Eliminating Higher-Multiplicity Intersections, II. The Deleted Product Criterion in the r-Metastable Range*

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January 6, 2016

Abstract

Motivated by Tverberg-type problems in topological combinatorics and by classical results about embeddings (maps without double points), we study the question whether a finite simplicial complex K can be mapped into $\mathbb{R}^d$ without higher-multiplicity intersections. We focus on conditions for the existence of *almost r-embeddings*, i.e., maps $f\colon K \to \mathbb{R}^d$ such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset$ whenever $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint simplices of K.

Generalizing the classical Haefliger-Weber embeddability criterion, we show that a well-known necessary *deleted product condition* for the existence of almost r-embeddings is sufficient in a suitable *r-metastable range* of dimensions: If $rd \geq (r+1)\dim K + 3$, then there exists an almost r-embedding $K \to \mathbb{R}^d$ if and only if there exists an equivariant map $(K)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$, where $(K)^r_\Delta$ is the deleted r-fold product of K and $\mathfrak{S}_r$ is the symmetric group. This significantly extends one of the main results of our previous paper (which treated the special case where $d = rk$ and $\dim K = (r-1)k$ for some $k \geq 3$), and settles an open question raised there.

Combining our result with a theorem of Čadek, Krčál, and Vokřínek on the homotopy classification of equivariant maps, we obtain the following corollary: If r is prime, then in the r-metastable range the existence of an r-almost embedding can be decided algorithmically (in polynomial time if r and d are fixed).

Contents

1 Introduction

Let K be a finite simplicial complex, and let $f\colon K \to \mathbb{R}^d$ be a continuous map.[1] Given an integer $r \geq 2$, we say that $y \in \mathbb{R}^d$ is an **r-fold point** or **r-intersection point** of f if it has r pairwise distinct preimages, i.e., if there exist $y_1, \ldots, y_r \in K$ such that $f(y_1) = \ldots = f(y_r) = y$ and $y_i \neq y_j$

*Research supported by the Swiss National Science Foundation (Project SNSF-PP00P2-138948).

We would like to thank Arkadiy Skopenkov as well as the anonymous referees of a previous version for many helpful comments.

[1] For simplicity, throughout most of the paper we use the same notation for a simplicial complex K and its *underlying topological space*, relying on context to distinguish between the two when necessary.

for $1 \le i < j \le r$. We will pay particular attention to r-fold points that are **global**[2] in the sense that their preimages lie in r *pairwise disjoint simplices* of K, i.e., $y \in f(\sigma_1) \cap \ldots \cap f(\sigma_r)$, where $\sigma_i \cap \sigma_j = \emptyset$ for $1 \le i < j \le r$.

We say that a map $f\colon K \to \mathbb{R}^d$ is an **r-embedding** if it has no r-fold points, and we say that f is an **almost r-embedding** if it has no *global* r-fold points.[3]

The most fundamental case $r = 2$ is that of **embeddings** (=2-embeddings), i.e., injective continuous maps $f\colon K \to \mathbb{R}^d$. Finding conditions for a simplicial complex K to be embeddable into $\mathbb{R}^d$ — a higher-dimensional generalization of graph planarity — is a classical problem in topology (see [34, 42] for surveys) and has recently also become the subject of systematic study from a viewpoint of algorithms and computational complexity (see [29, 28, 11]).

Here, we are interested in necessary and sufficient conditions for the existence of almost r-embeddings. One motivation are *Tverberg-type problems* in topological combinatorics (see the corresponding subsection below). Another motivation is that, in the classical case $r = 2$, embeddability is often proved in two steps: in the first step, the existence of an **almost embedding** (=almost 2-embedding) is established; in the second step this almost embedding is transformed into an honest embedding, by removing *local* self-intersections. Similarly, we expect the existence of an almost r-embedding to be not only an obvious necessary condition but a useful stepping stone towards the existence of r-embeddings and, in a further step, towards the existence of embeddings in certain ranges of dimensions.

The Deleted Product Criterion for Almost r-Embeddings. There is a well-known *necessary condition* for the existence of almost r-embeddings. Given a simplicial complex K and $r \ge 2$, the (combinatorial) **deleted r-fold product**[4] of K is defined as

$$(K)^r_\Delta := \{(x_1, \ldots, x_r) \in \sigma_1 \times \cdots \times \sigma_r \mid \sigma_i \text{ a simplex of } K, \sigma_i \cap \sigma_j = \emptyset \text{ for } 1 \le i < j \le r\}.$$

The deleted product is a regular polytopal cell complex (a subcomplex of the cartesian product), whose cells are products of r-tuples of pairwise disjoint simplices of K.

Lemma 1 (**Necessity of the Deleted Product Criterion**). *Let K be a finite simplicial complex, and let $d \ge 1$ and $r \ge 2$ be integers. If there exists an almost r-embedding $f : K \to \mathbb{R}^d$ then there exists an equivariant map*[5]

$$\widetilde{f}\colon (K)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1},$$

where $S^{d(r-1)-1} = \left\{(y_1, \ldots, y_r) \in (\mathbb{R}^d)^r \mid \sum_{i=1}^r y_i = 0, \sum_{i=1}^r \|y_i\|_2^2 = 1\right\}$, and the symmetric group $\mathfrak{S}_r$ acts on both spaces by permuting components.

Proof. Given $f\colon K \to \mathbb{R}^d$, define $f^r : (K)^r_\Delta \to (\mathbb{R}^d)^r$ by $f^r(x_1, \ldots, x_r) := (f(x_1), \ldots f(x_r))$. Then f is an almost r-embedding iff its image avoids the *thin diagonal* $\delta_r(\mathbb{R}^d) := \{(y, \ldots, y) \mid y \in \mathbb{R}^d\} \subset (\mathbb{R}^d)^r$. Moreover, $S^{d(r-1)-1}$ is the unit sphere in the orthogonal complement $\delta_r(\mathbb{R}^d)^\perp \cong \mathbb{R}^{d(r-1)}$, and there is a straightforward homotopy equivalence $\rho\colon (\mathbb{R}^d)^r \setminus \delta_r(\mathbb{R}^d) \simeq S^{d(r-1)-1}$. Both f^r and ρ are equivariant hence so is their composition $\widetilde{f} := \rho \circ f^r : (K)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$. □

Our main result is that the converse of Lemma 1 holds in a wide range of dimensions.

Theorem 2 (**Sufficiency of the Deleted Product Criterion in the r-Metastable Range**). *Let $m, d \ge 1$ and $r \ge 2$ be integers satisfying*

$$rd \ge (r + 1)m + 3. \tag{1}$$

[2] In our previous paper [24], we used the terminology "*r-Tverberg point*" instead of "*global r-fold point.*"

[3] We emphasize that the definitions of global r-fold points and of almost r-embeddings depend on the actual simplicial complex K (the specific triangulation), not just the underlying topological space.

[4] For more background on deleted products and the broader *configuration space/test map* framework, see, e.g., [27] or [48, 49].

[5] Here and in what follows, if X and Y are spaces on which a finite group G acts (all group actions will be from the right) then we will use the notation $F\colon X \to_G Y$ for maps that are **equivariant**, i.e., that satisfy $F(x \cdot g) = F(x) \cdot g$ for all $x \in X$ and $g \in G$).

Suppose that K is a finite m-dimensional simplicial complex and that there exists an equivariant map $F : (K)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$. Then there exists an almost r-embedding $f\colon K \to \mathbb{R}^d$.

Remarks 3. (a) When studying almost r-embeddings, it suffices to consider maps $f\colon K \to \mathbb{R}^d$ that are *piecewise-linear*[6] (*PL*) and *in general position.*[7]

(b) Theorem 2 is trivial for *codimension* $d-m \leq 2$. Indeed, if r, d, m satisfy (1) and, additionally, $d-m \leq 2$ then a straightforward calculation shows that $(r-1)d > rm$, so that a map $K \to \mathbb{R}^d$ in general position has no r-fold points.

(c) The special case $r = 2$ of Theorem 2 corresponds to the classical *Haefliger–Weber Theorem* [17, 46], which guarantees that for $2d \geq 3m+3$ the existence of an equivariant map $(K)^2_\Delta \to_{\mathfrak{S}_2} S^{d-1}$ guarantees the existence of an almost embedding $f\colon K \to \mathbb{R}^d$. An almost embedding can be then be turned into an embedding by a delicate construction of Skopenkov [41] or Weber [46]. The condition $2d \geq 3m + 3$ is often referred to as the **metastable range**; correspondingly, we call Condition (1) the **r-metastable range**[8].

(d) Theorem 2 significantly extends one of the main results of our previous paper ([25, Thm. 7] and [24, Thm. 3], which treated the special case $(r-1)d = rm$, $d - m \geq 3$), and settles one of the open questions raised there.

In our previous paper, we consider the special case when all the global r-intersection points are isolated (i.e., the r-intersections are 0-dimensional). The "elimination" of these isolated r-intersections is achieved in two steps:

(1) First, we obtain the algebraic cancellation of the r-intersection points by "finger moves": we modify a given map $f : K^m \to \mathbb{R}^d$ such that for each r-tuples of pairwise disjoint cells $\sigma_1, \dots, \sigma_r$ of K, the intersection $f\sigma_1 \cap \dots \cap f\sigma_r$ consists of *pairs of points of opposite intersection signs* (hence, algebraically, they "cancel").

(2) In a second step, we geometrically cancel each pair of r-intersection points of opposite sign, and for this, we use an r-fold version of the Whitney Trick. Hence, we obtain $f\sigma_1 \cap \dots \cap f\sigma_r = \emptyset$.

In other words, for the special case consider in our previous paper, the proof decomposes naturally into two steps: (1) first a "linking step" when we link cell together to introduce new r-intersection points (and therefore obtain the "algebraic cancellation" of the r-intersection points), (2) secondly, in an "unlinking step" we translate that algebraic cancellation into geometry (i.e., from intersection = 0, we obtain intersection = $\emptyset$).

In the present paper, these two steps are not so disjoint anymore: multiple cases of global r-intersection points can occur, resulting in singular set of various dimension (no only isolated points). Therefore, we will have to merge the two steps (1) and (2): In our construction, we will first "unlink" the r-intersection points of a given r-tuple of cells (i.e., remove their r-intersection points), and immediately after we will "link" this r-tuple in order to permit the unlinking of r-tuples of higher dimension. (See both parts of Lemma 11: Part 1 corresponds to the "unlinking", and Part 2 corresponds to the "linking").

Algorithms. By a result of Čadek, Krčál, and Vokřínek [11, Thm. 1.1] on the homotopy classification of equivariant maps (extending alier work [10, 9] in the nonequivariant case) , the existence of an equivariant map $(K)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$ can be decided algorithmically if $\dim (K)^r_\Delta \leq 2d(r-1)-3$ (a straightforward calculation shows that the former condition is satisfied in the nontrivial case $d - m \geq 3$ of the r-metastable range) and if $\mathfrak{S}_r$ acts *freely* on $S^{d(r-1)-1}$ (which is the case if and only if r is *prime*). Together with Theorem 2, this implies the following:

[6] Recall that f is PL if there is some subdivision K' of K such that $f|_\sigma$ is affine for each simplex σ of K'.

[7] Every continuous map $g\colon K \to \mathbb{R}^d$ can be approximated arbitrarily closely by PL maps in general position, and if g is an almost r-embedding, then the same holds for any map sufficiently close to g.

[8] Our r-metastable range is *not* the same as the "k-metastable range" of Haefliger defined in Annex 9.1 of [18].

Corollary 4. *Suppose $d, m \geq 1$ and $r \geq 2$ satisfy* (1) *and that r is prime. Then there exists an algorithm that, given a finite m-dimensional simplicial complex K, decides whether K admits an almost r-embedding in $\mathbb{R}^d$. Moreover, if d and r are fixed then the algorithm runs in polynomial time (in the size of the input, measured in terms of the number of simplices of K).*

Remark 5. In recent work, Filakovský and Vokřínek (personal communication) have extended the algorithm of Čadek, Krčál, and Vokřínek to the setting of non-free actions. Theorem 2 together with this more general algorithm implies that Corollary 4 holds without the assumption that r is prime.

Background and Motivation: Topological Tverberg-Type Problems. Tverberg's classical theorem [44] in convex geometry can be rephrased as follows: if $N = (d+1)(r-1)$ then any *affine* map from the N-dimensional simplex σ^N to $\mathbb{R}^d$ has a global r-fold point, i.e., there does not exist an *affine* almost r-embedding of σ^N in $\mathbb{R}^d$.

Bajmoczy and Bárány [2] and Tverberg [16, Problem 84] raised the question whether the conclusion holds true, more generally, for arbitrary continuous maps:

Conjecture 6 (Topological Tverberg Conjecture)**.** *Let $r \geq 2$, $d \geq 1$, and $N = (d+1)(r-1)$. Then there is no almost r-embedding $\sigma^N \to \mathbb{R}^d$.*

This was proved by Bajmoczy and Bárány [2] for $r = 2$, by Bárány, Shlosman, and Szűcs [5] for all primes r, and by Özaydin [31] for prime powers r, but the case of arbitrary r remained open and was considered a central unsolved problem of topological combinatorics.

There are numerous close relatives and other variants of (topological) Tverberg-type problems and results. These can be seen as *generalized nonembeddability results* or *problems* and typically state that a particular complex K (or family of complexes) does not admit an almost r-embedding into $\mathbb{R}^d$. Well-known examples are the *Colored Tverberg Problem* [3, 4, 50, 49, 7] and generalized *Van Kampen–Flores-type results* [38, 45]. Theorem 2 provides a general necessary and sufficient condition for topological Tverberg-type results in the r-metastable range.

The topological Tverberg conjecture and the subsequent developments played an important role in the introduction and use of methods from *equivariant topology* in discrete and computational geometry. The prime and prime power cases of Conjecture 6 were proved via Lemma 1, i.e., by showing that there exists no equivariant map $(\sigma^N)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$. However, this fails in the remaining cases: Özaydin [31, Thm. 4.2] showed that if r is *not* a prime power then there exists an equivariant map $F\colon (\Delta^N)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$.

In the extended abstract of our previous paper [24], we proposed a new approach to the conjecture, based on the idea of combining Özaydin's result with the sufficiency of the deleted product product ([24, Thm 3]) to construct counterexamples, i.e., almost r-embeddings $\sigma^N \to \mathbb{R}^d$, whenever r is not a prime power. At the time we suggested this in [24], there remained what seemed a very serious obstacle to completing this approach: Our theory required the assumption of *codimension* $d - \dim K \geq 3$, which is not satisfied for $K = \sigma^N$.

In a recent breakthrough, Frick [14] was the first to find a way to overcome this "codimension 3 barrier" and to construct counterexamples to the topological Tverberg conjecture for all parameters (d, r) with $d \geq 3r + 1$ and r not a prime power, by a clever reduction (using the *constraints method* of Blagojević–Frick–Ziegler [6]) to a suitable lower-dimensional skeleton for which the required almost r-embedding exists by Özaydin's result and ours).

A different solution to the codimension 3 obstacle (based on the notion of *prismatic maps*) is given in the full version of our paper [25], leading to counterexamples for $d \geq 3r$. In joint work with Avvakumov and Skopenkov [1], we recently improved this further and obtained counterexamples for $d \geq 2r$, using an extension (for $r \geq 3$) of [25, Thm. 7] to *codimension 2*.

In conclusion, methods from equivariant topology and the general framework of *configuration spaces* and *test maps* [48, 49] have been very successfully used in discrete and computational geometry. In particular, *equivariant obstruction theory* and, more generally *equivariant homotopy theory*, provide powerful tools for deciding whether suitable test maps exist. However in cases where the existence of a test map does not settle the problem (as with the topological Tverberg conjecture), further geometric ideas are needed. The general philosophy and underlying idea here and in the two companion papers [25, 1] is to complement equivariant methods by methods from *geometric topology*, in particular *piecewise-linear topology* to discrete geometry, and we hope that these will find further applications.

Remarks 7 (**Further Questions and Future Research**). (a) *Beyond the r-Metastable Range.* Is condition (1) in Theorem 2 necessary? In the case $r = 2$, it is known that for $d \geq 3$, the Haefliger–Weber Theorem fails outside the metastable range: for every pair (m, d) with $2d < 3m + 3$ and $d \geq 3$, there are examples [26, 40, 13, 39, 15] of m-dimensional complexes K such that $(K)^2_\Delta \to_{\mathfrak{S}_2} S^{d-1}$ but K does not embed into $\mathbb{R}^d$. Moreover, in the case $r = 2$, $m = 2$ and $d = 4$, the examples do not even admit an almost embedding into $\mathbb{R}^4$, see [1].

On the other hand, as remarked above, in [1] the following extension of [25, Thm. 7] is proved: if $r \geq 3$ $d = 2r$, and $m = 2(r-1)$, then a finite m-dimensional complex K admits an almost r-embedding if and only if there exists an equivariant map $(K)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$.

It would be interesting to know whether there is analogous extension (for $r \geq 3$) of Theorem 2 that is nontrivial in codimension $d - m = 2$.

(b) *The Planar Case and Hanani–Tutte.* In the classical setting ($r = 2$) of embeddings, the case $d = 2, m = 1$ of *graph planarity* is somewhat exceptional: the parameters lie outside the (2-fold) metastable range, but the existence of an equivariant map $F \colon (K)^2_\Delta \to_{\mathfrak{S}_2} S^1$ is sufficient for a graph K to be planar, by the *Hanani–Tutte Theorem*[9] [12, 43]. The classical proofs of that theorem rely on *Kuratowski's Theorem*, but recently [32, 33], more direct proofs have been found that do not use forbidden minors. It would be interesting to know whether there is an analogue of the Hanani–Tutte theorem for almost r-embeddings of 2-dimensional complexes in $\mathbb{R}^2$, as an approach to constructing counterexamples to the topological Tverberg conjecture in dimension $d = 2$. We plan to investigate this in a future paper.

Structure of the Paper. The remainder of the paper is devoted to the proof of Theorem 2. By Lemma 1, we only need to show that the existence of an equivariant map $(K)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$ implies the existence of an almost r-embedding $K \to \mathbb{R}^d$. Moreover, by Remarks 3 (b) and (d), we may assume, in addition to the parameters being in the r-fold metastable range, that the *codimension* $d - m$ of the image of K in $\mathbb{R}^d$ is at least 3, and that the intersection multiplicity r is also at least 3. Thus, we will work under the following hypothesis:

$$rd \geq (r+1)m + 3, \quad d - m \geq 3, \quad \text{and} \quad r \geq 3. \tag{2}$$

The proof of Theorem 2 is based on two main lemmas: Lemma 9 (*Reduction Lemma*) reduces the situation to a single r-tuple of pairwise disjoint simplices of K, and Lemma 11 (*generalized Weber–Whitney Trick*) solves that reduced situation. In Section 2, we give the precise (and somewhat technical) statements of these lemmas, along with some background, and prove the Reduction Lemma 9. In Section 3, we show how to prove Theorem 3 using these lemmas, before proving Lemma 11 (the core of the paper) in Section 4.

2 The Two Main Lemmas

In this section, we formulate the two main lemmas on which the proof of Theorem 2 rests.

We work in the *piecewise-linear* (*PL*) category (standard references are [47, 37]). All manifolds (possibly with boundary) are PL-manifolds (can be triangulated as locally finite simplicial complexes such that the link of every nonempty face is either a PL-sphere or a PL-ball), and

[9] The existence of an equivariant map implies, via standard equivariant obstruction theory, that there exists a map from the graph K into $\mathbb{R}^2$ such that the images of any two disjoint (*independent*) edges intersect an even number of times, which is the hypothesis of the Hanani–Tutte Theorem.

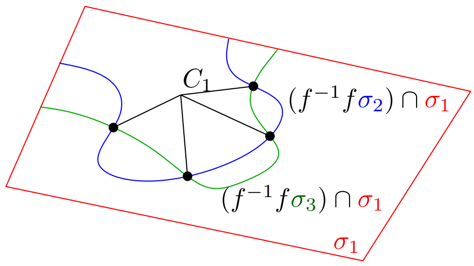

Figure 1: For $r = 3$, the construction of C_1 inside of σ_1. The collapsible polyhedron C_1 is a "cone" over the triple intersection set S_1 (which consists of four isolated points in the picture).

all maps between **polyhedra** (geometric realizations of simplicial complexes) are PL-maps (*i.e.*, simplicial on sufficiently fine subdivisions).[10] In particular, all balls are PL-ball and all spheres are PL-spheres (PL-homeomorphic to a simplex and the boundary of a simplex, respectively). A submanifold P of a manifold Q is **properly embedded** if $\partial P = P \cap \partial Q$. The **singular set** of a PL-map f defined on a polyhedron K is the closure in K of the set of points at which f is not injective.

One basic fact that we will use for the proofs of both Lemmas 9 and 11 is the following version of *engulfing* [47, Ch. VII]:

Theorem 8 (Engulfing, [47, Ch. VII, Thm. 20])**.** *Let M be an m-dimensional k-connected manifold with $k \le m-3$. Let X a compact x-dimensional subpolyhedron in the interior of M. If $x \le k$, then there exists a collapsible subpolyhedron C in the interior of M with $X \subseteq C$ and* $\dim(C) \le x+1$.

The collapsible polyhedron C can be thought of as an analogue of a "cone" over X.

Lemma 9 (Reduction Lemma)**.** *Let m, d, r be three positive integers satisfying* (2). *Suppose $f\colon K \to \mathbb{R}^d$ is a map in general position, and $\sigma_1, \dots, \sigma_r$ be pairwise disjoint simplices of K of dimension $s_1, \dots, s_r \le m$ such that $f^{-1}(f(\sigma_1) \cap \dots \cap f(\sigma_r))$ is contained in the* interior *of each simplex σ_i. Then there exists a ball B^d in $\mathbb{R}^d$ such that*

1. *B^d intersects each $f(\sigma_i)$ in a ball that is properly embedded in B^d, and that avoids the image of the singular set of $f|_{\sigma_i}$, as well as $f(\partial\sigma_i)$;*
2. *B^d contains $f(\sigma_1) \cap \dots \cap f(\sigma_r)$ in its interior; and*
3. *B^d does not intersect any other parts of the image $f(K)$.*

Proof. Let us consider $S_i := f^{-1}(f(\sigma_1) \cap \dots \cap f(\sigma_r)) \cap \sigma_i$. By general position [37, Thm 5.4] this is a polyhedron of dimension at most $s_1 + \dots + s_r - (r-1)d \le rm - (r-1)d$. By Theorem 8, we find $C_i \subseteq \sigma_i$ collapsible, containing S_i, and of dimension at most $rm - (r-1)d + 1$. Figure 1 illustrates the case $r = 3$.

The dimension of the singular set of $f|_{\sigma_i}$ is at most $2s_i - d$. Hence, C_i is disjoint from it since $(rm - (r-1)d + 1) + (2s_i - d) - s_i \le (r+1)m - rd + 1$, which is negative in the metastable range. Thus, f is injective in a neighbourhood of C_i.

Again by Theorem 8, we find in $\mathbb{R}^d$ a collapsible polyhedron $C_{\mathbb{R}^d}$ of dimension at most $rm - (r-1)d + 2$ and containing $f(C_1) \cup \dots \cup f(C_r)$. Figure 2 illustrates the construction for $r = 3$. By general position we have the following properties:

1. $C_{\mathbb{R}^d}$ intersects $f(\sigma_i)$ exactly in $f(C_i)$. Indeed, in the metastable range, $rm - (r-1)d + 2 + s_i - d \le (r+1)m - rd + 2 < 0$.
2. $C_{\mathbb{R}^d}$ does not intersect any other part of $f(K)$ (by a similar computation).

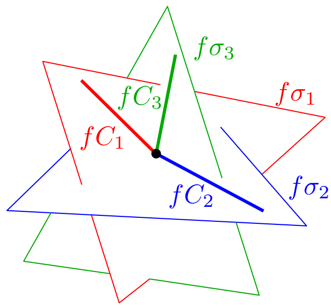

Figure 2: For $r = 3$, the polyhedron $C_{\mathbb{R}^d}$ is a "cone" over $fC_1 \cup fC_2 \cup fC_3$.

We take a small **regular neighbourhood** [37, Ch. 3] B of $C_{\mathbb{R}^d}$, which still avoids the singular set of each $f|_{\sigma_i}$ as well as other parts of $f(K)$. This regular neighbourhood is a ball, since $C_{\mathbb{R}^d}$ is collapsible. The intersection $B \cap f(\sigma_i)$ is a regular neighbourhood of $f(C_i)$ which is also a collapsible space, hence $B \cap f(\sigma_i)$ is a ball (properly contained in B). $\square$

An **ambient isotopy** H is of a PL-manifold X is a collection of homeomorphisms $H_t : X \to X$ for $t \in [0, 1]$, which vary continuously with t, and with $H_0 = \text{id}$. We say that an ambient isotopy H **throws** a subspace $Y \subseteq X$ onto Z if $H_1(Y) = Z$, see [47, Ch. V].

We say that an ambient isotopy H of X is **proper** if $H_t|_{\partial X} = \text{id}_{\partial X}$ for all t.

Definition 10. Let m, d, r be three positive integers satisfying (2). Let $\sigma_1, \ldots, \sigma_r$ be balls of dimensions $s_1, \ldots, s_r \leq m$. We define

$$s := s_1 + \ldots + s_r.$$

Let f be a continuous map, mapping the disjoint union of the σ_i to a d-dimensional ball B^d, i.e.,

$$f : \sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d.$$

We define the **Gauss map** $\widetilde{f}$ associated to f

$$\widetilde{f} : \sigma_1 \times \cdots \times \sigma_r \to B^d \times \cdots \times B^d, \quad \text{by} \quad (x_1, ..., x_r) \mapsto (fx_1, ..., fx_r),$$

If, for each $i = 1, ..., r$,

$$f\sigma_1 \cap \cdots \cap f\partial\sigma_i \cap \cdots \cap f\sigma_r = \emptyset.$$

then $\widetilde{f}\partial(\sigma_1 \times \cdots \times \sigma_r) \subset B^d \times \cdots \times B^d$, avoids the **thin diagonal** $\delta_r(B^d) = \{(x, \ldots, x) \mid x \in B^d\}$ of B^d. Thus,

$$\partial(\sigma_1 \times \cdots \times \sigma_r) \to (B^d \times \cdots \times B^d) \setminus \delta_r(B^d). \tag{3}$$

Observe that $\partial(\sigma_1 \times \cdots \times \sigma_r) \cong S^{s-1}$, where $s := \sum_i s_i$, and $(B^d \times \cdots \times B^d) \setminus \delta_r(B^d)$ is homotopy equivalent to $S^{d(r-1)-1}$. Therefore, the map (3) defines an element

$$\alpha(f) \in \pi_{s-1}(S^{d(r-1)-1}),$$

which we call **intersection class of** f.

Lemma 11 (Generalized Weber-Whitney Trick)**.** *Let m, d, r be three positive integers satisfying* (2)*.*

Let $\sigma_1, \ldots, \sigma_r$ be balls of dimensions $s_1, \ldots, s_r \leq m$ properly contained in a d-dimensional ball B and with $\sigma_1 \cap \cdots \cap \sigma_r$ in the interior *of B.*

[10] The PL assumption is no loss of generality: if K is a finite simplicial complex and $f : K \to \mathbb{R}^d$ is an almost r-embedding then f can be slightly perturbed to a PL map with the same property.

1. *Let us denote by α the intersection class of the map $\sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d$.*

   If $\alpha = 0$, then there exists $(r-1)$ proper ambient isotopies of B that we can apply to $\sigma_1, \dots, \sigma_{r-1}$, respectively, to remove the r-intersection set; i.e., there exist $(r-1)$ proper isotopies $H_t^1, \dots, H_t^{r-1}$ of B throwing σ_i onto $\sigma_i' := H_1^i \sigma_i$ and such that

$$\sigma_1' \cap \cdots \cap \sigma_{r-1}' \cap \sigma_r = \emptyset.$$

2. *Let us assume that $\sigma_1 \cap \cdots \cap \sigma_r = \emptyset$ and $\sigma_2 \cap \cdots \cap \sigma_r \neq \emptyset$, and let $z \in \pi_s(S^{d(r-1)-1})$.*

   There exists J_t a proper ambient isotopy of B such that

   - $J_1 \sigma_1 \cap \sigma_2 \cap \cdots \cap \sigma_{r-1} \cap \sigma_r = \emptyset$,
   - *The intersection class of f is z, where*

$$f : (\sigma_1 \times I) \sqcup \sigma_2 \sqcup \cdots \sqcup \sigma_r \to B^d$$

   is defined as the inclusion on σ_i for $i \geq 2$, and for $(x,t) \in \sigma_1 \times I$, $f(x,t) = J_t(x)$.

Remark 12.

- The proof of Lemma 11 is the technical core of the paper and will be given in Section 4. For $r = 2$, Lemma 11 already appears in Section 4 of Weber's thesis [46]. Our contribution in the present paper is to show that the result holds for any $r \geq 3$.
- Roughly speaking, Part 1 of Lemma 11 means that if the intersection class vanishes, then one can solve the r-intersection set.

  Part 2 means that each element of $\pi_s(S^{d(r-1)-1})$ can be obtained by moving from a fixed solution to a new solution.

3 Proof of Theorem 2

Here, we show how to use Lemmas 9 and 11 to prove the main theorem. The inductive argument used in the proof mirrors that of Section 5 in Weber's thesis [46], where Theorem 2 is proven for $r = 2$.

Proof of Theorem 2. We are given $F : (K)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$, and we want to construct $f : K \to \mathbb{R}^d$ without global r-intersection points.

We start with a map $f : K \to \mathbb{R}^d$ in general position. Inductively, we will redefine f on the skeleta of K as to get the desired property. There are two levels in the induction. To describe these, let us fix a total ordering of the simplices of K that extends the partial ordering by dimension, *i.e.*,

$$K = \{\tau_1, \dots, \tau_N\}, \qquad \dim \tau_i \leq \dim \tau_{i+1} \text{ for } 1 \leq i \leq N-1.$$

First, we give a very informal plan of the "double induction" that we are going to use in the proof: we go over the list of simplices $\tau_1, ..., \tau_N$, and for each simplex τ_i we consider all the global r-intersection of τ_i with all the simplices *before* τ_i in the list. More precisely, we consider the list l_i of all r-tuple of pairwise disjoint simplices containing τ_i and simplices *before* τ_i in the list $\tau_1, ..., \tau_N$. For each r-tuple in l_i, we need to eliminate its global r-intersection points.

Therefore, once τ_i is fixed, we have a *new list* l_i. We are going to *order* l_i (by a notion of dimension), and then inductively scan over it and remove the global r-intersections points for each r-tuple in l_i.

For the first level of the inductive argument, it suffices to prove the following: Suppose we are given a map $f : K \to \mathbb{R}^d$ in general position with the following two properties:

1. Restricted to the subcomplex $L = \{\tau_1, \dots, \tau_{N-1}\}$ the map $f|_L$ does not have r-intersection between disjoint r-tuples of simplices;

2. $\widetilde{f}$ restricted to $(L)^r_\Delta$ is $\mathfrak{S}_r$-equivariantly homotopic to F, where $\widetilde{f}$ is the map defined in Lemma 1.

Then we can redefine f as to have these two properties on the whole of K. This is the first level of induction.

For the second level of the induction, let us define the **dimension** of a finite set of simplices as the sum of their individual dimensions. For the the purposes of this proof, we use the terminology k**-collection** for a set of cardinality k. Consider those $(r-1)$-collections t of simplices of L that, together with τ_N, form an r-collection of pairwise disjoint simplices. We fix a total ordering of these $(r-1)$-collections that extends the partial ordering given by dimension, *i.e.*, we list them as

$$t_1, \ldots, t_M,$$

with $\dim t_i \leq \dim t_{i+1}$ for $1 \leq i < M$. (Thus, each t_i is an $(r-1)$-collection of simplices of L, and t_i joined with τ_N is a r-collection of pairwise disjoint simplices.) Once again, inductively, it suffices to prove the following: Assuming that f has the two properties

1. For each $(r-1)$-collection t_i in the list $t_1, \ldots, t_{M-1}$, the map f does not have any r-intersection with preimages in the r-collection formed by adjoining τ_N to t_i.

2. the map $\widetilde{f}$ is $\mathfrak{S}_r$-equivariantly homotopic to F on the complex

$$(L)^r_\Delta \cup \bigcup_{i \leq M-1} [t_i \cup \{\tau_N\}] \subseteq (K)^r_\Delta,$$

   where the operator $[-]$ converts an unordered r-collection of pairwise disjoint simplices of K into the set of its corresponding cells[11] in $(K)^r_\Delta$.

Then we can modify f as to have these two properties on the list $t_1, \ldots, t_M$.

In order to do so, let us consider the r-collection $t_M \cup \{\tau_N\}$. We rename its elements as

$$t_M \cup \{\tau_N\} = \{\sigma_1, \ldots, \sigma_r\}, \quad (\text{with } \tau_N = \sigma_r).$$

By the induction hypothesis (namely the order on the τ_i and the t_i), for each $i = 1, \ldots, r$, $f^{-1}(f\sigma_1 \cap \cdots \cap f\sigma_r) \cap \sigma_i$ is contained in the *interior* of σ_i (since the induction has already "worked" on the simplices in $\partial\sigma_i$). Furthermore, the map $\widetilde{f} : \partial(\sigma_1 \times \cdots \times \sigma_r) \to S^{d(r-1)-1}$ is homotopic to F, this also follows from the ordering on the τ_i and the t_i (the homotopy is already defined on all the cells of $\partial(\sigma_1 \times \cdots \times \sigma_r)$).

We are in position to apply Lemma 9: we find a ball B^d in $\mathbb{R}^d$ with the three properties listed in the Lemma. Let us call σ'_i the sub-ball in σ_i properly embedded into B^d, *i.e.*, $\sigma'_i \overset{f}{\hookrightarrow} B^d$, and $f\partial\sigma'_i = \partial B^d \cap f\sigma'_i$.

By the Combinatorial Annulus Theorem [8, 3.10], there exists an isotopy of σ_i in itself that progressively retracts σ_i to σ'_i. *I.e.*, there exists $G^i_t : \sigma_i \to \sigma_i$ with G^i_0 being the identity and G^i_1 being an homeomorphism between σ_i and σ'_i. We define a homotopy by

$$\begin{array}{cccc} G: & \partial(I \times \sigma_1 \times \cdots \times \sigma_r) & \xrightarrow{fG^1 \times \cdots \times fG^r} & \mathbb{R}^d \times \cdots \times \mathbb{R}^d \setminus \delta_r\mathbb{R}^d \\ & (t, x_1, \ldots, x_r) & \longmapsto & (fG^1_t x_1, \ldots fG^r_t x_r). \end{array} \tag{4}$$

By the induction hypothesis,

$$\partial(\sigma_1 \times \cdots \times \sigma_r) \xrightarrow{f \times \cdots \times f} \mathbb{R}^d \times \cdots \times \mathbb{R}^d \setminus \delta_r\mathbb{R}^d \tag{5}$$

is homotopic to F, and F is defined over $\sigma_1 \times \cdots \times \sigma_r$. Therefore, the homotopy class of

$$\partial(\sigma'_1 \times \cdots \times \sigma'_r) \xrightarrow{f \times \cdots \times f} B^d \times \cdots \times B^d \setminus \delta_r B^d$$

[11] E.g., $[\{\alpha, \beta, \gamma\}] = \{\alpha \times \beta \times \gamma, \alpha \times \gamma \times \beta, \beta \times \alpha \times \gamma, \beta \times \gamma \times \alpha, \gamma \times \alpha \times \beta, \gamma \times \beta \times \alpha\}$.

is trivial. Hence, we can use the first part of the Lemma 11 to find $(r-1)$ proper ambient isotopies of B, say $H_t^1, \ldots, H_t^{r-1}$, such that $H_1^1(f\sigma_1') \cap \cdots \cap H_1^{r-1}(f\sigma_{r-1}') \cap f\sigma_r' = \emptyset$. This removes the r-intersection set.

To finish the induction, we also need to extend the equivariant homotopy between $\widetilde{f}$ and F on the cell $\sigma_1 \times \cdots \times \sigma_r$, as the homotopy is already defined on $\partial(\sigma_1 \times \cdots \times \sigma_r)$. This is when the second part of Lemma 11 becomes useful.

We define a map on $\partial(I \times \sigma_1 \times \cdots \times \sigma_r) \to \mathbb{R}^d \times \cdots \times \mathbb{R}^d \setminus \delta_r \mathbb{R}^d$ in the following way:

1. on $\{0\} \times \sigma_1 \times \cdots \times \sigma_r$, we use F,
2. on $[0, \frac{1}{3}] \times \partial(\sigma_1 \times \cdots \times \sigma_r)$, we use the homotopy from F to (5),
3. on $[\frac{1}{3}, \frac{2}{3}] \times \partial(\sigma_1 \times \cdots \times \sigma_r)$, we use G,
4. on $[\frac{2}{3}, 1] \times \partial(\sigma_1 \times \cdots \times \sigma_r)$, we use $(H_t^1 \times \cdots \times H_t^{r-1} \times \mathrm{id}) \circ (fG_1^1 \times \cdots \times fG_1^r)$,
5. $\{1\} \times \sigma_1 \times \cdots \times \sigma_r$, we use $(H_1^1 \times \cdots \times H_1^{r-1} \times \mathrm{id}) \circ (fG_1^1 \times \cdots \times fG_1^r)$.

This defines a class $\theta \in \pi_{\sum \dim \sigma_i}(S^{d(r-1)-1})$. To conclude, we need to have $\theta = 0$ (this is the condition to be able to extend to homotopy between $\widetilde{f}$ and F).

By the second part of Lemma 11, we can[12] performs a "second move" on σ_1 with an ambient isotopy J_t of B such that

$$\partial(I \times \sigma_1 \times \cdots \times \sigma_r) \xrightarrow{(J_t \times \mathrm{id} \times \cdots \times \mathrm{id}) \circ (H_1^1 \times \cdots \times H_1^{r-1} \times \mathrm{id}) \circ (fG_1^1 \times \cdots \times fG_1^r)} \mathbb{R}^d \times \cdots \times \mathbb{R}^d \setminus \delta_r \mathbb{R}^d$$

represents exactly $-\theta$. Therefore, by using this last move, we can assume that $\theta = 0$, *i.e.*, we can extend the equivariant homotopy between $\widetilde{f}$ and F, as needed for the induction. □

4 Proof of Lemma 11

Throughout this section, we assume that m, d, r are positive integers satisfying (2). Furthermore, we will denote the sum of the dimensions s_i of the balls σ_i by

$$s := s_1 + \ldots + s_r.$$

The proof of Lemma 11 is essentially inductive: we reduce from r balls to $(r-1)$ balls. The trick is to consider the intersection pattern of the first $(r-1)$ balls $\sigma_1, \ldots, \sigma_{r-1}$ on σ_r. If each of the intersections $\sigma_i \cap \sigma_r$, $1 \leq i \leq r-1$, were a ball properly embedded in σ_r, then we could solve the situation first at the level of σ_r (*i.e.*, remove the $(r-1)$-intersections between the $\sigma_i \cap \sigma_r$), and then extend the solution to B, thus completing the induction.

However, the intersections $\sigma_i \cap \sigma_r$ need not be balls, so our first task is to move $\sigma_1, \ldots, \sigma_{r-1}$ inside B as to modify their intersection with σ_r. As it will turn out, if we manage to increase sufficiently the connectedness of the intersections $\sigma_i \cap \sigma_r$, then Theorem 8 becomes useful to reduce the situation (as in the proof of Lemma 9) in such a way that the intersections $\sigma_i \cap \sigma_r$ do become balls. For this to work, $\sigma_i \cap \sigma_r$ needs to be $\dim(\sigma_1 \cap \cdots \cap \sigma_r)$-connected.

[12] We can always obtain the assumption $\sigma_2 \cap \cdots \cap \sigma_r \neq \emptyset$ by modifying the map f as follows [25, "Finger moves" in the proof of Lemma 43]: we pick $r-1$ spheres $S^{s_2}, \ldots, S^{s_r}$ in the interior of B^d of dimension $s_2, \ldots, s_r$ in general position and such that $S^{s_2} \cap \cdots \cap S^{s_r}$ is a sphere S. Then, for $i = 2, \ldots, r$, we pipe σ_i' to S^{s_i}. The resulting map has the desired property.

This "piping" change can be absorbed by a slight modification (and renumbering) of the H_t^i. The support of these modifications is a collection of regular neighborhoods of 1-polyhedra (= paths used for piping).

Also, note that the cases when, by general position, $\dim S < 0$ corresponds the trivial cases $\theta = 0$. Indeed, $\dim S < 0$ corresponds to $(d - s_2) + \cdots + (d - s_r) > d$, i.e., $(r-1)d + s_1 - d > \sum s_i$, and since $s_1 - d \leq -3$, we have $(r-1)d - 1 > \sum s_i$, and so $\pi_{\sum \dim \sigma_i}(S^{d(r-1)-1}) = 0$.

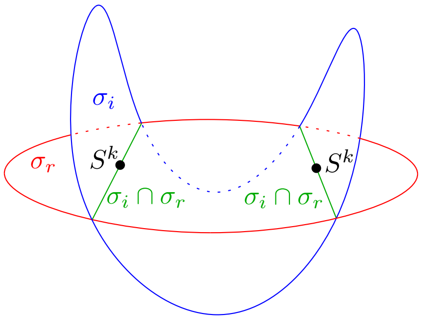

Figure 3: S^k represents a non-zero element of the homotopy group $\pi_k(\sigma_i \cap \sigma_r)$.

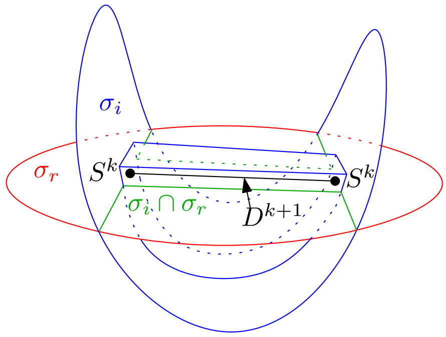

Figure 4: By moving a sub-ball of σ_i inside of B, we modify the intersection of σ_i and σ_r as to "kill" by surgery the homotopy class represented by $S^r \subseteq \sigma_i \cap \sigma_r$.

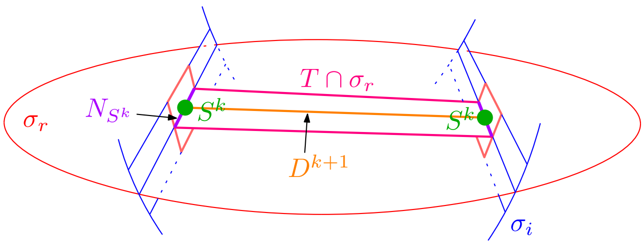

Figure 5: The different steps in the construction of the handle used for the ambient surgery.

4.1 Increasing the connectivity of the intersections

Proposition 13. *With the same notations as in Lemma 11, for each $i = 1, \ldots, r-1$, there exists a proper ambient isotopy H_t of B such that $H_1(\sigma_i) \cap \sigma_r$ is $\dim(\sigma_1 \cap \cdots \cap \sigma_r)$-connected, and such that*

$$I \times \partial(\sigma_1 \times \cdots \times \sigma_i \times \cdots \times \sigma_r) \xrightarrow{incl \times \cdots \times H_t \times \cdots \times incl} (B^d \times \cdots \times B^d) \setminus \delta_r(B^d) \tag{6}$$

is well-defined, i.e., its image is disjoint from the diagonal $\delta_r(B^d)$.

Proof. Proposition 13 follows directly by inductively using the Lemma 14 (below), as in [30, Lemma 2]. □

Lemma 14. *(a) With the same notation as above, for all $1 \leq k \leq \dim(\sigma_i \cap \cdots \cap \sigma_r)$ and $S^k \to \sigma_i \cap \sigma_r$ representing a homotopy class in $\pi_k(\sigma_i \cap \sigma_r)$, there exists a proper ambient isotopy H_t of B such that, for $j < k$,*

$$\pi_j(H_1(\sigma_i) \cap \sigma_r) \cong \pi_j(\sigma_i \cap \sigma_r),$$

and

$$\pi_k(H_1(\sigma_i) \cap \sigma_r) \cong \pi_k(\sigma_i \cap \sigma_r)/\text{a subgroup containing } [S^k].$$

(b) An analoguous statement holds for $k = 0$: If $\sigma_i \cap \sigma_r$ has more than one connected component, then there exists a proper ambient isotopy H_t of B such that $H_1(\sigma_i) \cap \sigma_r$ has one less connected component.

In both cases (a) and (b) with have the following additional property of H_t: the map (6) defined using H_t avoids the diagonal.

Here, we only present the proof of the part (a), i.e., for $k \geq 1$. For $k = 0$, the construction is similar, and is aready presented in [25, Sec. 3.2] as *piping and unpiping*.

Our main technique in the proof is to use *surgery* (as presented by Milnor [30]) to increase the connectivity of $\sigma_i \cap \sigma_r$. The precise definition of surgery used in our situation is given later (Definition 23).

Figure 3 illustrates the situation, and Figure 4 tries to illustrate how we intend to 'kill' a homotopy class of $S^k \in \pi_l(\sigma_i \cap \sigma_r)$ by surgery.

Remark 15. We decompose the proof of Lemma 14 into a series of Lemmas.

For the first two Lemmas, we need to use a PL analogous of vector bundles for smooth manifold. In the PL category, this analogous notion is called *block bundles*. See [8] for a rapid introduction, or the original [35]. In the present paper, we need to use results from [35, 36].

Since we only work in the PL category, we sometimes only say *bundle* instead of *block bundle*.

First we render, once and for all, the intersections transverse:

Lemma 16. *With the same notations as in Lemma 11, we can assume that σ_r is unknotted in B^d, i.e.,*

$$B^d = \sigma_r \times [-1, 1]^{d-s_r},$$

and we can also assume that σ_i intersects σ_r transversely in the sense of [36], i.e., for $\varepsilon > 0$ small enough, $\sigma_r \times \varepsilon[-1, 1]^{d-s_r}$ is a normal block bundle to σ_r in B^d, and we have

$$\sigma_i \cap (\sigma_r \times \varepsilon[-1, 1]^{d-s_r}) = (\sigma_i \cap \sigma_r) \times \varepsilon[-1, 1]^{d-s_r}.$$

Proof. The first statement follows from Zeeman's Unknotting of balls. The second statement follows by [36, Theorem 1.1 (a)]: there exists an ε-isotopy of B carrying σ_i locally transverse to σ_r. Using a collar on ∂B, we can furthermore assume that this isotopy is fixed on ∂B. □

Remark 17. In the sequence of lemmas that follows, $S^k \to \sigma_i \cap \sigma_r$ represents an homotopy class in $\pi_k(\sigma_i \cap \sigma_r)$, which we want to "kill".

Lemma 18. *In the situation given by Lemma 16, let $a : S^k \to \sigma_i \cap \sigma_r$ represents an homotopy class in $\pi_k(\sigma_i \cap \sigma_r)$. Then there exists an embedded copy of $S^k \subset \sigma_i \cap \sigma_r$ such that its inclusion map is homotopic to a, and with the two additional properties:*

(1) the normal block bundle of $S^k \subset \sigma_i \cap \sigma_r$ is trivial.

(2) Let N_{S^k} be a regular neighborhood of S^k inside $\sigma_i \cap \sigma_r$. Then

$$N_{S^k} \cong S^k \times [-1, 1]^{s_i+s_r-d-k},$$

containing S^k as $S^k \times 0$.

Proof. The existence of the embedded copy of S^k follows by general position[13].

The first property follows from [36]: By the previous Lemma, the normal bundle of $\sigma_i \cap \sigma_r$ in σ_i is trivial. Hence its tangent bundle is stably trivial [36, Corollary 5.6], i.e., $\sigma_i \cap \sigma_r$ is a π-manifold, in the sense of Milnor [30]. To complete the proof, we simply recast in the PL category the proof of Theorem 2 from Milnor's paper.

Using again [36, Corollary 5.6], the Whitney sum

$$(\text{normal bundle of } S^k \text{ in } \sigma_i \cap \sigma_r) \oplus (\text{tangent bundle of } S^k)$$

[13]Indeed, this is the case if

$$k \leq s - (r-1)d < \frac{s_i + s_r - d}{2},$$

and, after rearrangement and using $s_i \leq m$, we get the sufficient condition

$$2(r-1)m < (2r-3)d, \quad \text{which is the case since} \quad \frac{2(r-1)}{2r-3} m \leq \frac{r+1}{r} m < d,$$

where the first inequality is true for $r \geq 3$, and the second follows from the metastable range.

is equal to the tangent bundle of $\sigma_i \cap \sigma_r$ restricted to S^k, which is stably trivial. Hence there exists $j \geq 0$ such that

$$\text{(normal bundle of } S^k \text{ in } \sigma_i \cap \sigma_r) \oplus \text{(tangent bundle of } S^k) \oplus \varepsilon^j$$

is the trivial bundle over S^k. Since

$$\text{(tangent bundle of } S^k) \oplus \varepsilon^1$$

is trivial, the normal bundle of S^k in $\sigma_i \cap \sigma_r$ is stably trivial. Finally, by [36, Corollaries 5.2 & 5.3], this normal bundle must already be trivial.

The second property about N_{S^k} follows by the correspondence between regular neighborhoods and normal block bundles [35, Theorems 4.3 & 4.4]. □

Lemma 19. *In the situation given by Lemma 18, with $S^k \subset \sigma_i \cap \sigma_r$ and the two additional properties. There exists a ball D^{k+1} in σ_r with*

$$D^{k+1} \cap \sigma_i = \partial D^{k+1} = S^k,$$

and which avoids the other σ_j.

Furthermore, the trivialisation of N_{S^k} can be extended to D^{k+1}, i.e., there exists in σ_r

$$N_{D^{k+1}} \cong D^{k+1} \times [-1,1]^{s_i+s_r-d-k} \tag{7}$$

containing D^{k+1} as $D^{k+1} \times 0$ and with

$$N_{D^{k+1}} \cap \sigma_i = N_{S^k} \cong S^k \times [-1,1]^{s_i+s_r-d-k},$$

and this last homeomorphism is the restriction of (7).

Remark 20. For proving the second part of Lemma 19, we could use the PL-analogue of Stiefel manifolds [36, p. 274]: the obstruction to extending the trivialisation of S^k is always trivial in the metastable range. But, to avoid entering more deeply into the theory of block bundles, we rather use the following unknotting theorem of Hudson:

Theorem 21 ([22, Unknotting Theorem Moving the Boundary, 10.2, p. 199])**.** *If $f, g : M^m \to Q^q$ are proper PL embeddings between manifolds M and Q. Then f, g homotopic as maps of pairs $(M, \partial M) \to (Q, \partial Q)$ implies that f, g are ambient isotopic provided that*

- *M is compact*
- *$q - m \geq 3$*
- *$(M, \partial M)$ is $(2m - q + 1)$-connected*
- *$(Q, \partial Q)$ is $(2m - q + 2)$-connected*

Proof of Lemma 19. The first statement follows by general position and the metastable range hypothesis.

We use Theorem 21 to prove the existence of $N_{D^{k+1}}$.

First, we take a regular neighborhood V of D^{k+1} in σ_r. We can assume that

$$V \cap \sigma_i = N_{S^k} \cong S^k \times [-1,1]^{s_i+s_r-d-k}.$$

If N_{S^k} unknots in V in the sense of Theorem 21, then the existence of $N_{D^{k+1}}$ is immediate: we use an "standard" version of N_{S^k} to construct $N_{D^{k+1}}$, and move it to our situation by the isotopy given by the unknotting theorem.

So we are left with checking the connectivity hypothesis of Theorem 21. Trivially, $(V, \partial V)$ is sufficiently connected. So we only need to analyse the connectivity of the pair

$$(S^k \times [-1,1]^{s_i+s_r-d-k}, S^k \times S^{s_i+s_r-d-k-1})$$

which we need to be $(2(s_i + s_r - d) - s_r + 1)$-connected. Let us consider the exact sequence in homotopy for this pair

$$\cdots \to \pi_i(\underbrace{S^k \times S^{s_i+s_r-d-k-1}}_{:=\partial X}) \to \pi_i(\underbrace{S^k \times [-1,1]^{s_i+s_r-d-k}}_{:=X}) \to \pi_i(X, \partial X) \to \cdots$$

Since $s_i + s_n - d - k > k$ (i.e., $s_i + s_r - d > 2k$ that we already used to assume S^k embedded in $\sigma_i \cap \sigma_r$), the above sequence can be rewritten for $i < s_i + s_r - d - k$ as

$$\cdots \to \pi_i(S^k) \to \pi_i(S^k) \to \pi_i(X, \partial X) \to \cdots$$

Since the $\pi_i(S^k) \to \pi_i(S^k)$ is an isomorphism, we get $\pi_i(X, \partial X) = 0$ as long as $i < s_i + s_r - d - k$. So we are left with checking

$$s_i + s_r - d - k - 1 > 2s_i + s_r - 2d + 1$$

which reduces to $d - s_i - 2 > k$, which is true if $d - s_i - 2 > s - (r-1)d$, and this is implied by $rd \geq (r+1)m + 3$, i.e., the metastable range hypothesis. □

Lemma 22 (Existence of the surgery-handle)**.** *In the situation given by Lemma 19, there exists in B a handle*

$$T := D^{k+1} \times [-1,1]^{s_i+s_r-d-k} \times [-1,1]^{d-s_r}$$

such that

- *T contains D^{k+1} as $D^{k+1} \times 0$,*
- *T intersects σ_r as $D^{k+1} \times [-1,1]^{s_i+s_r-d-k} \times 0$,*
- *T intersects σ_i as $S^k \times [-1,1]^{s_i+s_r-d-k} \times [-1,1]^{d-s_r}$.*

Figure 5 illustrates the handle T.

Proof. This follows from the construction of D_{k+1} (Lemma 19) and the transversality of the intersection of σ_i and σ_r (Lemma 16). □

Definition 23 (Ambient surgery)**.** Let S^k be an embedded sphere in (the interior of) σ_i with a trivialized regular neighborhood $S^k \times [-1,1]^{s_i-k}$, and let T is a **handle based on** S^k, i.e.,

$$T = D^{k+1} \times [-1,1]^{s_i-k} \subseteq B^d$$

for a ball D^{k+1} with

$$T \cap \sigma_i = \partial D^{k+1} \times [-1,1]^{s_i-k} = S^{k+1} \times [-1,1]^{s_i-k}.$$

Using T, we perform a **ambient surgery** on $\sigma_i \subset B^d$ by constructing the new manifold

$$\sigma_i^* := (\sigma_i \setminus (S^k \times [-1,1]^{s_i-k})) \cup (D^{k+1} \times \partial[-1,1]^{s_i-k}) \subset B^d \tag{8}$$

In order to attach the handle T we made choices on σ_i: the choice of S^k, its regular neighborhood $S^k \times [-1,1]^{s_i-k}$, the 'core' D^{k+1}, etc. In the next Lemma, we show that, up to isotopy, there is only one way to attach a handle T to σ_i:

Lemma 24. *If S^k and $\widetilde{S}^k$ are embedded spheres in σ_i with a trivialized regular neighborhoods and handles T and $\widetilde{T}$ as in Definition 23.*

Then performing a surgery on σ_i using T or $\widetilde{T}$ produces two homeomorphic manifolds σ_i^* and $\widetilde{\sigma}_i^*$ that are connected by a proper ambient isotopy of B^d.

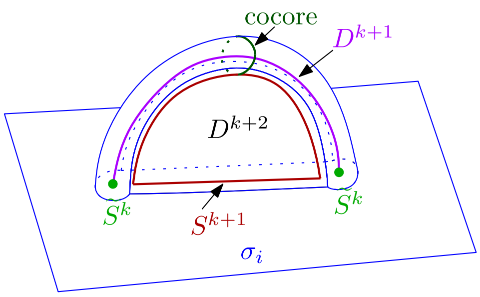

Figure 6: We perform two complementary surgeries on σ_i such that the resulting manifold σ_i' is a ball homeomorphic to σ_i.

Proof. By Irwin's Theorem [47, Ch. VIII, Theorem 24], there exists a proper isotopy of σ_i "throwing" $\widetilde{S}^k$ onto S^k, so we can assume $S^k = \widetilde{S}^k$ (since this isotopy can be extended to B^d [21]).

Then, by the uniqueness of regular neighborhoods, we can assume that the trivialisation of the normal block bundle are identical [35, Theorem 4.4].

We have reached the situation where $T \cap \sigma_i = \widetilde{T} \cap \sigma_i$.

Let us take a cone C in B^d over $D^{k+1} \cup \widetilde{D}^{k+1}$. By general position[14], this cone avoids σ_i (except on $S^k = \widetilde{S}^k$). Let us take a regular neighborhood V of the collapsible space C in B^d *relative* to $S^k = \widetilde{S}^k$. Hence, V is a d-ball, and

$$V \cap \sigma_i = S^k, \quad [23, \text{p. } 719, \text{(iii)}].$$

Inside of V, we can find an ambient isotopy (fixed on the boundary) 'throwing' $\widetilde{D}^{k+1}$ to D^{k+1}, hence, we can assume that $D^{k+1} = \widetilde{D}^{k+1}$.

We have reached the situation where both T and $\widetilde{T}$ are equal on σ_i and have the same 'core' $D^{k+1} = \widetilde{D}^{k+1}$.

To conclude, let us take a regular neighborhood N of D^{k+1}. We can assume that

- $N \cap \sigma_i = S^k \times [-1,1]^{s_r - k} = T \cap \sigma_i = \widetilde{T} \cap \sigma_i$, and
- $T, \widetilde{T} \subseteq N$ (after, possibly, shrinking the handles).

We have that

$$\sigma_i^* \cap N = D^{k+1} \times [-1,1]^{s_i - k} \cong \widetilde{\sigma}_i^* \cap N$$

and

$$\partial(\sigma_i^* \cap N) = \partial(\widetilde{\sigma}_i^* \cap N) = S^k \times \partial[-1,1]^{s_i - k}.$$

Hence, to conclude, we only have to check that $D^{k+1} \times \partial[-1,1]^{s_i-k}$ unknots inside of N (keeping the boundary fixed). First, we observe that two proper maps $D^{k+1} \times S^{s_i-k-1} \to B^d$ that are equal on the boundary are always homotopic (by a straight-line homotopy). Hence, by Irwin's Theorem [47, Ch. VIII, Theorem 24], we only need to check

$$2s_i - d + 1 \leq s_i - k + 1, \quad \text{i.e.,} \quad k + 3 \leq d - s_i$$

which is true if $rm - (r-1)d + 3 \leq d - s_i$, and this is implied by $(r+1)m + 3 \leq rd$ (the metastable range hypothesis). ☐

[14] We have to check, e.g., $(k+2) + s_r - d < 0$, i.e., $k < d - s_r - 2$, which is true if $s - (r-1)d < d - s_r - 2$, and this is implied by the metastable hypothesis $(r+1)m + 3 \leq rd$.

Lemma 25 (Existence of a complementary handle)**.** *Let S^k and T be as in Definition 23. Then there exists an handle (see Figure 6)*

$$T^c = D^{k+2} \times [-1,1]^{s_i-k-1} \subseteq B^d$$

with

- $T^c \cap \sigma_i^* = S^{k+1} \times [-1,1]^{s_i-k-1}$ *for a $(k+1)$-sphere S^{k+1} such that*
- S^{k+1} *intersects the cocore of T*

$$0^{k+1} \times \partial[-1,1]^{s_r-k} \subseteq \partial T$$

  at exactly one point.

- *Furthermore, we can assume that S^{k+1} is at positive distance of σ_r.*

Proof. By the previous lemma, there exists, up to proper isotopy of B^d, an unique way to to perform a surgery by the handle T on σ_i. From this fact, the existence of the complementary handle T^c is immediate.

For the last property, we need to shift S^{k+1} to general position[15]. □

Remark 26. We call T^c the 'complementary handle' to T.

Lemma 27. *Let $\alpha \in \pi(\sigma_i \cap \sigma_r)$. Then there exists a sphere $S^k \subset \sigma_i \cap \sigma_r$ and a handle T as in Defintion 23 such that performing a surgery on σ_i by the handle T, followed by a surgery by the handle T^c produces a manifold σ_i^{**} which is a s_i-ball. Furthermore for $j < k$*

$$\pi_j(\sigma_i^{**} \cap \sigma_r) \cong \pi_j(\sigma_i \cap \sigma_r)$$

and

$$\pi_k(\sigma_i^{**} \cap \sigma_r) \cong \pi_k(\sigma_i \cap \sigma_r)/\textit{a subgroup containing } \alpha.$$

Proof. The existence of S^k is given by Lemma 18. The existence of T is given by Lemma 22. The existence of T^c is given by Lemma 27.

To conclude, one notices

- By the first surgery using the handle T, we have 'killed' the homotopy class $\alpha = [S^k] \in \pi_k(\sigma_i \cap \sigma_r)$, i.e., by construction,

$$\sigma_i^* \cap \sigma_r = ((\sigma_i \cap \sigma_r) \setminus (S^k \times [-1,1]^{s_i+s_r-d-k} \times 0)) \cup (D^{k+1} \times \partial[-1,1]^{s_i+s_r-d-k} \times 0)$$

  and so we have killed $[S^k]$ in the sense of [30, Lemma 2].

- The effect of two surgeries by complementary handles cancels, hence σ_i^{**} is a s_i-ball [37, Lemma 6.4]. □

Proof of Lemma 14. One combines Lemma 27 with Zeeman's Unknotting of balls. □

4.2 Proof for balls

Proposition 28. *The first part of Lemma 11 is true if we add the following hypothesis: for each $i = 1, \dots, r-1$,*

$$\sigma_i \cap \sigma_r \textit{ is a } (s_i + s_r - d)\textit{-ball properly contained in } \sigma_r.$$

Before proving Proposition 28, we need two Definitions and two Lemmas.

[15]We want $(k+1) + (s_i + s_r - d) - s_i < 0$, i.e., $k < d - s_r - 1$. But $k \le s - (r-1)d$, so we only need $s - (r-1)d < d - s_r - 1$, i.e., $s + s_r + 1 < rd$, and this is true in the metastable range $(r+1)m + 3 \le rd$.

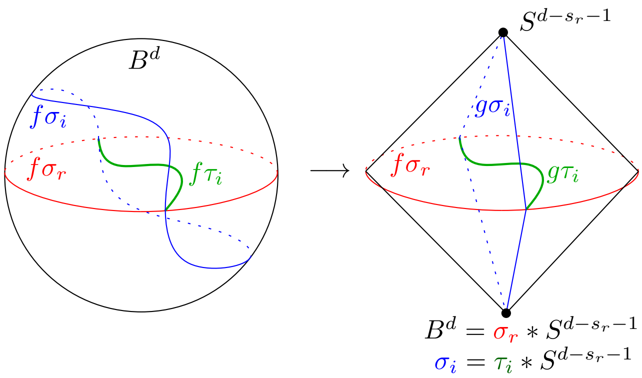

Figure 7: Using that σ_r unknots in B^d and that $\sigma_i \cap \sigma_r$ unknots in σ_r, we change the setting to a suspension over σ_r.

Definition 29. Let $g\colon \sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d$ be balls properly mapped inside B^d, with the dimensional restriction of Lemma 11.

We say that g is a **suspended map** if it has the following structure

- $g\sigma_r$ is an embedded and unknotted ball inside B^d, hence we can assume that

$$B^d = (g\sigma_r) * S^{d-s_r-1},$$

  for some S^{d-s_r-1}.

- For $i = 1, ..., r-1$, the preimage by $g|_{\sigma_i}$ of $g\sigma_r \subset B^d$ is a ball properly embedded and unknotted inside σ_i. I.e.,

$$\sigma_i = g|_{\sigma_i}^{-1}(g\sigma_r) * S^{d-s_r-1},$$

  for some S^{d-s_r-1}.

  Notation: $\tau_i := g|_{\sigma_i}^{-1}(g\sigma_r) \subset \sigma_i$.

- For $i = 1, ..., r-1$, g is defined as follows:
  - the sphere $S^{d-s_r-1} \subset \sigma_i$ is mapped homeomorphically to $S^{d-s_r-1} \subset B^d$,
  - the ball $\tau_i \subset \sigma_i$ is properly map to $g\sigma_r$.
  - g is defined elsewhere on σ_i by interpolating in the obvious way between the two joins

$$\sigma_i = \tau_i * S^{d-s_r-1} \quad \text{and} \quad B^d = (g\sigma_r) * S^{d-s_r-1}.$$

Figure 7 shows on the right a suspended map.

Lemma 30 (Suspended maps, Figure 7)**.** *Let $f\colon \sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d$ be balls properly embedded inside B^d in general position, with the dimensional restriction of Lemma 11, and with the additional hypothesis of Proposition 28, i.e., for each $i = 1, \ldots, r-1$, $\sigma_i \cap \sigma_r$ is a $(s_i + s_r - d)$-ball properly contained in σ_r.*

Then there exists a suspended map $g\colon \sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d$, such that

- *the intersection classes of f and g are equal*
- $f|_{\sigma_r} = g|_{\sigma_r}$
- *For $i = 1, ..., r-1$, we have $f|_{\sigma_i}^{-1}(\sigma_r) = g|_{\sigma_i}^{-1}(\sigma_r) =: \tau_i$.*
- $f|_{\tau_i} = g|_{\tau_i}$.

Proof. To simplify notation, during the proof we assume that f is an inclusion map, i.e., $\sigma_i \subset B^d$.

The existence of g will follow from the facts that

- σ_r unknots in B^d,
- $\sigma_i \cap \sigma_r$ unknots inside of σ_r,
- the modifications applied during the unknotting on $\sigma_1, \ldots, \sigma_{r-1}$ do not change the homotopy class that we are interested into.

More precisely, since $\sigma_i \cap \sigma_r$ unknots inside of σ_r, we can represent σ_r as

$$\sigma_r = (\sigma_i \cap \sigma_r) * S^{d-s_i-1}, \quad \text{and so} \quad B^d = (\sigma_i \cap \sigma_r) * S^{d-s_i-1} * S^{d-s_r-1}$$

Hence, we define a retraction from

$$B^d \setminus (\emptyset * S^{d-s_i-1} * \emptyset) \quad \text{onto} \quad (\sigma_i \cap \sigma_r) * \emptyset * S^{d-s_r-1},$$

and, using this retraction on σ_i, we can assume that $\sigma_i \subseteq (\sigma_i \cap \sigma_r) * S^{d-s_r-1}$.

If B^{d-s_i} is the "standard ball" in σ_r with boundary S^{d-s_i-1}, then $\sigma_i \cap \sigma_r$ intersects this ball precisely once, and this translates into the fact that $\partial\sigma_i$ is a generator of the homotopy group $\pi_{s_i-1}(\partial(\sigma_i \cap \sigma_r) * S^{d-s_r-1}) \cong \mathbb{Z}$.

Hence, we can assume that $\sigma_i = (\sigma_i \cap \sigma_r) * S^{d-s_r-1}$, after an homotopy of σ_i inside of $(\sigma_i \cap \sigma_r) * S^{d-s_r-1}$ (keeping $\partial\sigma_i$ on ∂B^d). □

Lemma 31 (Commuting Square for Suspended Maps)**.** *Let $f\colon \sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d$ be a suspended map. Then there exists a diagram commuting up to homotopy*

$$\begin{array}{ccc} \partial(\sigma_1 \times \cdots \times \sigma_r) & \xrightarrow{\simeq} & \Sigma^{d(r-1)-s_r(r-2)}\partial(\tau_1 \times \cdots \times \tau_{r-1}) \\ \downarrow & & \downarrow \\ B \times \cdots \times B \setminus \delta_r(B) & \xrightarrow{\simeq} & \Sigma^{d(r-1)-s_r(r-2)}((\sigma_r \times \cdots \times \sigma_r) \setminus \delta_{r-1}(\sigma_r)) \end{array} \qquad (9)$$

where

- *the map on the left is the obvious one, representing an element $\alpha \in \pi_{s-1}(S^{d(r-1)-1})$,*
- *the map on the right is the suspension Σ applied $(d(r-1) - s_r(r-2))$ times to the map*

  $$\partial(\tau_1 \times \cdots \times \tau_{r-1}) \to (\sigma_r \times \cdots \times \sigma_r) \setminus \delta_{r-1}(\sigma_r),$$

  and this map represents an element

  $$\beta \in \pi_{(s_1+s_r-d)+\cdots+(s_{r-1}+s_r-d)-1}(S^{s_r(r-2)-1}).$$

- *The two horizontal maps are defined within the proof.*

We defer the proof of Lemma 31 to Section 4.2.2.

Proof of Proposition 28. We apply Lemma 30, to get a suspended map $f\colon \sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d$ with the same intersection class as our initial map.

From Diagram (9) in Lemma 31

$$\Sigma^{d(r-1)-s_r(r-2)}\beta = \alpha.$$

But $\alpha = 0$, and we are in the stable range of the suspension homomorphism[16], hence $\beta = 0$. Therefore, using the third property in Lemma 30, we have reduced the problem to that of removing the $(r-1)$-intersection set between

$$\sigma_1 \cap \sigma_r, \dots, \sigma_{r-1} \cap \sigma_r \subseteq \sigma_r,$$

which are $(r-1)$ balls embedded in σ_r in the metastable range for $r-1$.

Thus, we are in position to work inductively: since σ_r unknots in B^d, we have $B^d = \sigma_r * S^{d-s_r-1}$, so proper ambient isotopies of σ_r can be extended to B^d.

The beginning of the induction (for $r = 3$) reduces to the classical case of two balls intersecting inside a third ball, and is solved in Weber [46, Prop. 1 & 2]. □

We are left with proving Lemma 31, this is what the rest of this section is devoted to. Before starting the proof (that will be split into three Lemmas in Section 4.2.2), we introduce another kind of configuration space that will be useful for us during that proof.

4.2.1 Deleted Joins

Let K be a simplicial complex. We define the **k-fold k-wise *topological* deleted join** of K

$$K^{*r} \setminus \delta_r^* K := K * \cdots * K \setminus \left\{ \frac{1}{r}x + \cdots + \frac{1}{r}x \;\middle|\; x \in K \right\},$$

and the **k-fold k-wise *simplicial* deleted join** of K

$$(K)^{*r}_{\delta} := \{\tau_1 * \cdots * \tau_r \mid \tau_i \in K \text{ and } \tau_1 \cap \cdots \cap \tau_r = \emptyset\}.$$

Both spaces $K^{*r} \setminus \delta_r K$ and $(K)^{*r}_{\delta}$ have a natural $\mathfrak{S}_r$-action by permutation of the coordinates.

Lemma 32. *$K^{*r} \setminus \delta_r^* K$ can be $\mathfrak{S}_r$-equivariantly retracted onto $(K)^{*r}_{\delta}$.*

Proof. Our proof is modelled on the deleted *product* case [20, Lemma 10.1].
Warm up. We first show the main trick on a very simple case. *I.e.*, assuming Δ is the simplex on two vertices $\{x, y\}$, we construct an homeomorphism

$$\Delta * \Delta \cong (\Delta)^{*2}_{\delta} * \delta_2^*(\Delta), \quad \text{(see Figure 8)}.$$

Once we have this homeomorphism the conclusion is immediate.

First, we name the four vertices of $\Delta * \Delta$ as $\{x, y, x', y'\}$ (with $\{x, y\} \in \Delta * \emptyset$ and $\{x', y'\} \in \emptyset * \Delta$). Then every point of $\Delta * \Delta$ is represented as

$$x = ax + by + a'x' + b'y' \quad \text{with } a, a', b, b' \in [0, 1],$$

Assuming that $a \geq a'$, $b \geq b'$ and that a' or b' is non-zero, we rewrite x as

$$x = (a - a' + b - b') \underbrace{\left(\frac{a - a'}{a - a' + b - b'} x + \frac{b - b'}{a - a' + b - b'} y \right)}_{\in (\Delta)^{*2}_{\delta}} +$$

$$(2a' + 2b') \underbrace{\left(\frac{a'}{2a' + 2b'} (x + x') + \frac{b'}{2a' + 2b'} (y + y') \right)}_{\in \delta_2^*(\Delta)}.$$

[16]The suspension $\pi_i(S^n) \to \pi_{i+l}(S^{n+l})$ is an isomorphism if $i < 2n-1$ [19, Corollary 4.24]. For us this translates into

$$s + (r-2)s_r - d(r-1) - 1 < 2(s_r(r-2) - 1) - 1, \textit{i.e.},$$

i.e.,

$$(s_1 - s_r) + \cdots + (s_{r-2} - s_r) + s_{r-1} + 2 < d(r-1),$$

which is trivially true if $m \leq d - 3$.

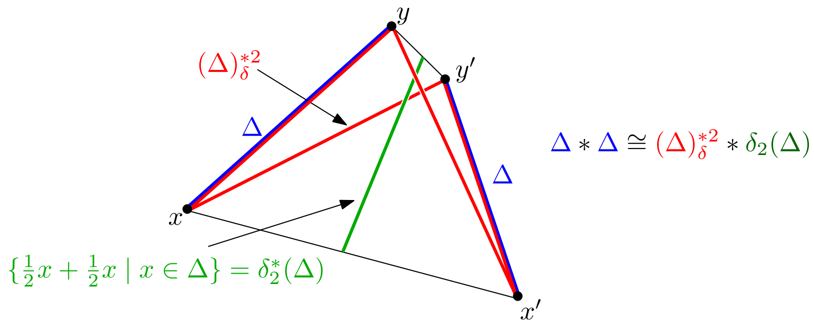

Figure 8: The k-fold k-wise topological deleted join can be retracted to the k-fold k-wise simplicial deleted join.

The other possible orders on a, b, a', b' can be worked on in a similar way, and will correspond to other faces of $(\Delta)^{*2}_\delta$.

The general case. Let K be a simplicial complex. We can write any simplex of K^{*r} as

$$\underbrace{(\Delta^1 * \omega^1)}_{\in K} * \cdots * \underbrace{(\Delta^r * \omega^r)}_{\in K} \quad \text{for some simplices } \Delta^i, \omega^i \in K,$$

with the condition

$$\Delta^1 = \cdots = \Delta^r \quad \text{and} \quad \omega^1 \cap \cdots \cap \omega^r = \emptyset.$$

Our goal is to build an homeomorphism

$$(\Delta^1 * \omega^1) * \cdots * (\Delta^r * \omega^r) \cong (\Delta)^{*r}_\delta * \delta^*_r(\Delta) * (\omega^1 * \cdots * \omega^r).$$

where Δ is any of the Δ^i. Once we have this homeomorphism the conclusion is immediate.

Let us name p^i_j the vertices spanning Δ^i, and q^i_j the vertices spanning ω_i. Then, any $x \in (\Delta^1 * \omega^1) * \cdots * (\Delta^r * \omega^r)$ can be written as

$$x = \sum_{i,j} p^i_j(x) p^i_j + \sum_{i,j} q^i_j(x) q^i_j, \quad \text{with } p^i_j(x), q^i_j(x) \geq 0 \text{ and } \sum_{i,j} p^i_j(x) + \sum_{i_j} q^i_j(x) = 1.$$

We assume that at least one of the $q^i_j(x)$ is non-zero (otherwise nothing has to be done). Then, we write x as

$$x = \sum p^i_j(x) \left(\frac{1}{\sum p^i_j(x)} \sum p^i_j(x) p^i_j \right) + \sum q^i_j(x) \left(\frac{1}{\sum q^i_j(x)} \sum q^i_j(x) q^i_j \right)$$

The first term lies in $\Delta^1 * \cdots * \Delta^r$, and the second in $\omega^1 * \cdots * \omega^r$. To further decompose the first term, we name p_j the minimum of $\{p^1_j(x), \ldots, p^r_j(x)\}$, then

$$\sum_{i,j} p^i_j(x) p^i_j = \sum_{i,j} (p^i_j(x) - p_j) p^i_j + \sum_i (p_1 p^i_1 + \cdots + p_r p^i_r)$$

Hence, we can write $\sum_{i,j} p^i_j(x) p^i_j$ as a point in the join of $(\Delta)^{*r}_\delta$ and $\delta^*_r \Delta$. □

4.2.2 Proof of Lemma 31

We split the proof of Lemma 31 in three steps.

Lemma 33 (A First square)**.** *Let $\sigma_1, \ldots, \sigma_r$ be balls properly mapped to B^d by $f\colon \sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d$, with the dimensional restrictions of Lemma 11.*

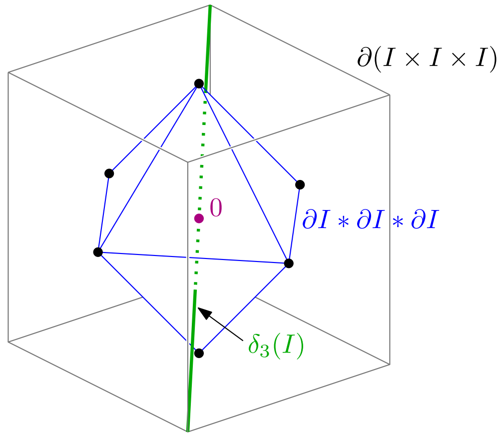

Figure 9: For $r = 3$ and $d = 1$: the cube $I \times I \times I$ contains the join $\partial I * \partial I * \partial I$.

Then the diagram

$$\begin{array}{ccc} \partial(\sigma_1 \times \cdots \times \sigma_r) & \xrightarrow{\cong} & \partial\sigma_1 * \cdots * \partial\sigma_r \\ \downarrow & & \downarrow \\ \partial(B \times \cdots \times B) \setminus \delta_r(B) & \xrightarrow{\cong} & \partial B * \cdots * \partial B \setminus \delta_r(B) \end{array} \tag{10}$$

commutes up to homotopy.

The map on the left is defined as before[17]*. The map on the right maps*

$$\emptyset * \cdots * \partial\sigma_i * \cdots * \emptyset \to \emptyset * \cdots * \partial B * \cdots * \emptyset$$

and extends linearly. The two horizontal homeomorphisms are obtained in the following way: we represent B^d as $I^d = [-1,1]^d$, then $\partial B * \cdots * \partial B$ can be formed inside of the cube $B \times \cdots \times B$, i.e.,

$$\partial B * \cdots * \partial B \subseteq B \times \cdots \times B \quad \textit{(Figure 9)}$$

and by radial projection from the center of the cube, we get that $\partial(B \times \cdots \times B)$ is homeomorphic with $\partial B * \cdots * \partial B$. This defines the bottom horizontal arrow, and the same construction work with the top horizontal arrow.

Proof. The top left-to-right arrow is defined as (where $|.|$ is the infinity-norm)

$$(x_1, \ldots, x_r) \mapsto \frac{|x_1|}{\sum |x_i|} \underbrace{\frac{x_1}{|x_1|}}_{\in \partial\sigma_1} \oplus \cdots \oplus \frac{|x_r|}{\sum |x_i|} \underbrace{\frac{x_r}{|x_r|}}_{\in \partial\sigma_r} \subset \partial\sigma_1 * \cdots * \partial\sigma_r$$

[17]It is easy to see that $\partial(\sigma_1 \times \cdots \times \sigma_r)$ maps into

$$\partial(B \times \cdots \times B) \setminus \delta_r(B) \subseteq B \times \cdots \times B \setminus \delta_r(B)$$

since the σ_i are *properly* mapped in B^d. Also, if B is represented as a cube $I^d = [-1,1]^d$, then

$$B \times \cdots \times B \setminus \{(0, \ldots, 0)\}$$

can be retracted onto its boundary, and this defines a retraction from

$$B \times \cdots \times B \setminus \delta_r(B) \quad \text{to} \quad \partial(B \times \cdots \times B) \setminus \delta_r(B).$$

with the convention that, if $|x_i| = 0$, then $\frac{x_i}{|x_i|}$ is undefined (but since its coefficient in the join is 0, this is not a problem). The inverse application divides a point $p \in \partial\sigma_1 * \cdots * \partial\sigma_r \subset \sigma_1 \times \cdots \times \sigma_r$ by its $|.|$-norm (as a point in $\sigma_1 \times \cdots \times \sigma_r$) to project the point on the boundary $\partial(\sigma_1 \times \cdots \times \sigma_r)$.

Starting from the top-left corner of Diagram (10), we follow the directions: right, down, left. We obtain a map $\partial(\sigma_1 \times \cdots \times \sigma_r) \to \partial(B \times \cdots \times B) \setminus \delta_r(B)$ defined as

$$(x_1, \ldots, x_r) \mapsto \frac{\left(\frac{|x_1|}{\sum |x_i|} f \frac{x_1}{|x_1|}, \cdots, \frac{|x_r|}{\sum |x_i|} f \frac{x_r}{|x_r|}\right)}{\left|\left(\frac{|x_1|}{\sum |x_i|} f \frac{x_1}{|x_1|}, \cdots, \frac{|x_r|}{\sum |x_i|} f \frac{x_r}{|x_r|}\right)\right|}. \tag{11}$$

To conclude, we must show that (11) is homotopic to $(x_1, \ldots, x_r) \mapsto (fx_1, \ldots, fx_r)$.

Let us assume, without loss of generality, that $x_1 \in \partial\sigma_1$. Then, $|x_1| = 1$, hence $\left|\frac{|x_1|}{\sum |x_i|} f \frac{x_1}{|x_1|}\right| = \frac{|fx_1|}{\sum |x_i|} = \frac{1}{\sum |x_i|}$. Therefore, the denominator in (11) must be $\frac{1}{\sum |x_i|}$, and so (11) becomes

$$\left(|x_1| f \frac{x_1}{|x_1|}, ..., |x_r| f \frac{x_r}{|x_r|}\right) \tag{12}$$

which is homotopic to $(x_1, \ldots, x_r) \mapsto (fx_1, \ldots, fx_r)$ by a straight-line homotopy. Indeed, by contradiction, let us assume that for a given $(x_1, ..., x_r) \in \partial(\sigma_1 \times \cdots \times \sigma_r)$ and a given $t \in (0, 1)$, the straight-line homotopy intersects the diagonal $\delta_r(B)$. Without loss of generality, $x_1 \in \partial\sigma_1$. But then, we must have $fx_2, ..., fx_r \in \partial B$, which implies that $x_2 \in \partial\sigma_2$, ..., $x_r \in \partial\sigma_r$. But (12) is the identify on such an r-tuple $(x_1, ..., x_r)$, so it cannot intersect the diagonal. □

Remark 34.
- We define $L := d(r-1) - s_r(r-2)$ to shorten the exponent in $\Sigma^{d(r-1)-s_r(r-2)}$.
- Recall that for g a suspended map (Definition 29), we define $\tau_i := g|_{\sigma_i}^{-1}\sigma_r$.

Lemma 35 (Second square.). *Let $\sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d$ be a suspended map. Then the diagram*

$$\begin{array}{ccc} \partial\sigma_1 * \cdots * \partial\sigma_r & \xrightarrow{\cong} & \Sigma^L(\partial\tau_1 * \cdots * \partial\tau_{r-1}) \\ \downarrow & & \downarrow \\ \partial B * \cdots * \partial B \setminus \delta_r(B) & \hookleftarrow & \Sigma^L(\partial\sigma_r * \cdots * \partial\sigma_r \setminus \delta_{r-1}(\sigma_r)) \end{array} \tag{13}$$

commutes. The map on the left is the obvious one, and the map on the right is the $(d(r-1) - s_r(r-2))$-suspension of the map

$$\partial\tau_1 * \cdots * \partial\tau_{r-1} \to \partial\sigma_r * \cdots * \partial\sigma_r \setminus \delta_{r-1}(\sigma_r).$$

The horizontal maps are obtained as rearrangements using

$$B^d = \sigma_r * S^{d-s_r-1} \quad \text{and} \quad \sigma_i = \tau_i * S^{d-s_r-1}, \text{ for } i \neq r.$$

More precisely, the top-horizontal homeomorphism is obtained in the following way

$$\begin{aligned} \partial\sigma_1 * \cdots * \partial\sigma_r &= \underbrace{\left(\partial\tau_1 * S^{d-s_r-1}\right)}_{\partial\sigma_1} * \cdots * \underbrace{\left(\partial\tau_{r-1} * S^{d-s_r-1}\right)}_{\partial\sigma_r} * \partial\sigma_r \\ &\cong \underbrace{\left(\partial\sigma_r * S^{(r-1)(d-s_r)-1}\right)}_{S^{(r-1)d-s_r(r-2)-1}} * \partial\tau_1 * \cdots * \partial\tau_{r-1}, \end{aligned}$$

and this last expression is the suspension applied $((r-1)d - s_r(r-2))$-times on $\partial\tau_1 * \cdots * \partial\tau_{r-1}$.

The bottom horizontal inclusion is derived, in a very similar fashion, as

$$\begin{aligned} &S^{(r-1)d-s_r(r-2)-1} * (\partial\sigma_r * \cdots * \partial\sigma_r) \\ &\qquad \cong \left(\partial\sigma_r * S^{d-s_r-1}\right) * \cdots * \left(\partial\sigma_r * S^{d-s_r-1}\right) * \left(S^{s_r-1} * \emptyset\right) \\ &\qquad \cong \partial B * \cdots * \partial B * (\partial\sigma_r * \emptyset) \\ &\qquad \subseteq \partial B * \cdots * \partial B * \partial B. \end{aligned} \tag{14}$$

Proof. It follows easily that (13) commutes. We show in the next step that the bottom horizontal inclusion is an homotopy equivalence, using Diagram (15). □

Lemma 36 (Third square)**.** *Let $\sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d$ be a suspended map. Then the diagram*

$$\begin{array}{ccc} \partial B * \cdots * \partial B \setminus \delta_r(B) & \hookleftarrow & \Sigma^L(\partial\sigma_r * \cdots * \partial\sigma_r \setminus \delta_{r-1}(\sigma_r)) \\ \simeq \big\uparrow & & \simeq \big\uparrow \\ (\partial B)^{*r}_{\delta} & \xleftarrow{\ \simeq\ } & \Sigma^L(\partial\sigma_r)^{*r-1}_{\delta} \end{array} \tag{15}$$

commutes, and the three arrows with the symbol '$\simeq$' are homotopy equivalences.

Proof. Here, $(-)^{*k}_{\delta}$ is the *k-fold k-wise (simplicial) deleted join.* The definition is given in Section 4.2.1, where we also prove that both left and right vertical arrows are homotopy equivalences (Lemma 32). Hence, we are left with the bottom-horizontal map.

We are going to use the two following facts that are easy to check:

1. for any simplicial complexes $L_1, \ldots, L_n$, $(L_1 * \cdots * L_n)^{*k}_{\delta} \cong (L_1)^{*k}_{\delta} * \cdots * (L_n)^{*k}_{\delta}$,
2. $(\partial I)^{*k}_{\delta}$ collapses simplicially onto $\partial I * \cdots * \partial I * \emptyset * \partial I * \cdots * \partial I$ (i.e., the k-join of ∂I where one of the factor is replaced by $\emptyset$).

Therefore, if we represent σ_r as I^{s_r}, we have

$$(\partial\sigma_r)^{*r-1}_{\delta} \cong (\partial I * \cdots * \partial I)^{*r-1}_{\delta} = (\partial I)^{*r-1}_{\delta} * \cdots * (\partial I)^{*r-1}_{\delta}.$$

We collapse each of the $(\partial I)^{*r-1}_{\delta}$ to $\partial I * \cdots * \partial I * \emptyset$, hence $(\partial\sigma_r)^{*r-1}_{\delta}$ collapses to $\partial\sigma_r * \cdots * \partial\sigma_r * \emptyset$. The suspension of this space in $(\partial B)^{*r}_{\delta}$ is, by equation (14),

$$\big(\partial\sigma_r * S^{d-s_r-1}\big) * \cdots * \big(\partial\sigma_r * S^{d-s_r-1}\big) * \big(\emptyset * S^{d-s_r-1}\big) * (S^{s_r-1} * \emptyset),$$

We can collapse $(\partial B)^{*r}_{\delta}$ onto this last space. Indeed,

$$(\partial B)^{*r}_{\delta} = (\partial\sigma_r * S^{d-s_r-1})^{*r}_{\delta} = (\partial\sigma_r)^{*r}_{\delta} * (S^{d-s_r-1})^{*r}_{\delta}.$$

The first term $(\partial\sigma_r)^{*r}_{\delta}$ can be factors into terms $(\partial I)^{*r}_{\delta}$, that we all collapse onto $\partial I * \cdots * \partial I * \emptyset * \partial I$. For the second term $(S^{d-s_r-1})^{*r}_{\delta}$, we collapse onto $\partial I * \cdots * \partial I * \emptyset$.

Then, since both $(\partial B)^{*r}_{\delta}$ and $\Sigma^L(\partial\sigma_r)^{*r-1}_{\delta}$ can be collapsed onto the same sub-sphere, it follows that the bottom inclusion in (15) is an homotopy equivalence. □

Proof of Lemma 31. Combining the all the previous Lemmas in this section, we get the following commuting diagram

$$\begin{array}{ccc} \partial(\sigma_1 \times \cdots \times \sigma_r) & \xrightarrow{\ \simeq\ } & \Sigma^L(\partial\tau_1 * \cdots * \partial\tau_{r-1}) \\ \big\downarrow & & \big\downarrow \\ \partial(B \times \cdots \times B) \setminus \delta_r(B) & \xrightarrow{\ \simeq\ } & \Sigma^L(\partial\sigma_r * \cdots * \partial\sigma_r \setminus \delta_{r-1}(\sigma_r)) \end{array}$$

Reusing the first square (10) on

$$\partial\tau_1, \cdots, \partial\tau_{r-1},$$

we form

$$\begin{array}{ccc} \Sigma^L(\partial\tau_1 * \cdots * \partial\tau_{r-1}) & \xrightarrow{\ \simeq\ } & \Sigma^L(\partial\tau_1 \times \cdots \times \partial\tau_{r-1}) \\ \big\downarrow & & \big\downarrow \\ \Sigma^L(\partial\sigma_r * \cdots * \partial\sigma_r \setminus \delta_{r-1}(\sigma_r)) & \xrightarrow{\ \simeq\ } & \Sigma^L(\partial(\sigma_r \times \cdots \times \sigma_r) \setminus \delta_{r-1}(\sigma_r)) \end{array}$$

Combining the last two diagrams, we get the diagram (9), as wanted. □

4.3 The complete proof

Proof of Lemma 11. **First Part.** We apply Proposition 13 to make each of the $\sigma_i \cap \sigma_r$ $(s - d(r - 1))$-connected. Then there exists, by Theorem 8, for each $i = 1, \ldots, r-1$, a collapsible subspace C_i of $\sigma_i \cap \sigma_r$ of dimension at most $s - d(r-1) + 1$ such that $C_i \supseteq \sigma_1 \cap \cdots \cap \sigma_r$.

In σ_r there exists a collapsible space C of dimension at most $s - d(r-1) + 2$ containing $C_1, \ldots, C_r$. Furthermore, by general position, C intersects $\sigma_i \cap \sigma_r$ only on C_i. We take a regular neighbourhood N of C in B^d. By construction, N intersects $\sigma_i \cap \sigma_r$ in a regular neighbourhood of C_i, which must be a ball (C_i is disjoint for the other σ_j for $j \neq i, r$ by general position). Hence, "retracting" from B^d to the ball N (as we did in the proof of Theorem 2, equation 4), we are reduced to the situation of Proposition 28, which we can then directly apply.

Second Part. For $r = 2$, the result already appeared in Weber [46, Proposition 3 & proof of Lemma 1]. *I.e.*, if σ^s and τ^t are two balls properly contained in B^d in the metastable range ($m \geq s, t$ with $d \geq \frac{3}{2}m + 3$) and without intersection. Then for every $\alpha \in \pi_{s+t}(S^{d-1})$ there exists a proper isotopy J_t of B such that $J_1\sigma \cap \tau = \emptyset$, and the homotopy class defined by

$$\partial(I \times \sigma \times \tau) \xrightarrow{J_t \mathrm{incl}_\sigma \times \mathrm{incl}_\tau} B^d \times B^d \setminus \delta_2 B^d$$

represents α (after identifications).

Hence, we can work inductively: we assume that the part 2 of the Lemma is already true for $(r-1)$ balls, and we show how construct the isotopy $J_t : B^d \to B^d$ for r balls.

Let $\sigma_1, ..., \sigma_r$ be the r balls properly contained in B^d as in the hypothesis of part 2 of the Lemma. In particular,

$$\sigma_1 \cap \cdots \cap \sigma_r = \emptyset \quad \text{and} \quad \sigma_2 \cap \cdots \cap \sigma_r \neq \emptyset,$$

and we can assume that σ_r is unknotted in B^d, i.e., $B^d = \sigma_r * S^{d-s_r-1}$.

Claim 37. *We can assume that for $i = 1, ..., (r-1)$, $\sigma_i \cap \sigma_r$ are balls properly contained inside σ_r. Furthermore, we can assume that $\sigma_2 \cap \cdots \cap \sigma_r$ is also a ball properly contained in σ_r.*

Proof. Let us pick $x \in \sigma_1 \cap \sigma_r$ and $y \in \sigma_2 \cap \cdots \cap \sigma_r$, that we join by a path $\lambda \subseteq \sigma_r$ in general position. We take a regular neighborhood of λ in B^d, and restrict ourselves to this neighborood. □

By the induction hypothesis applied on

$$\sigma_1 \cap \sigma_r, ..., \sigma_{r-1} \cap \sigma_r \subseteq \sigma_r,$$

for every homotopy class $\alpha \in \pi_{s+(r-2)s_r-(r-1)d+1}(S^{(r-2)s_r-1})$, there exists an isotopy J_t of σ_r such that J_t applied to the ball $\sigma_1 \cap \sigma_r \subseteq \sigma_r$ represents α.

The isotopy J_t can be extended to B^d (we still denote it by J_t), hence this isotopy applied to the ball $\sigma_1 \subseteq B^d$ represents an homotopy class $\beta \in \pi_s(S^{d(r-1)-1})$. We are done if we can show that β is a suspension of α (we are in the stable range of the suspension isomorphism).

The problem is similar to Lemma 31. Indeed we have r balls

$$J(\sigma_1 \times [-1, 1]), \sigma_2, ..., \sigma_r$$

that are mapped into B^d, and we would like to form a diagram as in (9) with the ball '$\sigma_1 \times [-1, 1]$' instead of σ_1.

Hence, to conclude, we only need to prove a version of the Suspended Map Lemma 30 for our present situation.

Note that $\sigma_1 \times [-1, 1]$ is not embedded inside B^d, and is not even properly mapped (the boundary is not mapped to the boundary).

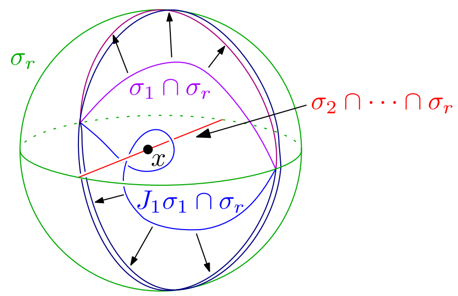

Figure 10: Retraction of the sphere $(B_1 \cap \sigma_r) \cup (B_2 \cap \sigma_r)$ on the boundary of σ_r.

Claim 38. *Let $\widetilde{\sigma}_1$ be a (s_1+1)-ball mapped into B^d with its boundary mapped as follows: $\partial\widetilde{\sigma}_1$ is decomposed into two balls B_1 and B_2 such that B_1 is mapped onto $J_0\sigma_1$ and B_2 is mapped onto $J_1\sigma_1$ (and $\partial B_1 = \partial B_2$ mapped onto $J_0\partial\sigma_1 = J_1\partial\sigma_1$), then*

$$J(\sigma_1 \times [-1,1]) \times \sigma_2 \times \cdots \times \sigma_r \quad \text{and} \quad \widetilde{\sigma}_1 \times \sigma_2 \times \cdots \times \sigma_r$$

define the same element $\beta \in \pi_s(S^{d(r-1)-1})$.

Proof. This is immediate by using a straight line homotopy between $J(\sigma_1 \times [-1,1])$ and $\widetilde{\sigma}_1$ □

So we are reduced with working with a (s_1+1)-ball $\widetilde{\sigma}_1$ instead of $J(\sigma_1 \times [-1,1])$, and the way that we 'fill' this ball does not matter (only the boundary decides of the homotopy class β).

We can decompose $\partial\widetilde{\sigma}_1$ as two balls B_1 and B_2 both homeomorphic with $(\sigma_1 \cap \sigma_r) * S^{d-s_r-1}$, and with $B_1 \cap B_2 \simeq \partial(\sigma_1 \cap \sigma_r) * S^{d-s-r}$.

Claim 39. *We can assume that B_1 is mapped to $(\sigma_1 \cap \sigma_r) * S^{d-s_r-1}$ and that B_2 is mapped to $(J_1\sigma_1 \cap \sigma_r) * S^{d-s_r-1}$.*

Proof. This follows by an argument identical to that of Lemma 30 (we work with the two balls seperately). □

Claim 40. *We can assume that $B_1 = (\sigma_1 \cap \sigma_r) * S^{d-s_r-1}$ and $B_2 = (J_1\sigma_1 \cap \sigma_r) * S^{d-s_r-1}$ are mapped onto the boundary of B^d, and that $\widetilde{\sigma}_1 = (\widetilde{\sigma}_1 \cap \sigma_r) * S^{d-s_r-1}$.*

Proof. Figure 10 illustrate the construction inside σ_r.

We pick a point x in the interior of the ball $\sigma_2 \cap \cdots \cap \sigma_r$. Since this ball unknots in σ_r, there exists a retraction r_t of $\sigma_r \setminus x$ to $\partial\sigma_r$ such that $r_1^{-1}\ \partial(\sigma_2 \cap \cdots \cap \sigma_r) = \sigma_2 \cap \cdots \cap \sigma_r$.

Using that $B^d = \sigma_r * S^{d-s_r-1}$, we extend r_t to B_d (which now retract $B^d \setminus x$ to ∂B^d). We can then use r_t to conclude. (Note that B_1 and B_2 stop to be embedded, but this is not a problem for us). □

We can now apply Lemma 31 to the balls $\widetilde{\sigma}_1, \sigma_2, ..., \sigma_r$, and thus conclude. □

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