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SOME NOTES ON THE REGULAR GRAPH DEFINED BY SCHMIDT AND SUMMERER AND UNIFORM APPROXIMATION

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ABSTRACT. Within the study of parametric geometry of numbers W. Schmidt and L. Summerer introduced so-called regular graphs. Roughly speaking the successive minima functions for the classical simultaneous Diophantine approximation problem have a very special pattern if the vector $\underline{\zeta}$ induces a regular graph. The regular graph is in particular of interest due to a conjecture by Schmidt and Summer concerning classic approximation constants. This paper aims to provide several new results on the behavior of the successive minima functions for the regular graph. Moreover, we improve the best known upper bounds for the classic approximation constants $\widehat{w}_n(\zeta)$, valid uniformly for all transcendental ζ, provided that the Schmidt-Summerer conjecture is true.

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1. Introduction

1.1. **Outline.** This paper aims on the one hand to give a better understanding of the regular graph defined by Schmidt and Summerer, and on the other hand to give a connection to the uniform approximation constants $\widehat{w}_n$. Theorem 2.6 and Theorem 2.8 can be considered the main results concerning the first, Theorem 3.3 the main result for the latter topic. The paper is structured in a way that the main new results appear rather late in the paper since a lot of (notational) preparation will be needed for their formulation. We provide the preparatory results, based on the work [22],[24] of Schmidt and Summerer, in Section 1 and Section 2.1. We recommend the reader to look at the illustrations of combined graphs and in particular the regular graph in [24]. See also [20] for Matlab plots of the combined graph for special choices of real vectors.

1.2. **Parametric geometry of numbers.** We first recall the notion and results concerning parametric geometry of numbers following Schmidt and Summerer [22]. Our notation will partially deviate from that in [22] for technical reasons. Let $n \geq 1$ be an integer, Λ be a lattice and K be a convex body in $\mathbb{R}^{n+1}$. Then for $1 \leq j \leq n+1$ the j-th successive minimum $\tau_{n,j}$ of K with respect to Λ is defined as the minimum real number ν such that νK contains j linearly independent lattice points of Λ. Minkowski's first lattice point theorem asserts that if $\det(\Lambda) \leq \mathrm{vol(K)}$ then the first successive minimum is at most 1, that is K contains a non-trivial lattice point. Moreover, Minkowski's second lattice point theorem asserts

$$\frac{2^{n+1}}{(n+1)!}\frac{\det(\Lambda)}{\mathrm{vol(K)}} \leq \tau_{n,1}\tau_{n,2}\cdots\tau_{n,n+1} \leq 2^{n+1}\frac{\det(\Lambda)}{\mathrm{vol(K)}}.$$

Consider $\underline{\zeta} = (\zeta_1, \ldots, \zeta_n)$ in $\mathbb{R}^n$. Schmidt and Summerer apply Minkowski's theorems to a particular lattice and convex bodies parametrized by $Q > 1$ connected to simultaneous approximation. Concretely they consider the lattice of points $(x, \zeta_1 x - y_1, \ldots, \zeta_n x - y_n)$ with $(x, y_1, \ldots, y_n) \in \mathbb{Z}^{n+1}$ and the convex body $K = K(Q)$ defined by $|x| \leq Q$ and $|y_j| \leq Q^{-1/n}$ for $1 \leq j \leq n$, where $Q > 1$ is a parameter. Let $\tau_{n,j}(Q)$ be the j-th successive minimum for the parameter Q and put $\psi_{n,j}(Q) = \log\tau_{n,j}(Q)/\log Q$. Further let

$$\underline{\psi}_{n,j} = \liminf_{Q\to\infty} \psi_j(Q), \qquad \overline{\psi}_{n,j} = \limsup_{Q\to\infty} \psi_j(Q).$$

In this situation Minkowki's theorems give $\psi_1(Q) < 0$ for all $Q > 1$ and

$$\left|\sum_{j=1}^{n+1} \psi_{n,j}(Q)\right| \leq \frac{c}{\log Q}$$

for a constant c. Moreover it is not hard to see that

$$-1 \leq \psi_{n,j}(Q) \leq \psi_{n,2}(Q) \leq \cdots \leq \psi_{n,n+1}(Q) \leq \frac{1}{n}, \qquad Q > 1,$$

and in particular

$$-1 \leq \underline{\psi}_{n,j} \leq \overline{\psi}_{n,j} \leq \frac{1}{n}, \qquad 1 \leq j \leq n+1.$$

We also want to introduce the inferred successive minima functions $L_{n,j}(q)$, where $q = \log Q$, derived from the functions $\psi_{n,j}$ basically by taking logarithms

$$L_{n,j}(q) = \log(\tau_{n,j}(Q)) = q\psi_{n,j}(Q), \qquad 1 \leq j \leq n+1. \tag{1}$$

The functions $L_{n,j}(q)$ are piecewise linear with slopes among $\{-1, 1/n\}$. It follows that the j-th successive minimum $\tau_{n,j}$ of the above defined lattice point problem tends to infinity if and only if $L_{n,j}(q)$ tends to infinity, and to 0 if and only if $L_{n,j}(q)$ tends to $-\infty$.

The new results in this paper will mostly be formulated and proved in terms of closely connected classical approximation constants we define now. For given $\underline{\zeta} = (\zeta_1, \ldots, \zeta_n)$ in $\mathbb{R}^n$ let $\lambda_{n,j} = \lambda_{n,j}(\underline{\zeta})$ be the supremum ν such that there are arbitrarily large X for which

$$|x| \leq X, \qquad \max_{1\leq j\leq k} |\zeta_j x - y_j| \leq X^{-\nu} \tag{2}$$

has j linearly independent solutions $(x, y_1, \ldots, y_n)$ in $\mathbb{Z}^{n+1}$. Moreover let $\widehat{\lambda}_{n,j} = \widehat{\lambda}_{n,j}(\underline{\zeta})$ be the supremum of ν such that the system (2) has j linearly independent integer vector solutions $(x, y_1, \ldots, y_n)$ for all large X. For $\lambda_{n,1}$ we will also simply write λ_n and similarly $\widehat{\lambda}_n$ for $\widehat{\lambda}_{n,1}$. Observe that Minkowski's first lattice point theorem (or Dirichlet's Theorem) implies for all $\underline{\zeta} \in \mathbb{R}^n$ the estimates

$$\lambda_n \geq \widehat{\lambda}_n \geq \frac{1}{n}. \tag{3}$$

Minkowski's second Theorem more generally yields

$$\frac{1}{n} \leq \lambda_n \leq \infty, \tag{4}$$

$$\frac{1}{n} \leq \lambda_{n,2} \leq 1, \tag{5}$$

$$0 \leq \lambda_{n,j} \leq \frac{1}{j-1}, \qquad 3 \leq j \leq n+1. \tag{6}$$

and similarly

$$\frac{1}{n} \leq \widehat{\lambda}_n \leq 1, \tag{7}$$

$$0 \leq \widehat{\lambda}_{n,j} \leq \frac{1}{j}, \qquad 2 \leq j \leq n, \tag{8}$$

$$0 \leq \widehat{\lambda}_{n,n+1} \leq \frac{1}{n}. \tag{9}$$

This is [20, (14)-(18)]. Moreover $\lambda_{n,j} \geq \widehat{\lambda}_{n,j-1}$ holds for $2 \leq j \leq n+1$ as pointed out in [22]. Furthermore by [20, (13)], which generalizes [22, Theorem 1.4], we have

$$(1+\lambda_{n,j})(1+\underline{\psi}_{n,j}) = (1+\widehat{\lambda}_{n,j})(1+\overline{\psi}_{n,j}) = \frac{n+1}{n}, \qquad 1 \leq j \leq n+1.$$

In particular for $1 \leq j \leq n+1$ we have the equivalence

$$\underline{\psi}_{n,j} < 0 \iff \lambda_{n,j} > \frac{1}{n}, \qquad \overline{\psi}_{n,j} < 0 \iff \widehat{\lambda}_{n,j} > \frac{1}{n}. \tag{10}$$

This is of interest since $\overline{\psi}_{n,j} < 0$ implies that the j-th successive minimum function $\tau_{n,j}$ related to the above lattice point problem tends to 0, whereas $\underline{\psi}_{n,j} > 0$ implies it tends to infinity. For $\underline{\psi}_{n,j} = 0$ and $\overline{\psi}_{n,j} = 0$ respectively, it is not possible to decide whether this is true and one has to look closer at the functions $L_{n,j}(q)$. This will be of importance in Section 2.3.

We now introduce the dual problem studied in [22] as well. Let Λ^* be the dual lattice of λ above, defined as the set of points $(x-\zeta_1 y_1 - \cdots - \zeta_n y_n)$ with $(x, y_1, \ldots, y_n) \in \mathbb{Z}^{n+1}$. Further consider the convex body $K^* = K^*(Q)$ defined by the convex body of points that arises from the body $\sum_{1\leq i\leq n+1} |x_i| \leq 1$ by the transformation $T_Q : (p_1, \ldots, p_{n+1}) \to (p_1^{-1}, p_2^{1/n}, \ldots, p_{n+1}^{1/n})$. Let $\tau_{n,j}^*(Q)$ denote the j-th successive minimum in the above context. Again put $q = \log Q$ and $\psi_{n,j}^*(q) = \log \tau_{n,j}^*(Q)/q$ for $1 \leq j \leq n+1$. Similarly as above we will rather use classic approximation constants for our results in this paper. Define the classic approximation constant $w_{n,j}$ and $\widehat{w}_{n,j}$ respectively as the supremum of ν such that

$$|x| \leq X, \qquad |x + \zeta_1 y_1 + \cdots + \zeta_n y_n| \leq X^{-\nu}$$

has j linearly independent integer vector solutions for arbitrarily large X and all large X respectively. Again we also write w_n instead of $w_{n,1}$ and $\widehat{w}_n$ instead of $\widehat{w}_{n,1}$. Observe that again Minkoski's first lattice point theorem (or Dirichlet's Theorem) implies

$$w_n \geq \widehat{w}_n \geq n. \tag{11}$$

Then again we have

$$(1+w_{n,j})\left(\frac{1}{n}+\underline{\psi}^{*}_{n,j}\right)=(1+\widehat{w}_{n,j})\left(\frac{1}{n}+\overline{\psi}^{*}_{n,j}\right)=\frac{n+1}{n}, \qquad 1\leq j\leq n+1. \tag{12}$$

Moreover $\underline{\psi}^{*}_{n,j} = -\underline{\psi}_{n,n+2-j}$ and $\overline{\psi}^{*}_{n,j} = -\overline{\psi}_{n,n+2-j}$ for $1 \leq j \leq n+1$ by Mahler's relations [12], which was in fact used to infer (12). From the above, as already mentioned in [19, (1.24)], for the special case of successive powers, it can be deduced that

$$w_{n,j}=\frac{1}{\widehat{\lambda}_{n,n+2-j}}, \qquad 1\leq j\leq n+1, \tag{13}$$

such as

$$\lambda_{n,j}=\frac{1}{\widehat{w}_{n,n+2-j}}. \qquad 1\leq j\leq n+1, \tag{14}$$

Together with the bounds in (4)-(9) we obtain

$$n\leq w_{n}\leq\infty, \tag{15}$$

$$n+2-j\leq w_{n,j}\leq\infty, \qquad 2\leq j\leq n, \tag{16}$$

$$1\leq w_{n,n+1}\leq n. \tag{17}$$

and

$$n+1-j\leq \widehat{w}_{n,j}\leq\infty, \qquad 1\leq j\leq n-1, \tag{18}$$

$$1\leq \widehat{w}_{n,n}\leq n, \qquad 2\leq j\leq n, \tag{19}$$

$$0\leq \widehat{w}_{n,n+1}\leq n. \tag{20}$$

In Section 3 we will deal with the heavily studied case $\underline{\zeta}=(\zeta,\zeta^2,\ldots,\zeta^n)$ for ζ some real number. In this case we will write $w_{n,j}(\zeta)$ for $w_{n,j}(\zeta,\zeta^2,\ldots,\zeta^n)$ and similarly for $\widehat{w}_{n,j}(\zeta),\lambda_{n,j}(\zeta),\widehat{\lambda}_{n,j}(\zeta)$. On the other hand if no variable appears in $w_{n,j},\widehat{w}_{n,j},\lambda_{n,j},\widehat{\lambda}_{n,j}$ we will assume arbitrary $\underline{\zeta}\in\mathbb{R}^n$. Observe $w_n(\zeta)=w_{n,1}(\zeta)$ and $\widehat{w}_n(\zeta)=\widehat{w}_{n,1}(\zeta)$ are the supremum of ν such that

$$H(P)\leq X, \qquad 0<|P(\zeta)|\leq X^{-\nu} \tag{21}$$

has a solution $P\in\mathbb{Z}[X]$ of degree at most n for arbitrarily large X and all large values X respectively, where $H(P)$ is the height of P, that is the maximum modulus among the coefficients of P.

1.3. **The regular graph and the Schmidt-Summerer Conjecture.** For fixed $n\geq 1$ and a parameter $\rho\in[1,\infty]$ in [24] Schmidt and Summerer define what the call the regular graph. This geometrically describes a special pattern of the combined graph of the successive minima functions $L_{n,j}(q)=L_{n,j}(\log Q)$ from Section 1.2. Roughly speaking the integers $(x_k)_{k\geq 1}$ that induce a longer falling period of all $L_{n,j}$ simultaneously have the property that the logarithmic quotients $\log x_{k+1}/\log x_k$ tend to some constant which coincides with $\lambda_n/\widehat{\lambda}_n$. We refer to [24, page 90] and [20, page 72] for idealized illustrations of the functions $L_{n,j}(q)$ for the regular graph connected to approximation of three and two numbers respectively, i.e. $n=3$ and $n=2$ in our notation. The parameter $\rho\in[1,\infty]$ in Schmidt-Summerer notation coincides with the value $\lambda_n/\widehat{\lambda}_n$. In particular all $\lambda_{n,j},\widehat{\lambda}_{n,j}$

are determined by one real parameter $\lambda \geq 1/n$ and by (13) also all $w_{n,j}$ and $\widehat{w}_{n,j}$. We will use a different parametrization. We consider the equivalent situation that the constant λ_n is prescribed in the interval $[1/n, \infty]$. Any such choice again uniquely determines a regular graph in dimension n and vice versa. Thus we have the assignment

$$(n, \lambda) \to (\lambda_n, \lambda_{n,2}, \ldots, \lambda_{n,n+1}, \lambda_{n,n+2}), \qquad \lambda \in [1/n, \infty], \tag{22}$$

where $\lambda_n = \lambda$ and we let $\lambda_{n,n+2} := \widehat{\lambda}_{n,n+1}$ here and everywhere it occurs in the sequel. It will follow from (27) in Section 2.1 that the right hand side depends continuously on λ. Moreover the regular graph satisfies

$$\lambda_{n,j} = \widehat{\lambda}_{n,j-1}, \qquad 2 \leq j \leq n+2, \tag{23}$$

such that any pair (n, λ) in (22) determines all classical approximation constants $\lambda_{n,j}, \widehat{\lambda}_{n,j}$, and by (13) and (14) also all $w_{n,j}, \widehat{w}_{n,j}$. We will also write $\lambda_{n,j}(\lambda)$ and $\widehat{\lambda}_{n,j}(\lambda)$ for the quantity $\lambda_{n,j}$ in the regular graph in dimension n and parameter λ, and call the graph with the above assignment *the regular graph in dimension n with parameter* λ. For $n = 2$ the graphs of the functions $\lambda_{2,j}$ are illustrated below.

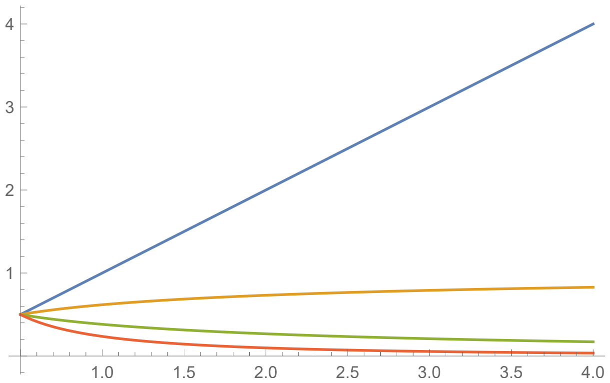

FIGURE 1. The functions $\lambda_{2,1}(\lambda), \lambda_{2,2}(\lambda), \lambda_{2,3}(\lambda), \lambda_{2,4}(\lambda)$ in the interval $\lambda \in [1/2, 4]$

For $n = 2$ Jarník's equality $\widehat{\lambda}_2 = (\widehat{w}_2 - 1)/\widehat{w}_2$ from [10] can be seen with (14) to be equivalent to $\widehat{\lambda}_2 + \lambda_{2,3} = 1$. Consequently $\lambda_{2,2}(\lambda) + \lambda_{2,3}(\lambda) = 1$ for the regular graph in dimension two and all parameters $\lambda \in [1/2, \infty]$, which can be observed in Figure 1.

It is worth noting that for $\lambda = 1/n$ all constants in (22) take the value $1/n$, which is an elementary fact due to Minkoski's second theorem and does not require the regular graph. Moreover, in the other degenerate case of the regular graph $\lambda = \infty$, it is not hard to see that we have $\lambda_{n,2}(\infty) = 1$ and $\lambda_{n,j}(\infty) = 0$ for $3 \leq j \leq n+2$, see also Proposition 2.5 below.

It was proved by Roy [17] that for any pair (n, λ) as in (22) there exist $\mathbb{Q}$-linearly independent vectors $\underline{\zeta}$ that induce the corresponding regular graph. For $n = 2$ and the special choice $\lambda_{2,1}(\underline{\zeta}) = \lambda = 1$, vectors (ζ, ζ^2) with ζ a so-called extremal number, see Roy [16], provide explicit examples of the regular graph. The existence of the regular graph for the special "degenerate" case $\lambda_n = \infty$ had already been constructively proved before by the author [20, Theorem 4].

The importance of the regular graph comes in particular from the conjecture by Schmidt and Summerer [24] that the value $\widehat{\lambda}_n$ is maximized among all $\underline{\zeta}$ that are $\mathbb{Q}$ linearly independent with 1 and have a prescribed value of λ_n in case of the regular graph. The conjecture also states that $\widehat{w}_n$ is maximized for given value of w_n. Recall λ_n or w_n already determine all approximation constants in the regular graph. For convenience we introduce some notation.

Definition 1. Let ϕ_n be the function that expresses $\widehat{w}_n$ in terms of $w_n \in [n,\infty]$ in the regular graph, which is the unique solution to (33) in the interval $[n, w_n)$ unless $w_n = n$. Further denote by ϑ_n the value $\widehat{\lambda}_n$ in terms of λ_n in the regular graph.

Note that $\vartheta_n(\lambda)$ coincides with $\widehat{\lambda}_n(\lambda) = \lambda_{n,2}(\lambda)$ defined above. Then the Schmidt Summerer Conjecture can be stated in the following way.

Conjecture 1.1 (Schmidt, Summerer)**.** For any positive integer n and every $\underline{\zeta} \in \mathbb{R}^n$ which is $\mathbb{Q}$-linearly independent together with $\{1\}$, we have $\widehat{w}_n(\underline{\zeta}) \leq \phi_n(w_n(\underline{\zeta}))$ and $\widehat{\lambda}_n(\underline{\zeta}) \leq \vartheta_n(\lambda_n(\underline{\zeta}))$. In particular for any real transcendental ζ and all $n \geq 1$ we have $\widehat{w}_n(\zeta) \leq \phi_n(w_n(\zeta))$ and $\widehat{\lambda}_n(\zeta) \leq \vartheta_n(\lambda_n(\zeta))$.

For $n \in \{2,3\}$ Schmidt and Summerer settled the conjecture in [23] and [24], see also [9]. For $n \geq 4$ it is open. The conjectured bounds are best possible since as mentioned above Roy [17] proved that equality holds for suitable $\underline{\zeta}$.

2. Structural study of the regular graph

2.1. **Preliminary results.** In this section we start to investigate how the coordinate functions in (22) behave in dependence of the dimension n and the parameter λ. We will apply the results we gather below in Sections 2.2, 2.3 and 3. The results are based predominately on the fact that for the regular graph the identity

$$\frac{(\lambda_n+1)^{n+1}}{\lambda_n} = \frac{(\widehat{\lambda}_{n,n+1}+1)^{n+1}}{\widehat{\lambda}_{n,n+1}} \tag{24}$$

holds. This is just [20, (95) in Section 3] in our notation. In view of (24) define the auxiliary functions

$$f_n(x) := \frac{(1+x)^{n+1}}{x}. \tag{25}$$

It is easily verified that f_n decays on $(0,1/n)$ and increases on $(1/n,\infty)$. By these facts, more precisely for given $\lambda_n \in [1/n,\infty]$ the constant $\widehat{\lambda}_{n,n+1}$ is the unique solution of (24) in the interval $[0,1/n]$.

Moreover as also mentioned already in [20, Section 3] for the regular graph all quotients $\lambda_{n,j}/\lambda_{n,j+1} = \lambda_{n,j}/\widehat{\lambda}_{n,j}$ coincide for $1 \leq j \leq n+1$. That is

$$\frac{\lambda_n}{\lambda_{n,2}} = \frac{\lambda_{n,2}}{\lambda_{n,3}} = \cdots = \frac{\lambda_{n,n+1}}{\lambda_{n,n+2}} = \frac{\widehat{\lambda}_n}{\widehat{\lambda}_{n,2}} = \cdots = \frac{\widehat{\lambda}_{n,n}}{\widehat{\lambda}_{n,n+1}}. \tag{26}$$

Observe by (24) and the constant quotients (26), the constants $\lambda_n = \lambda$ and $\lambda_{n,j}(\lambda)$ satisfy the implicit equation

$$\frac{(1+\lambda)^{n+1}}{\lambda} = \frac{\left(1+\lambda^{1-\frac{n+1}{j-1}}\lambda_{n,j}(\lambda)^{\frac{n+1}{j-1}}\right)^{n+1}}{\lambda^{1-\frac{n+1}{j-1}}\lambda_{n,j}(\lambda)^{\frac{n+1}{j-1}}}. \tag{27}$$

Moreover from (24) and (26) we infer

$$\widehat{\lambda}_n = \lambda_n^{\frac{n}{n+1}}\widehat{\lambda}_{n,n+1}^{\frac{1}{n+1}} = \lambda_n\left(\frac{\widehat{\lambda}_{n,n+1}}{\lambda_n}\right)^{\frac{1}{n+1}}. \tag{28}$$

We point out that from (28) and $\widehat{\lambda}_{n,n+1} \leq 1/n$ which is (9) we get

$$\widehat{\lambda}_n \leq n^{-\frac{1}{n+1}}\lambda_n^{\frac{n}{n+1}}.$$

With the stronger Khintchine inequality $\widehat{\lambda}_{n,n+1} = w_n^{-1} \leq (n\lambda_n + n - 1)^{-1}$ we obtain the stronger result

$$\widehat{\lambda}_n \leq (n\lambda_n + n - 1)^{-\frac{1}{n+1}}\lambda_n^{\frac{n}{n+1}}.$$

We derive an implicit equation involving λ_n and $\widehat{\lambda}_n$ by combining (24) with (28) of the form

$$\widehat{\lambda}_n^{n+1}(\lambda_n+1)^{n+1} - \lambda_n^{n+1}\left(1+\frac{\widehat{\lambda}_n^{n+1}}{\lambda_n^n}\right)^{n+1} = 0. \tag{29}$$

Now we want to establish the dual results. One can either proceed similarly as in [20] for (24), or immediately apply (13) and (14) to (24), to derive

$$\frac{(1+w_n)^{n+1}}{w_n^n} = \frac{(1+\widehat{w}_{n,n+1})^{n+1}}{\widehat{w}_{n,n+1}^n} \tag{30}$$

for the regular graph. Observe that $\widehat{w}_{n,n+1} = 1/\lambda_n \in [0,n]$ by (13) and (3), whereas $w_n \in [n,\infty]$ by (11). In particular it is not hard to see that for given $w_n \in [n,\infty]$ the approximation constant $\widehat{w}_{n,n+1}$ is the unique real solution of (30) in the interval $[0,n]$.

Moreover again for the regular graph all quotients $w_{n,j}/w_{n,j+1} = w_{n,j}/\widehat{w}_{n,j}$ coincide for $1 \leq j \leq n+1$, where we put $w_{n,n+2} := \widehat{w}_{n,n+1}$. This yields

$$\widehat{w}_n = w_n^{\frac{n}{n+1}}\widehat{w}_{n,n+1}^{\frac{1}{n+1}} = w_n\left(\frac{\widehat{w}_{n,n+1}}{w_n}\right)^{\frac{1}{n+1}}. \tag{31}$$

From (31) and (20) we obtain

$$\widehat{w}_n \leq n^{\frac{1}{n+1}}w_n^{\frac{n}{n+1}}, \tag{32}$$

where equality holds only in case of $\widehat{w}_{n,n+1} = n$ which is equivalent to $w_n(\zeta) = n$. The stronger inequality

$$\widehat{w}_n \leq ((n-1)w_n^n + nw_n^{n-1})^{\frac{1}{n+1}}$$

is obtained by combining (31) with Khintchine's inequality

$$\widehat{w}_{n,n+1} = \frac{1}{\lambda_n} \leq \frac{(n-1)w_n + n}{w_n}.$$

Alternatively, expressing $\widehat{w}_n$ in terms of $w_n, \widehat{w}_n$ by rearranging (31) and inserting in (30) some rearrangements lead to the nice implicit equation

$$w_n - \widehat{w}_n + 1 = \left(\frac{w_n}{\widehat{w}_n}\right)^n. \tag{33}$$

Thus we have (basically) established the following.

Proposition 2.1. *The function ϕ_n coincides with the unique solution of $\widehat{w}_n$ in* (33) *in terms of w_n in the interval $[n, w_n)$, unless $w_n = \phi_n(w_n) = \widehat{w}_n = n$. The function ϑ_n coincides with the unique solution of $\widehat{\lambda}_n$ in* (29) *in terms of λ_n in the interval $[1/n, \lambda_n)$, unless $\lambda_n = \vartheta_n(\lambda_n) = \widehat{\lambda}_n = 1/n$.*

Proof. The asserted uniqueness can be easily proved. It has been established that (33) and (29) are satisfied and the claim on the intervals follows from (3) and (11). ☐

We remark that similarly one can obtain an implicit equation involving $w_n(\zeta)$ and $w_{n,j} = \widehat{w}_{n,j-1}$ for $2 \le j \le n+2$ (where $w_{n,n+2} := \widehat{w}_{n,n+1}$) as

$$w_n^{\frac{n+j-2}{j-1}}(1+w_n)^{n+1} = \frac{\left(w_n^{\frac{n+j-2}{j-1}} + w_{n,j}^{\frac{n+1}{j-1}}\right)^{n+1}}{w_{n,j}^{\frac{n(n+1)}{j}}}.$$

For $j = 2$ this results in (33), the choice $j = n+2$ again yields (30).

In the proof of Theorem 3.3 we will use that the functions ϕ_n are increasing. This seems rather obvious from the definition of the regular graph and Conjecture 1.1, but it is not trivial and we want to carry this out in Section 5.2.

Lemma 2.2. *For every $n \ge 1$, the function ϕ_n that expresses $\widehat{w}_n$ in terms of $w_n \in [n, \infty]$ for the regular graph defined as above is monotonic increasing on $[n, \infty]$.*

Theorem 2.6 and Remark 2 below more generally suggest that precisely for $1 \le j \le \lceil n/2 \rceil + 1$ the functions $w_{n,j}(w)$ defined by the regular graph with parameter $w_n = w$ (similar to $\lambda_{n,j}(\lambda)$) should increase on $w \in [n, \infty]$. Indeed Theorem 2.6 shows the weaker claim $w_{n,j}(w) > w_{n,j}(n) = n$ for these values of j and $w > n$, and disproves it for larger j. Observe $w_{n,2}(w)$ coincides with $\phi_n(w)$. Moreover, the analogue result should be true for the functions $\vartheta_n = \lambda_{n,2}$, but certainly for no function $\lambda \to \lambda_{n,j}(\lambda)$ if $j \ge 3$ by Proposition 2.5.

2.2. **Fixed λ.** In this section let $\lambda > 0$ be given. We investigate constants $\lambda_{n,j}$ in the regular graph for prescribed value $\lambda_n = \lambda$ in dependence of n, for which obviously it is necessary and sufficient to assume $n \ge \lceil \lambda^{-1} \rceil$. Recall the notation $\lambda_{n,j}(\lambda)$ and $\widehat{\lambda}_{n,j}(\lambda)$ for the constants $\lambda_{n,j}, \widehat{\lambda}_{n,j}$ obtained in the regular graph in dimension n and the parameter $\lambda_{n,1} = \lambda_n = \lambda$. Our first result shows roughly speaking that for fixed $\lambda_n = \lambda$ the remaining constants $\lambda_{n,j}(\lambda)$ for fixed $j \ge 2$ are decreasing as the dimension n increases.

Proposition 2.3. *Let $\lambda > 0$ be fixed and $n_1 > n_2 \ge j-1 \ge 1$ be integers such that $n_2 \ge \lceil \lambda^{-1} \rceil$. Then the constants $\lambda_{n_i,j}(\lambda), i \in \{1,2\}$ in the regular graphs in dimensions n_1 and n_2 respectively and parameter λ satisfy $\lambda_{n_1,j}(\lambda) < \lambda_{n_2,j}(\lambda)$.*

Remark 1. The proposition can be used to show that for every fixed n for the regular graph in dimension $n \geq 2$ we have

$$\lim_{\lambda\to\infty} \lambda + 1 - \frac{\lambda}{\widehat{\lambda}_n(\lambda)} = 0, \tag{34}$$

with $\lambda_n = \lambda$ and $\widehat{\lambda}_n(\lambda)$ as in Section 1.3. This was remarked but not proved in [20]. Indeed Proposition 2.3 and $\lambda_n + 1 - \lambda_n/\widehat{\lambda}_n \geq 0$, which is [20, Proposition 5], reduces the problem to the case $n = 2$. In this case (24) can be explicitly solved and (34) can be verified with some elementary computations. Observe that (34) in particular yields

$$\lim_{\lambda\to\infty} \widehat{\lambda}_n(\lambda) = 1.$$

This property can be roughly seen in Figure 1.

The following corollary from Proposition 2.3 will be proved in Section 5.1 as well.

Corollary 2.4. *Let $j \geq 2$ an integer and $\lambda > 0$ a parameter fixed. Consider the regular graphs in all dimensions $n \geq \lceil \lambda^{-1} \rceil$ with $\lambda_n = \lambda$ as in (22), which are well-defined. Then we have the asymptotic behavior*

$$\lim_{n\to\infty} \widehat{\lambda}_{n,j-1}(\lambda) = \lim_{n\to\infty} \lambda_{n,j}(\lambda) = \frac{\lambda}{(1+\lambda)^{j-1}}.$$

2.3. **Fixed n and Schmidt's conjecture.** Now we investigate the regular graph in fixed dimension n in dependence of the parameter $\lambda \geq 1/n$. We are particularly interested in parameters λ in some small interval $(1/n, 1/n + \epsilon)$. We ask what is the largest index j such that $\lambda_{n,j}$ is larger than $1/n$ in such intervals. Due to (10) this is closely connected to a conjecture of W. Schmidt [21] where he conjectured that for any integers $1 \leq T \leq n-1$ there exist vectors $\underline{\zeta}$ that are $\mathbb{Q}$ linearly independent together with $\{1\}$, and for which the T-th successive minimum $\tau_{n,T}$ of the lattice point problem from Section 1.2 tends to 0 whereas the $(T+2)$-nd $\tau_{n,T+2}$ tends to infinity. As mentioned in Section 1.2 this is equivalent to the fact that the function $L_{n,T}(q)$ tends to $-\infty$ whereas $L_{n,T+2}(q)$ tends to $+\infty$ as $q \to \infty$, and for this $\underline{\psi}_{n,T} < 0$ and $\overline{\psi}_{n,T+2} > 0$ is sufficient. Thus for convenience we introduce the following notation.

Definition 2. Let n, T be integers with $1 \leq T \leq n-1$. We say $\underline{\zeta} \in \mathbb{R}^n$ satisfies Schmidt's property for (n, T) if $\underline{\zeta}$ is $\mathbb{Q}$-linearly independent together with $\{1\}$ and the induced functions $L_{n,j}(q)$ from Section 1.2 satisfy $\lim_{q\to\infty} L_{n,T}(q) = -\infty$ and $\lim_{q\to\infty} L_{n,T+2}(q) = \infty$.

By (1) and (10) a sufficient condition for $\underline{\zeta}$ to satisfy Schmidt's property for (n, T) is given by $\lambda_{n,T+2} < 1/n < \lambda_{n,T}$. The existence conjecture of Schmidt was proved by Moshchevitin [14] in a complicated non-constructive way. Moreover in case of T not too close to n, where the condition $T < n/\log n$ is sufficient, it was reproved constructively in [20]. We should remark that the modified Schmidt's property for the pair $T, T+1$ instead of $T, T+2$ cannot be satisfied if $\underline{\zeta}$ is $\mathbb{Q}$ linearly independent together with $\{1\}$, since then $L_{n,j}(q) = L_{n,j+1}(q)$ has arbitrarily large solutions q for all $1 \leq j \leq n$, see [22]. On the other hand if one drops the linear independence condition the conjecture would be true for $T, T+1$ as well by a rather easy argument, as carried out in [14]. We want to discuss how the regular graph relates to this problem. Our main result in this context will be Theorem 2.8. First we state an easy but important observation.

Proposition 2.5. *Let $n \geq 2$ and $3 \leq j \leq n+2$. Then the quantities $\lambda_{n,j}(\lambda) = \widehat{\lambda}_{n,j-1}(\lambda)$ for the regular graph in dimension n with parameter λ tend to 0 as λ tends to infinity.*

Proof. Observe $\widehat{\lambda}_n(\lambda) = \lambda_{n,2}(\lambda) \leq 1$ always holds by (7). Together with the constant quotients property (26) we have $\lambda_{n,j}(\lambda) = \widehat{\lambda}_n(\widehat{\lambda}_n/\lambda)^{j-2} \leq \lambda^{2-j}$, which clearly tends to 0 for $j \geq 3$ as $\lambda \to \infty$. □

The next theorem provides more detailed information on the functions $\lambda_{n,j}(\lambda)$ in (22).

Theorem 2.6. *Let $j \geq 3$ and $n \geq j-2$ be integers. If $n \geq 2j-2$, then there exist $\tilde{\lambda} \in (1/n, n)$ with the following properties. The regular graph in dimension n with parameter λ satisfies $\lambda_{n,j}(\lambda) > 1/n$ for $\lambda \in (1/n, \tilde{\lambda})$, $\lambda_{n,j}(\lambda) = 1/n$ for $\lambda \in \{1/n, \tilde{\lambda}\}$ and $\lambda_{n,j}(\lambda) < 1/n$ for $\lambda \in (\tilde{\lambda}, \infty]$. If on the other hand $n \leq 2j-3$, then for all $\lambda \in (1/n, \infty]$ the regular graph in dimension n with parameter λ satisfies $\lambda_{n,j}(\lambda) < 1/n$.*

It is easy to check the following consequence of Theorem 2.6.

Corollary 2.7. *Precisely in case of $n \leq 3$ none of the functions $\lambda_{n,j}(\lambda) - 1/n$ changes sign on $\lambda \in (1/n, \infty)$.*

Remark 2. For $j \in \{1, 2\}$ and $n \geq 2$ clearly we have $\lambda_{n,j}(\lambda) > 1/n$ for all $\lambda \in (1/n, \infty]$ by (4) and (7), with equality in both inequalities only for $\lambda = 1/n$. See also Lemma 2.2 and Proposition 2.3. A similar dual argument shows $\lambda_{n,j}(\lambda) < 1/n$ for $j \in \{n+1, n+2\}$, as we will carry out in the proof. In particular for $n = 2$ it is clear that $\lambda_{2,1}(\lambda) > \lambda_{2,2}(\lambda) > 1/2 > \lambda_{n,3}(\lambda) > \lambda_{n,4}(\lambda)$ for all $\lambda > 1/2$, and it can be shown easily that all functions $\lambda_{2,i}(\lambda)$ are monotonic on $[1/n, \infty]$, see also Figure 1. For $n = 3$ on the other hand the above argument is already too weak to yield $\lambda_{3,3}(\lambda) < 1/3$ for all $\lambda > 1/3$, as Theorem 2.6 does. A Mathematica plot howover shows that the graph of $\lambda_{3,3}(\lambda)$ decreases and has an inflection point somewhere in the interval $(1/2, 1)$.

Moreover it should be true that the derivative of $\lambda_{n,j}(\lambda)$ with respect to the parameter λ changes sign at most once, and precisely for $3 \leq j < \frac{n+3}{2}$ somewhere in the interval $(1/n, \tilde{\lambda})$. However, a proof seems to demand some cumbersome estimates and we do not study this further. For given n, j with the formula (27) the constants $\tilde{\lambda}$ can be (for large n, j only numerically) computed with Mathematica. For example for $n = 4, j = 3$ we get $\tilde{\lambda} = 3/4 + \sqrt{2}/2 \approx 1.4571$, for $n = 8, j = 3$ we have $\tilde{\lambda} = 3 + 2\sqrt{2} \approx 5.8284$ for $n = 8, j = 5$ we have $\tilde{\lambda} \approx 0.2719$. From Theorem 2.6 it is not hard to deduce explicit examples for Schmidt's property if j is not larger than roughly $n/2$.

Theorem 2.8. *Let $n \geq 2$ be an integer. Then for any $1 \leq T \leq \lfloor n/2 \rfloor$ there exists a non-empty subinterval $I = I(T)$ of $(1/n, n)$ such that for all $\lambda \in I$ the regular graph in dimension n with parameter λ satisfies*

$$\widehat{\lambda}_{n,T}(\lambda) > \frac{1}{n}, \qquad \lambda_{n,T+2}(\lambda) < \frac{1}{n}.$$

In other words for any pair (n,T) with $1\leq T\leq \lfloor n/2\rfloor$ there exist $\underline{\zeta}$ that induce the regular graph and satisfy Schmidt's property for (n,T).

Proof. Let $j\leq \lfloor n/2\rfloor+1$. Then the first case of Theorem 2.6 applies and yields $\lambda_{n,j}(\tilde{\lambda})=1/n$ and $\lambda_{n,j}(t)>1/n$ for some $\tilde{\lambda}>1/n$ and $t\in(1/n,\tilde{\lambda})$. Since $\lambda_{n,j+1}<\lambda_{n,j}$ unless both are equal to $\lambda=1/n$, we have $\lambda_{n,j+1}(\tilde{\lambda})<1/n$. Hence by continuity of the function $\lambda_{n,j+1}(\lambda)$ in the parameter λ there exists some non-empty interval $J=J(j)=(\delta-\epsilon,\delta)$ such that for $t_0\in J$ the inequalities $\lambda_{n,j+1}(t_0)<1/n<\lambda_{n,j}(t_0)$ are satisfied. Since in the regular graph $\widehat{\lambda}_{n,j-1}=\lambda_{n,j}$ holds by (23), the claim follows with $T=j-1$ and the fact that $\underline{\zeta}$ inducing the corresponding regular graphs exist as mentioned above. □

3. Implications of Conjecture 1.1 to uniform approximation

We restrict to the case of successive powers $(\zeta,\zeta^2,\ldots,\zeta^n)$ in the sequel. In this section we conditionally improve the following Theorem 3.1, established in [6, Theorem 2.1] where the last claim also incorporates [6, Theorem 2.3], under assumption of Conjecture 1.1. Moreover we will treat a weaker assumption in Theorem 3.4.

Theorem 3.1 (Bugeaud, Schleischitz)**.** *Let $n\geq 2$ be an integer and ζ a real transcendental number. Then*

$$\widehat{w}_n(\zeta)\leq n-\frac{1}{2}+\sqrt{n^2-2n+\frac{5}{4}}. \tag{35}$$

For $n=3$ we have the stronger estimate

$$\widehat{w}_3(\zeta)\leq 3+\sqrt{2}\approx 4.4142... \tag{36}$$

In case of $w_{n-1}(\zeta)=w_n(\zeta)$ we have $\widehat{w}_n(\zeta)\leq 2n-2$ and in case of $w_{n-2}(\zeta)=w_{n-1}(\zeta)=w_n(\zeta)$ we have $\widehat{w}_n(\zeta)\leq 2n-3$.

For $n\to\infty$ the right hand side is of order $2n-3/2+o(1)$ with positive remainder term for any fixed n. This improved the earlier result $\widehat{w}_n(\zeta)\leq 2n-1$ by Davenport and Schmidt [7]. For $n=2$ Theorem 3.1 is best possible as proved by Roy, see [18]. We will also need special cases of [6, Theorem 2.2, 2.3 and 2.4] comprised in Theorem 3.2 below. Before we can state Theorem 3.2 we need the definition of the exponents of approximation by algebraic numbers which is closely related to $w_n(\zeta),\widehat{w}_n(\zeta)$.

Definition 3. The constant $w_n^*(\zeta)$ is defined as the supremum of ν such that

$$0<|\zeta-\alpha|\leq H(\alpha)^{-\nu-1} \tag{37}$$

has a real algebraic solution α of degree at most n, where $H(\alpha)$ is the maximum modulus of the coefficients of the irreducible minimal polynomial P of α over $\mathbb{Z}[X]$. The uniform constant $\widehat{w}_n^*(\zeta)$ is defined as the supremum of ν such that the system

$$H(\alpha)\leq X,\qquad 0<|\zeta-\alpha|\leq H(\alpha)^{-1}X^{-\nu} \tag{38}$$

has a solution for all large values of X.

For all $n\geq 1$ and all real ζ, the estimates

$$w_n^*(\zeta)\leq w_n(\zeta)\leq w_n^*(\zeta)+n-1,\qquad \widehat{w}_n^*(\zeta)\leq \widehat{w}_n(\zeta)\leq \widehat{w}_n^*(\zeta)+n-1, \tag{39}$$

are well-known, see [2, Lemma A8].

Theorem 3.2 (Bugeaud, Schleischitz). *Let $n \geq 2$ and ζ be real transcendental. We have*

$$\widehat{w}_n^{*}(\zeta) \leq \frac{n w_n(\zeta)}{w_n(\zeta) - n + 1}.$$

If $w_n(\zeta) > w_{n-1}(\zeta)$ then we have the stronger estimate

$$\widehat{w}_n(\zeta) \leq \frac{n w_n(\zeta)}{w_n(\zeta) - n + 1}.$$

If otherwise $m < n$ is such that $w_m(\zeta) = w_n(\zeta)$ then

$$\widehat{w}_n(\zeta) \leq m + n - 1 \leq 2n - 2.$$

In fact Theorem 3.2 was used to prove Theorem 3.1 in combination with bounds from Summerer and Schmidt [23, page 48, last formula], and [24, (1.2)] for $n = 3$. Under the stronger bounds implied by assumption of Conjecture 1.1, for $n \geq 4$ Theorem 3.1 can be slightly improved. The main result of this section is the following.

Theorem 3.3. *Suppose Conjecture 1.1 holds for every $n \geq 2$. Let $\tau \approx 0.5693$ be the solution $y \in (0, 1)$ of $ye^{1/y} = 2\sqrt{e}$, where e is Euler's number. Then for any $D < \log(2/\tau) + 1 \approx 2.2564$ there exists $n_0 = n_0(D)$ such that for all real transcendental numbers ζ we have*

$$\widehat{w}_n^{*}(\zeta) \leq 2n - D, \qquad n \geq n_0. \tag{40}$$

The same bound holds for $\widehat{w}_n(\zeta)$ unless $w_{n-2}(\zeta) < w_{n-1}(\zeta) = w_n(\zeta)$. In any case we have

$$\widehat{w}_n(\zeta) \leq 2n - 2, \qquad n \geq 10. \tag{41}$$

In fact the bound $\log(2/\tau) + 1$ in Theorem 3.3 seems to be very close to best possible from what we can get by combining the results of Theorem 3.2 with estimates for the regular graph from Section 2.1. We want to present some numeric results indicating this obtained with Mathematica. It follows from Theorem 3.1 and Theorem 3.2 that for the solution $\widetilde{w}_n(\zeta)$ of

$$\phi_n(\widetilde{w}_n(\zeta)) = \frac{\widetilde{w}_n(\zeta)}{\widetilde{w}_n(\zeta) - n + 1} \tag{42}$$

the corresponding value $\phi_n(\widetilde{w}_n(\zeta))$ is an upper bound for $\widehat{w}_n^{*}(\zeta)$ under the assumption of Conjecture 1.1, and if $\phi_n(\widetilde{w}_n(\zeta)) \geq 2n - 2$ also for $\widehat{w}_n(\zeta)$. For $n \in \{2, 3\}$ this procedure leads precisely to the bounds $(3 + \sqrt{5})/2$ and $3 + \sqrt{2}$ in (35) and (36), respectively. For $n \geq 4$ Mathematica can numerically solve the problem if n is not too large. Examples are given by

$$\begin{aligned} \widetilde{w}_4(\zeta) &\approx 8.2460, & \phi_4(\widetilde{w}_4(\zeta)) &\approx 6.2874 \\ \widetilde{w}_5(\zeta) &\approx 10.2481, & \phi_5(\widetilde{w}_5(\zeta)) &\approx 8.2096 \\ \widetilde{w}_6(\zeta) &\approx 12.2495, & \phi_6(\widetilde{w}_6(\zeta)) &\approx 10.1382 \\ \widetilde{w}_7(\zeta) &\approx 14.2505, & \phi_7(\widetilde{w}_7(\zeta)) &\approx 12.0906 \\ \widetilde{w}_{10}(\zeta) &\approx 20.2523, & \phi_{10}(\widetilde{w}_{10}(\zeta)) &\approx 17.9984 \\ \widetilde{w}_{20}(\zeta) &\approx 40.2544, & \phi_{20}(\widetilde{w}_{20}(\zeta)) &\approx 37.8786 \\ \widetilde{w}_{30}(\zeta) &\approx 60.2551, & \phi_{30}(\widetilde{w}_{30}(\zeta)) &\approx 57.8355 \\ \widetilde{w}_{50}(\zeta) &\approx 100.2556, & \phi_{50}(\widetilde{w}_{50}(\zeta)) &\approx 97.7996. \end{aligned}$$

Hence for example under Conjecture 1.1 we get for all real transcendental numbers ζ

$$\widehat{w}_4(\zeta) < 6.2875, \quad \widehat{w}_5(\zeta) < 8.2097, \quad \widehat{w}_6(\zeta) < 10.1383, \quad \widehat{w}_7(\zeta) < 12.0907$$
$$\widehat{w}_{10}^*(\zeta) < 17.9985 \quad \widehat{w}_{20}^*(\zeta) < 37.8787 \quad \widehat{w}_{30}^*(\zeta) < 57.8356 \quad \widehat{w}_{50}^*(\zeta) < 97.7996.$$

Again unless ζ satisfies $w_{n-2}(\zeta) < w_{n-1}(\zeta) = w_n(\zeta)$ the above bounds for $n \in \{10, 20, 30, 50\}$ are valid for $\widehat{w}_n(\zeta)$ as well and we believe the additional condition is not necessary. The data suggests that $2n - \phi_n(\widetilde{w}_n(\zeta))$ converges to some constant not much larger than $\log(2/\tau) + 1 \approx 2.2564$ from Theorem 3.3, and even if this is not the case it seems unlikely one can improve the asymptotic bound of the form $2n - C$.

We want to close this section with a weaker result under a weaker assumption than Conjecture 1.1, which still yields better bounds than the unconditioned Theorem 3.1. The involved assumption will play a role later in Theorem 4.3.

Theorem 3.4. *Assume* (32) *is satisfied for all* $n \geq 1$ *and real transcendental* ζ*, which is in particular true if Conjecture 1.1 holds. Then for every* $\epsilon > 0$ *there exists* $n_0 = n_0(\epsilon)$ *such that*

$$\widehat{w}_n(\zeta) \leq 2n - 1 - \log 2 + \epsilon, \qquad n \geq n_0$$

is satisfied.

For $n = 100$ numerical calculations approximately yield the bound 198.3245, which is slightly larger than $2n - 1 - \log 2 \approx 198.3068$.

4. Conditioned results under another conjecture

4.1. **Uniform approximation.** Let $n \geq 1$ an integer and ζ be a real number. We call $P \in \mathbb{Z}[X]$ of degree at most n a best approximation for (n, ζ) if there is no $Q \in \mathbb{Z}[X]$ of degree at most n with $H(Q) < H(P)$ and $|Q(\zeta)| < |P(\zeta)|$. Every real transcendental ζ induces a sequence of best approximation polynomials $P_1, P_2, \ldots$ that satisfy $|P_1(\zeta)| > |P_2(\zeta)| > \cdots$ and $H(P_1) \leq H(P_2) \leq \cdots$.

Conjecture 4.1. For any $n \geq 1$ and any real transcendental ζ, there exist infinitely many k such that $n + 1$ successive best approximations $P_k, P_{k+1}, \ldots, P_{k+n}$ for (n, ζ) are linearly independent (i.e. their span equals the entire space $\mathbb{R}^{n+1}$).

Remark 3. The claim is known to hold for $n = 2$. Moshchevitin [13] proved that for $n > 2$ there exist counterexamples for the analogue claim for vectors $\underline{\zeta} \in \mathbb{R}^n$ that are together with $\{1\}$ $\mathbb{Q}$-linearly independent. The vector can even be chosen such that the $(n+1) \times (n+1)$-matrix with columns $n+1$ successive best approximation vectors has rank at most 3 for all large k. However, it seems plausible that such vectors cannot lie on the Veronese curve.

Theorem 4.2. *If (n, ζ) satisfies the assumption of Conjecture 4.1 then*

$$w_{n,n+1}(\zeta) \geq \frac{\widehat{w}_n(\zeta)^n}{w_n(\zeta)^{n-1}} \tag{43}$$

and moreover

$$w_n(\zeta) \geq \widehat{w}_n(\zeta) \left(\frac{\widehat{w}_n(\zeta) - 1}{n-1} \right)^{\frac{1}{n-1}}. \tag{44}$$

For $n = 2$ estimate (44) yields the inequality $w_2(\zeta) \geq \widehat{w}_2(\zeta)(\widehat{w}_2(\zeta) - 1)$ known by Laurent [11]. Again this is sharp for extremal numbers. See [15, Section 3] for related results. For us the main purpose of Theorem 4.2 is the connection to uniform approximation in the following theorem.

Theorem 4.3. *Assume Conjecture 4.1 is true for all (n, ζ) with $n \geq 1$ and ζ a transcendental number. Then the claim of Theorem 3.4 holds.*

Proof. By assumption and Theorem 4.2 inequality (44) holds, which is stronger than (32). Thus the claim follows from Theorem 3.4. □

There is numeric evidence that the bound cannot be improved much by using (44) instead of (32). For example Mathematica calculates the bounds

$$\widehat{w}_4(\zeta) \leq 6.4575, \quad \widehat{w}_{10}(\zeta) \leq 18.366, \quad \widehat{w}_{100}(\zeta) \leq 198.313.$$

For $n = 100$ the value is still larger than $2n - 1 - \log 2$ and only slightly better than the bound 198.3245 derived directly from (32), or more precisely from (68) below.

4.2. **Wirsing's problem.** Finally we want to briefly point out the implications of Conjecture 4.1 to Wirsing's problem, which is to decide whether $w_n^*(\zeta) \geq n$ holds for all $n \geq 1$ and real transcendental ζ. The conjecture has so far only been settled for $n \in \{1, 2\}$.

Theorem 4.4. *Assume Conjecture 4.1 is true. Then there is an increasing sequence of real numbers $(\gamma_n)_{n \geq 1}$ with $\lim \gamma_n > 2.31$ such that $w_n^*(\zeta) > n/2 + \gamma_n$ for all real transcendental ζ.*

This is stronger than the bound $w_n^*(\zeta) \geq n/4 + \sqrt{n^2 + 16n - 8}/4$ by Bernik and Tishchenko [1], which has the asymptotic behavior $n/2 + 2 - o(1)$. However, it is weaker than the result in the very technical paper [25] where it was shown that the sequence γ_n in context of Theorem 4.4 can be chosen with limit 3. Nevertheless the method of Theorem 4.4 is of some interest and some new idea might lead to new insights.

5. Proofs

5.1. Proofs of Section 2. First we prove Lemma 2.2.

Proof of Lemma 2.2. It follows from (32) and (33) by implicit function theorem that ϕ_n is a C^1-function in a neighborhood of every w with $nw^n \neq \phi_n(w)^{n+1}$. Hence by (32) we infer ϕ_n is a C^1-function on (n,∞). Moreover (33) with $w = w_n(\zeta)$ readily implies

$$\lim_{w\to\infty} \phi_n(w) = \infty. \tag{45}$$

Assume ϕ_n is not monotonic increasing, that is for some $n \le u < v$ we have $\phi(u) > \phi(v)$. Then there exists a local maximum of ϕ_n in $[n,v)$. This maximum cannot be n since $\phi_n(n) = n$ which is the global minimum for $\widehat{w}_n(\zeta) = \phi_n(w_n(\zeta))$ on $[n,\infty)$ (recall the regular graph can be attained for any dimension and parameter for some $\underline{\zeta} \in \mathbb{R}^n$ such that indeed n is the lower bound by Dirichlet Theorem which is (18)). Without loss of generality assume this local maximum is attained at u. Then since $u \neq n$ we deduce $\phi_n'(u) = 0$. On the other hand from (45) and our assumption we obtain some local minimum z of ϕ_n strictly larger than u for which obviously $\phi_n(z) < \phi_n(u)$ and $\phi_n'(z) = 0$ hold. Summing up our hypothesis would imply the existence of $u < z$ in $[n,\infty)$ such $\phi_n(z) < \phi_n(u)$ and $\phi_n'(u) = \phi_n'(z) = 0$. We lead this to a contradiction. Implicit differentiation of (33) for $w = w_n(\zeta)$ with the replacement $\widehat{w}_n(\zeta) = \phi(w_n(\zeta)) = \phi(w)$ and rearranging shows that we have

$$1 - \phi_n'(w) = nw^{n-1}\phi(w)^n \frac{1 - w\phi_n(w)\phi_n'(w)}{\phi_n(w)^{2n}}$$

such that in case of $\phi_n'(w) = 0$ we derive

$$\phi_n(w) = n^{\frac{1}{n}} w^{\frac{n-1}{n}}$$

where the right hand side is obviously monotonic increasing in w. Since $u < z$ and $\phi(u) > \phi(z)$ this leads to a contradiction and the proof is finished. □

For the next proof recall the functions f_n from (25) and their properties.

Proof of Proposition 2.3. By the assumptions the regular graphs with parameter λ in dimension n_1, n_2 are well-defined (and exist due to Roy [17]). Since in the regular graph the quotients (26) coincide, it suffices to prove that $\lambda_{n,2}(\lambda) = \widehat{\lambda}_n(\lambda)$ decreases for fixed λ as n increases.

Recall the functions f_n defined in Section 2. We have $f_{n+1}(\lambda)/f_n(\lambda) = 1+\lambda$ and hence in view of (24) also

$$\frac{f_{n+1}(\widehat{\lambda}_{n+1,n+2}(\lambda))}{f_n(\widehat{\lambda}_{n,n+1}(\lambda))} = 1+\lambda. \tag{46}$$

On the other hand we claim that

$$\widehat{\lambda}_{n+1,n+2}(\lambda) < \widehat{\lambda}_{n,n+1}(\lambda). \tag{47}$$

In case of $\widehat{\lambda}_{n,n+1}(\lambda) > 1/(n+1)$ this is trivial since $\widehat{\lambda}_{n+1,n+2}(\lambda) \le 1/(n+1)$. If otherwise $\widehat{\lambda}_{n,n+1}(\lambda) \le 1/(n+1)$ then (47) follows from the decay of the function f_{n+1} on $(0, 1/(n+1))$ and $f_{n+1}(\widehat{\lambda}_{n,n+1}(\lambda))/f_n(\widehat{\lambda}_{n,n+1}(\lambda)) = 1 + \widehat{\lambda}_{n,n+1}(\lambda) \le 1+\lambda$ in combination with (46).

From (47) we deduce

$$(1+\widehat{\lambda}_{n+1,n+2}(\lambda))^{n+2} \leq (1+\widehat{\lambda}_{n,n+1}(\lambda))^{n+2} = (1+\widehat{\lambda}_{n,n+1}(\lambda))^{n+1}\frac{f_{n+1}(\widehat{\lambda}_{n,n+1}(\lambda))}{f_n(\widehat{\lambda}_{n,n+1}(\lambda))}$$

Observe the involved quantities are the nominators of $f_{n+1}(\widehat{\lambda}_{.,.}(\lambda))$. Together with (46) for the according denominators we infer

$$\frac{\widehat{\lambda}_{n,n+1}(\lambda)}{\widehat{\lambda}_{n+1,n+2}(\lambda)} > \frac{1+\lambda}{\widehat{\lambda}_{n,n+1}(\lambda)}. \tag{48}$$

The identities (28) for $n, n+1$ yield

$$\widehat{\lambda}_n(\lambda) = \lambda^{n/(n+1)}\widehat{\lambda}_{n,n+1}(\lambda)^{1/(n+1)}$$
$$\widehat{\lambda}_{n+1}(\lambda) = \lambda^{(n+1)/(n+2)}\widehat{\lambda}_{n+1,n+2}(\lambda)^{1/(n+2)}.$$

Taking quotients with $(n+1)/(n+2)-n/(n+1) = 1/(n+1)-1/(n+2) = (n+1)^{-1}(n+2)^{-1}$ we get

$$\frac{\widehat{\lambda}_n(\lambda)}{\widehat{\lambda}_{n+1}(\lambda)} \geq \lambda^{-\frac{1}{(n+1)(n+2)}}\widehat{\lambda}_{n,n+1}(\lambda)^{\frac{1}{(n+1)(n+2)}}\left(\frac{\widehat{\lambda}_{n,n+1}(\lambda)}{\widehat{\lambda}_{n+1,n+2}(\lambda)}\right)^{\frac{1}{n+2}}.$$

Inserting the bound (48) for the last expression we obtain

$$\frac{\widehat{\lambda}_n(\lambda)}{\widehat{\lambda}_{n+1}(\lambda)} \geq \lambda^{-\frac{1}{(n+1)(n+2)}}\widehat{\lambda}_{n,n+1}(\lambda)^{\frac{1}{(n+1)(n+2)}}\left(\frac{1+\lambda}{\widehat{\lambda}_{n,n+1}(\lambda)}\right)^{\frac{1}{n+2}}. \tag{49}$$

One readily checks that the right hand side in (49) equals 1, since this is equivalent to $f_k(\lambda) = f_k(\widehat{\lambda}_{n,n+1}(\lambda))$, which is (24). This finishes the proof. □

Proof of Corollary 2.4. As mentioned in Remark 1 we have $\widehat{\lambda}_n(\lambda)/\lambda \geq (\lambda+1)^{-1}$ in the regular graph with parameter $\lambda_n = \lambda$. On the other hand the quotients $\lambda_{n,j}/\lambda_{n,j+1}$ are identical for all $1 \leq j \leq n+1$ with $\lambda_{n,n+2} := \widehat{\lambda}_{n,n+1}$ by (26). Hence

$$\widehat{\lambda}_{n,j-1}(\lambda) = \lambda_{n,j}(\lambda) = \lambda\left(\frac{\widehat{\lambda}_n(\lambda)}{\lambda}\right)^{j-1} \geq \frac{\lambda}{(1+\lambda)^{j-1}}.$$

In Proposition we proved that the values $\widehat{\lambda}_{n,j-1}(\lambda) = \lambda_{n,j}(\lambda)$ decay as n increases, hence the limit of $\lambda_{n,j}(\lambda)$ as $n \to \infty$ exists and is at least the given quantity. We have to show equality. Again for all the quotients $\lambda_{n,j}/\lambda_{n,j+1}$ are identical it obviously suffices to show this for $j = 2$. For $\lambda, \widehat{\lambda}_n(\lambda)$ as above define $\alpha(n)$ implicitly by

$$\widehat{\lambda}_n(\lambda) = \alpha(n)\frac{\lambda}{1+\lambda}. \tag{50}$$

Then the sequence $\alpha(n) \geq 1$ decreases to some limit at least 1 and we have to show $\lim_{n\to\infty}\alpha(n) = 1$. Observe a rearrangement of (28) and (50) yields

$$\widehat{\lambda}_{n,n+1}(\lambda) = \lambda\left(\frac{\widehat{\lambda}_n(\lambda)}{\lambda}\right)^{n+1} = \lambda\left(\frac{\alpha(n)}{1+\lambda}\right)^{n+1}.$$

Inserting the right hand side in the identity (24), elementary rearrangements yield

$$\alpha(n) = 1 + \lambda \left(\frac{\alpha(n)}{1+\lambda} \right)^{n+1}. \tag{51}$$

If we had $\lim_{n\to\infty} \alpha(n) \geq \lambda+1$ then $\widehat{\lambda}_n(\lambda) \geq \lambda = \lambda_n(\lambda)$, contradiction. Thus $\lim_{n\to\infty} \alpha(n) < \lambda + 1$. Hence the right hand side of (51) converges to 1 as $n \to \infty$ and thus the left hand side as well. This completes the proof. $\square$

Proof of Theorem 2.6. Clearly $\lambda_{n,j} = 1/n$ for all $1 \leq j \leq n+2$ if $\lambda = 1/n$. Further observe that by (15), (18), (13) and (14) we have $\lambda_{n,n+1}(\lambda) = \widehat{w}_n(\lambda)^{-1} \leq 1/n$ and $\lambda_{n,n+2} = w_n(\lambda)^{-1} \leq 1/n$ with equality only if the quantities equal $1/n$ anyway, where we put $w_n(\lambda)$ for the value w_n induced for the regular graph with parameter $\lambda_{n,1} = \lambda$. Thus we can restrict to $2 \leq j \leq n$.

So let $n \geq 1$ and $2 \leq j \leq n$ arbitrary but fixed. Write $\lambda_n = \lambda = \alpha/n$ for $\alpha > 1$, where we consider only α slightly larger than 1. Then (27) becomes

$$\frac{\left(1+\frac{\alpha}{n}\right)^{n+1}}{\frac{\alpha}{n}} = \frac{\left(1+\left(\frac{\alpha}{n}\right)^{1-\frac{n+1}{j-1}} \lambda_{n,j}\left(\frac{\alpha}{n}\right)^{\frac{n+1}{j-1}}\right)^{n+1}}{\left(\frac{\alpha}{n}\right)^{1-\frac{n+1}{j-1}} \lambda_{n,j}\left(\frac{\alpha}{n}\right)^{\frac{n+1}{j-1}}}. \tag{52}$$

We ask for which values of j it is possible to have $\lambda_{n,j}(\frac{\alpha}{n}) = 1/n$ for some $\alpha > 1$. Inserting $\lambda_{n,j}(\frac{\alpha}{n}) = 1/n$ in (52), multiplying with α/n and dividing through the nominator of the right hand side and taking the $(n+1)$-st root yields after simplification and rearrangements the equivalent assertion

$$n = \frac{\alpha - \alpha^{\frac{j-n-1}{j-1}}}{\alpha^{\frac{1}{j-1}} - 1}. \tag{53}$$

Let $\theta := \alpha^{\frac{1}{j-1}}$. Clearly $\theta > 1$ is equivalent to $\alpha > 1$ and (53) is further equivalent to

$$n = \theta^{j-1-n}\frac{\theta^n - 1}{\theta - 1} = \theta^{j-2} + \theta^{j-3} + \cdots + \theta^{j-1-n} =: \chi_{n,j}(\theta). \tag{54}$$

By construction $\chi_{n,j}(1) = n$. First consider $n \leq 2j-3$ or equivalently $j \geq \frac{n+3}{2}$. We calculate

$$\chi_{n,j}'(t) = (j-2)t^{j-3} + (j-3)t^{j-4} + \cdots + (j-n-1)t^{j-n-2}.$$

and

$$\chi_{n,j}''(t) = (j-2)(j-3)t^{j-4} + (j-3)(j-4)t^{j-5} + \cdots + (j-1-n)(j-2-n)t^{j-n-3}.$$

It is easy to verify $\chi_{n,j}''(t) > 0$ for all $t > 0$, since any expression in the sum is non-negative, and for $j \geq 4$ the first and for $j = 3$ the last is strictly positive. Hence it suffices to show $\chi_{n,j}'(1) > 0$ to see that $\chi_{n,j}(t) > n$ for all $t > 1$. Indeed for $j \geq \frac{n+3}{2}$ we verify

$$\chi_{n,j}'(1) = (j-2) + (j-3) + \cdots + (j-n-1) = nj - \sum_{i=2}^{n+1} i = nj - \frac{n^2+3n}{2} \geq 0. \tag{55}$$

We conclude $\lambda_{n,j}(\lambda) \neq 1/n$ for all $\lambda > 1/n$. By the continuity of $\lambda_{n,j}$ we must have either $\lambda_{n,j}(\lambda) < 1/n$ for all $\lambda > 1/n$ or $\lambda_{n,j}(\lambda) > 1/n$ for all $\lambda > 1/n$. However, since $j \geq 3$ we can exclude the latter since in Proposition 2.5 we showed

$$\lim_{\lambda\to\infty} \lambda_{n,j}(\lambda) = 0, \qquad j \geq 3. \tag{56}$$

We have proved all claims for $j \geq \frac{n+3}{2}$.

Now let $j < \frac{n+3}{2}$, which is equivalent to $n \geq 2j-2$. Then

$$\chi'_{n,j}(1) = nj - \frac{n^2+3n}{2} < 0.$$

Hence since $\chi''_{n,j}(t) > 0$ for all $t > 0$ there exists precisely one value $\mu_0 > 1$ for which $\chi_{n,j}(\mu_0) = n$, or equivalently precisely one $\tilde{\lambda} > 1/n$ with $\lambda_{n,j}(\tilde{\lambda}) = 1/n$. Again by (56) and continuity we must have $\lambda_{n,j}(\lambda) < 1/n$ for $\lambda > \tilde{\lambda}$. Moreover again by intermediate value theorem either $\lambda_{n,j}(\lambda) > 1/n$ for all $\lambda \in (1/n, \tilde{\lambda})$ or $\lambda_{n,j}(\lambda) < 1/n$ for all $\lambda \in (1/n, \tilde{\lambda})$. Suppose conversely to the claim of the theorem the latter is true. Recall the implicit equation (27) involving $\lambda_n = \lambda$ and $\lambda_{n,j}(\lambda)$. Denote

$$F(x) = \frac{(1+x)^{n+1}}{x}, \qquad G(x,y) = \frac{(1+xy)^{n+1}}{xy},$$

such that (27) becomes $F(\lambda) = G(\lambda^{1-\frac{n+1}{j-1}}, \lambda_{n,j}(\lambda)^{1-\frac{n+1}{j-1}})$. Proceeding as above we show next that for λ close to $1/n$ we have

$$F(\lambda) = G(\lambda^{1-\frac{n+1}{j-1}}, \lambda_{n,j}(\lambda)^{\frac{n+1}{j-1}}) < G(\lambda^{1-\frac{n+1}{j-1}}, (1/n)^{\frac{n+1}{j-1}}). \tag{57}$$

Indeed, with $\lambda = \alpha/n$ inequality (57) is equivalent to

$$n > \frac{\alpha - \alpha^{\frac{j-n-1}{j-1}}}{\alpha^{\frac{1}{j-1}} - 1}, \tag{58}$$

Proceeding as above subsequent to (53) we see that for (58) the condition $\chi'_{n,j}(1) > 0$ is sufficient, which is true for $j < \frac{n+3}{2}$ and α sufficiently close to 1 by a very similar calculation as in (55). Thus we have showed (57). Hence if $\lambda_{n,j}(\lambda) < 1/n$ for such λ then by intermediate value theorem of differentiation we must have

$$\frac{dG}{dy}(\lambda^{1-\frac{n+1}{j-1}}, \eta) > 0 \tag{59}$$

for some pair (λ, η) with $\lambda \geq 1/n$ and $\eta \in (\lambda_{n,j}(\lambda)^{\frac{n+1}{j-1}}, (1/n)^{\frac{n+1}{j-1}})$. We disprove this. We calculate

$$\frac{dG(x,y)}{dy} = (nxy-1)(1+xy)^n \frac{1}{xy^2}.$$

Hence the sign of the partial derivative of G in (59) equals that of $nxy-1$. Our hypothesis yields

$$n\lambda\eta \leq n\left(\frac{\alpha}{n}\right)^{1-\frac{n+1}{j-1}} \left(\frac{1}{n}\right)^{\frac{n+1}{j-1}} = \alpha^{1-\frac{n+1}{j-1}} < 1$$

since $\alpha > 1$ and the exponent is negative. Hence $dG(\lambda^{1-\frac{n+1}{j-1}}, \eta)/dy < 0$ for all $\eta \in (\lambda_{n,j}(\lambda)^{\frac{n+1}{j-1}}, (1/n)^{\frac{n+1}{j-1}})$. This contradicts (59). Hence the hypothesis was wrong and we must have $\lambda_{n,j}(\lambda) > 1/n$ for all $\lambda \in (1/n, \tilde{\lambda})$.

Finally the fact that $\tilde{\lambda} < n$ follows from combination of $\lambda_{n,j}(\tilde{\lambda}) = 1/n$ and $\lambda_{n,j}(\tilde{\lambda}) < \tilde{\lambda}^{2-j} \leq \tilde{\lambda}^{-1} < n$ for $\tilde{\lambda} > 1/n$ and $j \geq 3$, see the proof of Proposition 2.5. □

5.2. **Proofs of Section 3.** We turn towards the proof of Theorem 3.3. We will frequently use the well-known fact that

$$\lim_{n\to\infty} (1 + x/n)^n = e^x \tag{60}$$

for real x, where the sequence $(1+x/n)^n$ is monotonic increasing. In particular for every integer $n \geq 1$ and real $\theta > 1$ we have $\theta^{1/(n+1)} > 1 + \log(\theta)/(n+1)$ since the sequence $(1 + \log(\theta)/n)^n$ converges monotonically increasing to θ. Thus for $n \geq 1, \theta > 1$ we have

$$\theta^{-1/(n+1)} < \frac{1}{1 + \frac{\log(\theta)}{n+1}} = 1 - \frac{\log(\theta)}{\log(\theta) + n + 1}. \tag{61}$$

Together with (30) and (31), this allows for establishing upper bounds for the evaluations of $\phi_n(w_n(\zeta))$ in terms of $n, w_n(\zeta)$, which is in turn an upper bound for $\widehat{w}_n(\zeta)$ by Conjecture 1.1 and Proposition 2.1. Combination with the bounds from Theorem 3.2 yields the estimate (40). The estimate (41) will follow similarly from (33) and Theorem 3.2.

Proof of Theorem 3.3. First we show (40). From the assumption of Conjecture 1.1 together with Proposition 2.1 and (39) we obtain

$$\widehat{w}_n^*(\zeta) \leq \widehat{w}_n(\zeta) \leq \phi_n(w_n(\zeta)). \tag{62}$$

Together with Theorem 3.2 we derive

$$\widehat{w}_n^*(\zeta) \leq \min\left\{\frac{nw_n(\zeta)}{w_n(\zeta) - n + 1}, \phi_n(w_n(\zeta))\right\}. \tag{63}$$

Let $D > 1$ be fixed number to be specified later and consider large n, in particular $n > 3D$. Let

$$\kappa_n := \frac{(2n - D)(n-1)}{n - D}.$$

First assume $w_n(\zeta) \geq \kappa_n$. Then $nw_n(\zeta)/(w_n(\zeta) - n + 1) \leq 2n - D$ such that (40) follows from (63). Since all ϕ_n are increasing by Lemma 2.2 it only remains to be shown that $\phi_n(\kappa_n) \leq 2n - D$ for large n to derive (63) also in case of $n \leq w_n(\zeta) < \kappa_n$. Hence we may assume $w_n(\zeta) = \kappa_n$. It is easy to check

$$\kappa_n = 2n - 2 + (2 - 2/n)D + O(1/n) = 2n + 2D - 2 + O(1/n). \tag{64}$$

In particular $\kappa_n = 2n + o(n)$. Let

$$\varphi_n(x) = \frac{(x+1)^{n+1}}{x^n} = (1 + 1/x)^n(1 + x).$$

With (60) we infer

$$\varphi_n(\kappa_n) = \left(1 + \frac{1}{\kappa_n}\right)^n (\kappa_n + 1) = \left(1 + \frac{1}{2n + o(n)}\right)^n (2n + o(n)) = (2\sqrt{e} + o(1))n.$$

From (30) we further deduce

$$\varphi_n(\widehat{w}_{n,n+1}(\zeta)) = \varphi_n(w_n(\zeta)) = \varphi_n(\kappa_n) = (2\sqrt{e}+o(1))n.$$

We noticed preceding the theorem that $\widehat{w}_{n,n+1}(\zeta) \leq n$. Thus if we write $\widehat{w}_{n,n+1}(\zeta) = bn$ then $b = b(n) \in [0,1]$ and again (60) yields that b satisfies $be^{1/b} = 2\sqrt{e}+o(1)$ as $n \to \infty$. This yields $b(n) = \tau + o(1)$ as $n \to \infty$ where $\tau \approx 0.5693$ is the solution $y \in (0,1)$ to $ye^{1/y} = 2\sqrt{e}$. Together with (64) we infer

$$\phi(\kappa_n) = w_n(\zeta)\left(\frac{\widehat{w}_{n,n+1}(\zeta)}{w_n(\zeta)}\right)^{1/(n+1)} = (2n+2D-2+o(1))\left(\frac{\tau}{2}+o(1)\right)^{1/(n+1)}. \tag{65}$$

Inserting (61) with $\theta := 2/\tau \approx 3.5128$ in (65) yields

$$\phi(\kappa_n) \leq (2n+2D-2+o(1))\left(1-\frac{\log(2/\tau+o(1))}{\log(2/\tau+o(1))+n+1}\right). \tag{66}$$

One checks that if $D < \log(2/\tau)+1$ and n is large then the right hand side of (66) is smaller than $2n-D$. This finishes the proof of (40).

Now we treat the case of $\widehat{w}_n(\zeta)$. In case of $w_{n-2}(\zeta) = w_n(\zeta)$ from Theorem 3.2 with $m = n-2$ we derive that $\widehat{w}_n(\zeta) \leq 2n-3 < 2n-D$ which proves the claim. In case of $w_{n-1}(\zeta) < w_n(\zeta)$ we can again apply Theorem 3.2 and obtain the same bounds for $\widehat{w}_n$ as in (63), and can proceed as in the proof of (40). Hence only possibly in case of $w_{n-2}(\zeta) < w_{n-1}(\zeta) = w_n(\zeta)$ the bounds may fail, as asserted. Finally concerning (41) we need preciser error terms in dependence of n. First observe (62) and Theorem 3.2 imply

$$\widehat{w}_n(\zeta) \leq \min\left\{\max\left\{2n-2, \frac{nw_n(\zeta)}{w_n(\zeta)-n+1}\right\}, \phi_n(w_n(\zeta))\right\}. \tag{67}$$

To derive (41) we use (33) directly. As above with $D = 2$ we see that $w_n(\zeta) \geq 2(n-1)^2/(n-2)$ implies $nw_n(\zeta)/(w_n(\zeta)-n+1) \leq 2n-2$ and hence (67) implies (41). Hence again since ϕ_n are monotonic increasing by Lemma 2.2 it remains to be checked that $\phi_n(w) \leq 2n-2$ for $n \geq 10$ where $w := 2(n-1)^2/(n-2)$. Let

$$H(x,y) = x-y+1-\left(\frac{x}{y}\right)^n.$$

Observe $(w_n(\zeta), \phi_n(w_n(\zeta)))$ solves (33), in particular $H(w,\phi(w)) = 0$ or $\phi_n(w)$ is the solution $y_0 < w$ of

$$H(w,y_0) = \frac{2(n-1)^2}{n-2} - y_0 + 1 + \left(\frac{2(n-1)^2}{(n-2)y_0}\right)^n = 0.$$

Some elementary calculation shows

$$H(w,2n-2) = \frac{2}{n-2}+3-\left(\frac{n-1}{n-2}\right)^n = \frac{2}{n-2}+3-\left(1+\frac{1}{n-2}\right)^{n-2}\left(1+\frac{1}{n-2}\right)^2$$

which with (60) and some computation for small n can be easily checked to be positive for $n \geq 10$. On the other hand we have

$$\frac{dH}{dy}(w,y) = -1+nw^ny^{-n-1},$$

which is positive for any $y < \phi_n(w)$ by (32). Thus indeed the root $y_0 = \phi_n(w)$ of $H(w, y_0) = 0$ must be smaller than $2n - 2$. This finishes the proof. □

Finally we sketch the proof of Theorem 3.4, where we omit technical calculations.

Proof of Theorem 3.4. From (32) and Theorem 3.1 we deduce

$$\widehat{w}_n(\zeta) \leq \min\left\{\max\left\{\frac{nw_n(\zeta)}{w_n(\zeta)-n+1}, 2n-2\right\}, n^{\frac{1}{n+1}} w_n(\zeta)^{\frac{n}{n+1}}\right\}. \tag{68}$$

First consider only the minimum without the expression $2n - 2$. It is checked easily that the equality condition $n\widetilde{w}_n(\zeta)/(\widetilde{w}_n(\zeta) - n + 1) = n^{1/(n+1)}\widetilde{w}_n(\zeta)^{n/(n+1)}$ maximizes this minimum and can be equivalently written $\widetilde{w}_n(\zeta) - n + 1 = \widetilde{w}_n(\zeta)^{1/(n+1)} n^{n/(n+1)}$. With (60) it can be further shown that this solution $\widetilde{w}_n(\zeta)$ satisfies $2n - 1 < \widetilde{w}_n(\zeta) < 2n$ for all n and is of the form $2n - 1 + \log 2 - \rho_n$ as $n \to \infty$ with positive ρ_n decreasing monotonically to 0. For this value $\widetilde{w}_n(\zeta)$ the evaluation in (68) lies in the interval $(2n - 2, 2n - 1)$ and is of the form $2n - 1 - \log 2 + \epsilon_n \approx 2n - 1.6931 + \epsilon_n$ with positive ϵ_n decreasing monotonically to 0. Since this is larger than $2n - 2$ we can drop $2n - 2$ in (68) and the claim follows. □

5.3. **Proofs of Section 4.** In the proof of Theorem 4.2 we will apply the transference inequality

$$\widehat{\lambda}_n(\underline{\zeta}) \geq \frac{\widehat{w}_n(\underline{\zeta}) - 1}{(n-1)\widehat{w}_n(\underline{\zeta})} \tag{69}$$

due to German [8], valid for all $\underline{\zeta}$ that are $\mathbb{Q}$-linearly independent together with $\{1\}$.

Proof of Theorem 4.2. Let $\epsilon > 0$. By definition of $\widehat{w}_n(\zeta)$ for any sufficiently large k we have

$$|P_{k+1}(\zeta)| < |P_k(\zeta)| < H(P_{k+1})^{-\widehat{w}_n(\zeta)+\epsilon}. \tag{70}$$

On the other hand, it follows from the definitions of $w_n(\zeta)$ and $\widehat{w}_n(\zeta)$ that for two successive best approximations P_l, P_{l+1} when l is sufficiently large we have $\log H(P_{l+1})/\log H(P_l) \leq w_n(\zeta)/\widehat{w}_n(\zeta) + \epsilon$ or equivalently $\log H(P_l)/\log H(P_{l+1}) \geq \widehat{w}_n(\zeta)/w_n(\zeta) - \tilde{\epsilon}$ where $\tilde{\epsilon}$ tends to 0 as ϵ does. This same argument applied repeatedly for l from $k+1$ to $k+n-1$ shows that

$$\frac{\log H(P_{k+1})}{\log H(P_{k+n})} \geq \left(\frac{\widehat{w}_n(\zeta)}{w_n(\zeta)}\right)^{n-1} - \tilde{\epsilon}_1 \tag{71}$$

for some $\tilde{\epsilon}_1$ which depends on ϵ and tends to 0 as ϵ tends to 0. Combination of (70) and (71) yields

$$-\frac{\log|P_k(\zeta)|}{\log H(P_{k+n})} = -\frac{\log|P_k(\zeta)|}{\log H(P_{k+1})} \cdot \frac{\log H(P_{k+1})}{\log H(P_{k+n})} \geq (\widehat{w}_n(\zeta) - \epsilon)\left(\left(\frac{\widehat{w}_n(\zeta)}{w_n(\zeta)}\right)^{n-1} - \tilde{\epsilon}_1\right).$$

Since $|P_k(\zeta)| > |P_{k+1}(\zeta)| > \cdots > |P_{k+n}(\zeta)|$ we infer that

$$-\frac{\log|P_{k+j}(\zeta)|}{\log H(P_{k+n})} \geq \widehat{w}_n(\zeta)\left(\frac{\widehat{w}_n(\zeta)}{w_n(\zeta)}\right)^{n-1} + \tilde{\epsilon}_2, \qquad 0 \leq j \leq n,$$

for some $\tilde{\epsilon}_2$ which again depends on ϵ and tends to 0 as ϵ does. Thus by our assumption that we can find arbitrarily large k such that the polynomials $P_k, P_{k+1}, \ldots, P_{k+n}$ are linearly independent and since $H(P_{n+k}) \geq H(P_{n+j})$ for $0 \leq j \leq k$, we obtain (43) as we may take ϵ arbitrarily small.

Finally (44) follows from (43) combined with

$$w_{n,n+1}(\zeta) = \frac{1}{\widehat{\lambda}_n(\zeta)} \leq \frac{(n-1)\widehat{w}_n(\zeta)}{\widehat{w}_n(\zeta)-1}$$

by elementary rearrangements, where the right inequality is obtained from (69) by taking reciprocals. ☐

Now we turn to Wirsing's conjecture. It was shown by the author [19] that

$$w_n^*(\zeta) \geq w_{n,n+1}(\zeta) \tag{72}$$

for all $n \geq 1$ and real transcendental ζ. On the other hand Bugeaud and Laurent [5] proved

$$w_n^*(\zeta) \geq \frac{\widehat{w}_n(\zeta)}{\widehat{w}_n(\zeta)-n+1}. \tag{73}$$

Moreover within the proof of [6, Theorem 2.7] it was shown that

$$w_n^*(\xi) \geq \min\Big\{\widehat{w}_n(\xi), \frac{w_n(\xi)+1}{2} + \widehat{w}_n(\xi) - n\Big\}.$$

The last result implies that either $w_n^*(\zeta) \geq \widehat{w}_n(\zeta) \geq n$ anyway or

$$w_n^*(\xi) \geq \max\Big\{\frac{\widehat{w}_n(\zeta)}{\widehat{w}_n(\zeta)-n+1}, \frac{w_n(\xi)}{2} + \widehat{w}_n(\xi) - n + \frac{1}{2}\Big\}. \tag{74}$$

The above observations will be combined to proof Theorem 4.4. We will again only sketch the proof and skip the calculations that occur, which are technical but not difficult.

Proof of Theorem 4.4. Clearly the claim holds for $w_n^*(\zeta) \geq n$. In case of $w_n^*(\zeta) < n$, combination of (43), (72), (73) and (74) yields

$$w_n^*(\zeta) \geq \max\left\{\frac{\widehat{w}_n(\zeta)}{\widehat{w}_n(\zeta)-n+1}, \frac{w_n(\zeta)}{2} + \widehat{w}_n(\zeta) - n + \frac{1}{2}, \frac{\widehat{w}_n(\zeta)^n}{w_n(\zeta)^{n-1}}\right\} \tag{75}$$

First consider only the last two expressions in (75). For given $\widehat{w}_n(\zeta)$ one can consider the function $w_n(\zeta) = \Phi(\widehat{w}_n(\zeta))$ which minimizes the maximum of these. This is the point where the expressions are equal since again the middle expression increases and the most right expression decreases as $w_n(\zeta)$ increases. Then determining the intersection of the functions $\widehat{w}_n(\zeta)/(\widehat{w}_n(\zeta)-n+1)$ and $\widehat{w}_n(\zeta)^{n+1}/w_n(\zeta)^n$ where $w_n(\zeta) = \Phi(\widehat{w}_n(\zeta))$, one checks that the minimum value of the maximum of all three expressions in (75) is attained for some pair $(\widehat{w}_n(\zeta), w_n(\zeta))$ with $w_n(\zeta) = n+1-\epsilon_n$ for some positive sequence ϵ_n which converges monotonically to 0. Moreover and more important, the evaluation of $\widehat{w}_n(\zeta)^{n+1}/w_n(\zeta)^n = \widehat{w}_n(\zeta)^{n+1}/\Phi(\widehat{w}_n(\zeta))^n$ (or another expressions in (75)) is of the form $n/2+\gamma_n$ for some increasing sequence γ_n which converges to some value, and numerical calculations show γ_{300} is already larger than 2.31. ☐

References

[1] V. I. BERNIK and K. TISHCHENKO. Integral polynomials with an overfall of the coefficient values and Wirsing's problem. Dokl. Akad. Nauk Belarusi 37 (1993), no 5, 9–11 (in Russian).

[2] Y. BUEGAUD. Approximation by algebraic numbers. Cambridge tracts in Mathematics, Cambridge 2004.

[3] Y. BUGEAUD. On simultaneous rational approximation to a real numbers and its integral powers. *Ann. Inst. Fourier (Grenoble)* (6) 60 (2010), 2165–2182.

[4] Y. BUGEAUD and M. LAURENT. Exponents in Diophantine approximation. *Diophantine Geometry Proceedings, Scuola Normale Superiore Pisa*, Ser. CRM, 4 (2007), 101–121.

[5] Y. BUGEAUD and M. LAURENT. Exponents of Diophantine approximation and Sturmian continued fractions. *Ann. Inst. Fourier* (Grenoble) 55 (2005), 773–804. *Diophantine Geometry Proceedings, Scuola Normale Superiore Pisa*, Ser. CRM, 4 (2007), 101–121.

[6] Y. BUGEAUD and J. SCHLEISCHITZ. On uniform approximation to real numbers. *arXiv: 1512.00780.*

[7] H. DAVENPORT and W. M. SCHMIDT. Approximation to real numbers by algebraic integers. *Acta Arith.* 15 (1969), 393–416.

[8] O. GEMRAN. On Diophantine exponents and Khintchine's transference principle. *Mosc. J. Comb. Number Theory* 2 (2012), 22–51.

[9] O. GERMAN and N. G. MOSHCHEVITIN. A simple proof of Schmidt-Summerer's inequality. Monatsh. Math. 170 (2013), no. 3–4, 361–370.

[10] V. JARNIK. Zum Khintchinschen "Übertragungssatz". *Trav. Inst. Math. Tbilissi* 3 (1938), 193–212.

[11] M. LAURENT. Exponents of Diophantine approximation in dimension two. *Canadian J. Math.* 61 (2009) , no. 1, 165–189.

[12] K. MAHLER. The successive minima in the geometry of numbers and the distinction between algebraic and transcnedental numbers. *J. Number Theory* 22 (1986), 147–160.

[13] N. G. MOSHCHEVITIN. Best Diophantine approximations: the phenomenon of degenerate dimension. *London Math. Soc. Lecture Note Ser.* 338, Cambridge Univ. Press, Cambridge 2007, 158–182.

[14] N. G. MOSHCHEVITIN. Proof of W. M. Schmidt's conjecture concerning successive minima of a lattice. *J. London Math. Soc. (2)* 86 (2012), no. 1, 129–151.

[15] N. G. MOSHCHEVITIN. On some open problems in Diophantine approximation. *arXiv 1202.4539*

[16] D. ROY. Approximation to real numbers by cubic algebraic integers I. *Proc. London Math. Soc.* 88 (2004), 42–62.

[17] D. ROY. Construction of points realizing the regular systems of Wolfgang Schmidt and Leonhard Summerer, *to appear in Journal de Théorie de Nombres de Bordeaux, arXiv:1406.3669*

[18] D. ROY. On the continued fraction expansion of a class of numbers. *in: Diophantine approximation, Festschrift for Wolfgang Schmidt, Developments in Math.* vol. 16, Eds: H. P. Schlickewei, K. Schmidt and R. Tichy, Springer-Verlag, 2008, 347–361.

[19] J. SCHLEISCHITZ. Two estimates concerning classical Diophantine approximation constants, *Publ. Math. Debrecen* 84/3-4 (2014), 415–437.

[20] J. SCHLEISCHITZ. Diophantine approximation and special Liouville numbers, *Comm. Math.* 21 (2013), 39–76.

[21] W. M. SCHMIDT. Open problems in Diophantine approximation. Diophatine approximation and transcendental numbers, Luminy 1982. *Progr. Math.* 31, Birkhäuser 271–289 (1983).

[22] W.M. SCHMIDT, L. SUMMERER. Parametric geometry of numbers and applications, *Acta Arith.* 140 (2009), no. 1, 67–91.

[23] W.M. SCHMIDT, L. SUMMERER. Diophantine approximation and parametric geometry of numbers, *Monatsh. Math.* 169 (2013), no. 1, 51–104.

[24] W.M. SCHMIDT, L. SUMMERER. Simultaneous approximation to three numbers, *Mosc. J. Comb. Number Theory* 3 (2013), No. 1, 84–107.

[25] K. TISHCHENKO. On approximation of real numbers by algebraic numbers of bounded degree. *J. Number Theory* 123 (2007), no. 2, 290–314.

[26] E. WIRSING. Approximation mit algebraischen Zahlen beschränkten Grades. *J. Reine Angew. Math.* 01/1961, 67–77.