Minimal state space realizations in Jacobson normal form

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Abstract

We derive a procedure for a minimal state space realization of a rational transfer matrix over an arbitrary field. The procedure is based on the Smith-McMillan form and leads to a state transition matrix in Jacobson normal form.

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1 Introduction

Realization theory provides tools and techniques for a wide range of applications of mathematical systems theory. In particular, state space realizations are used for systems identification (Kalman and Declaris 1970), linear sequential circuits (Gill 1966) and amplifier circuit synthesis (Newcomb 1967). During the last decade realization techniques of algebraic systems theory have been playing an increasing role in convolutional coding (Rosenthal 2001). Various types of realizations serve as first order representations for convolutional codes and are the basis for the construction of new codes of Rosenthal and York (1999).

Overviews over the literature by De Schutter (2000) and Datta (1980) show three distinct approaches to construct minimal realizations of a strictly proper rational transfer matrix $T(s)$. The starting point for the first approach developed by Ho and Kalman (1966), Silverman (1971), Eising and Hautus (1981), is the impulse response written as

$$\sum_{\nu=1}^{\infty} C_\nu s^{-\nu} = T(s).$$

A block Hankel matrix containing the Markov parameters C_ν is then transformed in such a way that it produces a triple (F, G, H) of a minimal realization

$$H(sI - F)^{-1}G = T(s). \tag{1.1}$$

The algorithms of the second approach, e.g., of Mayne (1968), Rosenbrock (1970), Datta (1980), take advantage of the fact that, according to Kailath (1980: Chapter 2), it is fairly easy to write down a non-minimal controllable (or observable) realization by inspection. A minimal realization is then obtained by extracting the unobservable (or uncontrollable) parts. With the exception of Datta (1980) the two methodes described above are not designed to give a matrix F in (1.1) in a canonical form. In general, this can only be achieved by a third class of approaches which employ factorizations and transformations of the transfer matrix $T(s)$. Kalman's (1965) pioneering paper belongs to this group, and also Pace and Barnett (1974), Montes (1976) and Coppel (1981). By transforming the partial fraction components of a complex (or real) transfer matrix $T(s)$ into Smith-McMillan form and then using Taylor expansions, Kalman produced a minimal realization (1.1) with F being in Jordan normal form (or in real Jordan normal form). That procedure is restricted to algebraically (or real) closed fields.

In this paper we are dealing with transfer matrices over an arbitrary field K. We will adapt Kalman's approach to obtain a realization (1.1) where

F is in Jacobson normal form. The motivation for our study comes from applications of systems over finite fields such as linear sequential circuits (Gill 1966). According to Massey and Sain (1967), Forney (1970), Rosenthal et al. (1996), and Rosenthal (2001), convolutional codes can be interpreted as linear sequential circuits. Therefore we have developed our realization with the prospect of new constructions of codes in the spirit of Rosenthal and York (1999).

2 The Jacobson normal form

Let us briefly recall how the Jacobson normal form extends the concept of Jordan normal form. Throughout this paper $p \in K[s]$ will be a fixed monic irreducible polynomial,

$$p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0.$$

Let

$$C = C(p) = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{pmatrix} \tag{2.1}$$

be the companion matrix associated with p. In particular, if $p = s - \lambda$, then $C(p) = (\lambda)_{1\times 1}$. Define

$$V = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}_{n\times n} = e_n e_1^T, \tag{2.2}$$

where $e_1 = (1, 0 \ldots, 0)^T$ and $e_n = (0, \ldots, 0, 1)^T$are unit vectors of K^n. We call

$$J = J(p^k) = \begin{pmatrix} C & V & & & \\ & C & \ddots & & \\ & & \ddots & \ddots & \\ & & & C & V \\ & & & & C \end{pmatrix}_{nk\times nk} \tag{2.3}$$

a *Jacobson block* corresponding to p^k. The Jordan block

$$J[(s-\lambda)^k] = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & & \cdot \\ \vdots & & & \ddots & \\ \cdot & & & & 1 \\ 0 & \cdot & \cdot & \cdots & \lambda \end{pmatrix}_{k\times k}$$

is a special case of (2.3). The fact that the $(i, i+1)$-entries of J are equal to 1 implies that J is nonderogatory and that the Smith form of $sI - J$ is $\mathrm{diag}(1, \dots 1, p^k)$. Hence, if $A \in K^{\ell\times\ell}$ has p^k as its only elementary divisor, then A is similar (over K) to $J = J(p^k)$. The following, more general result can be traced back to Krull's (1921) Ph.D. thesis. More easily accessible references are the books of Jacobson (1953), Ayres (1962) or Cohn (1974).

Theorem 2.1. *Let $p_1, \dots, p_m$ be the distinct irreducible factors of the characteristic polynomial of a matrix $A \in K^{\ell\times\ell}$ and let*

$$p_1^{k_{11}}, \dots, p_1^{k_{1\tau_1}}, \dots, p_m^{k_{m1}}, \dots, p_m^{k_{m\tau_m}}, \qquad k_{11} \le \cdots \le k_{1\tau_1}, \dots, k_{m1} \le \cdots \le k_{m\tau_m}, \tag{2.4}$$

be the corresponding elementary divisors. Then A is similar to

$$\mathrm{diag}\Big(J\big(p_1^{k_{11}}\big), \dots, J\big(p_m^{k_{m\tau_m}}\big)\Big) . \tag{2.5}$$

The matrix (2.5) is called the *Jacobson normal form* of A.

3 Notation

Let $K(s)$ be the field of rational functions over K. An element $f \in K(s)$ is called *strictly proper* if $f = 0$ or $f = g/h$, $g, h \in K[s]$, $gh \neq 0$ and $\deg g < \deg h$. Let $K_{sp}(s)$ be the K-vector space of strictly proper rational functions over K. Then each $f \in K(s)$ can be decomposed uniquely as

$$f = w + y$$

such that $w \in K_{sp}(s)$ and $y \in K[s]$. If we set $\pi_- f = w$, then π_- is the projection of $K(s)$ onto $K_{sp}(s)$. In a natural way these definitions extend elementwise to vectors and matrices of rational functions. For a nonzero polynomial vector $h = (h_1, \dots h_r)^T \in K^r[s]$ we define

$$\deg h = \max\{\deg h_i,\ h_i \neq 0,\ i = 1, \dots r\} .$$

We set $\deg h = -\infty$ if $h = 0$.

Let I_k denote the $k \times k$ identity matrix and define

$$N_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \cdot & 0 & 1 & & \cdot \\ \vdots & & & \ddots & \\ \cdot & & & & 1 \\ 0 & \cdot & \cdot & \cdots & 0 \end{pmatrix}_{k \times k} .$$

According to Horn and Johnson (1991: Chapter 4) the Kronecker product of two matrices $A = (a_{ij})$ and B is the block matrix

$$A \otimes B = (a_{ij}\, B).$$

Note that the Jacobson block (2.3) can be written as $J = I_k \otimes C + N_k \otimes V$. If the products AC and BD exist then

$$(A \otimes B)(C \otimes D) = (AC \otimes BD)\ . \tag{3.1}$$

4 A special case

In this section we shall focus on a particular type of transfer matrices. The general realization problem will then be reduced to that special case. Let $W \in K^{q \times t}(s)$, $W \neq 0$, be of rank 1,

$$W = h \frac{1}{p^k} g^T \tag{4.1}$$

where $h \in K^q[s]$, $g \in K^t[s]$ are polynomial vectors. We want to construct a realization of $\pi_- W$. In addition to the companion matrix C associated with $p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$ we shall need the matrix

$$M = M(p) = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & 1 & 0 \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} . \tag{4.2}$$

Note that M satisfies

$$MC = C^T M\ . \tag{4.3}$$

Let h have the p-adic expansion

$$h = h_0 + h_1 p + \cdots + h_{k-1} p^{k-1} + \dots \tag{4.4}$$

where

$$h_i \in K^q[s],\ \deg h_i < n = \deg p,\ i \geq 0\ . \tag{4.5}$$

We define $H_i \in K^{q \times n}$, $i \geq 0$, by

$$h_i(s) = H_i \begin{pmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{pmatrix} . \tag{4.6}$$

Using the expansion

$$g = g_0 + g_1 p + \cdots + g_{k-1} p^{k-1} + \dots \tag{4.7}$$

with

$$g_i \in K^t[s],\ \deg g_i < n,\ i \geq 0\ , \tag{4.8}$$

we define matrices $G_i \in K^{t \times n}$ by

$$g_i(s) = G_i M \begin{pmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{pmatrix} . \tag{4.9}$$

Theorem 4.1. *Let $h \in K^q[s]$ and $g \in K^t[s]$ be given. Assume that*

$$W = h \frac{1}{p^k} g^T \tag{4.10}$$

is a coprime factorization. Let H_i and G_i, $i = 0, 1, \dots, k-1$, be defined by (4.4) – (4.6) *and* (4.7) – (4.9)*. Set*

$$H = (H_0, \dots, H_{k-1})$$

and

$$G = \begin{pmatrix} G_{k-1}^T \\ \vdots \\ G_0^T \end{pmatrix} .$$

Then

$$\pi_- W = H\Big(sI - J\big(p^k\big)\Big)^{-1} G\ , \tag{4.11}$$

and the realization in (4.11) *is minimal.*

Let us briefly describe how in the case of $p = s - \lambda$ the realization (4.11) reduces to the realization of Kalman (1965: 532-533). Consider (4.1) with

$$W = h \, \frac{1}{(s-\lambda)^k} \, g^T$$

and

$$h = \sum_{i \geq 0} h_i (s-\lambda)^i, \; h_i \in K^q, \quad \text{and} \quad g = \sum_{i \geq 0} g_i (s-\lambda)^i, \; g_i \in K^t.$$

Because of $\deg p = 1$ the matrix M in (4.2) reduces to $M = I_1$. Furthermore, in (4.6) and (4.9) we have $h_i(s) = h_i$ and $g_i(s) = g_i$. Therefore (4.11) yields

$$\pi_- h \frac{1}{(s-\lambda)^k} g^T = (h_0, \ldots, h_{k-1}) \Big((s-\lambda) I_k - N_k \Big)^{-1} \begin{pmatrix} g_{k-1}^T \\ \vdots \\ g_0^T \end{pmatrix} .$$

The proof of Theorem 4.1 is based on two lemmas.

Lemma 4.2. *Let the polynomial vector $b \in K^n[s]$ be defined as*

$$b(s) = (1, s, \ldots, s^{n-1})^T .$$

Let $C = C(p)$ be the companion matrix for the polynomial p and let $M = M(p)$ be given by (4.2). Then

$$(sI - C)^{-1} = \pi_- p^{-1} b b^T M \tag{4.12}$$

and

$$\big(sI - J(p^k) \big)^{-1} = \pi_- \big(p I_k - N_k \big)^{-1} \otimes b b^T M \; . \tag{4.13}$$

Proof. Obviously $(sI - C) b = p \, e_n$ is equivalent to

$$(sI - C)^{-1} e_n = p^{-1} b. \tag{4.14}$$

From (4.14) and (4.3) we obtain

$$e_1^T (sI - C)^{-1} = e_n^T M (sI - C)^{-1} e_n^T (sI - C^T)^{-1} M = p^{-1} b^T M \; . \tag{4.15}$$

It is easy to see that

$$s^j (sI - C)^{-1} = (s^{j-1} I + \cdots + C^{j-1}) + C^j (s^{-1} I + s^{-2} C + \cdots)$$

implies

$$\pi_- s^j (sI - C)^{-1} = C^j (sI - C)^{-1} \; . \tag{4.16}$$

Therefore,

$$\pi_- p^{-1} b b^T M = \pi_- b e_1^T (sI - C)^{-1} =$$

$$= \pi_- \sum_{\nu=1}^{n} s^{\nu-1} e_\nu e_1^T (sI - C)^{-1} = \sum_{\nu=1}^{n} e_\nu e_1^T C^{\nu-1} (sI - C)^{-1} =$$

$$= \left(\sum_{\nu=1}^{n} e_\nu e_\nu^T \right) (sI - C)^{-1} = (sI - C)^{-1} \, . \tag{4.17}$$

To verify (4.13) we note that

$$\big(pI_k - N_k\big)^{-1} = \begin{pmatrix} p^{-1} & p^{-2} & \cdots & p^{-k} \\ 0 & p^{-1} & \cdots & p^{-(k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p^{-1} \end{pmatrix} . \tag{4.18}$$

Put $Q = V(sI - C)^{-1}$ where V is given by (2.2). Then

$$(sI - J)^{-1} = \begin{pmatrix} (sI - C)^{-1} & (sI - C)^{-1} Q & \cdots & (sI - C)^{-1} Q^{k-1} \\ O & (sI - C)^{-1} & \cdots & (sI - C)^{-1} Q^{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & (sI - C)^{-1} \end{pmatrix} .$$

Now (4.14), (4.15) and

$$e_1^T (sI - C)^{-1} e_n = p^{-1}$$

imply

$$\begin{aligned} &(sI - C)^{-1} Q^{i-1} = (sI - C)^{-1} \big[e_n e_1^T (sI - C)^{-1} \big]^{i-1} = \\ &= (sI - C)^{-1} e_n \, \big[e_1^T (sI - C)^{-1} e_n \cdots e_1^T (sI - C)^{-1} e_n \big] \, e_1^T (sI - C)^{-1} = \\ &= (sI - C)^{-1} e_n \, [p^{-i+2}] \, e_1^T (sI - C)^{-1} = p^{-1} b \, p^{-i+2} \, p^{-1} b^T M = \\ &\qquad\qquad = p^{-i} b b^T M = \pi_- \, p^{-i} b b^T M, \; i = 2, \ldots, k \, . \end{aligned}$$

Hence, (4.12) and (4.18) together with the definition of the Kronecker product yield

$$(sI-J)^{-1} = \pi_-\begin{pmatrix} p^{-1}bb^TM & \cdots & p^{-k}bb^TM \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p^{-1}bb^TM \end{pmatrix} =$$

$$= \pi_-\big((pI_k - N_k)^{-1} \otimes bb^TM\big) .$$

□

For the following well known result on the dimension of minimal realizations we refer to Coppel (1974).

Lemma 4.3. *Let P, S, A be polynomial matrices such that*

$$R = PA^{-1}S$$

is a coprime factorization. Then the dimension of a minimal realization of π_-R is equal to the degree of $\det A$.

Proof of Theorem 4.1:
From (4.4) and (4.7) follows

$$\pi_-W = \pi_-(h_0,\ldots,h_{k-1})\begin{pmatrix} p^{-1} & p^{-2} & \cdots & p^{-k} \\ 0 & p^{-1} & \cdots & p^{-(k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p^{-1} \end{pmatrix}\begin{pmatrix} g_{k-1}^T \\ \vdots \\ g_0^T \end{pmatrix} .$$

Hence the relations

$$(h_0,\ldots,h_{k-1}) = (H_0,\ldots,H_{k-1})(I_k \otimes b)$$

and

$$(g_{k-1},\ldots,g_0) = (G_{k-1},\ldots,G_0)(I_k \otimes Mb) ,$$

the identity (4.18), and the product formula (3.1) imply

$$\pi_-W = \pi_-\, H(I\otimes b)(pI-N)^{-1}(I\otimes b^TM)G =$$

$$= H\pi_-\Big[\big(pI-N\big)^{-1} \otimes bb^TM\Big]G .$$

Now (4.11) follows immediately form (4.13). According to Lemma 4.3 the realization (4.11) is minimal. □

5 Reduction to realizations with a single Jacobson block

Let $T \in K^{q\times t}(s)$, $T \neq 0$, be a strictly proper rational matrix, let $d \in K[s]$ be the monic least common denominator of all elements of T, and let $p_1, \dots, p_m$ be monic irreducible polynomials such that $p_1^{\ell_1} \dots p_m^{\ell_m}$ is a prime factorization of d. To build a realization of T based on Theorem 4.1 we carry out two steps. First we take a partial fraction decomposition of each entry of T. Then we decompose T accordingly as

$$T = \sum_{\mu=1}^{m} T_{p_\mu} \tag{5.1}$$

such that each component T_{p_μ} is strictly proper having only powers of p_μ as denominators of its entries. If

$$H_\mu(sI - F_\mu)^{-1}G_\mu = T_{p_\mu}(s),\ \mu = 1, \dots, m\ , \tag{5.2}$$

are minimal realizations and if we set

$$F = \mathrm{diag}(F_1, \dots, F_m),\ G = \begin{pmatrix} G_1 \\ \vdots \\ G_m \end{pmatrix},\ H = (H_1, \dots, H_m)\ ,$$

then

$$H(sI - F)^{-1}G = T(s) \tag{5.3}$$

is a minimal realization. We call (5.3) the *direct sum* of the realizations (5.2).

At this point we may restrict ourselves to a strictly proper rational matrix T where the least common denominator of its entries is a power of an irreducible polynomial p. In the second reduction step we want to decompose such a matrix T into a sum of rank 1 matrices. Assume rank $T = r$ and let

$$\Sigma = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

be the Smith-McMillan form of T with

$$D = \mathrm{diag}\Big(\frac{a_1}{p^{k_1}}, \dots, \frac{a_r}{p^{k_r}}\Big)\ , k_1 \geq \cdots \geq k_r \geq 0\ ,$$

and

$$a_i \in K[s],\ \gcd(p, a_i) = 1,\ i = 1, \dots, r,\ a_1 | \dots | a_r\ .$$

Let $U = (u_1, \ldots, u_q) \in K^{q\times q}[s]$ and $V = (v_1, \ldots, v_t) \in K^{t\times t}[s]$ be unimodular matrices such that

$$T = U\Sigma V^T \ . \tag{5.4}$$

It follows from Lemma 4.3 that a minimal realization of T has dimension equal to $\sum_{i=1}^{r} nk_i$. Now let $\widetilde{a}_i \in K[s]$ be such that

$$\pi_-\left(a_i p^{-k_i}\right) = \widetilde{a}_i p^{-k_i} \ ,$$

and define

$$w_i = \left(u_i\widetilde{a}_i\right)\frac{1}{p^{k_i}}\, v_i^T,\ i = 1, \ldots, r \ . \tag{5.5}$$

Then (5.4) and $T = \pi_- T$ imply

$$T = \sum_{i=1}^{r} \pi_- w_i \ . \tag{5.6}$$

Clearly, $\pi_- w_i = 0$ if $k_i = 0$. If $k_i > 0$ then (5.5) is a coprime factorization since u_i and v_i are columns of unimodular matrices and $\gcd(p, \tilde{a}_i) = 1$. In that case nk_i is the dimension of a minimal realization of $\pi_- w_i$. Therefore a direct sum of minimal realizations of the matrices $\pi_- w_i$ yields a minimal realization of T. We remark that (5.6) puts us in the position to apply Theorem 4.1. □

It has been pointed out by Gill (1966) that the resolvent

$$T(s) = (sI - A)^{-1} \tag{5.7}$$

of a matrix $A \in K^{\ell\times\ell}$ is a special case of transfer matrix. Thus our realization algorithm applied to (5.7) yields $(sI - A)^{-1} = H\,(sI - F)^{-1}\,G$ and $G = H^{-1}$. Hence $A = H\,F\,H^{-1}$, and H transforms A into Jacobson normal form. A different approach to derive the Jacobson normal form from the resolvent is due to Della Dora and Jung (1996).

6 An example

In the following example the underlying field is $K = \mathbb{Z}_5$. We consider the transfer matrix

$$T(s) =$$
$$\begin{pmatrix} \dfrac{2\,s^6 + 3\,s^3 + 2\,s^2 + s + 4}{(s^2 + s + 2)^2\,(s^3 + 3\,s^2 + s + 1)} & \dfrac{s^6 + 4\,s^3 + s^2 + 2\,s + 2}{(s^2 + s + 2)^2\,(s^3 + 3\,s^2 + s + 1)} \\ \dfrac{2\,s^6 + 3\,s^3 + 2\,s^2 + s + 1}{(s^2 + s + 2)^2\,(s^3 + 3\,s^2 + s + 1)} & \dfrac{2\,(3\,s^6 + 2\,s^3 + 3\,s^2 + s + 3)}{(s^2 + s + 2)^2\,(s^3 + 3\,s^2 + s + 1)} \end{pmatrix} \tag{6.1}$$

with entries in $\mathbb{Z}_5(s)$. We shall proceed along the lines of Section 5 and Section 4.

(1.) Partial fraction decomposition of T

Let $p_1 = s^2 + s + 2$ and $p_2 = s^3 + 3\,s^2 + s + 1$. Then

$$T = T_{p_1} + T_{p_2}$$

and

$$T_{p_1}(s) = \begin{pmatrix} \dfrac{3\,s^3 + 4\,s^2 + s}{(s^2+s+2)^2} & \dfrac{3\,s^3 + 2\,s^2 + 3\,s + 4}{(s^2+s+2)^2} \\ \dfrac{s+3}{(s^2+s+2)^2} & \dfrac{2\,s^3 + 4\,s^2 + 3\,s}{(s^2+s+2)^2} \end{pmatrix}$$

and

$$T_{p_2}(s) = \begin{pmatrix} \dfrac{4\,s^2 + 4\,s + 1}{s^3 + 3\,s^2 + s + 1} & \dfrac{3\,s^2 + 3\,s + 2}{s^3 + 3\,s^2 + s + 1} \\ \dfrac{2\,s^2 + s + 2}{s^3 + 3\,s^2 + s + 1} & \dfrac{4\,s^2 + 2\,s + 4}{s^3 + 3\,s^2 + s + 1} \end{pmatrix}.$$

(2.) Realization of T_{p_1}

The Smith-McMillan form of T_{p_1} is

$$\Sigma = \operatorname{diag}(\frac{a_1}{p^2}, \frac{a_2}{p}) = \begin{pmatrix} \dfrac{1}{(s^2+s+2)^2} & 0 \\ 0 & \dfrac{(s+1)\,(s^3 + 3\,s^2 + 4)}{s^2+s+2} \end{pmatrix}.$$

We have $T_{p_1} = U\Sigma V^T$, and the unimodular matrices U and V are given by

$$U = (u_1, u_2) = \begin{pmatrix} s\,(3\,s^2 + 4\,s + 1) & 4\,s^2 + 3 \\ s + 3 & 3 \end{pmatrix},$$

and

$$V = (v_1,\, v_2) = \begin{pmatrix} 1 & 0 \\ 2\,s^5 + 4\,s^4 + s^3 + 4\,s^2 + 2 & 1 \end{pmatrix}.$$

Note that

$$\frac{a_2}{p} = \frac{(s+1)(s^3 + 3s^2 + 4)}{s^2 + s + 2}$$

is not strictly proper. We calculate $\tilde{a}_2$ and obtain

$$\pi_- \frac{a_2}{p} = \frac{\tilde{a}_2}{p} = \frac{3}{s^2 + s + 2}\ .$$

Set

$$w_1 = u_1 \frac{\tilde{a}_1}{p^2} v_1^T =$$
$$\begin{pmatrix} s\,(3\,s^2+4s+1) \\ 3+s \end{pmatrix} \frac{1}{(s^2+s+2)^2} \begin{pmatrix} 1 \\ 2\,s^5+4\,s^4+s^3+4\,s^2+2 \end{pmatrix}^T$$

and

$$w_2 = u_2 \frac{\tilde{a}_2}{p} v_2^T = \begin{pmatrix} 4\,s^2+3 \\ 3 \end{pmatrix} \frac{3}{s^2+s+2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T .$$

Then $T_{p_1} = \pi_- w_1 + \pi_- w_2$.

(2.1) Realization of $\pi_- w_1$.

We set

$$h = u_1\,\tilde{a}_1 = \begin{pmatrix} s\,(3\,s^2+4s+1) \\ 3+s \end{pmatrix}$$

and

$$g = v_1 = \begin{pmatrix} 1 \\ 2\,s^5+4\,s^4+s^3+4\,s^2+2 \end{pmatrix}.$$

Then $h = h_0 + h_1 p$ with

$$h_0 = \begin{pmatrix} 4s+3 \\ s+3 \end{pmatrix},\ h_1 = \begin{pmatrix} 3s+1 \\ 0 \end{pmatrix}.$$

Similarly, $g = g_0 + g_1\,p + g_2\,p^2$ with

$$g_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ g_1 = \begin{pmatrix} 0 \\ s \end{pmatrix},\ g_2 = \begin{pmatrix} 0 \\ 2\,s \end{pmatrix}.$$

This leads to

$$H_0 = \begin{pmatrix} 3 & 4 \\ 3 & 1 \end{pmatrix},\ H_1 = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix},$$

and

$$(H_0 \,|\, H_1) = \left(\begin{array}{cc|cc} 3 & 4 & 1 & 3 \\ 3 & 1 & 0 & 0 \end{array}\right) = H.$$

The matrix M in (4.2) is given by

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence

$$G_0 = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix},\ G_1 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

such that

$$\left(\frac{G_1^T}{G_0^T}\right) = \left(\begin{array}{cc} 0 & 1 \\ 0 & -1 \\ \hline 0 & 0 \\ 1 & 2 \end{array}\right) = G\ .$$

Finally, for $p = s^2 + s + 2$, we have

$$J(p^2) = \begin{pmatrix} 0 & 1 & & \\ 3 & 4 & 1 & \\ & & 0 & 1 \\ & & 3 & 4 \end{pmatrix} = F\ .$$

(2.2) <u>Realization of $\pi_- w_2$</u>
Set

$$h = u_2\,\tilde{a}_2 = \begin{pmatrix} 4\,s^2 + 3 \\ 3 \end{pmatrix} \times 3 = \begin{pmatrix} 2\,s^2 + 4 \\ 4 \end{pmatrix}$$

and

$$g = v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Then $\quad h = h_0 + h_1\,p \quad$ with

$$h_0 = \begin{pmatrix} 3\,s \\ 4 \end{pmatrix},\ h_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}\ .$$

Hence

$$H_0 = \begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix} = H\ .$$

From

$$g_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and (4.9) we obtain

$$G_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = G\ .$$

The corresponding state space matrix is

$$J(p^1) = C(p) = \begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix} = F\ .$$

(3.) <u>Realization of T_{p_2}</u>

The Smith-McMillan form of T_{p_2} is

$$\Sigma = \begin{pmatrix} \frac{1}{s^3+3\,s^2+s+1} & 0 \\ 0 & 0 \end{pmatrix} .$$

The unimodular matrices U, V in the decomposition $U\Sigma V^T = T_{p_2}$ are

$$U = (u_1, u_2) = \begin{pmatrix} 4\,s^2+4\,s+1 & s \\ 2\,s^2+s+2 & 3\,s+1 \end{pmatrix}, \; V = (v_1, v_2) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} .$$

Set

$$p = s^3 + 3\,s^2 + s + 1.$$

Then

$$T_{p_2} = u_1 \frac{1}{p} v_1^T = \begin{pmatrix} 4\,s^2+4\,s+1 \\ 2\,s^2+s+2 \end{pmatrix} \frac{1}{s^3+3s^2+s+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T .$$

It is easy to see that one can obtain the minimal realization of T_{p_2} directly from Theorem 4.1. Note that T_{p_2} is of the form (4.10) with $h = u_1$, $g = v_1$, and $k = 1$. Moreover, $\deg h < \deg p$ and $\deg g < \deg p$ imply $H = H_0$ and $G = G_0$. Thus $h = h_0$ yields

$$H_0 = \begin{pmatrix} 1 & 4 & 4 \\ 2 & 1 & 2 \end{pmatrix} = H .$$

From

$$M = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{6.2}$$

and $g = g_0$ follows

$$G_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \end{pmatrix} = G .$$

Finally, we have

$$J(p^1) = C(p) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 4 & 2 \end{pmatrix} = F .$$

(4.) <u>Realization of T</u>

Taking the direct sum of the realizations of $\pi_- w_1$, $\pi_- w_2$ and T_{p_2} we obtain

$$F = \left(\begin{array}{cccc|cc|ccc} 0 & 1 & & & & & & & \\ 3 & 4 & 1 & & & & & & \\ 0 & 0 & 0 & 1 & & & & & \\ 0 & 0 & 3 & 4 & & & & & \\ \hline & & & & 0 & 1 & & & \\ & & & & 3 & 4 & & & \\ \hline & & & & & & 0 & 1 & 0 \\ & & & & & & 0 & 0 & 1 \\ & & & & & & 4 & 4 & 2 \end{array}\right),$$

$$H = \left(\begin{array}{cccc|cc|ccc} 3 & 4 & 1 & 3 & 0 & 3 & 1 & 4 & 4 \\ 3 & 1 & 0 & 0 & 4 & 0 & 2 & 1 & 2 \end{array}\right) \quad \text{and} \quad G = \left(\begin{array}{cc} 0 & 1 \\ 0 & -1 \\ 0 & 0 \\ 1 & 2 \\ \hline 0 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \\ 1 & 2 \end{array}\right).$$

Then the transfer matrix T in (6.1) has a minimal realization $H(sI-F)^{-1}G = T(s)$ where the matrices F, G, H are the ones displayed above, and F is in Jacobson normal form.

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