mathcal{B}_0), \pi^* D^b(\mathcal{S}) \otimes O_{\mathcal{X}/\mathcal{S}}(1), \ldots, \pi^* D^b(\mathcal{S}) \otimes O_{\mathcal{X}/\mathcal{S}}(2k-2)\right\rangle$$

in the sense that a left orthogonal to the category generated by $\pi^* D^b(\mathcal{S}) \otimes O_{\mathcal{X}/\mathcal{S}}(i), i = 1, \ldots, 2k-2$ is equivalent to $D^b(\mathcal{S}, \mathcal{B}_0)$. Kuznetsov's arguments apply in our slightly more general situation since his construction is functorial.

To compare $D^b(\mathcal{X})$ with the category $D_B(K, c; \Sigma)$ we need to consider a relative and equivariant version of the main theorem of Orlov [Orl09, Theorem 16]. In this particular setting it is provided by the idea of [BDFIK14]. We thank Matt Ballard for providing this reference.

The idea is to consider a fan Σ_{new} obtained from Σ by blowup at $\deg^\vee$. One can view passing from the original fan to the new one as an inverse of a toric flip which allows one to get a comparison of two categories.

More specifically, we subdivide every maximum cone of Σ into $2k$ cones by replacing one of s_i by $\deg^\vee$. The resulting open subset[10]

$$U_{\Sigma_{new}} \subset \mathbb{C}^{K^\vee_{(1)}\sqcup \deg^\vee} = \mathbb{C}^{K^\vee_{(1)}} \times \mathbb{C}$$

lies in $U_\Sigma \times \mathbb{C}$. In fact, it is given by

$$(U_\Sigma - U_{\overline{\Sigma}}) \times \mathbb{C}$$

where we think of $U_{\overline{\Sigma}}$ embedded via the zero section. We also consider the polynomial

$$C_{new}(\mathbf{z}) = uC(\mathbf{z}).$$

where u is the coordinate that corresponds to the last factor. The natural analog of the group $\hat{G}$ is simply $\hat{G} \times \mathbb{C}^*$ with the second factor acting on u.

We consider the derived category of singularities of $C_{new} = 0$ on two open subspaces $U_{\Sigma_{new}}$ and $U_\Sigma \times \mathbb{C}^*$, with the group action of $G_{new} = \hat{G} \times \mathbb{C}^*$, as in [BFK12].

On $U_{\Sigma_{new}} = (U_\Sigma - U_{\overline{\Sigma}}) \times \mathbb{C}$, the G_{new}-equivariant category of singularities is equal to the $\hat{G}$-equivariant category for the singular locus of $C_{new} = 0$ which is precisely the $\hat{G}$ equivariant category of $\{C = 0\} \cap (U_\Sigma - U_{\overline{\Sigma}})$. Since

$$\mathcal{X} = [\{C = 0\} \cap (U_\Sigma - U_{\overline{\Sigma}})/\hat{G}]$$

we get

$$D^b(\mathcal{X}) = D^b_{sg}(U_{\Sigma_{new}}, \hat{G} \times \mathbb{C}^*; C_{new}). \tag{6.2}$$

On the other side we have

$$D^b_{sg}(U_\Sigma \times \mathbb{C}^*, \hat{G} \times \mathbb{C}^*; C_{new}) = D^b_{sg}(U_\Sigma, \hat{G}; C). \tag{6.3}$$

Indeed, we have $[U_\Sigma \times \mathbb{C}^*/\hat{G} \times \mathbb{C}^*] = [U_\Sigma/\hat{G}]$.

Now the results of [BFK12] allow us to compare the two categories. We recall that one needs to consider the parameter μ which is the weight of the normal bundle to the fixed point locus of a one-parameter subgroup of $\hat{G} \times \mathbb{C}^*$. Specifically, in the notations of Theorem 3.5.2 of [BFK12], we have

$$X_+ = U_\Sigma \times \mathbb{C}^*,\ X_- = U_{\Sigma_{new}},\ X = U_\Sigma \times \mathbb{C}.$$

The group G_{BFK}[11] is given by

$$G_{BFK} = \hat{G} \times \mathbb{C}^*.$$

The G_{BFK}-line bundle $\mathcal{L}$ is trivial with the action given by the character of $\hat{G} \times \mathbb{C}^*$ such that C_{new} gives an invariant section.

[10]If $k = 1$, then $\deg^\vee$ lies in $K^\vee_{(1)}$ and we consider an additional copy of it.

[11]The notation G has a different meaning in our paper.

The parameter μ of [BFK12] is calculated by considering a subgroup $\lambda_{BFK} : \mathbb{C}^* \to \hat{G}$ which corresponds to the linear relation

$$0 = 2\deg^\vee - \sum_{i=1}^{2k} s_i.$$

Explicitly, this subgroup lies in $H \times \mathbb{C}^*$ and is given by (t, t^{-2}) in it. Thus the parameter μ is given by

$$\mu = 2 - \sum_{i=1}^{2k} 1 = 2k - 2.$$

Then [BFK12, Theorem 3.5.2] combined with (6.2) and (6.3) leads to the semiorthogonal decomposition for any d

$$D^b(\mathcal{X}) = \big\langle (\Upsilon_-)_{d+1}, \dots, (\Upsilon_-)_{d+2k-2}, D^b_{sg}([U_\Sigma/\hat{G}], C)\big\rangle \tag{6.4}$$

where the fully faithful functors

$$(\Upsilon_-)_j : D^b([U_{\overline{\Sigma}}/\hat{G}])_j \to D^b(\mathcal{X})$$

are defined in [BFK12]. It remains to argue that the images of these these functors are simply the pullbacks twisted by integers. This follows from the definition of the functors Υ in [BFK12, Section 3] and is left to the reader.

Now the two decompositions of $D^b(\mathcal{X})$ given by equations (6.1) and (6.4) show the desired equivalence of categories. □

As the consequence of Theorem 6.1 we get the derived equivalence of double mirrors.

Theorem 6.2. *Suppose that a complete intersection $\mathcal{X}$ and a Clifford non-commutative variety $\mathcal{Y}$ are given by different decompositions of the degree element* $\deg^\vee$ *of a reflexive Gorenstein cone $K^\vee$ and the appropriate regular simplicial fans in $K^\vee$. Then the bounded derived categories of $\mathcal{X}$ and $\mathcal{Y}$ are equivalent, provided the centrality and the flatness assumptions on $\mathcal{Y}$ hold.*

Proof. By Theorem 6.1 and Theorem 3.4 the derived categories in question are equivalent to two derived categories of singularities defined by different fans in $K^\vee$. These are then equivalent by Theorem 3.2. □

7. Generalization to Clifford algebras over complete intersections.

The goal of this section is to indicate a generalization of the Clifford algebra construction which would encompass both complete intersections and Clifford algebras over toric bases.

As always, we consider a pair of reflexive Gorenstein cones K and $K^\vee$ in lattices M and N respectively. Let us denote the degree elements by

$\deg \in K$ and $\deg^\vee \in K^\vee$ and introduce the index $k = \langle \deg, \deg^\vee \rangle$. In addition, we consider generic coefficient function

$$c : K_{(1)} \to \mathbb{C}.$$

As explained in Section 2, Calabi-Yau complete intersections in toric varieties appear as a consequence of a decomposition of $\deg^\vee$ into a linear combination of elements of $K^\vee_{(1)}$ with coefficients 1. As we saw in Section 5 the Clifford algebras over toric bases arise in the context of decomposition of $\deg^\vee$ into a linear combination with coefficients $\frac{1}{2}$. We will now consider a more general case where both types of coefficients may appear.

Suppose that we have

$$\deg^\vee = \frac{1}{2}(s_1 + \cdots + s_{2r}) + t_1 + \cdots + t_{k-r}$$

for some $0 \le r \le k$. The $(k + r)$ elements s_i and t_j are supposed to be linearly independent. In addition, there should exist (and be chosen) a regular simplicial fan Σ with support $K^\vee$ such that the following centrality condition holds.

(7.1)
All maximum dimensional cones of Σ *contain* $\{s_i, t_j\}$ *as ray generators.*

Remark 7.1. *The motivation for the above condition is our philosophy of large Kähler limits of the families of* $N = (2, 2)$ *SCFTs. There is a large class of such limits given by different regular simplicial cones* Σ*. We are interested in the situation where maximum cones of* Σ *contain* $\deg^\vee$*. While other limits may be of interest as well, they are likely to lead to more complicated geometric descriptions of the triangulated category of boundary conditions. Given such fan* Σ *it is natural to try to describe the minimum cone that* $\deg^\vee$ *lies in, which is the intersection of all the maximum cones of* Σ*. The element* $\deg^\vee$ *is a positive rational combination of the generators of this cone, and in this paper we consider the case when these coefficients are* $\frac{1}{2}$ *or* 1.

Remark 7.2. *The case* $r = k$ *was considered in Section 5 and led to (sheaves of) Clifford algebras over toric bases. The case* $r = 0$ *is the usual complete intersection case reviewed in Section 2.*

As in Section 5 we consider the open subset U_Σ of $\mathbb{C}^{K^\vee_{(1)}}$ of functions

$$\mathbf{z} : K^\vee_{(1)} \to \mathbb{C}$$

such that the preimage of 0 is a subset of Σ. Similarly, we consider the subset

$$U_{\overline{\Sigma}} \subset \mathbb{C}^{K^\vee_{(1)} - \{s_1, \ldots, s_{2r}, t_1, \ldots, t_{k-r}\}}$$

that corresponds to the stacky fan $\overline{\Sigma}$ for the group

$$\overline{N} = N/\mathbb{Z}s_1 + \cdots + \mathbb{Z}s_{2r} + \mathbb{Z}\deg^\vee + \mathbb{Z}t_1 + \cdots + \mathbb{Z}t_{k-r}.$$

We have
$$U_\Sigma = U_{\overline{\Sigma}} \times \mathbb{C}^{2r} \times \mathbb{C}^{k-r}$$
where the last coordinates correspond to values of $\mathbf{z}$ at s_i and t_i.

There is a group $\hat{G}$ defined as usual by
$$\hat{G} := \{\lambda : K_{(1)}^\vee \to \mathbb{C}^* \Big| \prod_{n\in K_{(1)}^\vee} \lambda(n)^{\langle m,n\rangle} = 1, \text{ for all } m \in \mathrm{Ann}(\deg^\vee)\}$$
as in Section 5. The analog of the subgroup H of $\hat{G}$ which will be denoted by the same name is given by $\mathbb{C}^*$ with
$$\lambda(s_i) = t, \lambda(t_i) = t^2, \lambda(v) = 1, \text{ for all } v \in K_{(1)}^\vee - \{s_1, \ldots, s_{2r}, t_1, \ldots, t_{k-r}\}.$$
As in Section 5, we see that the toric DM stack that corresponds to $(\overline{N}, \overline{\Sigma})$ can be realized as the quotient of $U_{\overline{\Sigma}}$ by $\overline{G} = \hat{G}/H$.

Note that the homogeneous polynomial
$$C(\mathbf{z}) = \sum_{m\in K_{(1)}} c(m) \prod_{n\in K_{(1)}^\vee} z(n)^{\langle m,n\rangle}$$
$$C(\mathbf{z}) = \sum_{m\in K_{(1)}} c(m) \prod_{n\in K_{(1)}^\vee} z(n)^{\langle m,n\rangle}$$
has total degree 2 with respect to H. It has terms linear in $\mathbf{z}(t_i)$ and terms that have no $\mathbf{z}(t_i)$ but are quadratic in $\mathbf{z}(s_i)$. We define the quadratic term by
$$C_2(\mathbf{z}) = \sum_{m\in K_{(1)}\cap \mathrm{Ann}(t_1,\ldots,t_{k-r})} \prod_{n\in K_{(1)}^\vee} z(n)^{\langle m,n\rangle}.$$
We use the linear terms to define the complete intersection $Y \subset U_{\overline{\Sigma}}$ given by
$$Y = \bigcap_{i=1}^{k-r} \Big\{ \sum_{m\in K_{(1)}, \langle m,t_i\rangle = 1} c(m) \prod_{n\in K_{(1)}^\vee - \{s_1,\ldots,s_{2r},t_1,\ldots,t_{k-r}\}} z(n)^{\langle m,n\rangle} = 0 \Big\}$$
where this intersection may assumed to be transversal if the coefficient function c is general. We then define the sheaf of Clifford algebras $\mathcal{B}_0$ on $\mathcal{S} = [Y/\overline{G}]$ as the pullback of the sheaf of Clifford algebras on $[U_{\overline{\Sigma}}/\overline{G}]$ under the natural inclusion. The aforementioned sheaf of Clifford algebras is defined by using the quadratic part $C_2(\mathbf{z})$ of $C(\mathbf{z})$. We will formulate this definition along the lines of Remark 5.7.

Definition 7.3. *The category of coherent sheaves on $(\mathcal{S}, \mathcal{B}_0)$ is defined as the category of $\overline{G}$-equivariant sheaves over the even part of the (locally constant) sheaf of Clifford algebras over the (reduced) scheme $Y \subseteq U_{\overline{\Sigma}}$ given by*
$$\Big(\mathcal{O}_Y\{y_1, \ldots, y_{2r}\}/\langle (\sum_{i=1}^{2r} z_i y_i)^2 + C_2(\mathbf{z}), \text{ for all } z_1, \ldots, z_{2r}\rangle\Big)_{even}$$

where $y_1, \ldots, y_{2r}$ *are free noncommuting variables.*

Remark 7.4. *The subscript* even *refers to the parity of the number of* y_i*. Here the action of* $\overline{G} = \hat{G}/H$ *on* $y_1, \ldots, y_{2k}$ *is defined as follows. We have the group* $\hat{G}$ *act by scaling on all* $z(n)$*, in particular on* $z_1, \ldots, z_{2r}$*. We will define its action on products of even number of* y_i *as follows. For* $\lambda \in \hat{G}$ *such that its image in* $\hat{G}/G = \mathbb{C}^*$ *is* $\varphi(\lambda)$ *we define*

$$\lambda(y_i y_j) = \lambda_i^{-1}\lambda_j^{-1}\varphi(\lambda) y_i y_j$$

to be the inverse of the action on corresponding z_i *twisted by* $\varphi(\lambda)$*. This ensures that* $\lambda\Big((\sum_{i=1}^{2r} z_i y_i)^2\Big) = \varphi(\lambda)(\sum_{i=1}^{2r} z_i y_i)^2$*. On the other hand,* $C_2(\mathbf{z})$ *is also semi-invariant with respect to* $\hat{G}$ *with the character* $\varphi(\lambda)$*. Thus the ideal in Definition 7.3 is preserved under* $\hat{G}$*. Note that an element* $t \in H = \mathbb{C}^*$ *acts trivially on* $\mathcal{O}_{U_{\overline{\Sigma}}}$ *and on* $y_i y_j$*, since* $\lambda_i = \lambda_j = t$ *and* $\varphi(\lambda) = t^2$*. Thus the action of* $\hat{G}$ *descends to the action of* $\hat{G}/H = \overline{G}$*.*

It is now reasonable to conjecture that these more sophisticated Clifford limits give the same triangulated categories as any other limits we have considered.

Conjecture 7.5. *Under the centrality and appropriate flatness assumptions, the bounded derived category of coherent sheaves on* $(\mathcal{S}, \mathcal{B}_0)$ *is equivalent to the category* $D_B(K, c, \Sigma)$*.*

8. Combinatorics of Clifford decompositions.

In the first part, we explore the combinatorics associated to a decomposition

$$\deg^\vee = \frac{1}{2}(s_1 + \cdots + s_{2r}) + t_1 + \cdots + t_{k-r}. \tag{8.1}$$

Large parts of this section can be read independently from the more technical parts of the paper that discuss derived equivalences and Clifford algebras.

Just as before, we have a pair of reflexive Gorenstein cones $K \subset M_\mathbb{R}, K^\vee \subset N_\mathbb{R}$ of index k. Suppose there exists a decomposition (8.1) where $s_i, t_j \in K^\vee_{(1)}$ are linearly independent lattice elements. We define a lattice

$$\overline{N}_{free} = N/(N \cap (\sum_{i=1}^{2r} \mathbb{R}\cdot s_i + \sum_{j=1}^{k-r} \mathbb{R}\cdot t_j)).$$

Notice that $\overline{N}_{free}$ is the quotient of $\overline{N}$ defined in Section 2 by its torsion subgroup. Induced from the pairing between M and N, the dual lattice of $\overline{N}_{free}$ is

$$\begin{aligned}\overline{M} &= \mathrm{Ann}(s_1, \cdots, s_{2r}, t_1, \cdots, t_{k-r}) \\ &= \{m \in M \mid \langle m, s_i\rangle = \langle m, t_j\rangle = 0\ \forall\ 1 \le i \le 2r, 1 \le j \le k-r\}.\end{aligned}$$

We also use $\langle -, -\rangle$ to denote the induced perfect pairing $\overline{M} \times \overline{N}_{free} \to \mathbb{Z}$.

Let Θ be the image of the convex hull of $K^\vee_{(1)}$ in $\overline{N}$. It is a lattice polytope by the definition of Gorenstein cone, and it contains the origin as an interior element. Indeed, any linear function on $\overline{N}_{free}$ lifts to a linear function on N which is zero on $\deg^\vee$. Since $\deg^\vee$ is in the interior of $K^\vee$, this function takes positive and negative values on some rays of $K^\vee$. Therefore, any linear function on $\overline{N}_{free}$ takes positive and negative values on Θ, which implies that the origin lies in the interior of Θ.

We introduce a polytope in $\overline{M}_\mathbb{R}$ defined by

$$T := \{x \in K \mid \langle x, s_i\rangle = \langle x, t_j\rangle = 1 \ \forall\ i, j\} - \deg\,. \tag{8.2}$$

We should point out that T may not be a lattice polytope, however, we will show that its dual polytope is the lattice polytope Θ.

Lemma 8.1. *The polytope T contains origin in its interior. There holds*

$$T^\vee = \{y \in \overline{N}_\mathbb{R}, \text{such that } \langle T, y\rangle \geq -1\} = \Theta.$$

Proof. Let us investigate the dual of Θ. By definition of the dual polytope and of Θ, the dual $\Theta^\vee$ is a subset of $\overline{M}_\mathbb{R}$ which consists of x such that $\langle x, K^\vee_{(1)}\rangle \geq -1$. In other words, this is a subset of $M_\mathbb{R}$ such that $\langle x, K^\vee_{(1)}\rangle \geq -1$ and $\langle x, s_i\rangle = \langle x, t_j\rangle = 0$ for all i and j.

Equivalently, $x + \deg$ can be characterized by

$$\langle x + \deg, K^\vee_{(1)}\rangle \geq 0,\ \langle x + \deg, s_i\rangle = \langle x + \deg, t_j\rangle = 1,\ \text{for all } i, j$$

which is precisely the definition of $T + \deg$. This shows that $T = \Theta^\vee$, which implies $T^\vee = \Theta$ and $\mathbf{0} \in T^\circ$. □

We will now define the following three sets of polytopes:

$$A_i = \text{Conv}\{x \in K_{(1)} \mid \langle x, t_i\rangle = 1\},\ 1 \leq i \leq k - r$$

and

$$T_{i,i} = \text{Conv}\{x \in K_{(1)} \mid \langle x, s_i\rangle = 2\},\ 1 \leq i \leq 2r$$
$$T_{i,j} = \text{Conv}\{x \in K_{(1)} \mid \langle x, s_i\rangle = \langle x, s_j\rangle = 1\},\ 1 \leq i, j \leq 2r \text{ and } i \neq j.$$

Because a lattice point $x \in K_{(1)}$ pairs with $\deg^\vee$ at 1, x must uniquely lie in one of $A_i, T_{i,i}$ and $T_{i,j}$. In other words, the primitive elements of rays of K can be classified according to the polytopes $A_i, T_{i,i}$ or $T_{i,j}$ they lie in. Note that elements of these sets pair by 0 with the other s and t points.

We now define

$$D((T_{i,j})_{i,j}) := \text{Conv}\left(\bigcup_{\sigma\in S_{2r}} \sum_{1\leq i\leq 2r} T_{i,\sigma(i)}\right)$$

be a lattice polytope, where S_{2r} denotes the $2r$-symmetric group. A priori, $D((T_{i,j})_{i,j})$ could be empty, but it is a consequence of Theorem 8.4 that this can never happen in our setting. Let

$$S := \sum_{i=1}^{2r} A_i + \frac{1}{2} D((T_{i,j})_{i,j}) - \deg$$

be a polytope. By pairing with s_i, t_j, one can verify directly that $S \subset \overline{M}$, and we will show in Theorem 8.4 that it coincides with T, that is $S = \Theta^\vee$. In order to prove it we need to first recall the following so called Birkhoff–von Neumann theorem.

Definition 8.2. *A permutation matrix is the matrix with exactly one entry* 1 *in each row and column and* 0 *elsewhere.*

Lemma 8.3. *(Birkhoff–von Neumann theorem) Suppose B_n is an $n \times n$ matrix whose entries are non-negative real numbers and whose rows and columns each add up to* 1*. Then B_n lies in the convex hull of the set of $n \times n$ permutation matrices.*

We now state the following key result:

Theorem 8.4. *The polytope*

$$S := \sum_{i=1}^{k-r} A_i + \frac{1}{2} \operatorname{Conv}\left(\bigcup_{\sigma \in S_{2r}} \sum_{1 \le i \le 2r} T_{i,\sigma(i)} \right) - \deg$$

is equal to T. In particular, its dual polytope is Θ.

Proof. By definition of T (see equation (8.2)), we only need to show

$$\begin{aligned} &\sum_{i=1}^{k-r} A_i + \frac{1}{2} \operatorname{Conv}\left(\bigcup_{\sigma \in S_{2r}} \sum_{1 \le i \le 2r} T_{i,\sigma(i)} \right) \\ &= \{x \in K \mid \langle x, s_i \rangle = \langle x, t_j \rangle = 1 \ \forall \ i, j\}. \end{aligned}$$

The inclusion $\subseteq$ is obtained by definition, thus we only need to show the inverse inclusion.

Suppose $x \in \{x \in K \mid \langle x, s_i \rangle = \langle x, t_j \rangle = 1 \ \forall \ i, j\}$, then

$$x = \sum_{v \in K_{(1)}} \lambda_v v, \quad \lambda_v \ge 0.$$

Because the generators of K can be classified according to the polytopes $A_i, T_{i,i}$ or $T_{i,j}$ they lie in, we can rewrite the summation as

$$x = \sum_i \sum_{v \in A_i} \lambda_v v + \sum_{i,j} \sum_{v \in T_{i,j}} \lambda_v v.$$

Notice that when $i \neq j$, we have $T_{i,j} = T_{j,i}$.

Let

$$x_t = \sum_i \sum_{v \in A_i} \lambda_v v,$$
$$x_s = \sum_{i,j} \sum_{v \in T_{i,j}} \lambda_v v = \sum_{i,j} b_{i,j} y_{i,j}, \tag{8.3}$$

with $y_{i,j} \in T_{i,j}$. Besides, we require $y_{i,j} = y_{j,i}$ and $b_{i,j} = b_{j,i}$.

We have $x = x_t + x_s$, moreover, by pairing with t_i, we have

$$1 = \langle x, t_i \rangle = \langle x_t, t_i \rangle = \sum_{v \in A_i} \lambda_v.$$

This implies that $\sum_{v \in A_i} \lambda_v v \in A_i$, thus $x_t \in \sum_{i=1}^{k-r} A_i$. Hence, all we need to show is $x_s \in \frac{1}{2} D((T_{i,j})_{i,j})$.

By pairing with s_j, we have

$$1 = \langle x, s_j \rangle = \langle x_s, s_j \rangle = 2b_{j,j} + \sum_{\substack{1 \leq i \leq 2s \\ i \neq j}} b_{i,j} + \sum_{\substack{1 \leq i \leq 2s \\ i \neq j}} b_{j,i} = 2 \sum_{1 \leq i \leq 2s} b_{i,j}. \tag{8.4}$$

Let $B_{2r} = (b_{i,j})_{i,j}$ be a $2r \times 2r$-symmetric matrix. Then by (8.4), we have

$$\sum_{1 \leq i \leq 2r} b_{i,j} = \sum_{1 \leq j \leq 2r} b_{i,j} = \frac{1}{2}.$$

According to the Birkhoff–von Neumann theorem (Lemma 8.3), B_{2r} is a convex linear combination of permutation matrices.

There is a 1-1 correspondence between elements of $2r$-symmetric group S_{2r} and $2r \times 2r$ permutation matrices under the map

$$\sigma \mapsto (\delta_{j,\sigma(i)})_{i,j},$$

where $\delta_{j,\sigma(i)}$ is the Kronecker delta. Because of this, there exist $r_\sigma \geq 0$, such that

$$b_{i,j} = \sum_{\sigma \in S_{2r}} r_\sigma \delta_{j,\sigma(i)}.$$

For fixed j, because $\sum_{1 \leq i \leq 2r} b_{i,j} = \frac{1}{2}$, we have

$$\sum_{1 \leq i \leq 2r} \sum_{\sigma \in S_{2r}} r_\sigma \delta_{j,\sigma(i)} = \sum_{\sigma \in S_{2r}} \sum_{\substack{j = \sigma(i) \\ 1 \leq i \leq 2r}} r_\sigma = \sum_{\sigma \in S_{2r}} r_\sigma = \frac{1}{2}. \tag{8.5}$$

We use the equation (8.3) to get

$$\begin{aligned} x_s &= \sum_{i,j} b_{i,j} y_{i,j} = \sum_{i,j} (\sum_{\sigma \in S_{2r}} r_\sigma \delta_{j,\sigma(i)}) y_{i,j} \\ &= \sum_{\sigma \in S_{2r}} r_\sigma (\sum_{\substack{j = \sigma(i) \\ i,j}} y_{i,j}) = \sum_{\sigma \in S_{2r}} r_\sigma (\sum_{1 \leq i \leq 2r} y_{i,\sigma(i)}) \end{aligned}$$

Combining this with (8.5), we have

$$x_s \in \frac{1}{2}\operatorname{Conv}\left(\bigcup_{\sigma\in S_{2r}} \sum_{1\le i\le 2s} T_{i,\sigma(i)}\right),$$

which finishes the proof. □

Corollary 8.5. *The polytope $2T$ has lattice vertices.*

Proof. Indeed, by Theorem 8.4 we see that

$$2T = 2S = \sum_{i=1}^{k-r} 2A_i + \operatorname{Conv}\left(\bigcup_{\sigma\in S_{2r}} \sum_{1\le i\le 2r} T_{i,\sigma(i)}\right) - 2\deg$$

is a lattice polytope, since Minkowski sums, convex hulls and lattice shifts of lattice polytopes are again lattice polytopes. □

Remark 8.6. *The geometric meaning of the above corollary is that the toric variety $\mathbb{P}$ defined by T is Fano $\mathbb{Q}$-Gorenstein with $(-2K_{\mathbb{P}})$ an ample Cartier divisor.*

The data of the coefficient function

$$c : K_{(1)} \to \mathbb{C}$$

may be equivalently encoded as a collection of $(k-r)$ Laurent polynomials f_i with Newton polytopes A_i and an $(2r)\times(2r)$ symmetric matrix R of Laurent polynomials with Newton polytopes $T_{i,j}$. These data allow us to consider a double cover of the complete intersection of $f_i = 0$ in the toric variety defined by T, ramified over $\deg(R) = 0$. The resulting variety is Calabi-Yau. However, it has singularities other than the ones coming from the ambient toric varieties due to the loci of corank two or higher of the matrix R. One can view the Clifford variety of Section 5 as a noncommutative crepant resolution of this double cover (with a certain Brauer class).

In many (though not all) cases we have the important technical centrality assumption (7.1) that there exists a regular simplicial fan Σ all of whose maximum dimensional cones contain all s_i and all t_j.

As a consequence of the assumption (7.1), there is a (stacky) fan $\overline{\Sigma}$ on the quotient

$$\overline{N} = N/\mathbb{Z}s_1 + \cdots + \mathbb{Z}s_{2r} + \mathbb{Z}\deg^\vee + \mathbb{Z}t_1 + \cdots + \mathbb{Z}t_{k-r}$$

obtained by removing s_i and t_j from the sets of Σ. Then we have natural line bundles $\mathcal{L}'_1, \ldots, \mathcal{L}'_{2r}$ and $\mathcal{L}_1, \ldots, \mathcal{L}_{k-r}$ on the stack $\mathbb{P}_{\overline{\Sigma}}$ with the property that

$$\bigotimes_{i=1}^{2r} \mathcal{L}'_i \otimes \bigotimes_{j=1}^{k-r} \mathcal{L}_j^{\otimes 2}$$

is the square of the anticanonical bundle on $\mathbb{P}_{\overline{\Sigma}}$, as considered in Section 7.

Remark 8.7. *While $\mathcal{L}_i$ are pullbacks of the Cartier divisors from the coarse moduli space, the same can not be guaranteed for $\mathcal{L}'_j$. The data of the coefficient function $c : K_{(1)} \to \mathbb{C}$ amounts to a choice of $(k-r)$ sections $f_1, \dots, f_{k-r}$ of $\mathcal{L}'_1, \dots, \mathcal{L}'_{k-r}$ respectively.*

Then we can view A_i as the Newton polytopes that support the sections of $\mathcal{L}_i$, and the polytopes $T_{i,j}$ are the ones that support sections of $\mathcal{L}'_i \otimes \mathcal{L}'_j$ under appropriate linearizations.

Remark 8.8. *In the absence of the centrality condition* (7.1)*, we still obtain a singular Calabi-Yau variety which is a double cover of the complete intersection in a toric $\mathbb{Q}$-Gorenstein Fano variety given by the polytope Θ. It would be interesting to see if one can find noncommutative resolutions of its singularities and whether there is still a derived equivalence statement. A priori, one no longer has the vector bundle structure on U_Σ, which prevents one from directly using the work of Kuznetsov.*

Remark 8.9. *It would be interesting to investigate possible torsion in*

$$\overline{N} = N/\mathbb{Z}s_1 + \cdots + \mathbb{Z}s_{2r} + \mathbb{Z}\deg^\vee + \mathbb{Z}t_1 + \cdots + \mathbb{Z}t_{k-r}$$

If the sets T_{ii} are nonempty, then we one can show that this is at most 2*-torsion. However, we don't know if $T_{ii} \neq \emptyset$ holds in general.*

9. More examples.

We will describe some examples of Clifford double mirrors in the literature as well as new examples. Each of these examples consists of a pair of Calabi-Yau varieties, and the evidence for the double mirror property comes from equivalence of derived categories.

9.1. **Example: $(2,2,2)$-complete intersections in $\mathbb{CP}^5$.** This example is given by Mukai in [Muk88] (see Examples (1.5)(1.6)(2.2)). Let $q_i, 0 \leq i \leq 2$ be quadratic equations in $\mathbb{CP}^5$, and Q_i be the corresponding quadric hypersurfaces. Suppose these Q_i intersect transversally in $\mathbb{CP}^5$, then their complete intersection X is a K3 surface. On the other hand, let A_i be the symmetric 6×6 matrix corresponding to the quadratic form q_i. Then a quadric $\{a_0q_0 + a_1q_1 + a_2q_2 = 0\}$ is smooth if and only if the matrix $a_0A_0 + a_1A_1 + a_2A_2$ is regular. Hence, their singular members are parameterized by the degree 6 curve $D := \{\det(a_0A_0 + a_1A_1 + a_2A_2) = 0\}$ in $\mathbb{CP}^2$ (where a_0, a_1, a_2 are variables). The double cover ramified along D is a K3 surface, and we denote it by Y.

We further assume that every quadric containing X is of rank ≥ 5. Let $h \in H^2(X, \mathbb{Z})$ be the cohomology class of hyperplane sections of X, then the moduli space of stable (with respect to polarization $X \to \mathbb{CP}^5$) rank 2 vector bundle with $c_1 = h$ and $c_2 = 4$ is canonically isomorphic to Y. By [Cal00a, Section 5.5], there exists an $\alpha \in Br(Y)$ in the Brauer group of Y such that $D^b(X) \cong D^b(Y, \alpha)$.

This example is a particular case of Kuznetsov's construction for $k = 4$ and can consequently be reconstructed by our method. The reader can either follow the discussion of Section 5.2 or the description below along the lines of Section 8.

Let Δ be the polytope which is the convex hull of $\{e_1, \ldots, e_5, e_6\}$, where $e_i, 1 \leq i \leq 5$ are standard basis of $\mathbb{Z}^5$, and $e_6 = -\sum_{i=1}^5 e_i$. The normal fan of Δ is the fan of $\mathbb{CP}^5$. Let $\Delta_j = \mathrm{Conv}\{e_{2j-1}, e_{2j}, \mathbf{0}\}\ \forall\ 1 \leq j \leq 3$. Then $\{\Delta_i \mid 1 \leq i \leq 3\}$ is a nef-partition of Δ.

We can define a reflexive Gorenstein cone associated to Δ_i by

$$K^\vee =: \{(a, b, c; a\Delta_1 + b\Delta_2 + c\Delta_3) \mid a, b, c \in \mathbb{R}_{\geq 0}\} \subset \mathbb{R}^8.$$

This is exactly how one can associate a reflexive Gorenstein cone to a nef-partition (see [BatBor94]). The $(2, 2, 2)$-complete intersection X is just the complete intersection defined by decomposition $\deg^\vee = (1, 0, 0; \mathbf{0}) + (0, 1, 0; \mathbf{0}) + (0, 0, 1; \mathbf{0}))$.

On the other hand, $\deg^\vee$ can also be presented by

$$\deg^\vee = \frac{1}{2}(s_1 + \cdots + s_6),$$

where

$$\begin{aligned} s_1 &= (1, 0, 0; e_1), \quad & s_2 &= (1, 0, 0; e_2) \\ s_3 &= (0, 1, 0; e_3), \quad & s_4 &= (0, 1, 0; e_4) \\ s_5 &= (0, 0, 1; e_5), \quad & s_6 &= (0, 0, 1; e_6). \end{aligned}$$

Then $\mathbb{Z}^8 \cap (\sum_{i=1}^6 \mathbb{R}s_i) = \mathbb{Z}\deg^\vee + \sum_{i=1}^6 \mathbb{Z}s_i$, hence

$$\overline{N}_{free} = \overline{N} = \mathbb{Z}^8/(\mathbb{Z}\deg^\vee + \sum_{i=1}^6 \mathbb{Z}s_i).$$

Again, let Θ be the image of the $K^\vee_{(1)}$ in $\overline{N} \cong \mathbb{Z}^2$. Then Θ is a convex hull of $\{v_1 := \overline{(1, 0, 0; \mathbf{0})}, v_2 := \overline{(0, 1, 0; \mathbf{0})}, v_3 := \overline{(0, 0, 1; \mathbf{0})}\}$, because all the other vertices of $K^\vee_{(1)}$ is zero in $\overline{N}$. Moreover, because $(1, 0, 0; \mathbf{0}) + (0, 1, 0; \mathbf{0}) + (0, 0, 1; \mathbf{0}) = 2\deg^\vee$, we have $v_1 + v_2 + v_3 = 0$. Thus, the normal fan of Θ is exactly the fan for $\mathbb{CP}^2$.

We can define $g = \mathbf{x}^{-2\deg}\det((g_{i,j})_{i,j})$ as above, where $g_{i,j}$ is the Laurent polynomial constructed from $T_{i,j}$. By Theorem 8.4, g is a global section of $H^0(\mathbb{CP}^2, \mathcal{O}(-2K_{\mathbb{CP}^2}))$, hence of degree 6. The Calabi-Yau variety Y in the construction is exactly the double cover ramified along the sextic $D := \{g = 0\}$.

9.2. **Double mirrors of Enriques surfaces.** We recall that Enriques surfaces are quotients of certain K3 surfaces by a fixed point free involution. One of the many constructions of these surfaces is provided by the quotients of $(2,2,2)$ complete intersections in $\mathbb{CP}^5$. Specifically, we need to consider an action of an involution on $V = \mathbb{C}^6$ that has trace 0 and take a complete intersection of three invariant quadrics on $\mathbb{P}V$. The involution fixes this surface and acts freely on it. The resulting quotient surface is Enriques.

The corresponding Gorenstein cones are given as in Section 5.3 as follows. We have

$$N = \Big((\bigoplus_{i=1}^{6} \mathbb{Z}s_i + \frac{1}{2}\mathbb{Z}(s_1+s_2+s_3)) \oplus \bigoplus_{j=1}^{3} \mathbb{Z}t_j \Big) / \mathbb{Z}(\sum_{i=1}^{6} s_i - 2\sum_{j=1}^{3} t_j).$$

The cone $K^\vee$ is the image of the nonnegative orthant. The dual lattice M is given by

$$M = \{ \sum_{i=1}^{6} a_i s_i^\vee + \sum_{j=1}^{3} b_j t_j^\vee \mid \sum_{i=1}^{6} a_i = 2\sum_{j=1}^{3} b_j \text{ and } \sum_{i=1}^{3} a_i \text{ is even} \}.$$

The cone K is the intersection of M with the nonnegative orthant.

The usual Kuznetsov's double mirror is the sheaf of Clifford algebras over $\mathbb{CP}^2$ whose center is the double cover of $\mathbb{CP}^2$ ramified at the union of two elliptic curves E_+ and E_- which are written as determinants of symmetric 3×3 matrices of linear forms. The action of the involution means that we need to consider the corresponding Clifford algebra over the gerbe $[\mathbb{CP}^2/\mathbb{Z}_2]$.

As in Section 5.3, we consider the semidirect product of the Kuznetsov's sheaf of Clifford algebras over $\mathbb{CP}^2$ and the group ring $\mathbb{C}[h]/\langle h^2-1 \rangle$ of $\mathbb{Z}_2$. Over the generic point of $\mathbb{CP}^2$, after diagonalization of the quadratic forms, we get the even part of the quotient of the free algebra over the field of rational functions F on $\mathbb{CP}^2$

$$F\{y_1^+, y_2^+, y_3^+, y_1^-, y_2^-, y_3^-, h\}$$

by the two-sided ideal generated by the relations

$$h^2-1, hy_i^+ - y_i^+h, hy_i^- + y_i^-h, (y_i^+)^2 + c_i^+, (y_i^-)^2 + c_i^-, y_i^+y_j^- + y_j^-y_i^+$$

for all i and j and

$$y_i^+y_j^+ + y_j^+y_i^+, y_i^-y_j^- + y_j^-y_i^-$$

for $i \neq j$. Again $c_i^\pm \in F$ may not be assumed to be 1, since F is not algebraically closed. In fact, up to squares, the products $\prod_{i=1}^3 c_i^\pm$ give equations of $E_\pm$.

The calculation of the center of the above algebra is done analogously to Section 5.3 but yields a different result. In fact, one simply gets F as the center.

Remark 9.1. *Some more delicate preliminary calculations seem to indicate that the structure of the double mirror of an Enriques surface is that of (smooth)* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *root stack over* $\mathbb{CP}^2$*, ramified over the union of two elliptic curves* E_+ *and* E_-*, presumably with a Brauer element. The corresponding orbifold Euler characteristics calculation is as follows. The complement of the union of two elliptic curves has* $\chi = 12$*. The elliptic curves contribute nothing, since the ages of the corresponding involutions are* $\frac{1}{2}$ *so the contribution of the twisted sector cancels that of the untwisted sector. Similarly, the contributions of the* 9 *intersection points are zero, since the* 4 *sectors cancel each other. Thus the Euler characteristics of the root stack matches that of the Enriques surface. We thank Howard Nuer for pointing out a likely relationship between these double mirrors and the construction of Enriques surfaces as logarithmic transformations of elliptic surfaces in [GH78, p.599].*

9.3. **Calabrese-Thomas' example.** We state the construction of Calabrese and Thomas' first example in [CT14].[12] Let V, W be complex vector spaces of dimension 3, and $\mathbb{P}(V \oplus W) = \mathbb{CP}^5$. Let

$$\pi : Z := \mathrm{Bl}_{\mathbb{P}(V)}(\mathbb{CP}^5) \to \mathbb{CP}^5$$

be the blowup along $\mathbb{P}(V)$ with exceptional divisor E. Then one can show that $\pi^*\mathcal{O}_{\mathbb{CP}^5}(3)(-E)$ is a base point free line bundle. Let f_1, f_2 be two general global sections of $\pi^*\mathcal{O}_{\mathbb{CP}^5}(3)(-E)$. Let

$$X := \{f_1 = f_2 = 0\} \subseteq Z$$

be their complete intersection. Since the anticanonical bundle of Z is equal to $\pi^*\mathcal{O}_{\mathbb{CP}^5}(6)(-2E)$, we see that X is a Calabi-Yau variety by the adjunction formula.

Let $\rho : Z \to \mathbb{P}(W)$ be the projection from the plane $\mathbb{P}(V) \subseteq \mathbb{CP}^5$ to $\mathbb{P}(W)$, then one can show that the universal hypersurface $\mathcal{H} := \{x_1 f_1 + x_2 f_2 = 0\} \subseteq Z \times \mathbb{CP}^1$ is a quadric fibration over $Y = \mathbb{P}(W) \times \mathbb{CP}^1$ under the morphism $\rho \times \mathrm{id}$. In particular, this quadric fibration corresponds to an even Clifford algebra sheaf $\mathcal{B}_0$. As a consequence of the relative version of homological projective duality, there is a derived equivalence $D^b(X) \cong D^b(Y, \mathcal{B}_0)$.

One can modify the right hand side of the equivalence by considering the relative Fano scheme of lines of $\rho \times id : \mathcal{H} \to \mathbb{P}(W) \times \mathbb{CP}^1$ (see [Kuz14]). Since $\rho \times \mathrm{id}$ is a quadric fibration, we can consider the loci D' where the rank of the quadratic form on the fibre is not of full rank (i.e. < 4). Then one can show that D' is a $(6, 4)$-bidegree divisor on $Y = \mathbb{P}(W) \times \mathbb{CP}^1 \cong \mathbb{CP}^2 \times \mathbb{CP}^1$. Let $F \to \mathbb{P}(W) \times \mathbb{CP}^1$ be the relative Fano scheme of lines of $\rho \times \mathrm{id}$, and $F \to X' \to Y$ be its Stein factorization, then $X' \to Y$ is a double cover ramified along D'. Finally, let $X'' \to X'$ be some small resolution (in the

[12]The second example of [CT14] falls into the framework of Batyrev-Nill double mirrors from Section 2.

analytic category), then there exists an $\alpha \in Br(X'')$, such that

$$D^b(X) \cong D^b(Y, \mathcal{B}_0) \cong D^b(X'', \alpha).$$

The reconstruction of this example is similar to the above cases, but a little bit involved:

We start by considering a $\mathbb{Z}^5$ with the usual $\mathbb{CP}^5$ fan on it, namely the one with $e_1 = (1,0,0,0,0), ..., e_5 = (0,0,0,0,1), e_6 = (-1, ... -1)$. Then the blowup of $\mathbb{P}(V) \cong \mathbb{P}^2$ introduces an additional vertex $e_0 = (1,1,1,0,0)$. Let Δ be the convex hull of $\{e_0, \cdots, e_6\}$, then it has a nef-partition with

$$\Delta_1 = \mathrm{Conv}(\{\mathbf{0}, e_0, e_1, e_2, e_6\}), \quad \Delta_2 = \mathrm{Conv}(\{\mathbf{0}, e_3, e_4, e_5\}).$$

Let $K^\vee \subseteq \mathbb{R}^7$ be the reflexive cone defined by

$$K^\vee := \{(a, b; a\Delta_1 + b\Delta_2 \mid a, b \in \mathbb{R}_{\geq 0})\} \subseteq \mathbb{R}^7.$$

Again, X is the complete intersection associated to $\deg^\vee = (1,0;\mathbf{0}) + (0,1;\mathbf{0})$, and one can verify that the nef divisors associated to $\Delta_i, i = 1,2$ are both linearly equivalent to $\pi^*\mathcal{O}_{\mathbb{CP}^5}(3)(-E)$ in Calabrese-Thomas' example.

Next, we write

$$\deg^\vee = \frac{1}{2}(s_1 + s_2 + s_3 + s_4),$$

where

$$s_1 = (1,0;e_0), \quad s_2 = (1,0;e_6), \quad s_3 = (0,1;e_4), \quad s_4 = (0,1;e_5).$$

One can show that

$$\mathbb{Z}^7 \cap \sum_{i=1}^{4} \mathbb{R} \cdot s_i = \mathbb{Z} \cdot \deg^\vee + \sum_{i=1}^{4} \mathbb{Z} \cdot s_i,$$

hence

$$\overline{N}_{free} = \overline{N} = \mathbb{Z}^7 / (\mathbb{Z} \cdot \deg^\vee + \sum_{i=1}^{4} \mathbb{Z} \cdot s_i).$$

Let Θ be the image of $K^\vee_{(1)}$ under the quotient map. Recall that $K^\vee_{(1)}$ has 9 generators

$$\begin{array}{lllll} (1,0;\mathbf{0}), & (1,0;e_0), & (1,0;e_1), & (1,0;e_2), & (1,0;e_6) \\ (0,1;\mathbf{0}), & (0,1;e_3), & (0,1;e_4), & (0,1;e_5). & \end{array}$$

After taking the quotient, 4 of them disappear, and the other 5 forms a fan of $Y = \mathbb{P}(W) \times \mathbb{CP}^1 \cong \mathbb{CP}^2 \times \mathbb{CP}^1$. Specifically,

$$(0,1;\mathbf{0}) + (1,0;\mathbf{0}) = \deg^\vee \ (= 0 \in \overline{N}),$$
$$(1,0;e_1) + (1,0;e_2) + (0,1;e_3) = \deg^\vee + (0,1;e_0) \ (= 0 \in \overline{N}).$$

By a computer search, there are 96 lattice points in $K_{(1)}$ which fall into 10 classes depending on the pairing with s_i. By pairing with

$$\overline{(1,0;\mathbf{0})}, \overline{(1,0;e_1)}, \overline{(1,0;e_2)} \in \overline{N},$$

we can find out the degree of these classes in $\mathbb{C}[\overline{N}]$. It turns out that the determinantal equation g is exactly of bidegree $(6,4)$. Hence the double cover of Y ramified along $D' = \{g = 0\}$ gives the Calabi-Yau variety X' which coincides with the construction in Calabrese-Thomas' example.

Remark 9.2. *Calabrese and Thomas consider analytic small resolutions of the singular double cover, similar to [Add09]. In contrast, our construction gives a noncommutative algebraic resolution by a sheaf of Clifford algebras. We also remark that the flatness assumption is satisfied in Calabrese-Thomas' example.*

9.4. **An example with $k = 1$.** As we mentioned before, even when the index $\langle \deg, \deg^\vee \rangle$ is equal to 1, there may exist double mirrors in our construction, in contrast to the Batyrev's original construction [Bat94]. In the following, we will give an example of such type in a 2 dimensional lattice. The resulting Calabi-Yau varieties are (elliptic) curves, therefore the Brauer class has to be trivial. This could be viewed as an almost trivial example of our construction, but it is informative to work it out in detail.

We consider the 2 dimensional reflexive polytope

$$\Delta = \mathrm{Conv}\{(1,1),(1,-1),(-1,-1),(-1,1)\},$$

whose dual polytope is

$$\Delta^\vee = \mathrm{Conv}\{(1,0),(0,-1),(-1,0),(0,1)\}.$$

Then we have a pair of reflexive Gorenstein cones

$$K = \{(a; a\cdot\Delta) \mid a \geq 0\} \subset M_{\mathbb{R}}, \quad K^\vee = \{(b; b\cdot\Delta^\vee) \mid b \geq 0\} \subset N_{\mathbb{R}}.$$

The picture of these cones is given in Section 2.1. We can decompose $\deg^\vee$ in two different ways:

$$\deg^\vee = (1;\mathbf{0}) = \frac{1}{2}(s_1 + s_2),$$

where $s_1 = (1,-1,0), s_2 = (1,1,0)$.

Let X be the elliptic curve defined by the decomposition $\deg^\vee = (1;\mathbf{0})$. This is exactly the Batyrev-Borisov variety as we just take the complete intersection and do not need to take the double cover. It is a hypersurface associated to the anticanonical divisor in the toric variety $\mathbb{CP}^1 \times \mathbb{CP}^1$. Hence, a global section f of the anticanonical divisor can be identified with $\{\sum a_v \mathbf{x}^v \in \mathbb{C}[M] \mid v \in \nabla\}$. Since a smooth curve is uniquely determined up to birational equivalence, X is uniquely determined by its intersection with $(\mathbb{C}^*)^2 \subset \mathbb{CP}^1 \times \mathbb{CP}^1$. Let

$$\begin{aligned} f \ &= a_{11}x^{-1}y + a_{12}y + a_{13}xy \\ &\quad + a_{21}x^{-1} + a_{22} + a_{13}x \\ &\quad + a_{31}x^{-1}y^{-1} + a_{32}y^{-1} + a_{33}xy^{-1} \end{aligned}$$

be the equation of X in $(\mathbb{C}^*)^2$.

The projection $(\mathbb{C}^*)^2 \to \mathbb{C}^*$ induces a projection $X \cap (\mathbb{C}^*)^2 \to \mathbb{C}^*$ given by $(x, y) \mapsto y$. Because $f = 0$ is the same as $xf = 0$ on $(\mathbb{C}^*)^2$, and

$$\begin{aligned} xf &= (a_{13}y + a_{23} + a_{33}y^{-1})x^2 \\ &\quad + (a_{12}y + a_{22} + a_{32}y^{-1})x \\ &\quad + (a_{11}y + a_{21} + a_{31}y^{-1}), \end{aligned}$$

this projection is a degree 2 morphism which is ramified along the discriminant

$$(a_{12}y + a_{22} + a_{32}y^{-1})^2 - 4(a_{11}y + a_{21} + a_{31}y^{-1})(a_{13}y + a_{23} + a_{33}y^{-1}) = 0.$$

Next, we will construct the double mirror X' associated to $\deg^\vee = \frac{1}{2}(s_1 + s_2)$.

Because $\overline{N} = \mathbb{Z}^2/\mathbb{Z}^2 \cap (\mathbb{R}s_1 + \mathbb{R}s_2) \cong \mathbb{Z}$, and $\Theta = \overline{K^\vee_{(1)}} = \mathrm{Conv}(-1, 1)$, the toric stack $\mathbb{P}_\Theta$ is actually the smooth toric variety $\mathbb{CP}^1$. One can find $T_{i,j}$ from $K_{(1)}$ by pairing with s_i. Specifically,

$$\begin{aligned} T_{1,1} \cap M &= \{(1, -1, 1), (1, -1, 0), (1. - 1, -1)\}, \\ T_{1,2} \cap M &= T_{2,1} \cap M = \{(1, 0, 1), (1, 0, 0), (1, 0, -1)\}, \\ T_{2,2} \cap M &= \{(1, 1, 1), (1, 1, 0), (1, 1, -1)\}. \end{aligned}$$

Hence a global section g in $H^0(\mathbb{CP}^1, \mathcal{O}(-2K_{\mathbb{CP}^1}))$ is

$$g = x^{-2} \cdot \det \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

where

$$\begin{aligned} g_{11} &= a_{11}xy^{-1}z + a_{21}xy^{-1} + a_{31}xy^{-1}z^{-1} \\ g_{12} &= \omega_{21} = \frac{1}{2}(a_{12}xz + a_{22}x + a_{32}xz^{-1}) \\ g_{22} &= a_{13}xyz + a_{23}xy + a_{33}xyz^{-1}. \end{aligned}$$

It is straightforward to compute that

$$g = (a_{11}z + a_{21} + a_{31}z^{-1})(a_{13}z + a_{23} + a_{33}z^{-1}) - \frac{1}{4}(a_{12}z + a_{22} + a_{32}z^{-1}).$$

Hence, the ramification loci (i.e. $g = 0$) on $\mathbb{CP}^1$ is exactly the same as those of X on $\mathbb{C}^* \subseteq \mathbb{CP}^1$. This shows that X, X' are isomorphic elliptic curves. In particular $D^b(X) \cong D^b(X')$.

Of course, this isomorphism is not unexpected and in fact follows from the results of our paper as follows. As a consequence of Theorem 6.1, the double mirrors should have equivalent derived categories. It is well known that derived equivalent smooth projective curves are isomorphic [Huy06, Corollary 5.46]. However, it is worthwhile to see this isomorphism explicitly.

Remark 9.3. *One can view this example as the $k = 2$ case of Kuznetsov's construction for complete intersections in of two quadrics in $\mathbb{CP}^3$, where we fix one of the quadrics and view it as a toric variety $\mathbb{CP}^1 \times \mathbb{CP}^1$.*

9.5. **An example of double mirrors without a central fan.** Let us consider the reflexive Gorenstein cone $K^\vee$ in $\mathbb{R}^3$ with the usual lattice $\mathbb{Z}^3$ generated by

$$(-1,-1,1),(2,-1,1),(-1,2,1).$$

The dual cone K is generated by

$$v_1 = (1,0,1), v_2 = (0,1,1), v_3 = (-1,-1,1)$$

in $\mathbb{Z}^3$. The degree elements are given by

$$\deg = (0,0,1),\ \deg^\vee = (0,0,1).$$

The coefficient function $c : K_{(1)} \to \mathbb{C}$ is determined by four values

$$c_1 = c(v_1),\ c_2 = c(v_2),\ c_3 = c(v_3),\ c_0 = c(\deg).$$

The corresponding hypersurface is the elliptic curve whose open part (in the big torus) is given by the equation

$$c_1 x + c_2 y + c_3 x^{-1} y^{-1} + c_0 = 0. \tag{9.1}$$

We can think of these curves as the mirrors to the usual elliptic curves in $\mathbb{CP}^2$.

Consider now the decomposition

$$\deg^\vee = \frac{1}{2}(0,-1,1) + \frac{1}{2}(0,1,1).$$

On the one hand, the points $s_1 = (0,-1,1)$ and $s_2 = (0,1,1)$ are in $K^\vee_{(1)}$. On the other hand, there is no simplicial fan in $K^\vee$ which has all maximum dimensional cones that contain both of these points. As such, the construction of Section 5 does not apply.

Nevertheless, we may consider the construction of Section 8. We sort the points of $K_{(1)}$ according to their pairings with s_1 and s_2 and take their convex hulls to get polytopes T_{ij}. We get

$$T_{1,1} = \{v_3\},\ T_{1,2} = T_{2,1} = \mathrm{Conv}\{v_1, \deg\},\ T_{2,2} = \{v_2\}.$$

The polytope $S = T$ is then given by

$$\frac{1}{2}\mathrm{Conv}\Big(v_3 + v_2, 2v_1, v_1 + \deg, 2\deg\Big) - \deg = \mathrm{Conv}\Big((-\frac{1}{2},0,0),(1,0,0)\Big).$$

The corresponding Calabi-Yau double cover of the toric variety $\mathbb{CP}^1$ that corresponds to $2T$ is given by the determinant of the symmetric matrix

$$\det \begin{pmatrix} c_3 x^{-1} y^{-1} & \frac{1}{2}(c_1 x + c_0) \\ \frac{1}{2}(c_1 x + c_0) & c_2 y \end{pmatrix}$$

which is, up to an invertible element, equal to

$$-4c_2 c_3 x^{-1} + (c_1 x + c_0)^2.$$

It remains to observe that the elliptic curve

$$z^2 = -4c_2c_3x^{-1} + (c_1x + c_0)^2$$

is isomorphic to the one given in (9.1) under a change of variables $z = 2c_2y + c_1x + c_0$.

Remark 9.4. *It appears that in $k = 1$ case the commutative variety that underlies the Clifford double mirror is always birational to the original toric hypersurface. Indeed, the set $K_{(1)}$ has "width two" and one can look at the corresponding double cover structure on the hypersurface. Moreover, in $k = 1$ (or more generally $r = 1$ case) the even part of Clifford algebra of a two-dimensional space is commutative, so one simply gets a double cover.*

9.6. **An example without flatness assumption.** In this subsection we discuss a case where the flatness assumption of Remark 5.5 does not hold.

The variety in question is the Clifford double mirror of the n-dimensional Calabi-Yau hypersurface H of bidegree $(2, n+1)$ in $\mathbb{CP}^1 \times \mathbb{CP}^n$ for $n \geq 3$. The corresponding reflexive Gorenstein cone $K^\vee$ in the lattice

$$\mathbb{Z}^2 \oplus (\bigoplus_{i=1}^{n+1} \mathbb{Z}e_i / \mathbb{Z}\sum_{i=1}^{n+1} e_i)$$

is generated by

$$(1, -1; \mathbf{0}),\ (1, 1; \mathbf{0}),\ (1, 0; e_i),\ i = 1, \ldots, n+1.$$

The degree element is given by $\deg^\vee = (1, 0; \mathbf{0})$ and is the only non-vertex lattice point in $K^\vee_{(1)}$. The dual cone K in the lattice

$$\mathbb{Z}^2 \oplus \{\sum_{i=1}^{n+1} a_i e_i^\vee, \sum_i a_i = 0\}$$

is generated by

$$(1, -1; (n+1)e_i^\vee - \sum_{j=1}^{n+1} e_j^\vee),\ (1, 1; (n+1)e_i^\vee - \sum_{j=1}^{n+1} e_j^\vee)$$

for $i = 1, \ldots, n+1$. The degree element is $\deg = (1, 0; \mathbf{0})$ so the index of the pair of cones is $k = 1$. The lattice points in $K_{(1)}$

$$(1, j, \sum_{l=1}^{n+1} a_l e_l^\vee - \sum_{l=1}^{n+1} e_l^\vee),\ -1 \leq j \leq 1, a_l \geq 0, \sum_{l=1}^{n+1} a_l = n+1$$

correspond to monomials $u_1^{j+1} u_2^{1-j} \prod_{l=1}^{n+1} v_l^{a_l}$ for the homogeneous coordinates $(u_1 : u_2, v_1 : \cdots : v_{n+1})$ on $\mathbb{CP}^1 \times \mathbb{CP}^n$. The coefficient function $c : K_{(1)} \to \mathbb{C}$ encodes the coefficients of the defining equation of H.

The Clifford limit corresponds to the decomposition

$$\deg^\vee = \frac{1}{2}(1, -1; \mathbf{0}) + \frac{1}{2}(1, 1; \mathbf{0}).$$

There is a fan Σ which satisfies (5.2) whose maximum dimensional cones are spanned by $(1,-1;\mathbf{0}),(1,1;\mathbf{0})$ and all but one of $(1,0;e_i)$. The construction of $(\mathcal{S},\mathcal{B}_0)$ shows that $\mathcal{S}=\mathbb{CP}^n$.

To describe $\mathcal{B}_0$ observe that $H\to\mathbb{CP}^n$ has fibers that are not two disjoint points over the locus which is the determinant of a symmetric 2×2 matrix of degree $n{+}1$ forms on $\mathbb{CP}^n$. This is a hypersurface $D\subset\mathbb{CP}^n$ of degree $(2n{+}2)$ which is singular at the locus of codimension 3 in $\mathbb{CP}^n$. The sheaf of algebras $\mathcal{B}_0$ is in this case commutative and is a pushforward of the structure sheaf of the double cover H_1 of $\mathbb{CP}^n$ ramified at D. While H is a small resolution of this cover, the map $H\to H_1$ is not an isomorphism under our assumption $n\geq 3$.

In this case, the Clifford double mirror construction produces the derived category of coherent sheaves on H_1. It is not equivalent to that of H because of the singularities of H_1. The reason for this failure is rooted in the absence of flatness condition of Remark 5.5. It fails precisely over the locus where the 2×2 matrix is identically zero, which is a complete intersection of three hypersurfaces of degree $n+1$ in $\mathbb{CP}^n$. The Kuznetsov's theorem is thus inapplicable, so the argument of Section 6 falls apart.

Remark 9.5. *We believe that even without the flatness or the central fan assumption, there is some kind of Calabi-Yau geometry associated to the decomposition of* $\deg^\vee$ *with coefficients* 1 *and* $\frac{1}{2}$. *However, its precise nature appears more complicated. This is another reason why the primary object of interest is the derived category* $D_B(K,c;\Sigma)$ *for a regular simplicial fan* Σ *in* $K^\vee$.

10. Concluding remarks and open questions.

There are non-toric double mirror examples which this paper does not address. For example the Pfaffian-Grassmannian double mirrors in [Rod00], whose derived equivalence is established in [BC09]; the Hosono-Takagi's construction of Reye congruence [HT14], whose derived equivalence is established in [HT13]. Note also the construction of Bak and Schnell [Bak09, Sch12] related to the Gross-Popescu Calabi-Yau varieties [GP01]. Besides, our method may not be able to cover some examples on derived equivalence between elliptic fibration and its relative Jacobian, see [Cal00a] for discussions on this direction.

Remark 10.1. *Much of the string-theoretic machinery that exists for commutative varieties and even for DM stacks does not exist in the general setting of noncommutative varieties. This paper is an indication that it should be possible to define a class of such mildly noncommutative smooth varieties, that would include smooth DM stacks and to try to extend the following definitions to this class.*

- *Non-linear sigma models with noncommutative targets.*
- *Gromov-Witten and Donaldson-Thomas invariants.*

- *String-theoretic Hodge numbers (to generalize orbifold Hodge numbers)*
- *Chiral de Rham complexes.*

Remark 10.2. *Perhaps the most important takeaway from our paper could be the definition of the derived category associated to reflexive Gorenstein cones. It clearly indicates the need for better understanding of the theories with potentials. One can pose and try to answer the same questions as in Remark 10.1 in this setting.*

Remark 10.3. *It would be interesting to see if there is a Berglund-Hübsch-Krawitz analog of our construction.*

Remark 10.4. *We believe that the double mirror construction for Enriques surfaces in Section 9.2 needs to be investigated further along the lines of Remark 9.1.*

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