# Volume of the steady-state space of financial flows in a monetary stock-flow-consistent model

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**Abstract**

We show that a steady-state stock-flow consistent macro-economic model can be represented as a Constraint Satisfaction Problem (CSP). The set of solutions is a polytope, which volume depends on the constraints applied and reveals the potential fragility of the economic circuit, with no need to specify the dynamics. Several methods to compute the volume are compared, inspired by operations research methods and the analysis of metabolic networks, both exact and approximate. We also introduce a random transaction matrix, and study the particular case of linear flows with respect to money stocks.

*Keywords:* economics, physics and society, constraint satisfaction, monte-carlo, random network, convex polytope, finance

## 1. Introduction

In this article we propose an approach to macro-economic modeling inspired by stock-flow consistent (SFC) models [1] and statistical physics, solving a Constraint Satisfaction Problem (CSP) in a way similar to recent works in the field of metabolic networks [2]. The formalism of DSGE (Dynamic Stochastic General Equilibrium) is dominant today in macro-economics, partly because the corresponding models can be written in the form of state-space models and estimated in a well-studied statistical framework[1]. Their usefulness has been widely debated among economists [4, 5] and physicists [6] because of their inability to predict crises. Many of their hypotheses have been criticized, such as representative rational agents, exogeneity of financial factors, clearing markets where offer always meet demand, etc. . .

Other approaches include SFC theories that see the economy as a whole, Agents-Based Models (ABM) [7], and dynamical systems models that consider crises as possible outcomes [8, 9] in an economy where production and finance may be coupled in instable ways. ABM models [10] are versatile and able to reproduce the dynamical behavior of a large number of heterogeneous and learning agents, in several countries, while representing debts, prices and investments. Computational requirements are high though when simulating, and theoretical understanding is limited so far. Calibration and validation are known to be difficult problems in that context.

We consider a simplified stock-flow consistent model developed by macro-economists, where the state of the economy is the set of all stocks and flows of money during a given time interval. We show in this article that one can compute the set of admissible steady-state financial flows established when agents interact. In this steady-state flow space, all flow configurations are equally weighted, thus allowing unusual states of the financial flows to be encompassed. The flow space can be computed independently of the stocks, i.e. of the amount of money deposited by agents on their bank accounts. The marginal probabilities of individual flows can be approximated over the whole solution space.

Our standpoint is inspired by the field of metabolic networks where steady-state fluxes have been studied as CSP and successfully compared to experimental data, as in the Red Blood Cell metabolism or the central

[1] see [3, §3.2] for a discussion.

metabolism of E.coli [11, 2, 12]). Such system-scale studies reveal some interesting features of metabolisms, for example the cooperation between pathways. It has been shown also that organisms such as E.coli do not necessarily optimize their metabolic fluxes.

The expected benefits of applying these methods in macro-economics include the analysis of fragilities, notably the sensitivity to arbitrary flow constraints, such as shortages. Indeed, the volume of the solution space evoked above is immediately impacted when constraints are added or removed, and can reveal the flexibility or rigidity of financial flows subject to perturbations.

In section 2, we detail the model of financial flows that will be used as a benchmark, and present its background from a macro-economic modeling point of view. Comparisons are made with ABM and econophysics. We present the different methods used to compute exactly and approximately the volume of the steady-state flow space. Then in section 3, the experimental results are explained, and the particular case of linear flows with a constant stock of money is examined. Finally, sections 4 and 5 are devoted to discussion and conclusion.

## 2. Background and methods

### *2.1. Steady-sate flow space in a stock-flow-consistent model*

Our analysis is based on stock-flow-consistent (SFC) monetary models at the macro-economic level [1]. Agents are grouped by categories: banks, firms, workers, state, central bank. For simplicity, the state and central bank are omitted in this model. Agents possess bank accounts, and transfer money between these accounts:

- Banks lend money to firms, which pay interests to the former.
- Banks pay interests on deposits made by workers and firms.
- Firms pay wages to workers.
- Workers and banks pay for consumption of goods to firms.

Transactions are balanced using a double-entry book-keeping representation: payments are functions of deposits; each payment corresponds to a receipt, and rows must sum to zero as shown, except those entitled "Record loan", corresponding to the column "Firm loan". Tab. 1 shows the different financial transactions occurring in a simple SFC model with one agent per class, after [8].

| Transaction | Bank vault $B_V$ | Bank transaction $B_T$ | Firm loan $F_L$ | Firm deposit $F_D$ | Worker deposit $W_D$ |
|---|---|---|---|---|---|
| Lend money | $-a$ | | | $a$ | |
| Record loan | | | $a$ | | |
| Compound debt | | | $b$ | | |
| Pay interest | | $c$ | | $-c$ | |
| Record payment | | | $-c$ | | |
| Deposit interest | | $-d$ | | $d$ | |
| Wages | | | | $-e$ | $e$ |
| Deposit interest | | $-f$ | | | $f$ |
| Consumption | | $-g$ | | $g+h$ | $-h$ |
| Repay loan | $i$ | | | $-i$ | |
| Record payment | | | $-i$ | | |

Table 1: Financial transactions in the SFC model.

The column "Firm loan" doesn't represent a bank account, but is a ledger to record the total amount of debt. The column sums of flows can be summarized in the following system of ordinary differential equations (ODE):

$$\begin{cases} \dfrac{dB_V}{dt} = i - a \\ \dfrac{dB_T}{dt} = c - d - f - g \\ \dfrac{dF_L}{dt} = a + b - c - i \\ \dfrac{dF_D}{dt} = a - c + d - e + g + h - i \\ \dfrac{dW_D}{dt} = e + f - h \end{cases} \tag{1}$$

where the flows $\{a, \dots, i\}$ are positive and bounded.

Restricting our study to non-equilibrium stationary state (NESS) of bank accounts $B_V, B_T, F_D, W_D$, the solution of the corresponding problem can be written in matrix form:

$$S = \{x \text{ s.t. } \xi x = 0, \forall i \; x_i \in [0, x_{max}]\} \tag{2}$$

where $x = [a, c, d, e, f, g, h, i]$ and:

$$\xi = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{bmatrix} \tag{3}$$

$\xi$ is an $m \times n$ matrix, $m$ being the number of stock accounts, and $n$ the number of flows. $S$ defines a convex bounded polytope, that appears in many disciplines, as will be discussed in section 2.2.

Constraints can be added or modified : for example, if a firm goes bankrupt, the loan will not be completely repaid, thus equation $a = i$ may be replaced by $a \leq i$. Agents may not spend more in consumption than the sum of their incomes, which can be written $h \leq d + e$, unless workers borrow money from banks.

The scope of this model can be extended to the case of multiple agents per class, replacing scalars by block matrices in Tab. 1. Each block can be designed to account not only for the flows but also for the connectivity between agents. For example, instead of $[-a, 0, 0, a, 0]$, the first row of the transaction table Tab. 1 may write $[-A, 0, 0, A, 0]$ where $A$ could encode the many-to-one relationship between firms and banks. Similarly, the first column of the same table would read $[-A^T, 0, \dots, 0, I^T, 0]^T$. One possible way to choose $A$ and $I$ is :

$$A_{nb,nf} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ 1 & 0 & \dots & 1 \end{bmatrix}, \; I = Id_{nf} \tag{4}$$

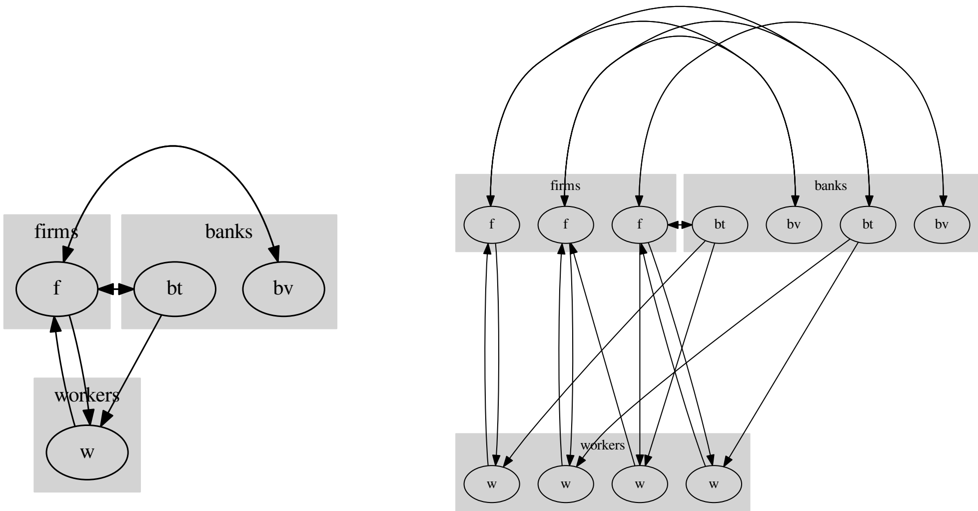
where $A_{i,j} = 1$ if the bank $i$ is lending money to the firm $j$. The number of banks, firms and workers are noted $nb, nf, nw$, and $Id$ is the identity matrix. In this example, the first and the last firm are client of the same bank.

The matrix $A$ can be randomly sampled efficiently, in some relevant ensemble of matrices consistent with empirical data. Following that example, we can design the others block matrices in the transaction table, and look for the corresponding solution space as in Eq.(2), with:

$$\xi = \begin{bmatrix} -A_{nb,nf} & 0 & 0 & 0 & 0 & 0 & 0 & A_{nb,nf} \\ 0 & A_{nb,nf} & -A_{nb,nf} & 0 & -F_{nb,nw} & -Id_{nb} & 0 & 0 \\ Id_{nf} & -Id_{nf} & Id_{nf} & -E_{nf,nw} & 0 & G_{nf,nb} & H_{nf,nw} & -Id_{nf} \\ 0 & 0 & 0 & Id_{nw} & Id_{nw} & 0 & -Id_{nw} & 0 \end{bmatrix} \tag{5}$$

where $E_{nf,nw}$, $F_{nb,nw}$, $G_{nf,nb}$, $H_{nf,nw}$ encode the connectivity patterns respectively between firms and workers (payment of wages), workers and banks (deposits), firms and banks (consumption of bankers), firms and workers (consumption of workers).

A graphical representation of monetary transactions with randomly sampled connections between agents is given in Fig. 1.




(a) (b)

Figure 1: Graph of monetary transactions. *bt,bv,w,f* stand for banks transaction accounts, banks vaults, workers deposit accounts, firms deposit accounts. (a) single agent in each class; (b) random connectivity, with 2 banks, 3 firms, 4 workers.

It is important to stress that the dependence of flows on stocks is voluntarily kept unspecified. The solutions to this class of under-specified problems in the space of steady-state flows will be examined in sections 2.3 and 2.4. The reader interested in the study of explicit specifications of flows may look at [8] and [9]. Lavoie and Godley examine many increasingly detailed models (open economy,...), as well as dynamic specifications of the flows[2].

To the best of our knowledge, the properties of random matrices that correspond to SFC models have not been established in the economics litterature. This representation calls for a closer examination following many works in the field of complex networks studies.

*2.2. Related works in economics, econophysics, and network science*

Double-entry book-keeping is used to establish National Accounts as a means to record estimated flows and stocks consistently, for each country, at several levels of aggregation. National accountants build balance sheets, income and product accounts, in order to compare economies, and to analyze the behavior of economies in time, such as growth. Such accounts are also used in macroeconomic modeling by policymakers to calibrate Computable General Equilibirum models. However, models of this type (e.g. DSGE) do not explicitly enforce stock-flow consistency.

Long before the break of the subprime crisis, many works have established ABM as an alternative, in a move to get rid of hypotheses perceived as unjustified. Equilibrium, rationality, and the hypothesis of the representative agent have been criticized by many economists [13, 3]. Among macro-economic ABM,

[2] see an implementation `https://github.com/kennt/monetary-economics`

stock-flow consistency is now a classical hypothesis that has been used by many authors [10, 14, 15]. ABM are praised for their flexibility, their ability to study large populations of heterogeneous and learning agents that interact in possibly non-linear ways. However, ABM need computer intensive simulations and face the problem of being over-parametrized. Calibration is thus difficult and unstable, all the more since empirical studies and reliable data are not abundant compared to the dimension of the parameter space. Under simplifying assumptions such as the aggregation of a subset of agents, theoretical results concerning some macro-economic ABM were recently obtained in a stock-flow-consistent framework, and phase diagrams established, for specific dynamic rules [16, 17].

The importance of the organization of interactions, embodied by networks, has been stressed in economics [18]. Theoretical works have studied their generic properties (supply chain [19], interbank network [20], trade credit [21]). Empirical studies have laid emphasis upon the topology of real economic networks, such as goods market, national inter-firm trading [22], world trade [23], global corporate control among transnational corporations [24], firm ownership networks. They must sometime be reconstructed, starting from limited information (of equity investments in the stock market [25], the interbank market [26]). At the level of individuals, detailed topological information about banking, employment, or consumption seems to be lacking.

Such empirical and theoretical material may serve as an input to shape the set of equations and inequalities discussed above.

### *2.3. Constraint satisfaction problem*

As shown in section 2.1, the steady-state flux space associated with Eq.( 2) is a bounded convex polytope. Without loss of generality, $S$ can be supposed to have full row rank. The polytope is a $n-m$ dimensional object embedded in the $n$-dimensional space of flows. Properties of convex polytopes such as their different representations are well studied [27]. Computing their volume exactly can be achieved by solving the vertex enumeration problem, which is $\#P$-hard. Existing implementations, such as *lrs* by Avis et al., allow to solve it in reasonable time when $n-m$ is close to 10. Exact computation methods are employed in linear programming and operations research to solve classical constraint satisfaction problems such as the map coloring problem, and have real-life applications, for example in resource allocation problems. More solutions can be obtained when relaxing hard constraints.

Approximate methods to determine the solution space were proposed by researchers studying metabolic networks and the metabolic steady-state flux space. These methods allow to sample the solution space, to estimate its volume, and to approximate probabilistic properties of the solutions such as the marginal densities [11, 2, 28, 29, 30]. They can be used to evaluate the sensitivity of the the solution space to new constraints. A restriction of this problem known as Flux-Balance-Analysis (FBA) consists in maximizing some objective constraint, which reduces the solution space to a finite set of points [12, 30]. A parallel may be mentioned with the field of random Constraint Satisfaction Problem (rCSP) that stands at the interface between theoretical computer science and statistical physics, and studies sets of solutions to a large number of random constraints, in a boolean space. Many important results such as phase transitions were developed [31]. Although our main focus is the continuous domain, we can take advantage of this theory.

### *2.4. Monte-Carlo sampling of the steady-state flow space*

In section 2.3 we mentionned the exact computation of the volume of the steady-state flow space using vertex enumeration, when $n-m$ is small. The result obtained is numeric, and not an analytic expression depending on the parameters of the problem in Eq. (2).

Another approach proposed in [11, 32], suited for larger problems, is Monte-Carlo sampling in the solution space, using a *hit-and-run* algorithm. The latter needs an initial point inside the convex polytope. Then, sampling from a hypersphere, a direction is selected at random. The half-line defined by the starting point and this direction intersects the boundary in a point. This intersection and the starting point form a segment that can be uniformly sampled to get the next point. The procedure defines a Markov Chain that converges to the uniform distribution over the polytope [33], in nondeterministic polynomial time $\mathcal{O}^*(n^k)$. The notation $\mathcal{O}^*()$ means that there are logarithmic factors that multiply $n^k$, and constants, but are neglected [34].

The *hit-and-run* method provides an estimate of the marginal probability density functions (pdf) denoted $P_i(x)$ for each transaction flow $i = 1, \ldots, N$. They characterise the shape of the polytope and are given by:

$$P_i(x) = Vol(S_i(x))/Vol(S),\ S_i(x) = \{x \in S\ s.t.\ x_i = x\} \tag{6}$$

They can also be written as an integral over all flows. As remarked by [11], the *hit-and-run* method will not permit us to estimate the absolute volume. Instead, approximations for relative volumes can be obtained, such as $r = Vol(S_1)/Vol(S)$, where $S$ is the same as in Eq.(2) while $S_1$ has additional constraints, such as a lower bound for flow $i$.

To sum up, using efficient Monte-Carlo sampling methods such as *hit-and-run*, we will be able to sample medium-sized problems (up to hundreds of flows), to approximate the pdfs, to compute relative volumes. However, we will get no analytic expression of theses quantities, nor approximate entropy. Furthermore, the mixing of the Markov Chain should be examined to ensure convergence. In section 3 we use the implementation by Tervonen et al. [35].

As evoked in section 2.3, researchers have also used the replica method [29], and message passing algorithms [2, 36, 30] to deal with the problem of estimating marginal densities.

## 3. Results

### *3.1. Comparison between lrs and hit-and-run in low dimensional systems*

First let us compare the relative volumes corresponding to small problems, when exact results can be obtained.

A cube in 3 dimensions can be defined by the set of constraints $S_0$. We also define $S_1$ where all boundaries remain the same, except one of them.

$$\begin{aligned} S_0 &= \{0 \le x_i \le c_0,\ \forall i \in [0, 2]\} \\ S_1 &= \{0 \le x_0 \le c_0,\ 0 \le x_1 \le c_0,\ 0 \le x_2 \le c_1\} \end{aligned}$$

The volume ratio is defined by $r = Vol(S_1)/Vol(S_0)$ and can be computed by several methods as shown in the first row of Tab. 2. Since the number of sampling points is high, the relative error is low.

The histogram associated to the marginal pdf of the first coordinate of the cube is represented in Fig. 2(top left). As expected, the frequency is flat on the interval $[0, c_0]$, which corresponds to a uniform sampling.

Next, the volume ratios associated with the steady-state monetary flow problem depicted in section 2.1 are computed. To do so, following the example of the cube above, we iteratively diminish the boundary constraint for each flow variable, by a constant amount. For example, the second row of Tab. 2 is obtained by setting all upper bounds to $c_0 = 10$, then by setting all bounds to $c_0$ except along the first dimension, which corresponds to LF (loans granted to firms). As expected, all the volume ratios obtained with *lrs* and *hit-and-run* are close to each other. This is also consistent with results found in metabolic network analysis [11] when $n$ is small .

The marginal histograms associated with the 5 first flow variables are represented in Fig. 2. Unlike in the case of the cube, this result can't be compared to some ground truth.

### *3.2. The influence of knock-outs on larger systems*

The method of relative volumes devised in section 3.1 was proposed to compare *lrs* and *hit-and-run* when the size of the problem is small. However, this method can also be used to study the impact of an external perturbation on the volume of the solution space, for example when a flow is constrained. This is termed *knock-out* in metabolic network analysis, but in the present context can be interpreted as a limitation imposed on monetary flows, such as a credit shortage. Let us define:

| Algorithm | | | *lrs* | | | *hit-and-run* |
|---|---|---|---|---|---|---|
| | $c_0$ | $c_1$ | $V_0$ | $V_1$ | $V_1/V_0$ | $V_1/V_0$ |
| cube $x$ | 10 | 5 | 1000 | 500 | 0.5 | 0.50055 |
| LF | 10 | 5 | 12500 | 6250 | 0.5 | 0.49 |
| IFL | 10 | 5 | 12500 | 1822 | 0.145 | 0.14 |
| IFD | 10 | 5 | 12500 | 10677 | 0.854 | 0.85 |
| W | 10 | 5 | 12500 | 8072 | 0.645 | 0.64 |
| IWD | 10 | 5 | 12500 | 11718 | 0.937 | 0.93 |
| CB | 10 | 5 | 12500 | 10677 | 0.854 | 0.85 |
| CW | 10 | 5 | 12500 | 4427 | 0.354 | 0.35 |
| LR | 10 | 5 | 12500 | 6250 | 0.5 | 0.49 |

Table 2: Absolute and relative volume of steady-state flow space when one boundary value is changed from $c_0$ to $c_1$. Only one constraint is changed to get volume ratio, and is identified by the label of the row. The absolute volume is computed using *lrs* in the two cases, as well as their ratio. The latter is compared to the approximate volume ratio found with the Monte-Carlo *hit-and-run* procedure with $k = 10^5$ sampling points. The first row corresponds to the constraints definig a cube. The others are associated with the steady-state monetary flow model.

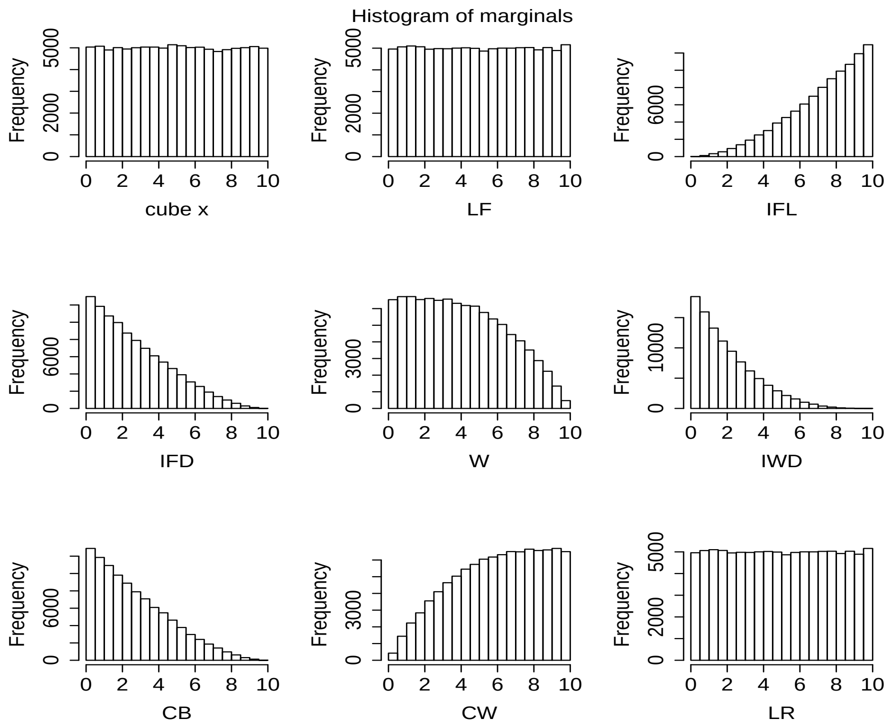



Figure 2: Histograms corresponding to marginal pdf (top left) cube along the $x$ axis (others) steady-state monetary flows labelled according to Tab.3. $c_0 = 10$

$$S_I = \{X \text{s.t } \xi X = 0,\ 0 \le x_i \le c_0,\ \forall i \in I\} \quad (7)$$
$$S^{\alpha}_{I\setminus I_0} = \{X \text{s.t } \xi X = 0,\ 0 \le x_i \le c_0,\ \forall i \in I \setminus I_0,\ 0 \le x_i \le \alpha c_0,\ \forall i \in I_0\} \quad (8)$$
$$\langle X \rangle_{S_I} = \int_{S_I} \sum_{i=1}^{n} x_i dX \quad (9)$$

where $X = [x_1, \ldots, x_n]$, $n$ is the number of columns of $\xi$, $I$ is the set indexing the space of flows, and $I_0$ the set of flows that will be partially knocked-out, by an amount $\alpha$. $\langle X \rangle_{S_I}$ is the mean flow over the solution space. Let us also define:

$$\begin{aligned} r_V &= Vol(S^{\alpha}_{I\setminus I_0})/Vol(S_I) \\ r_\phi &= \langle X \rangle_{S^{\alpha}_{I\setminus I_0}}/\langle X \rangle_{S_I} \end{aligned} \quad (10)$$

The ratios $r_V$ for volumes and $r_\phi$ for mean flows measure the impact of selective flow knock-out on the solution space.

We sample one random topology, as shown in Fig. 1(b), with 2 banks, 3 firms and 4 workers. Keeping this topology fixed for the rest of this section, we sample the solution space and summarize the results in Fig. 3. All ratios are functions of $1 - \alpha$. Firstly we note that when $1 - \alpha$ is close to one, that is when the selected flow is almost set to zero, then the variance of the estimator of the relative volume is increasing. Indeed, the number of samples that satisfy the constraint tends to zero, which raises the variance. This phenomenon can be seen on the right of Fig. 3(b,d).

Fig. 3(a) shows that the impact of a knock-out of the same type of flow can differ from one agent to the other. This reveals the influence of the random topology, that breaks the symmetry that should othewise prevail if agents belonged to completely distinct economic circuits. The effect of connectivity will be examined in future works.

Then, we remark in Fig. 3(a) that single flow knock-outs are able to reduce considerably the volume of the steady-state solution space. For example, a 33% knock-out on loans granted to firms (LF) by just one bank out of the two defined in this simple model entails a cut by two-thirds of the volume ratio $r_V$.

The four most influential variables appear to be the payment of interests on loans (IL), the payment of consumption by workers to firms (CW), and the payment of loans by banks to firms (LR, LF which share the same value in this model where loans are always repaid). This ranking is the same at the individual or group level, as shown by Fig. 3(a,d). The two least influential variables are the interests on deposits paid to workers and firms. The flows with marginal probability concentrated on the right of $[0, c_0]$, (such as IL, CW, LR and LF in Fig.2) undergo a large reduction because the large tail is cut off. Conversely, flows with a small tail on the right of $[0, c_0]$ show little reduction.

Comparing the left and right panels of Fig. 3, we remark that the influence of knock-outs is more important on volume ratios than on flow ratios. Also, the influence of group knock-out on volume and mean flow is larger than single knock-out. These remarks are left for further theoretical analysis.

### *3.3. Linear flows*

An interesting particular case is examined in [8], where flows are linear functions of stocks, as specified in Tab. 3. In this setting, loans granted to firms depend linearly on the level of the bank vault $B_V$. Admitting that the values of parameters are known, the value of stocks can be directly obtained. Furthermore, various marginal quantities can be computed, such as the loans to firm deposits ratio which characterizes the financial fragility of firms, or the ratio of the flow of wages to the total stock of money, which is related to the speed of money. Results are shown in Fig. 4, with the same experimental setup as in section 3.2.

The stock of money is gaussian shaped, which can be explained by a central-limit argument. The ratios in Fig. 4(middle,right) have a lognormal shape. The ratio of total loans to total firm deposits has a mean value of 1.3, which is consistent with what can be found in the ABM litterature [37]. However, the ratio of the total flow of wages to the total stock of money should be larger by several orders of magnitude. This

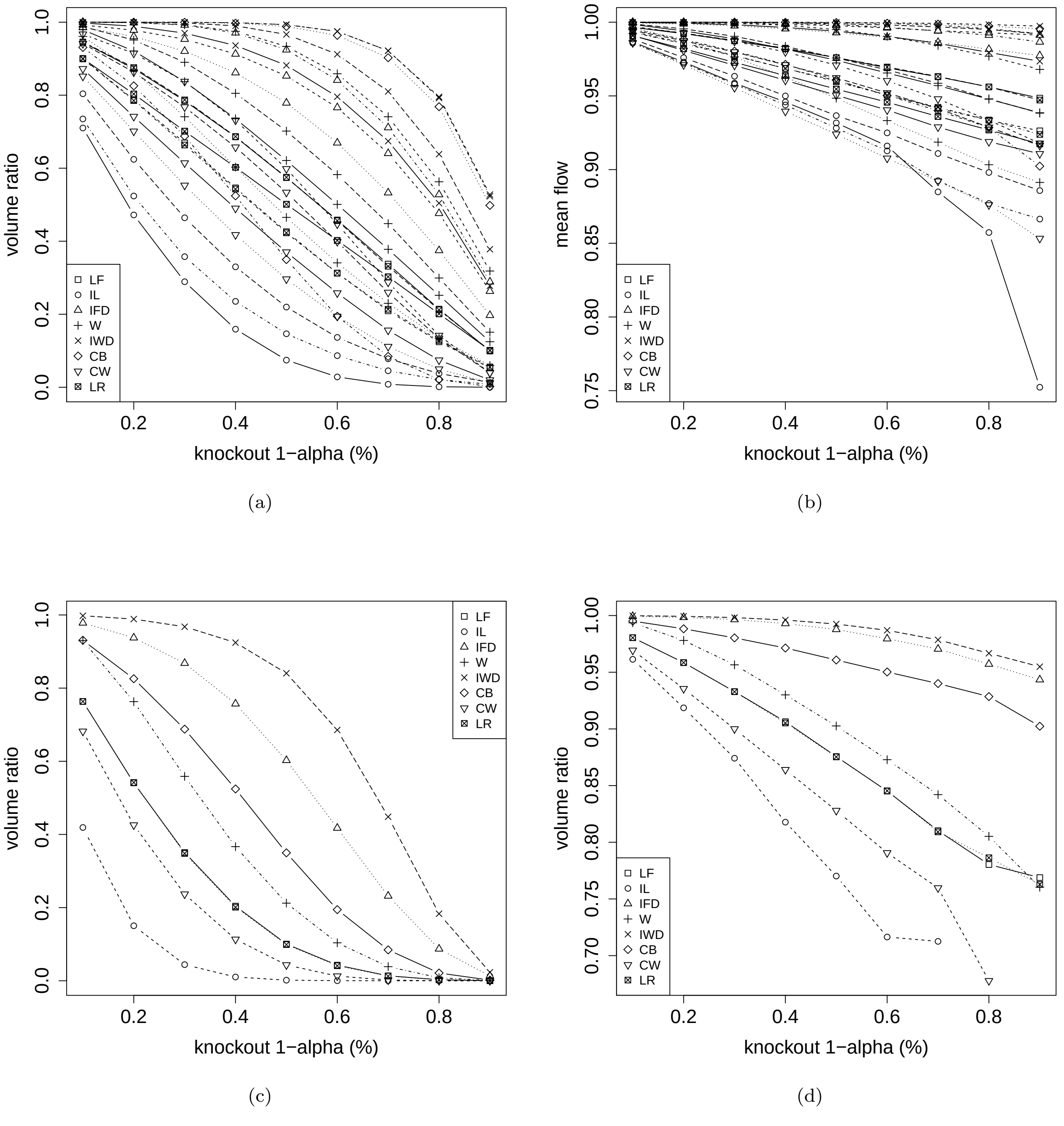



Figure 3: Impact of knock-out on volume and mean flow ratios, in the case of a single draw of the random connectivity model depicted in sec.2.1, with 2 banks, 3 firms and 4 workers. Volume and flow ratio appear as functions of $(1-\alpha)$. (top) single flow knock-out; (bottom) group flow knock-out. Flow labels are explained in Tab. 3. The number of sampled points is $n = 50000$. $c_0 = 10$

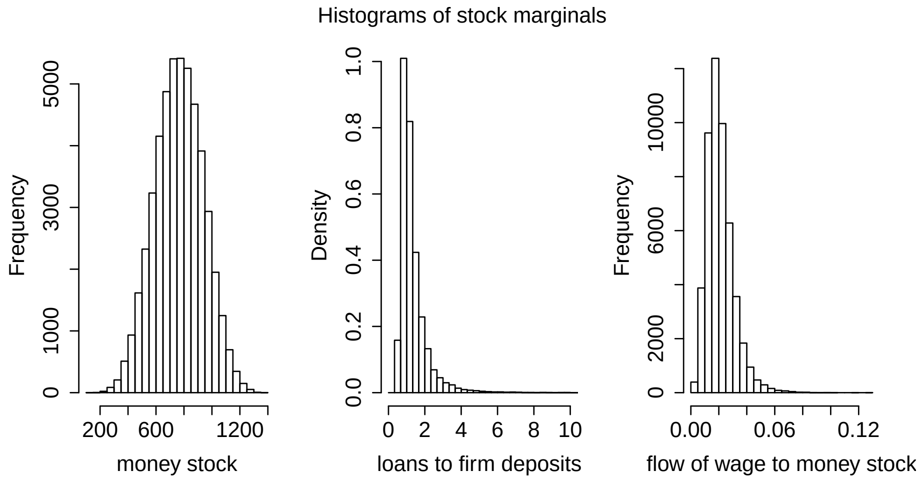



Figure 4: Histograms in the case of bounded linear flows $c_0 = 10$. (left) money stocks; (middle) ratio of total loans to total firm deposits; (right) ratio of the total flow of wages to the total stock of money. $nb = 2, nf = 3, nw = 4$, $n = 5.10^4$

can be explained because in the numerical experiments above, inspired by the framework developped in metabolic networks analysis, we impose an hypercubic constraint on flows. This is in contrast with SFC models where constraint are related to the initial stock of money.

With the hypotheses that flows are linear as summarized in Tab. 3 and $B_V + B_T + F_D + W_D = M$, we can write a new problem in a way more consistent with SFC models:

$$S = \left\{ x \text{ s.t.} \begin{bmatrix} \xi \\ v \end{bmatrix} x = \begin{bmatrix} 0 \\ M \end{bmatrix},\ B_V, B_T, F_D, W_D \in [0, M] \right\} \tag{11}$$

where :

$$\begin{aligned} x &= \begin{bmatrix} \beta_V B_V & r_L F_L & r_D F_D & \phi_D F_D & r_D W_D & \beta_T B_T & \omega_D W_D & \phi_L F_L \end{bmatrix}^T \\ v &= \begin{bmatrix} 1/\beta_V & 0 & 1/r_D & 0 & 1/r_D & 1/\beta_T & 0 & 0 \end{bmatrix} \end{aligned} \tag{12}$$

Eq.(11) can be solved exactly and approximately as seen in section 3.2. It is written in the special case $nf = nb = nw = 1$ but can be easily extended. Fig. 5 shows *hit-and-run* samples obtained setting the parameters values as in Tab. 4, with multiple agents per class.

The ratio of the total flow of wages to the total stock of money is now more consistent with econometric time series. However, the distribution of the stock of debt in Fig.5(left) is not realistic, since in many cases a large fraction of the stock of money is affected to debts. Consequently the ratio of total loans to total firm deposits is higher than in the case discussed at the beginning of this section, where flows were bounded.

More generally, these remarks show that the model depicted in this section has interesting properties, can be easily modified, but needs more work before being able to reproduce stylized facts. We discuss this topic in section 4.

## 4. Discussion

The first point we want to emphasize is that both the model and results in the sections above are preliminary. We do not claim that they can be of any use at the moment regarding economic analysis. More work is needed, in collaboration with economists, to improve their design, and to evaluate the empirical stylized facts they can reproduce. In our view, the capacity to account for economic crises can be seen

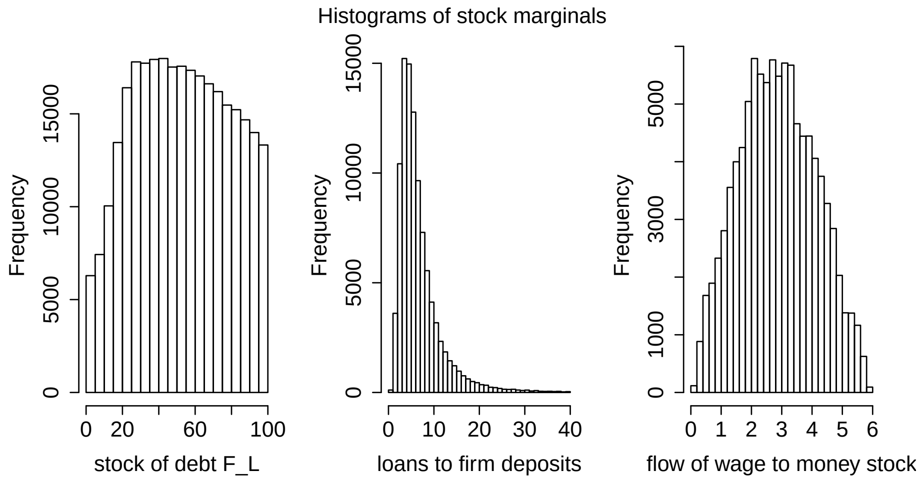



Figure 5: Histograms in the case of linear flows with constant money stock $M$. (left) debt stock; (middle) ratio of total loans to total firm deposits; (right) ratio of the total flow of wages to the total stock of money. $M = 100$, $nb = 2, nf = 3, nw = 4$, $n = 10^5$.

| Variable | Type of flow | Short label | Linear flows |
|---|---|---|---|
| $a$ | loans to firms | LF | $\beta_V B_V$ |
| $c$ | interest on loans | IL | $r_L F_L$ |
| $d$ | interest on firm deposits | IFD | $r_L F_L$ |
| $e$ | wages | W | $\phi_D F_d$ |
| $f$ | interest on worker deposits | IWD | $r_D W_D$ |
| $g$ | consumption of banks | CB | $\beta_T B_T$ |
| $h$ | consumption of workers | CW | $\omega_D W_D$ |
| $i$ | repayment of loans | LR | $\phi_L F_L$ |

Table 3: Labels associated with the different monetary flows, after [8]. The third column represents the particular case of linear flows.

| Parameter | Value (year$^{-1}$) |
|---|---|
| $\beta_V$ | 0.75 |
| $r_L$ | 0.05 |
| $r_D$ | 0.02 |
| $\phi_D$ | 2 |
| $\beta_T$ | 1 |
| $\omega_D$ | 26 |
| $\phi_L$ | 1/7 |

Table 4: Values of linear flows parameters, after [8] Tab.3

through the rapid decay of the volume ratio in response to modest fluctuations of the parameters, such as an increase in the knock-out factor. This can be seen independently of dynamical considerations.

In section 3.1 we compared two algorithms to compute the volume of the solution space. The first one, *lrs*, is exact but limited by the size of the problem because it relies on the vertex enumeration algorithm. The second belongs to the class of Monte-Carlo methods, and illustrates the uniform sampling property of the *hit-and-run* strategy. We didn't discuss convergence issues here, but they will become important as the size of the problem increases. Furthermore, alternative estimation methods exist, for example belief propagation that scales almost linearly in system size but needs the flow-space to be discretized.

None of these algorithms provide litteral expressions that could be compared to results presented in sections 3.1 and 3.2, such as the expression of absolute or relative volumes, or the mean flow. For example, the magnitude of volume reduction provoked by knock-outs is an important quantity from a system-scale point of view. Interestingly, some theoretical results have been developped in the statistical physics literature evoked earlier, and should be compared to our numerical experiments, in future works.

Concerning the complexity of the financial SFC model depicted in section 2.1, we started from a very simple setting, with a limited number of transaction types, one agent per class, no state nor central bank nor decentralized money creation. Flows were constrained in magnitude but were independent of stocks. Then, the number of agents was increased in section 3.2 which gave us a hint of the influence of topology. The accuracy of the model can be increased if random connectivity matrices are sampled in an random ensemble that corresponds to connectivity patterns observed empiricaly. Seeking inspiration in macro-economic literature (e.g. Lavoie and Godley), we may also add financial constraints on debt ratios at various levels. Transactions not covered in this article can also be added, related as bonds, investments, in an open economy. A major improvement would be to couple the financial side to a production model of the economy, in order to compute prices, demand, unemployment and profit, but many difficulties can be expected in that direction because of nonlinear relations that transform the linear CSP into a nonlinear one.

Similarly, in section 3.3 we proposed to rewrite the flow boundaries in the form of a constant money constraint, with the hypothesis that flows are linear functions of stocks, following [8]. We obtained interesting marginal quantities, such as the ratio of loans to firms deposits, or the ratio of money stock to the flow of wages. But this raises the issue of choosing the right parameter values. Various strategies can be implemented to address it, such as setting the parameters according to empirical data taken from public statistics. Another point of view is to consider the linear flows parameters as variables defined in specific intervals, and to include them in the sampling scheme used above. Although flows are linear functions of stocks, this problem is also a nonlinear CSP, harder to deal with than a linear one.

On the computational complexity side, a comparison should be made with other classical approaches such as DSGE models and ABM with respect to the number of agents, the sparseness of the network topology, the number of regions or countries, the richness of the financial mechanisms involved. We can remark that in the case of *hit-and-run* sampling, the main cost is polynomial in the system size during sampling. Then in order to compute all the ratios discussed above, it is not necessary to resample: basic thresholding is sufficient, and is linear in system size.

## 5. Conclusion

We proposed an original strategy to compute macro-economic flows in a financial economy, inspired by stock-flow consistent models and methods developed in the field of metabolic networks. We show that this approach can be efficiently transposed thanks to exact and approximate Monte-Carlo algorithms designed to solve Constraint Satisfaction Problems. The steady-state in flow space is seen as a polytope included in a large dimensional space, which weighs equally all configurations of the financial flows that are consistent with the constraints enforced by double-entry accounting.

Flow knock-outs can be used to model economic phenomena such as credit shortages. We show that different types of constraints have distinct effects on the volume of the solution space, that can be interpreted as characterizing the flexibility of the financial flows. Rapid decay of the volume ratio in response to modest fluctuations of the parameters can be interpreted as crises, independently of dynamical specifications.

We proposed an random connectivity extension of the linear flow model by [8], and have obtained a numerical approximation of the probability density of the ratio of loans to firms deposits, and the ratio of money stock to the flow of wages.

Our approach feeds a gap betwenn ABM on one hand, and system dynamics methods on the other hand, because the system size can scale up to thousands of agents while preserving the possibility of a theoretical analysis. This comes at the cost of a simplification of the model, notably the hypothesis of non-equilibrium stationary state. We discussed many potential improvements concerning the algorithms, the complexity of the model, and the relation to empirical data, and will deal with it in future works.

## Appendix A. Acknowledgements

Open-source software were used to perform this research: Python, R, the R *hit-and-run* package [35], *lrs*, graphviz, pygraphviz.

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