Jacobi weights, fractional integration, and sharp Ulyanov inequalities

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Abstract. We consider functions L^p-integrable with Jacobi weights on $[-1,1]$ and prove Hardy–Littlewood type inequalities for fractional integrals. As applications, we obtain the sharp (L_p, L_q) Ulyanov-type inequalities for the Ditzian–Totik moduli of smoothness and the K-functionals of fractional order.

1. Introduction

The following (L_p, L_q) inequalities of Ulyanov-type between moduli of smoothness of functions on $\mathbb{T}$ play an important role in approximation theory and functional analysis (see, e.g., [**7, 13, 15**]):

$$\omega^r(f,t)_q \leqslant C\left(\int_0^t \left(u^{-\sigma}\omega^r(f,u)_p\right)^{q_1}\frac{du}{u}\right)^{1/q_1}, \tag{1.1}$$

where $r\in\mathbb{N}$, $0<p\leqslant q\leqslant\infty$, $\sigma=\frac{1}{p}-\frac{1}{q}$, and $q_1=\begin{cases} q, & q<\infty\\ 1, & q=\infty\end{cases}$. Here the r-th moduli of smoothness of a function $f\in L_p(\mathbb{T})$ is given by

$$\omega^r(f,\delta)_p=\sup_{|h|\leqslant\delta}\left\|\Delta_h^r f(x)\right\|_{L^p(\mathbb{T})}, \quad 1\leqslant p\leqslant\infty,$$

where

$$\Delta_h^r f(x)=\Delta_h^{r-1}\left(\Delta_h f(x)\right) \quad \text{and} \quad \Delta_h f(x)=f(x+h)-f(x).$$

Recently ([**20, 23**]) the sharp version of (1.1) was proved in the case $1<p<q<\infty$:

$$\omega^r(f,t)_q \leqslant C\left(\int_0^t \left(u^{-\sigma}\omega^{r+\sigma}(f,u)_p\right)^{q_1}\frac{du}{u}\right)^{1/q}, \tag{1.2}$$

where $\omega^r(f,u)_p$ is the moduli of smoothness of the (fractional) order $r>0$. Moreover, it turned out that (1.2) also holds if $(p,q)=(1,\infty)$; see [**21**]. In this case $\sigma=1$ and one can work with the classical (not necessary fractional) moduli of smoothness. On the other hand, (1.2) is not true ([**21**]) for $1=p<q<\infty$ or $1<p<q=\infty$.

In the present paper, we consider a nonperiodic case, namely L_p spaces with Jacobi weights on an interval, and obtain inequalities similar to (1.2) for the fractional K-functionals and Ditzian–Totik moduli of smoothness. We start with notation.

Key words and phrases. Jacobi weights, Landau type inequalities, Hardy–Littlewood type inequalities, K-functionals, Ditzian–Totik moduli of smoothness, sharp Ulyanov inequality.

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Denote by $w^{(a,b)}(x)=(1-x)^a(1+x)^b$, $a,b>-1$, the Jacobi weight on $[-1,1]$. For $1\leqslant p<\infty$, let $L_p^{(a,b)}$ be the space of all functions f measurable on $[-1,1]$ with the finite norm

$$\|f\|_{p,(a,b)}=\left(\int_{-1}^{1}|f(x)|^p w^{(a,b)}(x)dx\right)^{1/p}.$$

If $a=b=0$, we write $L_p=L_p^{(a,b)}$, $\|\cdot\|_p=\|\cdot\|_{p,(0,0)}$. In the case $p=\infty$, we set $L_p^{(a,b)}:=C[-1,1]$ and

$$\|f\|_{\infty,(a,b)}=\|f\|_\infty=\max_{x\in[-1,1]}|f(x)|.$$

For an arbitrary interval $[x_1,x_2]$, we set

$$\|f\|_{L_p[x_1,x_2]}=\left(\int_{x_1}^{x_2}|f(x)|^p dx\right)^{1/p},\ 1\leqslant p<\infty,\quad \|f\|_{L_\infty[x_1,x_2]}=\max_{x\in[x_1,x_2]}|f(x)|.$$

For $\alpha,\ \beta>-1$, denote by $\psi_k^{(\alpha,\beta)}(x)$, $k=0,\,1,\,\ldots$, the system of Jacobi polynomials orthogonal on $[-1,1]$ with the weight $w^{(\alpha,\beta)}$ and normalized by the condition

$$\int_{-1}^{1}\left|\psi_k^{(\alpha,\beta)}(x)\right|^2 w^{(\alpha,\beta)}(x)dx=1.$$

The Jacobi polynomials are the eigenfunctions of the differential operator

$$\mathcal{D}=\mathcal{D}_2^{(\alpha,\beta)}=\frac{-1}{w^{(\alpha,\beta)}(x)}\frac{d}{dx}w^{(\alpha,\beta)}(x)(1-x^2)\frac{d}{dx},$$

$$\mathcal{D}\psi_k^{(\alpha,\beta)}=\left(\lambda_k^{(\alpha,\beta)}\right)^2\psi_k^{(\alpha,\beta)},\qquad \lambda_k^{(\alpha,\beta)}=\left(k(k+\alpha+\beta+1)\right)^{1/2}.$$

For a function $f\in L_p^{(\alpha,\beta)}$, $1\leqslant p\leqslant\infty$, the Fourier–Jacobi expansion is defined as follows:

$$f(x)\sim\sum_{k=0}^{\infty}\widehat{f}_k^{(\alpha,\beta)}\psi_k^{(\alpha,\beta)}(x), \tag{1.3}$$

where

$$\widehat{f}_k^{(\alpha,\beta)}=\int_{-1}^{1}f(x)\psi_k^{(\alpha,\beta)}(x)w^{(\alpha,\beta)}(x)dx,\quad k=0,\,1,\,2,\,\ldots$$

Let $\sigma>0$. If there exists a function $g\in L_1^{(\alpha,\beta)}$ such that its Fourier–Jacobi expansion has the form

$$g\sim\sum_{k=1}^{\infty}\left(\lambda_k^{(\alpha,\beta)}\right)^{\sigma}\widehat{f}_k^{(\alpha,\beta)}\psi_k^{(\alpha,\beta)},$$

then we use the notation

$$g=\mathcal{D}_\sigma^{(\alpha,\beta)}f$$

and we call $\mathcal{D}_\sigma^{(\alpha,\beta)}f$ the fractional derivative of order σ of the function f. If there exists a function $h\in L_1^{(\alpha,\beta)}$ such that its Fourier–Jacobi expansion has the form

$$h\sim\widehat{f}_0^{(\alpha,\beta)}+\sum_{k=1}^{\infty}\left(\lambda_k^{(\alpha,\beta)}\right)^{-\sigma}\widehat{f}_k^{(\alpha,\beta)}\psi_k^{(\alpha,\beta)},$$

then we use the notation

$$h=\mathcal{I}_\sigma^{(\alpha,\beta)}f$$

and we call $\mathcal{I}_\sigma^{(\alpha,\beta)}f$ the fractional integral of order σ of the function f. Notice that $\mathcal{I}_\sigma^{(\alpha,\beta)}$, $\sigma>0$, is a bounded linear operator on $L_1^{(\alpha,\beta)}$ (see, e.g., [**3**, Sec. 5, pp. 789–790]).

The K-functional corresponding to the differential operator $\mathcal{D}^{(\alpha,\beta)}$ and a real positive number r is defined by

$$K^r(f,\mathcal{D}_r^{(\alpha,\beta)},t)_{p,(\alpha,\beta)}=\inf\left\{\|f-g\|_{p,(\alpha,\beta)}+t^r\|\mathcal{D}_r^{(\alpha,\beta)}g\|_{p,(\alpha,\beta)}:\ g\in W^{r,(\alpha,\beta)}_{p,(\alpha,\beta)}\right\} \tag{1.4}$$

(see [**10**, (1.9)]), where $W^{r,(\alpha,\beta)}_{p,(\alpha,\beta)}=\left\{g:g,\,\mathcal{D}_r^{(\alpha,\beta)}g\in L_p^{(\alpha,\beta)}\right\}$.

The main result of this paper is the following

Theorem 1. *Let $1<p<q<\infty$, $r>0$, $\alpha\geqslant\beta>-1$, $\alpha\geqslant-1/2$. Suppose also that*

$$\sigma=(2\alpha+2)\left(\frac{1}{p}-\frac{1}{q}\right).$$

If $f\in L_p^{(\alpha,\beta)}$ and

$$\int_0^1\left(u^{-\sigma}K^{r+\sigma}(f,\mathcal{D}_{r+\sigma}^{(\alpha,\beta)},u)_{p,(\alpha,\beta)}\right)^q\frac{du}{u}<\infty,$$

then $f\in L_q^{(\alpha,\beta)}$ and

$$K^r(f,\mathcal{D}_r^{(\alpha,\beta)},t)_{q,(\alpha,\beta)}\leqslant C\left(\int_0^t\left(u^{-\sigma}K^{r+\sigma}(f,\mathcal{D}_{r+\sigma}^{(\alpha,\beta)},u)_{p,(\alpha,\beta)}\right)^q\frac{du}{u}\right)^{1/q}.$$

The rest of the paper is organized as follows. In Section 2 we obtain the key result to get sharp Ulyanov inequalities – the weighted inequalities of Hardy–Littlewood and Landau type for functions defined on the interval $[-1,1]$. Section 3 contains the definition of fractional K-functionals with Jacobi weights and sharp Ulyanov inequalities for K-functionals (Theorem 3). In Section 4 analogous results for the Ditzian–Totik moduli of smoothness are obtained. Namely, we study a relationship between these moduli and the corresponding K-functionals and prove sharp Ulyanov inequalities for the Ditzian–Totik moduli in the case of $1\leqslant p\leqslant q\leqslant\infty$ (Theorem 5).

2. Inequalities for fractional integrals with Jacobi weights

2.1. Landau-type inequalities. We will need the following Hardy-type inequality (see, e.g., [**5**] and [**19**, Theorem 6.2, Example 6.8]). We set $\frac{1}{q}:=0$ for $q=\infty$.

Theorem A. *Let $1\leqslant p\leqslant q\leqslant\infty$, $(p,q)\neq(\infty,\infty)$, $a>-\frac{1}{q}$, $\overline{x}\in(0,\infty)$. Then the inequality*

$$\|f(x)x^a\|_{L_q[0,\overline{x}]}\leqslant C(p,q,a,\overline{x})\left\|f'(x)x^{a+h}\right\|_{L_p[0,\overline{x}]}$$

holds for any locally absolutely continuous function f on $(0,\overline{x}]$ with the property $f(\overline{x})=0$ if and only if $h\leqslant 1-\left(\frac{1}{p}-\frac{1}{q}\right)$.

Let us mention that the quantity $C(p,q,a,\overline{x})$ is nondecreasing with respect to $\overline{x}$.

The following Landau–type inequality can be found in, e.g., [**6**, Ch. 2, Th. 5.6, p. 38].

Theorem B. *For $1\leqslant p\leqslant\infty$, $\ell\geqslant 2$, there is a constant $C(\ell)$ such that for all $r=0,\dots,\ell$ and any function f with $f^{(\ell-1)}$ absolutely continuous on $\left[-\frac{1}{2},\frac{1}{2}\right]$ and $f^{(\ell)}\in L_p\left[-\frac{1}{2},\frac{1}{2}\right]$ we have*

$$\left\|f^{(r)}\right\|_{L_p\left[-\frac{1}{2},\frac{1}{2}\right]}\leqslant C(\ell)\left(\|f\|_{L_p\left[-\frac{1}{2},\frac{1}{2}\right]}+\left\|f^{(\ell)}\right\|_{L_p\left[-\frac{1}{2},\frac{1}{2}\right]}\right).$$

As a corollary of Theorem A and Theorem B we get

Lemma 1. *Suppose that* $1 \leqslant p \leqslant q \leqslant \infty$, $(p,q) \neq (\infty,\infty)$, $a,b > -\frac{1}{q}$, $c,d > -\frac{1}{p}$, r *is a nonnegative integer,* k *is a positive integer, and*

$$h = k - \left(\frac{1}{p} - \frac{1}{q}\right).$$

Then, there exists a constant $C = C(p,q,a,b,c,d,r,k)$ *such that for any function* f *with* $f^{(r+k-1)}$ *absolutely continuous on* $(-1,1)$ *and* $f^{(r+k)}w^{(a+h,b+h)} \in L_p$ *we have*

$$\left\|f^{(r)}w^{(a,b)}\right\|_q \leqslant C\left(\left\|fw^{(c,d)}\right\|_p + \left\|f^{(r+k)}w^{(a+h,b+h)}\right\|_p\right). \tag{2.1}$$

Inequality (2.1) *is sharp in the following sense. If* $a - c < r + \left(\frac{1}{p} - \frac{1}{q}\right)$, *then for any* $\varepsilon > 0$ *there exists* $\{f_n\} \subset C^{k+r}[-1,1]$ *such that*

$$\left\|f_n^{(r)}w^{(a,b)}\right\|_q \cdot \left(\left\|f_n w^{(c,d)}\right\|_1 + \left\|f_n^{(r+k)}w^{(a+h+\varepsilon,b+h)}\right\|_p\right)^{-1} \to \infty \quad as \quad n \to \infty. \tag{2.2}$$

The analogous statement also holds with respect to the parameter b.

Proof of Lemma 1. It is enough to verify inequality (2.1) for $k=1$. The proof in the general case is by induction on k. Note that $f^{(r)}$ is continuous on $\left[-\frac{1}{2},\frac{1}{2}\right]$ by our assumption. We take $\overline{x} \in \left[-\frac{1}{2},\frac{1}{2}\right]$ such that

$$\left|f^{(r)}(\overline{x})\right| = \min\left\{\left|f^{(r)}(x)\right| : x \in \left[-\tfrac{1}{2},\tfrac{1}{2}\right]\right\}.$$

Let $g(x) = f^{(r)}(x) - f^{(r)}(\overline{x})$, then

$$\begin{aligned}\left\|f^{(r)}w^{(a,b)}\right\|_q &\leqslant \left\|gw^{(a,b)}\right\|_q + \left|f^{(r)}(\overline{x})\right| \left\|w^{(a,b)}\right\|_q \\ &\leqslant \left\|gw^{(a,b)}\right\|_{L_q[-1,\overline{x}]} + \left\|gw^{(a,b)}\right\|_{L_q[\overline{x},1]} + \left|f^{(r)}(\overline{x})\right| \left\|w^{(a,b)}\right\|_{L_q[-1,1]}.\end{aligned}$$

To estimate the first term, we apply Theorem A (for the interval $[-1,\overline{x}]$ instead of $[0,\overline{x}]$) with $h = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)$:

$$\begin{aligned}\left\|gw^{(a,b)}\right\|_{L_q[-1,\overline{x}]} &\leqslant 2^{|a|}\left\|g(x)(1+x)^b\right\|_{L_q[-1,\overline{x}]} \leqslant 2^{|a|}C\left\|g'(x)(1+x)^{b+h}\right\|_{L_q[-1,\overline{x}]} \\ &\leqslant 2^{|a|+|a+h|}C\left\|g'(x)(1-x)^{a+h}(1+x)^{b+h}\right\|_{L_p[-1,\overline{x}]} \\ &\leqslant 2^{|a|+|a+h|}C\left\|g'w^{(a+h,b+h)}\right\|_{L_p[-1,1]} = 2^{|a|+|a+h|}C\left\|f^{(r+1)}w^{(a+h,b+h)}\right\|_{L_p[-1,1]}.\end{aligned}$$

A similar estimate holds for $\left\|gw^{(a,b)}\right\|_{L_q[\overline{x},1]}$ as well.

To estimate $\left|f^{(r)}(\overline{x})\right|$, we apply Theorem B:

$$\begin{aligned}\left|f^{(r)}(\overline{x})\right| &\leqslant \left\|f^{(r)}\right\|_{L_1\left[-\frac{1}{2},\frac{1}{2}\right]} \leqslant C\left(\|f\|_{L_1\left[-\frac{1}{2},\frac{1}{2}\right]} + \left\|f^{(r+1)}\right\|_{L_1\left[-\frac{1}{2},\frac{1}{2}\right]}\right) \\ &\leqslant 2^{|c|+|d|+|a+h|+|b+h|}C\left(\left\|fw^{(c,d)}\right\|_{L_p[-1,1]} + \left\|f^{(r+1)}w^{(a+h,b+h)}\right\|_{L_p[-1,1]}\right),\end{aligned}$$

where C depends only on $r+1$. Thus, (2.1) follows.

Let us now show (2.2). Since for any $0 \leqslant \varepsilon_1 \leqslant \varepsilon_2$ the estimate

$$w^{(a+h+\varepsilon_2,b+h)}(x) \leqslant 2^{\varepsilon_2-\varepsilon_1}w^{(a+h+\varepsilon_1,b+h)}(x), \qquad x \in [-1,1],$$

holds, we can assume

$$0 < \varepsilon \leqslant c - a + r + 1/p - 1/q. \tag{2.3}$$

For $m > r + k$, consider the sequence of functions

$$f_n(x) = \big((x + 1/n - 1)_+\big)^m, \quad x \in [-1, 1], \qquad y_+ = \max\{y, 0\}.$$

It is easy to verify that if $\mu \geqslant 0$ and $\nu > -1/q$, then

$$\left\| \big((1/n - 1 + x)_+\big)^\mu (1 - x)^\nu \right\|_q \asymp \frac{1}{n^{\mu+\nu+1/q}} \quad \text{as} \quad n \to \infty.$$

Here $A_n \asymp B_n$ as $n \to \infty$ means that $B_n/C \leqslant A_n \leqslant CB_n$ for some positive constant C and all n. Using this, we get

$$\left\| f_n w^{(c,d)} \right\|_p \asymp \frac{1}{n^{m+c+1/p}}, \qquad \left\| f_n^{(r)} w^{(a,b)} \right\|_q \asymp \frac{1}{n^{m-r+a+1/q}},$$

$$\left\| f_n^{(r+k)} w^{(a+h+\varepsilon,b+h)} \right\|_p \asymp \frac{1}{n^{m-r-k+a+h+\varepsilon+1/p}} = \frac{1}{n^{m-r+a+\varepsilon+1/q}}.$$

Under assumption (2.3) we have

$$\left\| f_n w^{(c,d)} \right\|_p + \left\| f_n^{(r+k)} w^{(a+h+\varepsilon,b+h)} \right\|_p \asymp \frac{1}{n^{m-r+a+\varepsilon+1/q}},$$

and therefore,

$$\frac{\left\| f_n^{(r)} w^{(a,b)} \right\|_q}{\left\| f_n w^{(c,d)} \right\|_p + \left\| f_n^{(r+k)} w^{(a+h+\varepsilon,b+h)} \right\|_p} \asymp n^\varepsilon \quad \text{as} \quad n \to \infty,$$

concluding the proof. □

2.2. Hardy–Littlewood type inequalities. To prove Hardy–Littlewood type inequalities for the fractional integral $\mathcal{I}_\sigma^{(\alpha,\beta)}$, we will use the Muckenhoupt transplantation theorem [**18**, Collorary 17.11], which is written in our notation as follows.

Theorem C. *If* $1 < \overline{p} \leqslant \overline{q} < \infty$, $\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta} > -1$, $\overline{a}, \overline{b}, \overline{c}, \overline{d} > -1$,

$$s = \frac{1}{\overline{p}} - \frac{1}{\overline{q}},$$

$$\frac{\overline{a}}{\overline{q}} = \frac{\overline{c}}{\overline{p}} + \frac{\overline{\alpha} - \overline{\gamma}}{2} + \frac{1}{2}\left(\frac{1}{\overline{p}} - \frac{1}{\overline{q}}\right), \quad \frac{\overline{b}}{\overline{q}} = \frac{\overline{d}}{\overline{p}} + \frac{\overline{\beta} - \overline{\delta}}{2} + \frac{1}{2}\left(\frac{1}{\overline{p}} - \frac{1}{\overline{q}}\right),$$

the quantities $\overline{A} = (\overline{c} + 1)/\overline{p} - \overline{\gamma}$ *and* $\overline{B} = (\overline{d} + 1)/\overline{p} - \overline{\delta}$ *are not positive integers,* $M = \max\{0, [\overline{A}]\}$, $N = \max\{0, [\overline{B}]\}$, $f \in L_{\overline{p}}^{(\overline{c},\overline{d})}$,

$$\widehat{f}_k^{(\overline{\gamma},\overline{\delta})} = 0, \quad 0 \leqslant k \leqslant M + N - 1,$$

h is an integer, ν_k *has the form*

$$\nu_k = \sum_{j=0}^{J-1} c_j (k+1)^{-s-j} + O\left((k+1)^{-s-J}\right)$$

with $J \geqslant \overline{\alpha} + \overline{\beta} + \overline{\gamma} + \overline{\delta} + 6 + 2M + 2N$ *and* $0 \leqslant \rho < 1$, *then*

$$T_\rho f(x) = \sum_{k=0}^{\infty} \rho^k \nu_k \widehat{f}_k^{(\overline{\gamma},\overline{\delta})} \psi_{k+h}^{(\overline{\alpha},\overline{\beta})}(x)$$

converges for every $x \in (-1, 1)$,

$$\|T_\rho f\|_{\overline{q},(\overline{a},\overline{b})} \leqslant C \, \|f\|_{\overline{p},(\overline{c},\overline{d})},$$

where C is independent of ρ and f. Moreover, there is a function Tf in $L_{\overline{q}}^{(\overline{a},\overline{b})}$ such that $T_\rho f$ converges to Tf in $L_{\overline{q}}^{(\overline{a},\overline{b})}$ as $\rho \to 1-$. If it is also assumed that $\overline{a}+1 < (\overline{\alpha}+1)\overline{q}$ and $\overline{b}+1 < (\overline{\beta}+1)\overline{q}$, then

$$\widehat{Tf}_k^{(\overline{\alpha},\overline{\beta})} = \begin{cases} 0, & 0 \leqslant k \leqslant h-1 \\ \nu_{k-h}\widehat{f}_{k-h}^{(\overline{\gamma},\overline{\delta})}, & \max(0,h) \leqslant k. \end{cases}$$

The next Hardy–Littlewood inequality is a simple corollary of Theorem C.

Corollary 1. *Let $1 < p < q < \infty$, $-1/2 \geqslant a \geqslant b > -1$, $\alpha \geqslant \beta > -1$, $(a+1) < (\alpha+1)p$, $(b+1) < (\beta+1)p$, and*

$$\sigma \geqslant \frac{1}{p} - \frac{1}{q}.$$

Let also $f \in L_p^{(a,b)}$. Then there exists C independent of f such that

$$\left\| \mathcal{I}_\sigma^{(\alpha,\beta)} f \right\|_{q,(a,b)} \leqslant C \| f \|_{p,(a,b)}. \tag{2.4}$$

In the special case $(\alpha,\beta) = (a,b)$, the Hardy–Littlewood inequality (2.4) was studied by Askey and Wainger [**2**, Sec. J] (see also [**1**]) and later by Bavinck and Trebels [**3**, Theorem 5.4], [**4**, Theorems 1 and 1'].

Theorem D ([**2, 4**]). *Let $1 < p < q < \infty$, $a \geqslant b > -1$, $a+b \geqslant -1$, and*

$$\sigma \geqslant (2a+2)\left(\frac{1}{p} - \frac{1}{q}\right).$$

If $f \in L_p^{(a,b)}$, then $\mathcal{I}_\sigma^{(a,b)} f \in L_q^{(a,b)}$ and

$$\left\| \mathcal{I}_\sigma^{(a,b)} f \right\|_{q,(a,b)} \leqslant C(p,q,a,b) \| f \|_{p,(a,b)}.$$

For $(\alpha,\beta) \neq (a,b)$ we have the following result.

Theorem 2. *Let $1 < p < q < \infty$, $a \geqslant b > -1$, $a \geqslant -1/2$, $\alpha \geqslant \beta > -1$,*

$$p(\alpha-\beta) \leqslant 2(a-b) \leqslant q(\alpha-\beta), \tag{2.5}$$

the quantities $A = (a+1)/p - \alpha$ and $B = (b+1)/p - \beta$ be not positive integers, and either $\alpha = a$, or $\alpha > a$ and $q > 2$, or $\alpha < a$ and $p < 2$. Let

$$\sigma \geqslant (2a+2)\left(\frac{1}{p} - \frac{1}{q}\right), \tag{2.6}$$

$f \in L_p^{(a,b)} \cap L_1^{(\alpha,\beta)}$ and

$$\widehat{f}_k^{(\alpha,\beta)} = 0, \quad 0 \leqslant k \leqslant \max\{0,[A]\} + \max\{0,[B]\} - 1. \tag{2.7}$$

Then there exists C independent of f such that

$$\left\| \mathcal{I}_\sigma^{(\alpha,\beta)} f \right\|_{q,(a,b)} \leqslant C \| f \|_{p,(a,b)}. \tag{2.8}$$

Proof. It is sufficient to prove this theorem for polynomials. Indeed, suppose that (2.8) holds for polynomials. Consider a sequence of polynomials $\{Q_m\}$ convergent to f in $L_p^{(a,b)}$ and $L_1^{(\alpha,\beta)}$. Then $\{\mathcal{I}_\sigma^{(\alpha,\beta)} Q_m\}$ is a Cauchy sequence in $L_q^{(a,b)}$ and it converges to some function g in $L_q^{(a,b)}$. Without loss of generality we can assume that $\{\mathcal{I}_\sigma^{(\alpha,\beta)} Q_m\}$ converges to g a.e. on $[-1,1]$. Since the operator $\mathcal{I}_\sigma^{(\alpha,\beta)}$ is continuous in $L_1^{(\alpha,\beta)}$, the sequence $\{\mathcal{I}_\sigma^{(\alpha,\beta)} Q_m\}$ converges to $\mathcal{I}_\sigma^{(\alpha,\beta)} f$ in $L_1^{(\alpha,\beta)}$. There is a subsequence $\{\mathcal{I}_\sigma^{(\alpha,\beta)} Q_{m_j}\}$ convergent to $\mathcal{I}_\sigma^{(\alpha,\beta)} f$ a.e. on $[-1,1]$. Therefore, $g = \mathcal{I}_\sigma^{(\alpha,\beta)} f$.

Let f be a polynomial, i.e.,

$$f = \sum_{k=0}^{\infty} c_k \psi_k^{(\alpha,\beta)},$$

where $c_k = \widehat{f}_k^{(\alpha,\beta)}$ and $c_k = 0$ for $k > \deg(f)$.

Case 1. Consider $\alpha \geqslant a$, $q \geqslant 2$. More precisely, under assumption of the theorem, the following relations are possible: $\alpha > a$ and $q > 2$ or $\alpha = a$ and $q \geqslant 2$.

Now, we define α_1 and p_1. If $\alpha > a$, then we set

$$\alpha_1 = \frac{q\alpha - 2a}{q-2},$$
$$\frac{\alpha_1}{p_1} = \frac{a}{p} + \frac{\alpha_1 - \alpha}{2} + \frac{1}{2}\left(\frac{1}{p} - \frac{1}{p_1}\right).$$

In this case, we have

$$\frac{2\alpha_1 + 1}{p_1} = \frac{2a+1}{p} + \frac{2(\alpha - a)}{q-2}$$

and

$$(2\alpha_1 + 2)\left(\frac{1}{p_1} - \frac{1}{q}\right) + \frac{1}{p} - \frac{1}{p_1} = (2a+2)\left(\frac{1}{p} - \frac{1}{q}\right). \tag{2.9}$$

Notice that condition $\alpha > a$ implies that $\alpha_1 > \max\{a, \alpha, 0\}$ and $p < p_1 < q$.

If $\alpha = a$, then we set $\alpha_1 = \alpha$, $p_1 = p$.

We divide the rest of the proof in Case 1 into three steps.

Step 1.1. We apply Theorem C with $(\overline{q}, \overline{p}) = (p_1, p)$, $(\overline{\alpha}, \overline{\beta}) = (\alpha_1, \alpha_1)$, $(\overline{\gamma}, \overline{\delta}) = (\alpha, \beta)$, $(\overline{c}, \overline{d}) = (a, b)$, $h = 0$, $s = \sigma_1 = \frac{1}{p} - \frac{1}{p_1}$, and

$$\nu_k = \left(\lambda_k^{(\alpha_1,\alpha_1)}\right)^{-\sigma_1}.$$

Then we have $\overline{a} = \alpha_1$,

$$\frac{\overline{b}}{p_1} = \frac{b}{p} + \frac{\alpha_1 - \beta}{2} + \frac{1}{2}\left(\frac{1}{p} - \frac{1}{p_1}\right) = \frac{\alpha_1}{p_1} - \frac{2(a-b) - p(\alpha - \beta)}{2p}, \tag{2.10}$$

$$A = \frac{a+1}{p} - \alpha, \quad B = \frac{b+1}{p} - \beta.$$

Therefore, under condition (2.7) for any $\rho \in (0,1)$, we obtain the inequality

$$\left\| c_0 + \sum_{k=1}^{\infty} \rho^k \left(\lambda_k^{(\alpha_1,\alpha_1)}\right)^{-\sigma_1} c_k \psi_k^{(\alpha_1,\alpha_1)} \right\|_{p_1,(\alpha_1,\overline{b})} \leqslant C\|f\|_{p,(a,b)}, \tag{2.11}$$

where C is independent of f and ρ. Since f is a polynomial, the sum is finite, and we can rewrite (2.11) as

$$\left\| c_0 + \sum_{k=1}^{\infty} \left(\lambda_k^{(\alpha_1,\alpha_1)}\right)^{-\sigma_1} c_k \psi_k^{(\alpha_1,\alpha_1)} \right\|_{p_1,(\alpha_1,\overline{b})} \leqslant C\|f\|_{p,(a,b)}.$$

Relations (2.5) and (2.10) show that $\alpha_1 \geqslant \overline{b}$, and hence,

$$\left\| c_0 + \sum_{k=1}^{\infty} \left(\lambda_k^{(\alpha_1,\alpha_1)}\right)^{-\sigma_1} c_k \psi_k^{(\alpha_1,\alpha_1)} \right\|_{p_1,(\alpha_1,\alpha_1)} \leqslant C\|f\|_{p,(a,b)}. \tag{2.12}$$

Step 1.2. In view of (2.6) and (2.9), we have

$$\sigma - \sigma_1 \geqslant (2\alpha_1 + 2)\left(\frac{1}{p_1} - \frac{1}{q}\right),$$

we can apply Theorem D for the pair of spaces $L_q^{(\alpha_1,\alpha_1)}$ and $L_{p_1}^{(\alpha_1,\alpha_1)}$ to get

$$\left\| c_0 + \sum_{k=1}^{\infty} \left(\lambda_k^{(\alpha_1,\alpha_1)}\right)^{-\sigma} c_k \psi_k^{(\alpha_1,\alpha_1)} \right\|_{q,(\alpha_1,\alpha_1)} \leqslant C \left\| c_0 + \sum_{k=1}^{\infty} \left(\lambda_k^{(\alpha_1,\alpha_1)}\right)^{-\sigma_1} c_k \psi_k^{(\alpha_1,\alpha_1)} \right\|_{p_1,(\alpha_1,\alpha_1)}. \tag{2.13}$$

Step 1.3. We use Theorem C once again with $(\overline{q}, \overline{p}) = (q, q)$, $(\overline{\alpha}, \overline{\beta}) = (\alpha, \beta)$, $(\overline{\gamma}, \overline{\delta}) = (\alpha_1, \alpha_1)$, $(\overline{c}, \overline{d}) = (\alpha_1, \alpha_1)$, and

$$\nu_k = \left(\lambda_k^{(\alpha,\beta)} / \lambda_k^{(\alpha_1,\alpha_1)}\right)^{-\sigma}.$$

Then $s = 0$, $\overline{a} = a$,

$$\frac{\overline{b}}{q} = \frac{\alpha_1}{q} + \frac{\beta - \alpha_1}{2} = \frac{b}{q} - \frac{q(\alpha - \beta) - 2(a - b)}{2q}, \tag{2.14}$$

and

$$A = B = \frac{\alpha_1 + 1}{q} - \alpha_1 = \alpha_1 \left(\frac{1}{q} - 1\right) + \frac{1}{q} \leqslant -\frac{1}{2}\left(\frac{1}{q} - 1\right) + \frac{1}{q} < 1, \quad [A] = [B] = 0.$$

We have

$$\left\| c_0 + \sum_{k=1}^{\infty} \left(\lambda_k^{(\alpha,\beta)}\right)^{-\sigma} c_k \psi_k^{(\alpha,\beta)} \right\|_{q,(a,\overline{b})} \leqslant C \left\| c_0 + \sum_{k=1}^{\infty} \left(\lambda_k^{(\alpha_1,\alpha_1)}\right)^{-\sigma} c_k \psi_k^{(\alpha_1,\alpha_1)} \right\|_{q,(\alpha_1,\alpha_1)}.$$

Relations (2.5) and (2.14) show that $\overline{b} \leqslant b$, and hence,

$$\left\| c_0 + \sum_{k=1}^{\infty} \left(\lambda_k^{(\alpha,\beta)}\right)^{-\sigma} c_k \psi_k^{(\alpha,\beta)} \right\|_{q,(a,b)} \leqslant 2^{b-\overline{b}} \left\| c_0 + \sum_{k=1}^{\infty} \left(\lambda_k^{(\alpha,\beta)}\right)^{-\sigma} c_k \psi_k^{(\alpha,\beta)} \right\|_{q,(a,\overline{b})}. \tag{2.15}$$

Finally, combining (2.12), (2.13), and (2.15), we obtain inequality (2.8).

Case 2. Consider $\alpha \leqslant a$, $p \leqslant 2$. More precisely, under assumption of the theorem, the following relations are possible: $\alpha < a$ and $p < 2$ or $\alpha = a$ and $p \leqslant 2$.

Now, we define α_1 and q_1. If $\alpha < a$, then we set

$$\alpha_1 = \frac{2a - p\alpha}{2 - p},$$
$$\frac{a}{q} = \frac{\alpha_1}{q_1} + \frac{\alpha - \alpha_1}{2} + \frac{1}{2}\left(\frac{1}{q_1} - \frac{1}{q}\right).$$

In this case, we have

$$\frac{2\alpha_1 + 1}{q_1} = \frac{2a + 1}{q} + \frac{2(a - \alpha)}{2 - p}$$

and

$$(2\alpha_1 + 2)\left(\frac{1}{p} - \frac{1}{q_1}\right) + \frac{1}{q_1} - \frac{1}{q} = (2a + 2)\left(\frac{1}{p} - \frac{1}{q}\right). \tag{2.16}$$

Notice that condition $\alpha < a$ implies that $\alpha_1 > \max\{a, \alpha, 0\}$ and $p < q_1 < q$.

If $\alpha = a$, then we set $\alpha_1 = \alpha$, $q_1 = q$.

We can argue similarly to the proof in Case 1 dividing the rest of the proof into three steps.

Step 2.1. We are going to use Theorem C with $(\overline{q},\overline{p})=(p,p)$, $(\overline{\alpha},\overline{\beta})=(\alpha_1,\alpha_1)$, $(\overline{\gamma},\overline{\delta})=(\alpha,\beta)$, $(\overline{c},\overline{d})=(a,b)$, $h=0$, $s=0$, and $\nu_k=1$. Then $\overline{a}=\alpha_1$,

$$\frac{\overline{b}}{p}=\frac{b}{p}+\frac{\alpha_1-\beta}{2}=\frac{\alpha_1}{p}-\frac{2(a-b)-p(\alpha-\beta)}{2p}, \tag{2.17}$$

$$A=\frac{a+1}{p}-\alpha,\quad B=\frac{b+1}{p}-\beta.$$

Therefore, under condition (2.7) for any $\rho\in(0,1)$, we obtain the inequality

$$\left\|c_0+\sum_{k=1}^{\infty}\rho^k c_k\psi_k^{(\alpha_1,\alpha_1)}\right\|_{p,(\alpha_1,\overline{b})}\leqslant C\|f\|_{p,(a,b)}, \tag{2.18}$$

where C does not depend on f and ρ. Since f is a polynomial, the sum is finite. Taking into account (2.5) and (2.17), we conclude that $\alpha_1\geqslant\overline{b}$, and hence, and we can rewrite (2.18) as

$$\left\|c_0+\sum_{k=1}^{\infty}c_k\psi_k^{(\alpha_1,\alpha_1)}\right\|_{p,(\alpha_1,\alpha_1)}\leqslant C\|f\|_{p,(a,b)}. \tag{2.19}$$

Step 2.2. Set $\sigma_1=\sigma-\left(\frac{1}{q_1}-\frac{1}{q}\right)$. In view of (2.6) and (2.16), we have

$$\sigma_1\geqslant(2\alpha_1+1)\left(\frac{1}{p}-\frac{1}{q_1}\right).$$

We can apply Theorem D for the pair of spaces $L_{q_1}^{(\alpha_1,\alpha_1)}$ and $L_p^{(\alpha_1,\alpha_1)}$ to get

$$\left\|c_0+\sum_{k=1}^{\infty}\left(\lambda_k^{(\alpha_1,\alpha_1)}\right)^{-\sigma_1}c_k\psi_k^{(\alpha_1,\alpha_1)}\right\|_{q_1,(\alpha_1,\alpha_1)}\leqslant C\left\|c_0+\sum_{k=1}^{\infty}c_k\psi_k^{(\alpha_1,\alpha_1)}\right\|_{p,(\alpha_1,\alpha_1)}. \tag{2.20}$$

Step 2.3. We use Theorem C once again with $(\overline{q},\overline{p})=(q,q_1)$, $(\overline{\alpha},\overline{\beta})=(\alpha,\beta)$, $(\overline{\gamma},\overline{\delta})=(\alpha_1,\alpha_1)$, $(\overline{c},\overline{d})=(\alpha_1,\alpha_1)$, and

$$\nu_k=\left(\lambda_k^{(\alpha,\beta)}\right)^{-(\sigma-\sigma_1)}\left(\lambda_k^{(\alpha_1,\alpha_1)}/\lambda_k^{(\alpha,\beta)}\right)^{\sigma_1}.$$

Hence, $s=\sigma-\sigma_1=\frac{1}{q_1}-\frac{1}{q}$, $\overline{a}=a$,

$$\frac{\overline{b}}{q}=\frac{\alpha_1}{q_1}+\frac{\beta-\alpha_1}{2}+\frac{1}{2}\left(\frac{1}{q_1}-\frac{1}{q}\right)=\frac{b}{q}-\frac{q(\alpha-\beta)-2(a-b)}{2q}, \tag{2.21}$$

and

$$A=B=\frac{\alpha_1+1}{q_1}-\alpha_1=\alpha_1\left(\frac{1}{q_1}-1\right)+\frac{1}{q_1}\leqslant-\frac{1}{2}\left(\frac{1}{q_1}-1\right)+\frac{1}{q_1}<1,\quad [A]=[B]=0.$$

We have

$$\left\|c_0+\sum_{k=1}^{\infty}\left(\lambda_k^{(\alpha,\beta)}\right)^{-\sigma}c_k\psi_k^{(\alpha,\beta)}\right\|_{q,(a,\overline{b})}\leqslant C\left\|c_0+\sum_{k=1}^{\infty}\left(\lambda_k^{(\alpha_1,\alpha_1)}\right)^{-\sigma_1}c_k\psi_k^{(\alpha_1,\alpha_1)}\right\|_{q_1,(\alpha_1,\alpha_1)}.$$

Taking into account (2.5) and (2.21), we see that $\overline{b}\leqslant b$, and hence,

$$\left\|c_0+\sum_{k=1}^{\infty}\left(\lambda_k^{(\alpha,\beta)}\right)^{-\sigma}c_k\psi_k^{(\alpha,\beta)}\right\|_{q,(a,b)}\leqslant 2^{b-\overline{b}}\left\|c_0+\sum_{k=1}^{\infty}\left(\lambda_k^{(\alpha,\beta)}\right)^{-\sigma}c_k\psi_k^{(\alpha,\beta)}\right\|_{q,(a,\overline{b})}. \tag{2.22}$$

Finally, combining (2.19), (2.20), and (2.22), we obtain inequality (2.8).

□

3. Ulyanov-type inequalities for K-functionals

Definitions and facts, given in this section and in the next one, are based on the books [**14, 16**]; see also [**8, 10**] and the recent survey [**11**].

In this section, we assume that $1 \leqslant p \leqslant \infty$, $a, b > -1$, $\alpha, \beta > -1$ and

$$\frac{a+1}{p} - \alpha < 1, \quad \frac{b+1}{p} - \beta < 1. \tag{3.1}$$

Then, since $L_p^{(a,b)} \subset L_1^{(\alpha,\beta)}$, the Fourier–Jacobi expansion (1.3) is well-defined for any $f \in L_p^{(a,b)}$.

Denote by Π_n the set of all algebraic polynomials of degree at most n, $\Pi = \cup_{n \geqslant 0} \Pi_n$. Let $P_{n,f} = P_n(f)_{p,(a,b)}$, $P_{n,f} \in \Pi_n$, be a near best polynomial approximant of a function $f \in L_p^{(a,b)}$, that is,

$$\|f - P_{n,f}\|_{p,(a,b)} \leqslant C E_n(f)_{p,(a,b)}, \quad E_n(f)_{p,(a,b)} = \inf\left\{\|f - P\|_{p,(a,b)} : \ P \in \Pi_n\right\}. \tag{3.2}$$

The K-functional corresponding to the differential operator $\mathcal{D}^{(\alpha,\beta)}$ and a real positive number r is defined by

$$K^r(f, \mathcal{D}_r^{(\alpha,\beta)}, t)_{p,(a,b)} = \inf\left\{\|f - g\|_{p,(a,b)} + t^r \|\mathcal{D}_r^{(\alpha,\beta)} g\|_{p,(a,b)} : \ g \in W_{p,(a,b)}^{r,(\alpha,\beta)}\right\} \tag{3.3}$$

(see [**10**, (1.9)]), where $W_{p,(a,b)}^{r,(\alpha,\beta)} = \left\{g : g, \, \mathcal{D}_r^{(\alpha,\beta)} g \in L_p^{(a,b)}\right\}$. The following realization result holds:

$$K^r\left(f, \mathcal{D}_r^{(\alpha,\beta)}, 1/n\right)_{p,(a,b)} \asymp \|f - P_{n,f}\|_{p,(a,b)} + n^{-r} \|\mathcal{D}_r^{(\alpha,\beta)} P_{n,f}\|_{p,(a,b)}, \ 1 < p < \infty. \tag{3.4}$$

It is a corollary of Theorem 6.2 in [**10**]. To apply this theorem, we have to show that the Cesàro operator C_n^ℓ given by

$$C_n^\ell(f) = \sum_{k=0}^{n} \left(1 - \tfrac{k}{n+1}\right)\left(1 - \tfrac{k}{n+2}\right) \cdots \left(1 - \tfrac{k}{n+\ell}\right) \widehat{f}_k \psi_k^{(\alpha,\beta)}$$

is bounded in $L_p^{(a,b)}$ for some ℓ. This fact is mentioned in [**8**, Sec. 3]. Moreover, from [**18**, Theorem 1.10, p. 4] (see also [**8**, Theorem M]) it easily follows that the operator C_n^ℓ is bounded in $L_p^{(a,b)}$ for any

$$\begin{aligned} \ell > \max \Bigg\{ & \left|\tfrac{2(a+1)}{p} - \alpha - 1\right|, \left|\tfrac{2(b+1)}{p} - \beta - 1\right|, \\ & \left|\tfrac{2(a+1)}{p} - \alpha - \tfrac{1}{2} - \tfrac{1}{p}\right|, \left|\tfrac{2(b+1)}{p} - \beta - \tfrac{1}{2} - \tfrac{1}{p}\right|, \left|\tfrac{2}{p}(a-b) - (\alpha - \beta)\right|\Bigg\}. \end{aligned}$$

Note that one can equivalently consider the boundedness of the Riesz means, see [**22**, Theorem 3.19].

Now we formulate and prove the main result – Ulyanov type inequality for K-functionals with Jacobi weights. Theorem 3 contains Theorem 1, stated in Introduction, as a particular case.

THEOREM 3. *Let $1 < p < q < \infty$ and $r > 0$. Suppose that $\alpha, \beta > -1$, $a \geqslant b > -1$, $a \geqslant -1/2$, inequalities* (3.1) *hold, and either $(\alpha, \beta) = (a, b)$, or*

$$p(\alpha - \beta) \leqslant 2(a - b) \leqslant q(\alpha - \beta),$$

and $\alpha = a$, or $\alpha > a$, $q > 2$, or $\alpha < a$, $p < 2$.

Suppose also that

$$\sigma = (2a + 2)\left(\frac{1}{p} - \frac{1}{q}\right).$$

If $f \in L_p^{(a,b)}$ *and*

$$\int_0^1 \left(u^{-\sigma} K^{r+\sigma}(f, \mathcal{D}_{r+\sigma}^{(\alpha,\beta)}, u)_{p,(a,b)}\right)^q \frac{du}{u} < \infty,$$

then $f \in L_q^{(a,b)}$ *and*

$$K^r(f, \mathcal{D}_r^{(\alpha,\beta)}, t)_{q,(a,b)} \leqslant C \left(\int_0^t \left(u^{-\sigma} K^{r+\sigma}(f, \mathcal{D}_{r+\sigma}^{(\alpha,\beta)}, u)_{p,(a,b)}\right)^q \frac{du}{u} \right)^{1/q}. \tag{3.5}$$

Theorem 3 extends the results of [**13**, Theorem 11.2] and [**24**, Section 3.3.1] in two directions. First, our estimate involves the K-functional of order $r+\sigma$, i.e., we get the sharp estimate. Second, we consider the case when $(\alpha,\beta) \neq (a,b)$. We also remark that the sharp Ulyanov inequality for functions on $\mathbb{S}^{d-1}$ was recently proved in [**25**].

Proof. Using monotonicity properties of the K-functional, it is enough to verify inequality (3.5) for $t = 1/n$, $n \in \mathbb{N}$. We have

$$K^r(f, \mathcal{D}_r^{(\alpha,\beta)}, 1/n)_{q,(a,b)} \leqslant C \left(\|f - P_{n,f}\|_{q,(a,b)} + n^{-r} \|\mathcal{D}_r^{(\alpha,\beta)} P_{n,f}\|_{q,(a,b)} \right), \tag{3.6}$$

where $P_{n,f}$ is given by (3.2). To estimate the first term, we apply [**13**, Theorem 4.1, (4.6)'] to get

$$\|f - P_{n,f}\|_{q,(a,b)} \leqslant C \left(\sum_{k=n}^{\infty} k^{q\sigma-1} \|f - P_{k,f}\|_{p,(a,b)}^q \right)^{1/q}.$$

In view of the realization result (3.4), we obtain

$$\begin{aligned}
\|f - P_{n,f}\|_{q,(a,b)} &\leqslant C \left(\sum_{k=n}^{\infty} k^{q\sigma-1} \|f - P_{k,f}\|_{p,(a,b)}^q \right)^{1/q} \\
&\leqslant C \left(\sum_{k=n}^{\infty} k^{q\sigma-1} K^{r+\sigma}(f, \mathcal{D}_{r+\sigma}^{(\alpha,\beta)}, 1/k)_{p,(a,b)}^q \right)^{1/q} \\
&\leqslant C \left(\int_0^t \left(u^{-\sigma} K^{r+\sigma}(f, \mathcal{D}_{r+\sigma}^{(\alpha,\beta)}, u)_{p,(a,b)}\right)^q \frac{du}{u} \right)^{1/q}.
\end{aligned}$$

To estimate the second term in (3.6), we use Theorem D or Theorem 2 depending on whether $(\alpha,\beta) = (a,b)$ or $(\alpha,\beta) \neq (a,b)$:

$$n^{-r} \left\| \mathcal{D}_r^{(\alpha,\beta)} P_{n,f} \right\|_{q,(a,b)} \leqslant C n^{\sigma} n^{-(r+\sigma)} \left\| \mathcal{D}_{r+\sigma}^{(\alpha,\beta)} P_{n,f} \right\|_{p,(a,b)} \leqslant C n^{\sigma} K^{r+\sigma}(f, \mathcal{D}_{r+\sigma}^{(\alpha,\beta)}, 1/n)_{p,(a,b)}.$$

To complete the proof of (3.5), we have

$$n^{\sigma} K^{r+\sigma}(f, \mathcal{D}_{r+\sigma}^{(\alpha,\beta)}, 1/n)_{p,(a,b)} \leqslant C \left(\int_{1/2n}^{1/n} \left(u^{-\sigma} K^{r+\sigma}(f, \mathcal{D}_{r+\sigma}^{(\alpha,\beta)}, u)_{p,(a,b)}\right)^q \frac{du}{u} \right)^{1/q}.$$

□

4. Ulyanov-type inequalities for Ditzian–Totik moduli of smoothness

The (global) weighted modulus of smoothness of order $r \geqslant 1$ is given by

$$\begin{aligned}
\omega_\varphi^r(f,t)_{p,(a,b)} = \Omega_\varphi^r(f,t)_{p,(a,b)} &+ \inf_{P \in \Pi_{r-1}} \left\|(f-P)w\right\|_{L_p[-1,-1+4k^2t^2]} \\
&+ \inf_{P \in \Pi_{r-1}} \|(f-P)w\|_{L_p[1-4k^2t^2,1]},
\end{aligned}$$

where $w=\left(w^{(a,b)}\right)^{1/p}$,

$$\Omega_\varphi^r(f,t)_{p,(a,b)}=\sup_{0<h\leqslant t}\|\Delta_{h\varphi}^r f w\|_{L_p[-1+4k^2t^2,\,1-4k^2t^2]}$$

and

$$\Delta_{h\varphi}^r f(x)=\sum_{i=0}^r(-1)^i\binom{r}{i}f\left(x+\tfrac{r-2i}{2}h\varphi(x)\right).$$

Note that (see [**16**, (2.5.7)]) this definition is equivalent to the one given in [**14**, Chapter 6, Appendix B].

Let $K_\varphi^r(f,t)_{p,(a,b)}$, $r\in\mathbb{N}$, be the K-functional for the pair of spaces $\left(L_p^{(a,b)},W_{p,(a,b)}^r\right)$, where $W_{p,(a,b)}^r$ consists of functions $g\in L_p^{(a,b)}$ such that $g^{(r-1)}\in \mathrm{AC}_{\mathrm{loc}}$ and $\varphi^r g^{(r)}\in L_p^{(a,b)}$ (see [**14**, (6.1.1)]):

$$K_\varphi^r(f,t)_{p,(a,b)}=\inf\left\{\|f-g\|_{p,(a,b)}+t^r\|\varphi^r g^{(r)}\|_{p,(a,b)}:\ g\in W_{p,(a,b)}^r\right\}. \tag{4.1}$$

It is known that $K_\varphi^r(f,t)_{p,(a,b)}\asymp\omega_\varphi^r(f,t)_{p,(a,b)}$ for $a,b\geqslant 0$; see [**14**, Theorem 6.1.1]. Moreover, we have the following realization result:

$$\omega_\varphi^r(f,t)_{p,(a,b)}\asymp\|f-P_{n,f}\|_{p,(a,b)}+t^r\|\varphi^r P_{n,f}^{(r)}\|_{p,(a,b)},\qquad [1/t]=n. \tag{4.2}$$

The proof of this equivalence (cf. [**12**]) is based on the Jackson-type inequality and the estimate of $t^r\|\varphi^r\psi^{(r)}\|_{p,(a,b)}$ via $\omega_\varphi^r(f,t)_{p,(a,b)}$ (the Nikolskii–Stechkin type inequality). The Jackson-type inequality was obtained in [**14**, Theorem 7.2.1] for the unweighted case and in [**16**, Sec. 2.5.2, (2.5.17)] for the weighted case. The unweighted version of the Nikolskii–Stechkin type inequality was proved in [**14**, Theorem 7.3.1]. This argument can be used to show the weighted version.

The relation between K-functionals (4.1) and (3.3) in the case when r is positive integer follows from Corollary 2 below. Note that the case $(\alpha,\beta)=(a,b)$ is due to Dai and Ditzian [**8**, Theorem 7.1] and is based on the Muckenhoupt transplantation theorem. We follow the idea of their proof and first obtain the following result.

THEOREM 4. *Let $1<p<\infty$, r be a positive integer, and a, b, α, $\beta>-1$ be such that* (3.1) *holds. Then there exists a constant C such that for any $Q\in\Pi$, we have*

$$\left\|\varphi^r Q^{(r)}\right\|_{p,(a,b)}\leqslant C\left\|\mathcal{D}_r^{(\alpha,\beta)}Q\right\|_{p,(a,b)}, \tag{4.3}$$

$$\left\|\mathcal{D}_r^{(\alpha,\beta)}\left(Q-S_{r-1}^{(\alpha,\beta)}Q\right)\right\|_{p,(a,b)}\leqslant C\left\|\varphi^r Q^{(r)}\right\|_{p,(a,b)}, \tag{4.4}$$

where $S_{r-1}^{(\alpha,\beta)}Q$ is the $(r-1)$-th partial sum of the Fourier–Jacobi expansion of Q, i.e.,

$$S_{r-1}^{(\alpha,\beta)}Q=\sum_{k=0}^{r-1}\widehat{Q}_k^{(\alpha,\beta)}\psi_k^{(\alpha,\beta)}.$$

PROOF. The proof of (4.3) and (4.4) is based on Theorem C. Since $\widehat{Q}_k^{(\alpha,\beta)}=0$ starting from certain k, we obtain

$$\mathcal{D}_r^{(\alpha,\beta)}Q=\sum_{k=1}^\infty\left(\lambda_k^{(\alpha,\beta)}\right)^r\widehat{Q}_k^{(\alpha,\beta)}\psi_k^{(\alpha,\beta)}=\sum_{k=1-r}^\infty\left(\lambda_{k+r}^{(\alpha,\beta)}\right)^r\widehat{Q}_{k+r}^{(\alpha,\beta)}\psi_{k+r}^{(\alpha,\beta)},$$

$$Q^{(r)}=\sum_{k=r}^\infty\lambda_k\widehat{Q}_k^{(\alpha,\beta)}\psi_{k-r}^{(\alpha+r,\beta+r)}=\sum_{k=0}^\infty\lambda_{k+r}\widehat{Q}_{k+r}^{(\alpha,\beta)}\psi_k^{(\alpha+r,\beta+r)},$$

where

$$\lambda_k = \lambda_k(\alpha,\beta,r) = \lambda_k^{(\alpha,\beta)} \cdots \lambda_{k-r+1}^{(\alpha+r-1,\beta+r-1)}.$$

To prove inequality (4.3), we apply Theorem C with $(\overline{p},\overline{q}) = (p,p)$, $(\overline{\alpha},\overline{\beta}) = (\alpha+r,\beta+r)$, $(\overline{\gamma},\overline{\delta}) = (\alpha,\beta)$, $(\overline{c},\overline{d}) = (a,b)$, $h=-r$, and

$$\nu_k = \lambda_k / \left(\lambda_k^{(\alpha,\beta)}\right)^r.$$

Then $s=0$, $\left(\overline{a},\overline{b}\right) = (a+pr/2, b+pr/2)$, $A=(a+1)/p-\alpha$, and $B=(b+1)/p-\beta$. On account of (3.1), we conclude that $A<1$, $B<1$, and therefore, all conditions of Theorem C are satisfied. Hence, we get

$$\left\|\varphi^r Q^{(r)}\right\|_{p,(a,b)} = \left\|Q^{(r)}\right\|_{p,(a+pr/2,b+pr/2)} \leqslant C\left\|\mathcal{D}_r^{(\alpha,\beta)} Q\right\|_{p,(a,b)}.$$

Let us now obtain (4.4). We remark that $g = \mathcal{D}_r^{(\alpha,\beta)}\left(Q - S_{r-1}^{(\alpha,\beta)} Q\right)$ is a polynomial and its Fourier–Jacobi coefficients satisfy $\widehat{g}_k^{(\alpha,\beta)} = 0$ for $0\leqslant k\leqslant r-1$. We apply Theorem C with $(\overline{p},\overline{q}) = (p,p)$, $(\overline{\alpha},\overline{\beta}) = (\alpha,\beta)$, $(\overline{\gamma},\overline{\delta}) = (\alpha+r,\beta+r)$, $(\overline{c},\overline{d}) = (a+pr/2, b+pr/2)$, $h=r$, and

$$\nu_k = \left(\lambda_k^{(\alpha,\beta)}\right)^r / \lambda_k.$$

Then $s=0$, $\left(\overline{a},\overline{b}\right) = (a,b)$, $A=(a+1)/p-\alpha-r/2<1$, and $B=(b+1)/p-\beta-r/2<1$. Therefore, all conditions of Theorem C are satisfied, and we arrive at

$$\left\|\mathcal{D}_r^{(\alpha,\beta)}\left(Q - S_{r-1}^{(\alpha,\beta)} Q\right)\right\|_{p,(a,b)} \leqslant C\left\|Q^{(r)}\right\|_{p,(a+pr/2,b+pr/2)} = C\left\|\varphi^r Q^{(r)}\right\|_{p,(a,b)}.$$

□

Corollary 2. *Under assumptions of Theorem* 4, *there exists a constant* C *such that for any* $f\in L_p^{(a,b)}$ *and* $t\in(0,t_0)$ *we have*

$$K_\varphi^r(f,t)_{p,(a,b)} \leqslant C K^r(f,\mathcal{D}_r^{(\alpha,\beta)},t)_{p,(a,b)} \tag{4.5}$$

and

$$K^r(f,\mathcal{D}_r^{(\alpha,\beta)},t)_{p,(a,b)} \leqslant C\left(K_\varphi^r(f,t)_{p,(a,b)} + t^r\|f\|_{p,(a,b)}\right).$$

Proof. First, (4.3) and the realization result (4.2) yield that

$$\begin{aligned} K_\varphi^r(f,t)_{p,(a,b)} &\leqslant \|f-P_{n,f}\|_{p,(a,b)} + t^r\|\varphi^r P_{n,f}^{(r)}\|_{p,(a,b)} \\ &\leqslant C\left(\|f-P_{n,f}\|_{p,(a,b)} + t^r\|\mathcal{D}_r^{(\alpha,\beta)} P_{n,f}\|_{p,(a,b)}\right) \leqslant C K^r(f,\mathcal{D}_r^{(\alpha,\beta)},t)_{p,(a,b)}, \end{aligned}$$

which is (4.5).

Second, under condition (3.1), the operator $A:\Pi\to\Pi_{r-1}$ given by

$$A(Q) = \mathcal{D}_r^{(\alpha,\beta)} S_{r-1}^{(\alpha,\beta)} Q$$

is bounded in $L_p^{(a,b)}$, i.e.,

$$\|\mathcal{D}_r^{(\alpha,\beta)} S_{r-1}^{(\alpha,\beta)} Q\|_{p,(a,b)} \leqslant C(p,a,b,\alpha,\beta,r)\|Q\|_{p,(a,b)}. \tag{4.6}$$

Using this, we obtain

$$\begin{aligned} &K^r(f,\mathcal{D}_r^{(\alpha,\beta)},t)_{p,(a,b)} \leqslant \|f-P_{n,f}\|_{p,(a,b)} + t^r\|\mathcal{D}_r^{(\alpha,\beta)} P_{n,f}\|_{p,(a,b)} \\ &\leqslant \|f-P_{n,f}\|_{p,(a,b)} + t^r\|\mathcal{D}_r^{(\alpha,\beta)}(P_{n,f} - S_{r-1}^{(\alpha,\beta)} P_{n,f})\|_{p,(a,b)} + t^r\|\mathcal{D}_r^{(\alpha,\beta)} S_{r-1}^{(\alpha,\beta)} P_{n,f}\|_{p,(a,b)}. \end{aligned}$$

Finally, (4.4) and (4.6) imply

$$\begin{aligned} K^r(f,\mathcal{D}_r^{(\alpha,\beta)},t)_{p,(a,b)} &\leqslant C\left(\|f-P_{n,f}\|_{p,(a,b)}+t^{-r}\|\varphi^r P_{n,f}^{(r)}\|_{p,(a,b)}+t^r\|P_{n,f}\|_{p,(a,b)}\right)\\ &\leqslant C\left(K_\varphi^r(f,t)_{p,(a,b)}+t^r\|f\|_{p,(a,b)}\right). \end{aligned}$$

□

It is proved in [**13**, Theorem 11.2] that for $f\in L_p$, $0<p<q\leqslant\infty$, and integer $r\geqslant 1$ the following Ulyanov-type inequality holds:

$$\omega_\varphi^r(f,t)_q \leqslant C\left[\int_0^t \left(u^{-\sigma}\omega_\varphi^r(f,u)_p\right)^{q_1}\frac{du}{u}\right]^{1/q_1},$$

where $q_1=\begin{cases} q, & q<\infty\\ 1, & q=\infty\end{cases}$, $\sigma=2\left(\frac{1}{p}-\frac{1}{q}\right)$. The next theorem refines this result.

Theorem 5. *Let $1\leqslant p<q\leqslant\infty$, $a\geqslant b\geqslant 0$, r be a positive integer, and*

$$\sigma=(2a+2)\left(\frac{1}{p}-\frac{1}{q}\right).$$

Suppose that $f\in L_p^{(a,b)}$ and

$$\int_0^1\left(u^{-\sigma}\omega_\varphi^{r+[\sigma]}(f,u)_{p,(a,b)}\right)^{q_1}\frac{du}{u}<\infty.$$

Then $f\in L_q^{(a,b)}$ and

$$\omega_\varphi^r(f,t)_{q,(a,b)}\leqslant C\left[\int_0^t\left(u^{-\sigma}\omega_\varphi^{r+[\sigma]}(f,u)_{p,(a,b)}\right)^{q_1}\frac{du}{u}\right]^{1/q_1}+Ct^rE_{r-1}(f)_{p,(a,b)}, \tag{4.7}$$

where

$$q_1=\begin{cases} q, & q<\infty,\\ 1, & q=\infty.\end{cases}$$

Remark. (A). In particular, (4.7) implies

$$\omega_\varphi^r(f,t)_q\leqslant C\left[\int_0^t\left(u^{-1}\omega_\varphi^{r+1}(f,u)_p\right)^{q_1}\frac{du}{u}\right]^{1/q_1}+Ct^rE_{r-1}(f)_p,$$

when $\frac{1}{p}-\frac{1}{q}\geqslant\frac{1}{2}$, $1\leqslant p<q\leqslant\infty$, and

$$\omega_\varphi^r(f,t)_\infty\leqslant C\int_0^t u^{-2}\omega_\varphi^{r+2}(f,u)_1\frac{du}{u}+Ct^rE_{r-1}(f)_1.$$

(B). Corollary 2 shows that for $1<p<q<\infty$ and positive integer σ Theorem 5 follows from Theorem 3.

Proof. The proof is similar to the proof of Theorem 3. The only substantial difference is that we use Lemma 1 instead of Theorem D and Theorem 2.

Using monotonicity properties of the moduli of smoothness, it is enough to verify inequality (4.7) for $t=1/n$, where n is a positive integer. Let $P_{n,f}$ be defined by (3.2). Taking into account that $\omega_\varphi^r(f,t)_{q,(a,b)}\asymp K_\varphi^r(f,t)_{q,(a,b)}$, we obtain

$$\omega_\varphi^r(f,t)_{q,(a,b)}\leqslant C\left(\|f-P_{n,f}\|_{q,(a,b)}+n^{-r}\|\varphi^rP_{n,f}^{(r)}\|_{q,(a,b)}\right). \tag{4.8}$$

To estimate the first term, we apply Theorem 4.1 from [**13**]. Assumption (4.3) of this theorem is exactly the Nikol'skii inequality

$$\|P_n\|_{q,(a,b)} \leqslant Cn^{(2a+2)\left(\frac{1}{p}-\frac{1}{q}\right)}\|P_n\|_{p,(a,b)}, \quad P_n \in \Pi_n,$$

where $C = C(p, q, a, b)$, proved in [**9**, Theorem 4] (see also [**17**, Ch. 6, Theorem 1.8.4, 1.8.5]). Therefore, we have

$$\|f - P_{n,f}\|_{q,(a,b)} \leqslant C\left(\sum_{k=n}^{\infty} k^{q_1\sigma-1}\|f - P_{k,f}\|_{p,(a,b)}^{q_1}\right)^{1/q_1}.$$

Applying (4.2) and replacing the sum by the integral, we get

$$\begin{aligned}\|f - P_{n,f}\|_{q,(a,b)} &\leqslant C\left(\sum_{k=n}^{\infty} k^{q_1\sigma-1}\|f - P_{k,f}\|_{p,(a,b)}^{q_1}\right)^{1/q_1}\\ &\leqslant C\left(\sum_{k=n}^{\infty} k^{q_1\sigma-1}\omega_\varphi^{r+[\sigma]}(f,1/k)_{p,(a,b)}^{q_1}\right)^{1/q_1}\\ &\leqslant C\left(\int_0^t \left(u^{-\sigma}\omega_\varphi^{r+[\sigma]}(f,u)_{p,(a,b)}\right)^{q_1}\frac{du}{u}\right)^{1/q_1}.\end{aligned}$$

To estimate the second term in (4.8), we use Lemma 1:

$$\left\|\varphi^r P_n^{(r)}\right\|_{q,(a,b)} = \left\|\varphi^r (P_n - P_{r-1})^{(r)}\right\|_{q,(a,b)} \leqslant \|P_n - P_{r-1}\|_{p,(a,b)} + \left\|\varphi^{r+2[\sigma]-\sigma} P_n^{(r+[\sigma])}\right\|_{p,(a,b)}.$$

Further we need the following two-weight inequality proved in [**9**, Theorem 4]:

$$\left\|\varphi^{r+2[\sigma]-\sigma} P_n^{(r+[\sigma])}\right\|_{p,(a,b)} \leqslant C\,n^{\sigma-[\sigma]}\left\|\varphi^{r+[\sigma]} P_n^{(r+[\sigma])}\right\|_{p,(a,b)}.$$

Therefore, using monotonicity properties of moduli of smoothness, we get

$$\begin{aligned}n^{-r}\left\|\varphi^{r+2[\sigma]-\sigma} P_n^{(r+[\sigma])}\right\|_{p,(a,b)} &\leqslant C\,n^{\sigma}\omega_\varphi^{r+[\sigma]}(f,1/n)_{p,(a,b)}\\ &\leqslant C\left[\int_{1/2n}^{1/n}\left(u^{-\sigma}\omega_\varphi^{r+[\sigma]}(f,u)_{p,(a,b)}\right)^{q_1}\frac{du}{u}\right]^{1/q_1}.\end{aligned}$$

To complete the proof we note that $\|P_n - P_{r-1}\|_{p,(a,b)} \leqslant 2E_{r-1}(f)_{p,(a,b)}$. □

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