Gravity's Rainbow and Compact Stars

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A number of recent studies has focused on the implications of new physics at the Planck scale on the equilibrium of compact astrophysical objects such as white dwarf and neutron stars. Here we analyze the modification of the equilibrium configurations induced by the so-called Gravity's Rainbow that account for Planck scale deformation of the space-time.

1. Introduction

Compact stars, exotic stars, wormholes and black holes are astrophysical objects described by the Einstein's Field equations. For a perfect fluid and in case of spherical symmetry, these objects obey the Tolman-Oppenheimer-Volkoff (TOV) equation (in c.g.s. units)[1,2]

$$\frac{dp_r(r)}{dr} = -\left(\rho(r) + \frac{p_r(r)}{c^2}\right)\frac{4\pi G r^3 p_r(r)/c^2 + Gm(r)}{r^2\left[1 - 2Gm(r)/rc^2\right]} + \frac{2}{r}\left(p_t(r) - p_r(r)\right) \tag{1}$$

and

$$\frac{dm}{dr} = 4\pi\rho(r)\, r^2, \tag{2}$$

where c is the velocity of light, G is the gravitational constant, $\rho(r)$ is the macroscopic energy density measured in proper coordinates, $p_r(r)$ and $p_t(r)$ are respectively the radial pressure and the transverse pressure and $m(r)$ is an arbitrary function of the radial coordinate, r. The function $m(r)$ is the quasi-local mass, and is denoted as the mass function. It is clear that the knowledge of $\rho(r)$ allows to understand the astrophysical structure under examination. If we fix our attention on compact stars, ordinary General Relativity offers two kind of exact solutions for the isotropic TOV equation:

a) the constant energy density solution,

b) the Misner-Zapolsky energy density solution[3]

or the combination of a) and b), namely the Dev-Gleiser energy density profile[4]. Since compact stars are usually macroscopic objects, the Quantum Gravity contribution is expected to become important when the inner core of the star is considered, where the highest pressures and densities are reached. An attempt to include quantum gravitational effects in compact stars, besides those that are consequences of the standard Fermi degeneracy pressure, can be found in [5], where Planck scale modifications of the energy/momentum dispersion relations have been taken into the account, and in [6], where the TOV equation and the equation of state of zero temperature ultra-relativistic Fermi gas based on generalized uncertainty principle (GUP) have been used to see the quantum gravitational effects on the cores of compact stars. Gravity's Rainbow offers another opportunity to probe quantum gravitational effects into the core of a compact star. For simplicity we will fix our attention only on the isotropic case.

2. Gravity's Rainbow and the Equation of State

Basically, Gravity's Rainbow is a distortion of space-time induced by two arbitrary functions, $g_1(E/E_{\rm Pl})$ and $g_2(E/E_{\rm Pl})$, which have the following property

$$\lim_{E/E_{\rm Pl}\to 0} g_1(E/E_{\rm Pl}) = 1 \qquad \text{and} \qquad \lim_{E/E_{\rm Pl}\to 0} g_2(E/E_{\rm Pl}) = 1. \tag{3}$$

It has been introduced for the first time by Magueijo and Smolin [7], who proposed that the energy-momentum tensor and the Einstein's Field Equations were modified with the introduction of a one parameter family of equations[a]

$$G_{\mu\nu}(E/E_{\rm Pl}) = 8\pi G(E/E_{\rm Pl})\, T_{\mu\nu}(E/E_{\rm Pl}) + g_{\mu\nu}\Lambda(E/E_{\rm Pl}), \tag{4}$$

where $G(E/E_{\rm Pl})$ is an energy dependent Newton's constant and $\Lambda(E/E_{\rm Pl})$ is an energy dependent cosmological constant, defined so that $G(0)$ is the low-energy Newton's constant and $\Lambda(0)$ is the low-energy cosmological constant. For instance, the *rainbow* version of the Schwarzschild line element is

$$ds^2 = -\left(1 - \frac{2MG(0)}{r}\right)\frac{d\tilde{t}^2}{g_1^2(E/E_{\rm Pl})} + \frac{d\tilde{r}^2}{\left(1 - \frac{2MG(0)}{r}\right) g_2^2(E/E_{\rm Pl})} + \frac{\tilde{r}^2}{g_2^2(E/E_{\rm Pl})} d\Omega^2, \tag{5}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element of the unit sphere. It is immediate to generalize the metric (5) for any spherically symmetric spacetime

$$ds^2 = -\frac{e^{2\Phi(r)}}{g_1^2(E/E_{\rm Pl})}c^2 dt^2 + \frac{dr^2}{g_2^2(E/E_{\rm Pl})\left(1 - \frac{2Gm(r)}{rc^2}\right)} + \frac{r^2}{g_2^2(E/E_{\rm Pl})} d\Omega^2, \tag{6}$$

where $m(r)$ is the mass of the star inside the radius r and $\Phi(r)$ is the redshift function. Of course, the line element (6) has consequences on Eq.(1). To see what are these consequences, we consider the energy-momentum stress tensor describing a perfect-fluid of the form

$$T_{\mu\nu} = \left(\rho(r)c^2 + p_t\right) u_\mu u_\nu + p_t g_{\mu\nu} + (p_r - p_t)\, n_\mu n_\nu, \tag{7}$$

[a] Applications and implications of Gravity's Rainbow in Astrophysics and cosmology can be found in [8]

where u^μ is the four-velocity normalized in such a way that $g_{\mu\nu}u^\mu u^\nu = -1$, n_μ is the unit spacelike vector in the radial direction, i.e. $g_{\mu\nu}n^\mu n^\nu = 1$ with $n^\mu = \sqrt{1-2Gm(r)/rc^2}\delta^\mu_r$. $\rho(r)$ is the energy density, $p_r(r)$ is the radial pressure measured in the direction of n^μ, and $p_t(r)$ is the transverse pressure measured in the orthogonal direction to n^μ. Because of Gravity's Rainbow, the normalization of u^μ is modified and becomes

$$-1 = -\frac{e^{2\Phi(r)}}{g_1^2(E/E_{\rm Pl})}u^0u^0 \to \quad u^0 = g_1(E/E_{\rm Pl})e^{-\Phi(r)}, \tag{8}$$

and for n^μ, one gets

$$1 = \frac{n^1n^1}{g_2^2(E/E_{\rm Pl})\left(1-2Gm(r)/rc^2\right)} \to \quad n^1 = g_2(E/E_{\rm Pl})\sqrt{1-2Gm(r)/rc^2}. \tag{9}$$

Fixing our attention on the isotropic case, the Stress-Energy tensor becomes

$$\begin{array}{ll} T_{00} = \dfrac{\rho(r)c^2e^{2\Phi(r)}}{g_1^2(E/E_{\rm Pl})}c^2 & T_{11} = \dfrac{p(r)}{g_2^2(E/E_{\rm Pl})\left[1-2Gm(r)/rc^2\right]} \\ T_{22} = \dfrac{pr^2}{g_2^2(E/E_{\rm Pl})} & T_{33} = \dfrac{pr^2\sin^2\theta}{g_2^2(E/E_{\rm Pl})}. \end{array} \tag{10}$$

and the component of the Einstein tensor G_{00} reduces to

$$G_{00} = 2G\frac{e^{2\Phi(r)}}{r^2}\frac{g_2^2(E/E_{\rm Pl})}{g_1^2(E/E_{\rm Pl})}m'(r). \tag{11}$$

With the help of the first component of the Stress-Energy tensor (10) and Eq.(11), we can write the first Einstein's Field Equation, namely $G_{00} = \kappa T_{00}$ which assumes the form

$$m'(r) = \frac{\kappa\rho(r)r^2}{c^2g_2^2(E/E_{\rm Pl})}, \tag{12}$$

while for the second one, namely $G_{11} = \kappa T_{11}$, we get

$$\Phi'(r) = \frac{\kappa r^3p_r/g_2^2(E/E_{\rm Pl})c^4 + 2Gm(r)/c^2}{2r^2\left[1-2Gm(r)/rc^2\right]}. \tag{13}$$

It is important to say that the equilibrium equation

$$\frac{dp}{dr} + (\epsilon + p)\Phi'(r) = 0 \tag{14}$$

is not affected by Gravity's Rainbow. From Eq.(14), it follows that

$$\frac{dp_r}{dr} = -\left(\rho + \frac{p_r}{c^2}\right)\frac{\kappa r^3p_r/g_2^2(E/E_{\rm Pl})c^4 + 2Gm(r)/c^2}{2r^2\left[1-2Gm(r)/rc^2\right]}, \tag{15}$$

and

$$\frac{dm}{dr} = \frac{4\pi\rho(r)r^2}{g_2^2(E/E_{\rm Pl})}, \tag{16}$$

where ρ is the mass density. Eq.(15) represents the TOV equation modified by Gravity's Rainbow. We will fix our attention on the constant energy density case and to the variable case of the Misner-Zapolsky type.

2.1. *Isotropic pressure and the constant energy density case*

The constant energy density case, represents the simplest case to consider. With this assumption, equation (15) becomes

$$\frac{dp_r}{dr} = -\left(\rho + \frac{p_r(r)}{c^2}\right)\frac{4\pi G r^3 p_r(r)/c^2 g_2^2(E/E_{\rm Pl}) + Gm(r)}{r^2\left[1 - 2Gm(r)/rc^2\right]}, \tag{17}$$

while Eq.(16) can be easily solved to give

$$m(r) = \frac{4\pi\rho}{3g_2^2(E/E_{\rm Pl})} r^3, \tag{18}$$

where we have used the boundary condition $m(0) = 0$. It is important to observe that the mass density is constant in r, but it is not constant in E. It is also important to observe that Eqs.(17) and (18) work for the whole star included the external boundary R, where we can assume that the effects of Gravity's Rainbow have vanished. To this purpose, we analyze the problem into two fundamental regions[9]:

a) The boundary $R \ggg \alpha l_{\rm Pl}$, namely the boundary is very large compared to the size of the inner core.
b) The boundary $R \simeq \alpha l_{\rm Pl}$, that it means that we are exploring the possibility of the existence of stars of Planckian size. It is interesting to note that both cases respect the Buchdahl-Bondi bound which states that[10]

$$M < \frac{4}{9}\frac{c^2}{G}R. \tag{19}$$

The case *b*) can be interpreted as a star forming close to the Planck scale and stabilized by Gravity's Rainbow. This means that it is the distorted space-time which supports the existence of a star of Planckian size.

2.2. *Isotropic pressure and the variable energy density case*

The variable energy density case is represented by the Misner-Zapolsky solution[3]. To discuss the modification induced by Gravity's Rainbow, we consider a density energy profile of the following form

$$\rho = Ar^\alpha, \tag{20}$$

where A is a constant with dimensions of an energy density divided by a (length)$^\alpha$ with $\alpha \in \mathbb{R}$ to be determined. Solving Eq.(16) leads to

$$m(r) = \int_0^r \frac{4\pi A}{g_2^2(E/E_{\rm Pl})} r'^{2+\alpha} dr' = \frac{4\pi A}{g_2^2(E/E_{\rm Pl})\,(3+\alpha)} r^{3+\alpha}. \tag{21}$$

Plugging (21) into Eq.(15), one finds

$$\omega\frac{d\rho\left(r\right)}{dr}=-\rho\left(r\right)\left(\frac{c^{2}+\omega}{c^{2}}\right)\frac{4\pi Gr^{3}\omega\rho(r)+Gm(r)c^{2}g_{2}^{2}(E/E_{\rm Pl})}{r^{2}\left[1-2Gm(r)/rc^{2}\right]c^{2}g_{2}^{2}(E/E_{\rm Pl})}$$

$$\Downarrow \tag{22}$$

$$\alpha=-\left(\frac{c^{2}+\omega}{\omega c^{2}}\right)\frac{4\pi GAr^{2+\alpha}\left(\left(3+\alpha\right)\omega+c^{2}\right)}{\left[c^{2}g_{2}^{2}(E/E_{\rm Pl})\left(3+\alpha\right)-8\pi GAr^{2+\alpha}\right]}, \tag{23}$$

where we have used the following Equation of State $p_r(r)=\omega\rho\left(r\right)$. It is immediate to see that $\forall\alpha\neq-2$, there is a singularity into the TOV equation and a dependence on r still persists. Therefore fixing $\alpha=-2$ one gets the relationship

$$1=\frac{3\left(c^{2}+\omega\right)^{2}}{4\omega\left[7c^{2}g_{2}^{2}(E/E_{\rm Pl})-3\right]}, \tag{24}$$

where we have set $A=3c^{2}/\left(56\pi G\right)$. We find an identity when $\omega=1/3, \omega=3$, $c=1$ and $g_2(E/E_{\rm Pl})=1$, namely we get the ordinary GR solution of the undeformed TOV Equation. In particular for $\omega=1/3$

$$p_r=\omega\rho\left(r\right)=\omega\frac{3c^{2}}{56\pi Gr^{2}}=\frac{c^{2}}{56\pi Gr^{2}} \tag{25}$$

and

$$m(r)=\frac{3\pi c^{2}r}{14G}, \tag{26}$$

we reproduce the Misner-Zapolsky solution. On the other hand, when Gravity's Rainbow is switched on and $g_2(E/E_{\rm Pl})\neq1$, it is immediate to see that from Eq.(24) follows that ω is no longer a constant but it becomes a function of $E/E_{\rm Pl}$.

3. Summary and further comment

In this work, we have considered the possibility that a compact star is affected by Gravity's Rainbow. Since the action of Gravity's Rainbow is prevalently at Planckian length scales, we find that in case of isotropic pressure and constant energy density, a star of Planckian size if it is formed, and satisfies the usual Buchdahl-Bondi bound, is also stable. On the other hand, when the variable energy density case is considered and an equation of state is introduced, one finds that, from the relation $p_r=\omega\rho\left(r\right)$, ω becomes a function of $E/E_{\rm Pl}$, necessarily. It is interesting to note that the constant energy density and the Misner-Zapolsky energy density are two particular cases of the Dev-Gleiser potential which is of the form[4]

$$\rho\left(r\right)=\rho_{0}+\frac{A}{r^{2}}, \tag{27}$$

where ρ_0 is the parameter of the constant energy density case and $A=3c^{2}/\left(56\pi G\right)$. Note that in both cases, namely the constant and variable energy density case, also the mass becomes a function of $E/E_{\rm Pl}$. Here we have considered the simple case where $E/E_{\rm Pl}$ is not dependent on the radius r. Of course, other than introducing an anisotropy, the case in which $E/E_{\rm Pl}$ becomes $E\left(r\right)/E_{\rm Pl}$ will be a subject of a future investigation as well as the full examination of the Dev-Gleiser potential.

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