Probing Graph Proper Total Colorings With Additional Constrained Conditions

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Abstract

Graph colorings are becoming an increasingly useful family of mathematical models for a broad range of applications, such as *time tabling and scheduling*, *frequency assignment*, *register allocation*, *computer security* and so on. Graph proper total colorings with additional constrained conditions have been investigated intensively in the last decade year. In this article some new graph proper total colorings with additional constrained conditions are defined, and approximations to the chromatic numbers of these colorings are researched, as well as some graphs having these colorings have been verified.

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1 Introduction and concepts

A graph coloring/labelling is an assignments to vertices, edges or both by some certain requirements. The main reason is that graph colorings can divide a complex network into some smaller subnetworks such that each subnetwork has itself character differing from that of the rest subnetworks. Graph colorings/labellings have been applied in many areas of science and mathematics, such as in X-ray crystallographic analysis, to design communication network, in determining optimal circuit layouts and radio astronomy. In [10] the authors pointed out that graph theory provides important tools to capture various aspects of the network structure, and the analysis of such dynamical systems is facilitated by the development of some new tools in graph theory. It is interesting that the frequency assignment problem of communication networks is very similar with graph distinguishing colorings below. In the article [6], Burris and Schelp introduce that a proper edge-coloring of a simple graph G is called a *vertex distinguishing edge-coloring* (vdec) if for any two distinct vertices u and v of G, the set of the colors assigned to the edges incident to u differs from the set of the colors assigned to the edges incident to v. The minimum number of colors required for all vertex distinguishing colorings of G is denoted by $\chi'_s(G)$. Let $n_d = n_d(G)$ denote the number of all vertices of degree d in G. It is clear that $\binom{\chi'_s(G)}{d} \geq n_d$ for all d with respect to $\delta(G) \leq d \leq \Delta(G)$. Burris and Schelp [6] presented the following conjecture:

Conjecture 1. *Let G be a simple graph with no isolated edges and at most one isolated vertex, and let k be the smallest integer such that $\binom{k}{d} \geq n_d$ for all d with respect to $\delta(G) \leq d \leq \Delta(G)$. Then $k \leq \chi'_s(G) \leq k+1$.*

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A weak version of the vdec was introduced in [17], called the *adjacent vertex distinguishing edge coloring* (avdec). Zhang et al. [17] asked for every edge xy of G, the set of the colors assigned to the edges incident to x differs from the set of the colors assigned to the edges incident to y in an avdec, and use the notation $\chi'_{as}(G)$ to denote the least number of k colors required fir which G admits a k-avdec. They proposed: Every simple graph G having no isolated edges and at most one isolated vertex holds $\chi'_{as}(G) \leq \Delta(G) + 2$. Surprisingly, it is very difficult to settle down this conjecture, even for simple graphs (cf. [2]). In 2005, Zhang et al.[18] investigated the adjacent vertex distinguishing total coloring (avdtc) of graphs, and proposed a conjecture: $\chi''_{as}(G) \leq \Delta(G) + 2$, where $\chi''_{as}(G)$ is the smallest number of k colors for which G admits a k-avdtc. But, settling down these two conjectures is not a light work. Graph distinguishing colorings are investigated intensively within two decades years (cf. [7], [8], [9], [16]).

We use standard notation and terminology of graph theory. The shorthand symbol $[a, b]$ denotes an integer set $\{a, a+1, a+2, \ldots, b\}$ with integers $b > a \geq 1$. The set of vertices adjacent to a vertex u is denoted by $N(u)$, and the set of edges incident to the vertex u is denoted by $N_e(u)$. We call a graph G to be simple if the degree $\deg_G(u) = |N(u)|$ for every vertex $u \in V(G)$. Graphs mentioned here are simple, undirected and finite. Let f be a proper total k-coloring of a simple graph G. The colors of neighbors of the vertex u form the following color sets $C(f, u) = \{f(e) : e \in N_e(u)\} = \{f(ux) : x \in N(u)\}$, $C\langle f, u\rangle = \{f(x) : x \in N(u)\} \cup \{f(u)\}$, $C[f, u] = C(f, u) \cup \{f(u)\}$, and $N_2[f, u] = C(f, u) \cup C\langle f, u\rangle$. Notice that $\deg_G(u)+1 \leq |N_2[f, u]|$, where $\deg_G(u) = |N(u)|$ is the degree of the vertex u. These color sets gives rise to distinguishing total colorings of various types. So we have a set $A_{cc}(G)$ containing the following *additional constrained conditions*:

(C1) $C(f, u) \neq C(f, v)$ for distinct $u, v \in V(G)$;
(C2) $C(f, x) \neq C(f, y)$ for every edge $xy \in E(G)$;
(C3) $C\langle f, u\rangle \neq C\langle f, v\rangle$ for distinct $u, v \in V(G)$;
(C4) $C\langle f, x\rangle \neq C\langle f, y\rangle$ for every edge $xy \in E(G)$;
(C5) $C[f, u] \neq C[f, v]$ for distinct $u, v \in V(G)$;
(C6) $C[f, x] \neq C[f, y]$ for every edge $xy \in E(G)$;
(C7) $N_2[f, u] \neq N_2[f, v]$ for distinct $u, v \in V(G)$; and
(C8) $N_2[f, x] \neq N_2[f, y]$ for every edge $xy \in E(G)$.

We restate some known colorings again and define new colorings in Definition **??**.

Definition 1. Let f be a proper total k-coloring of a simple graph G having $n \geq 3$ vertices and no isolated edges as well as at most one isolated vertex. We have eight distinguishing total colorings with additional constrained conditions as follows:

Type 1: This total k-coloring f is called an *e-partially vertex distinguishing proper total k-coloring* (e-partially k-vdtc) if it holds (C1), and the smallest number of k colors required for which G admits an e-partially k-vdtc is denoted as $\chi''_{(s)}(G)$; this total coloring f is called an *e-partially adjacent vertex distinguishing proper total k-coloring* (e-partially k-avdtc) if it holds (C2), and the smallest number of k colors required for which G admits an e-partially k-avdtc is denoted as $\chi''_{(as)}(G)$.

Type 2: This total k-coloring f is called a *v-partially vertex distinguishing proper total k-coloring* (v-partially k-vdtc) if it holds (C3), and the smallest number of k colors required for which G admits a v-partially k-vdtc is denoted as $\chi''_{\langle s\rangle}(G)$; this total coloring f is called a

2

v-partially adjacent vertex distinguishing proper total k-coloring (v-partially k-avdtc) if it holds (C**4**), and the smallest number of k colors required for which G admits a v-partially k-avdtc is denoted as $\chi''_{\langle as\rangle}(G)$.

Type 3: This total k-coloring f is called a *vertex distinguishing proper total k-coloring* (k-vdtc) if it holds (C**5**), and the smallest number of k colors required for which G admits a k-vdtc is denoted as $\chi''_s(G)$; this total coloring f is called an *adjacent vertex distinguishing proper total k-coloring* (k-avdtc) if it holds (C**6**), and the smallest number of k colors required for which G admits a k-avdtc is denoted as $\chi''_{as}(G)$.

Type 4: This total k-coloring f is called a $\mu(k)$-*coloring* if it holds (C**7**), and the notation $\chi''_{2s}(G)$ stands for the least number of k colors required for which G admits a $\mu(k)$-coloring; this total coloring f is called a $\mu_e(k)$-*coloring* if it holds (C**8**), and the symbol $\chi''_{2as}(G)$ denotes the least number of k colors required for which G admits a $\mu_e(k)$-coloring. $\square$

Clearly, the degree $\deg_G(u)$ of a vertex u of a simple graph G holds: $\big|C(f,u)\big| = \deg_G(u)$, $2 \le \big|C\langle f,u\rangle\big| \le \deg_G(u)+1$, $\big|C[f,u]\big| = \deg_G(u)+1$ and $\deg_G(u)+1 \le \big|N_2[f,u]\big| \le 2\deg_G(u)+1$. Therefore, v-partially k-vdtcs, v-partially k-avdtcs, $\mu(k)$-colorings and $\mu_e(k)$-colorings may be complicated than e-partially k-vdtcs, e-partially k-avdtcs, k-vdtcs and k-avdtcs. In Figure 1(a) and (b), we can see $\chi''_{(s)}(G) < \chi''_{(s)}(H)$ although H is a proper subgraph of G.

Definition 2. Let f be a proper total coloring of a simple graph G having $n \ge 3$ vertices and no isolated edges as well as at most one isolated vertex. We call f an (8)-*distinguishing total coloring* if it holds each one of $A_{cc}(G)$. The minimum number of k colors required for which G admits an (8)-distinguishing total k-coloring is denoted as $\chi''_{(8)}(G)$. We call f a (6)-*distinguishing total coloring* if it holds each one of $A_{cc}(G) \setminus \{ (C\mathbf{3}), (C\mathbf{4})\}$. The minimum number of k colors required for which G admits a (6)-distinguishing total k-coloring is denoted as $\chi''_{(6)}(G)$. $\square$

The (8)-distinguishing total coloring has been discussed in [14] and [15]. Furthermore, we can define: A (4)-*avdtc* (resp. a (3)-avdtc) is a proper total coloring holding each one of $A_{cc}(G)\backslash\{(C\mathbf{1}),(C\mathbf{3}),(C\mathbf{5})(C\mathbf{7})\}$ (resp. $A_{cc}(G)\backslash\{ (C\mathbf{1}), (C\mathbf{3}), (C\mathbf{4}), (C\mathbf{5}) (C\mathbf{7})\}$). And $\chi''_{(4)as}(G)$ (resp. $\chi''_{(3)as}(G)$) is the minimum number of k colors required for which G admits a k-(4)-avdtc (resp. k-(3)-avdtc). A *mixed* (m)-*total coloring* with $m \in [2,7]$ is a proper total coloring holding m additional constrained conditions of $A_{cc}(G)$, herein.

Clearly, every graph admits (6)-distinguishing total k-colorings and (3)-avdtcs. In Figure 1(c), all color sets of the graph H are listed in the following Table-1.

Table-1:

m	$C(f,u_i)$	$C\langle f,u_i\rangle$	$C[f,u_i]$	$N_2[f,u_i]$
u_1	$\{1,5\}$	$\{3,4,5\}$	$\{1,4,5\}$	$\{1,3,4,5\}$
u_2	$\{1,2\}$	$\{1,4,5\}$	$\{1,2,5\}$	$\{1,2,4,5\}$
u_3	$\{2,3\}$	$\{1,2,5\}$	$\{1,2,3\}$	$\{1,2,3,5\}$
u_4	$\{3,4\}$	$\{1,2,3\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
u_5	$\{4,5\}$	$\{2,3,4\}$	$\{3,4,5\}$	$\{2,3,4,5\}$

Notice that $\chi''_{2s}(K_n) = \chi''_{2as}(K_n)$. In our memory, the exact value $\chi''_{2as}(K_n)$ is not determined for all integers $n \ge 3$ up to now. In the article [19], the authors introduce an adjacent vertex strong-distinguishing proper total colorings that is a $\mu(k)$-coloring defined in Definition **??**. They obtain the exact values of χ''_{2as} for cycles C_n, paths P_n and trees, and present
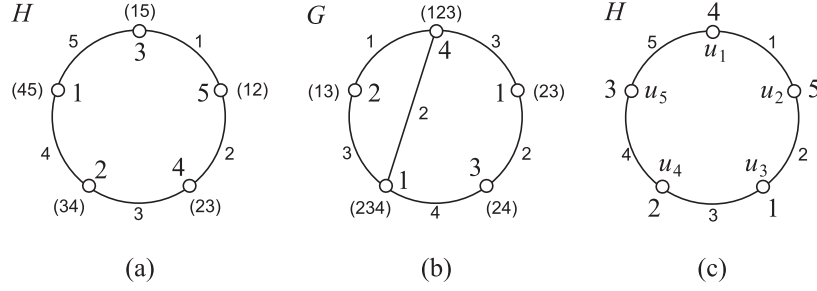
3

Figure 1: (a) H admits an e-partially 5-vdtc (also, an e-partially 5-avdtc), and $\chi''_{(s)}(H) = 5 = \chi''_{(as)}(H)$; (b) G admits an e-partially 4-vdtc (also, an e-partially 4-avdtc), and $\chi''_{(s)}(G) = 4 = \chi''_{(as)}(G)$; (c) H admits an (8)-distinguishing total 5-coloring f, and $\chi''_{(8)}(H) = 5$.
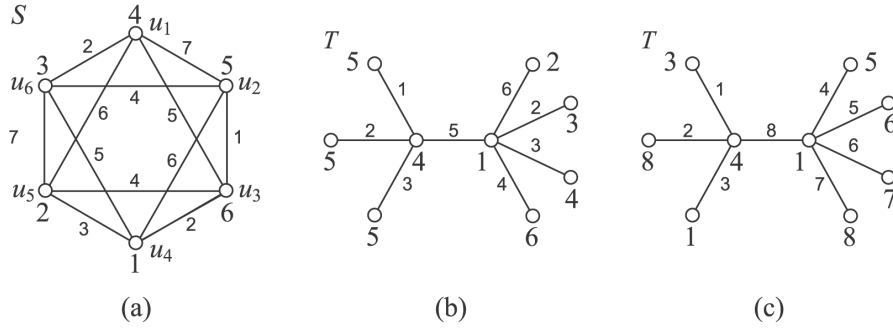

Figure 2: (a) S admits an (8)-distinguishing total coloring, and $\chi''_{(8)}(S) = 7$; (b) T admits a 6-(4)-avdtc, and $\chi''_{(4)as}(T) = 6$; (c) T admits an (8)-distinguishing total coloring, and $\chi''_{(8)}(T) = n_1(T) + 1 = 8$.

Conjecture 2. *Let G be a simple graph with $n \geq 3$ vertices and no isolated edges. Then $\chi''_{2as}(G) \leq n + \lceil \log_2 n \rceil + 1$.*

The procedure of "joining a vertex of a graph G and a vertex v out of G by an edge" is abbreviated as "adding a leaf v to G" here. The notations $\chi(G)$, $\chi'(G)$, $\chi''(G)$, denote the *proper chromatic number*, *proper chromatic index* and *proper total chromatic number* of G (cf. [5]), respectively.

2 Graphs having total colorings with additional constrained conditions

For the purpose of convenience, let $\mathcal{F}_{3s}(n)$ be the set of simple graphs with $n \geq 3$ vertices and no isolated edges as well as at most one isolated vertex. The following Lemma 1 follows from Definition **??**.

Lemma 1. *Let $G \in \mathcal{F}_{3s}(n)$. Then*
   (i) $\chi''_{(s)}(G) \geq \chi'_s(G)$, $\chi''_{(s)}(G) \geq \chi''_{(as)}(G)$ and $\chi''_{(as)}(G) \geq \chi'_{as}(G)$.
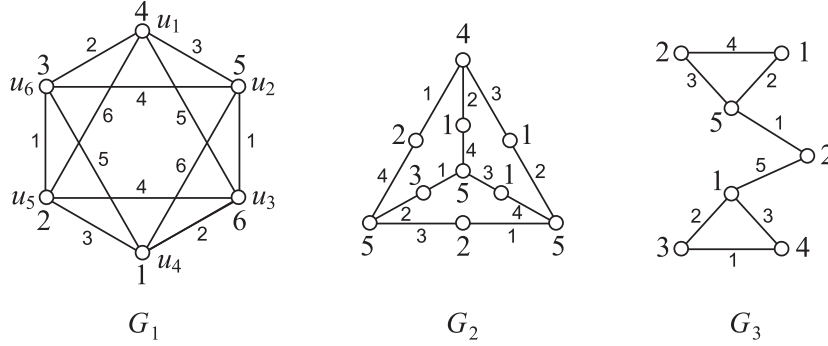   *(ii) $\chi''_{2s}(G) \geq \chi''_{2as}(G) \geq \chi''(G)$.*

4

Figure 3: G_1 admits a mixed (7)-total coloring; G_2 admits an (8)-distinguishing total coloring, and $\chi''_{(8)}(G_2) = 5$; G_3 admits a (3)-avdtc.

(iii) If $n_d \geq 2$ with $\delta(G) \leq d \leq \Delta(G)$, then $\chi''_{2s}(G) \geq d + 2$.

(iv) Let $\overline{G}$ be the complement of a graph $G \in \mathcal{F}_{3s}(n)$, and let $\overline{G} \in \mathcal{F}_{3s}(n)$. Then $\chi''_\varepsilon(G) + \chi''_\lambda(\overline{G}) \geq \chi''_\varepsilon(K_n)$ for $\varepsilon = (s), (as), 2s, 2as$.

It is noticeable, a complete graph K_n admits no v-partially k-vdtcs (also, v-partially k-avdtcs) at all.

Lemma 2. *A graph $G \in \mathcal{F}_{3s}(n)$ admits a v-partially k-vdtc (resp. a v-partially k-avdtc) if and only if $N(u) \cup \{u\} \neq N(v) \cup \{v\}$ for distinct $u, v \in V(G)$ (resp. every edge $uv \in E(G)$).*

Proof. To show the proof of 'if', we take a v-partially k-vdtc f of G. Then, $C\langle f, u \rangle \neq C\langle f, v \rangle$ for distinct $u, v \in V(G)$. If $uv \in E(G)$, $C\langle f, u \rangle \neq C\langle f, v \rangle$ means $\{f(x) : x \in N(u)\} \setminus \{f(u), f(v)\} \neq \{f(x) : x \in N(v)\} \setminus \{f(u), f(v)\}$, and furthermore $N(u) \cup \{u\} \neq N(v) \cup \{v\}$. If $uv \notin E(G)$, $C\langle f, u \rangle \neq C\langle f, v \rangle$ means $N(u) = N(v)$, or $N(u) \neq N(v)$. No mater one of two cases occurs, we have $N(u) \cup \{u\} \neq N(v) \cup \{v\}$.

Conversely, it is straightforward to provide a total coloring h of G for the proof of 'only if'. In fact, we can set h as a bijection from $V(G) \cup E(G)$ to $[1, n + q]$, where $q = |E(G)|$. Clearly, $C\langle h, u \rangle \neq C\langle h, v \rangle$ for distinct $u, v \in V(G)$ (including every edge $uv \in E(G)$), since $N(u) \cup \{u\} \neq N(v) \cup \{v\}$. $\square$

Lemma 3. (i) *Every graph $G \in \mathcal{F}_{3s}(n)$ admits (6)-total colorings and holds $\chi''_{(6)}(G) \geq \chi''_\lambda(G)$ for $\lambda = (s), (as), s, as, 2s, 2as$.*

(ii) *If a graph $G \in \mathcal{F}_{3s}(n)$ admits (8)-distinguishing total colorings, then $\chi''_{(8)}(G) \geq \chi''_\varepsilon(G)$ for $\varepsilon = (s), (as), \langle s \rangle, \langle as \rangle, s, as, 2s, 2as$. Furthermore,*

$$\chi''_{(8)}(G) \leq \min\{\chi'_s(G) + \chi''_{\langle s \rangle}(G), \ \chi''_{(s)}(G) + \chi''_{\langle s \rangle}(G)\}.$$

Proof. Notice that the assertion (ii) is an immediate consequence of Definition **??**. Thereby ,we show the assertion (i). Let $V(G) = \{u_i : i \in [1, n]\}$, $V(G) = \{e_j : j \in [1, q]\}$, where $q = |E(G)|$. It is sufficient to define a total coloring f of G as: $f(u_i) = i$ for $i \in [1, n]$, and $f(e_j) = n + j$ for $j \in [1, q]$. Clearly, f is a (6)-total coloring. Definition **??** leads to $\chi''_{(6)}(G) \geq \chi''_\varepsilon(G)$ for $\varepsilon = (s), (as), s, as, 2s, 2as$. $\square$

2.1 Connections between known chromatic numbers

We will build up some connections between known chromatic numbers (χ, χ', χ'', χ'_s, χ'_{as}, χ''_{as}) and the new chromatic numbers ($\chi''_{\langle s\rangle}$, $\chi''_{\langle as\rangle}$, χ''_{2s}, χ''_{2as}, $\chi''_{(as)}$, $\chi''_{(s)}$) defined in Definition 1 and Definition 2 in this subsection.

Lemma 4. *Let $G \in \mathcal{F}_{3s}(n)$.*
(i) $\chi''_{2s}(G) \leq \chi''_{(s)}(G) + \chi(G)$ and $\chi''_{2s}(G) \leq \chi''_{\langle s\rangle}(G) + \chi'(G)$; $\chi''_{2as}(G) \leq \chi''_{(as)}(G) + \chi(G)$ and $\chi''_{2as}(G) \leq \chi''_{\langle as\rangle}(G) + \chi'(G)$.
(ii) $\chi''_{\varepsilon}(G) \leq \chi'_s(G) + \chi(G)$ for $\varepsilon = (s), (as), s, as, 2s, 2as$ and $\chi''_{\mu}(G) \leq \chi'_{as}(G) + \chi(G)$ for $\mu = (as), as, 2as$.

Proof. (1) Notice that the method for showing $\chi''_{2s}(G) \leq \chi''_{(s)}(G) + \chi'(G)$ can be used to show the rest three inequalities in the assertion (i). Let f be an e-partially k-vdtc of G with $k = \chi''_{(s)}(G)$. Under this coloring we have $C(f, u) \neq C(f, v)$ for distinct $u, v \in V(G)$. Now we define another total coloring g of G as: $g(e) = f(e)$ for $e \in E(G)$, $g(u) = k + f'(u)$ for $u \in V(G)$, where f' is a proper vertex $\chi'(G)$-coloring of G. Notice that $\{g(x) : x \in V(G)\} \cap \{g(e) : e \in E(G)\} = \emptyset$. Hence, $C(f, u) \neq C(f, v)$ means that $N_2[g, u] \neq N_2[g, v]$ for distinct $u, v \in V(G)$.

(2) Since the proofs of two inequalities in the assertion (ii) are very similar, we show only the first one. Write $\chi'_s = \chi'_s(G)$ and $\chi = \chi(G)$. We color the edges of G with colors of $C = [1, \chi'_s]$ such that two incident edges e and e' of G are assigned distinct colors, and for any two vertices u and v of G the set of the colors assigned to the edges being incident to u is not equal to the set of the colors assigned to the edges being incident to v. We, now, define a coloring f for all vertices of G with colors of $C' = \{\chi'_s + 1, \chi'_s + 2, \ldots, \chi'_s + \chi\}$ such that the color assigned to u is different to the color assigned to v for arbitrary two vertices u and v of $V(G)$. From $C \cap C' = \emptyset$ and the coloring way used above, we obtain $C(f, u) \neq C(f, v)$, $C[f, u] \neq C2[f, v]$ and $N_2[f, u] \neq N_2[f, v]$ for distinct $u, v \in V(G)$. Hence, f is a mixed (6)-total coloring, and moreover $\chi''_{\varepsilon}(G) \leq \max\{f(x) : x \in E(G) \cup V(G)\} = \chi'_s + \chi$ for $\varepsilon = (s), (as), s, as, 2s, 2as$. $\square$

Let $K_{m,n}$ be a complete bipartite graph for $m \geq n \geq 2$. By Lemma 1 we have $\chi''_{2s}(K_{m,n}) \leq m + 4$ if $m = n$ and $\chi''_{2s}(K_{m,n}) \leq m + 3$ if $m > n \geq 2$, since $\chi(K_{m,n}) = 2$, and $\chi'_s(K_{m,n}) = m + 2$ for $m = n$ and $\chi'_s(K_{m,n}) = m + 1$ for $m > n \geq 2$ (cf. [6]). For example, $\chi''_{2s}(K_{3,2}) = \chi'_s(K_{3,2}) + \chi(K_{3,2}) = 4 + 2$. Notice that we have used the colors on the vertices being divided completely from the colors of the edges of G in the proof of Lemma 4, so the upper bounds of $\chi''_{2s}(G)$ in Lemma 4 are not optimal. A subset V^* of $V(G)$ is an edge-covering set if any edge uv of G holds $u \in V^*$ or $v \in V^*$. $G[V^*]$ indicates a vertex induced subgraph of $V(G)$ such that two ends of each edge of $G[V^*]$ both are in V^*.

Lemma 5. *Let V^* be a smallest edge-covering set of a graph $G \in \mathcal{F}_{3s}(n)$. Then there exists a bipartite subgraph H with bipartition $(V^*, V(G)\setminus V^*)$. We have $\chi''_{2s}(G) \leq \chi''(G[V^*]) + \chi'_s(H) + 1$ and $\chi''_{2as}(G) \leq \chi''(G[V^*]) + \chi'_{as}(H) + 1$ if $H \in \mathcal{F}_{3s}(n)$.*

Proof. We take a smallest edge-covering set V^* of a graph $G \in \mathcal{F}_{3s}(n)$. Then any $u \in V' = V(G) \setminus V^*$ is adjacent to a vertex v of V^*, and V' is an independent set of G by the definition of an edge-covering set.

Clearly, there exists a bipartite subgraph H of G that is defined as $V(H) = V(G)$ and $E(H) = \{uv : u \in V^*, v \in V'\}$. Write $m = \chi''(G[V^*])$ and $t = \chi'_s(H)$. Suppose that $H \in \mathcal{F}_{3s}(n)$. We use the colors of $S_1 = [1,m]$ to color properly every elements of the vertex-induced graph $G[V^*]$ and color every vertex of V' with color $m+1$. Consequently, we use the colors of $S_2 = \{m+2, m+3, \ldots, m+1+t\}$ to color properly all edges of H such that for any two vertices u and v of H, the set of the colors assigned to the edges incident to u differs to the set of the colors assigned to the edges incident to v. Hence, we obtain a proper total coloring f of G such that $N_2[f,u] \neq N_2[f,v]$ for distinct $u, v \in V(G)$ since $S_1 \cap S_2 = \emptyset$ and H is a spanning subgraph of G, and confirm $\chi''_{2s}(G) \leq m + t + 1$.

The above method is valid for proving the second inequality of the lemma. $\square$

2.2 New chromatic numbers and maximum degrees of graphs

Theorem 6. *For a bipartite graph $G \in \mathcal{F}_{3s}(n)$, we have $\chi''_{(as)}(G) \leq \Delta(G) + 3$; and $\chi''_{(s)}(G) \leq \chi'_s(G) + \varepsilon$ for $\varepsilon = 1$ if $\chi'_s(G) > \Delta(G)$, $\varepsilon = 2$ otherwise.*

Proof. In the article [3], a bipartite graph G holds $\chi'_{as}(G) \leq \Delta(G) + 2$. Let (X,Y) be the bipartition of the bipartite graph G and let f be an e-partially k-avdtc of G with $k = \Delta(G) + 2$. We define another total coloring g of G as: $g(xy) = f(xy)$ for $xy \in E(G)$, $g(y) = k+1$ for $y \in Y$, and $g(x) \in [1,k] \setminus \{f(xu) : u \in N(x)\}$ for $x \in X$. Clearly, g is an e-partially $(k+1)$-avdtc of G, which means $\chi''_{(as)}(G) \leq \Delta(G) + 3$. By the same way used above we can show the second inequality. $\square$

Theorem 7. *For a graph $G \in \mathcal{F}_{3s}(n)$ with the independent number $\alpha(G)$, we have*

(i) $\chi''_{2s}(G) \leq 4\Delta(G)$ *if* $G \notin \{C_{2m+1}, K_n\}$.

(ii) $\chi''_{2s}(G) \leq 2\Delta(G) + 5$ *if* $\delta(G) > \frac{n}{3}$ *and* $G \notin \{C_{2m+1}, K_n\}$.

(iii) $\chi''_{2s}(G) \leq n - \alpha(G) + \chi'_s(H) + 2$, *where H is a bipartite subgraph with bipartition $(V^*, V(G) \setminus V^*)$ generated from a smallest edge-covering set V^*.*

(iv) *If diameter $D(G) \geq 3$, then $\chi''_{2s}(G) \geq m$ where m is a the smallest integer such that $\binom{m}{\delta+1} \geq \deg_G(u) + \deg_G(v) + 2$ for each pair of vertices u, v holding $d(u,v) \geq 3$.*

(v) $\chi''_{2s}(G) \leq n - \alpha(G) + \Delta(G) + 7$ *if* $\delta(G) > \frac{2}{3}n$ *and* $G \notin \{C_{2m+1}, K_n\}$.

(vi) $\chi''_{2s}(G) \leq 2n - 1$ *if* $G = K_n$.

(vii) $\chi''_{2s}(G) \leq \chi''_{2s}(P_n) + \Delta(G)$ *if G is hamiltonian.*

(viii) $\chi''_{2s}(G) \leq n_1(T) + \Delta(G) + 1$ *if G contains a spanning tree T with $n_2(T) = 0$.*

Proof. (1) In the article [1], the authors show that a simple graph G without isolated edges holds $\chi'_s(G) \leq 3\Delta(G)$. By the Brooks' theorem, we obtain the assertion (i).

(2) Since $G \notin \{C_{2m+1}, K_n\}$, the assertion (ii) follows from $\chi(G) \leq \Delta(G)$ and a result of the article [4]: if $\delta(G) > \frac{1}{2}|G|$, then $\chi'_s(G) \leq \Delta(G) + 5$.

(3) Let $\alpha(G)$ and $\beta(G)$ be the vertex-independent number and the edge-covering number of G, respectively. Therefore, $\alpha(G) + \beta(G) = |G|$ (cf. [5]). Notice that $|G[V^*]| = \beta(G)$ and $\chi''(G[V^*]) \leq \beta(G) + 1$, since $\chi''(K_{2m}) = \chi''(K_{2m+1}) = 2m + 1$. By Lemma 5 we obtain the assertion (iii), that is, $\chi''_{2s}(G) \leq \beta(G) + \chi'_s(H) + 2$, or $\chi''_{2s}(G) \leq |V(G)| - \alpha(G) + \chi'_s(H) + 2$.

(4) Notice that $d(u,v) \geq 3$ for distinct $u,v \in V(G)$. It is clear that $|N(u) \cup N(v)| = \deg_G(u) + \deg_G(v)$, this shows that we need at least $\deg_G(u) + \deg_G(u) + 2$ distinct colors. The smallest case is $\binom{m}{\delta+1} \geq \deg_G(u) + \deg_G(v) + 2$, so the assertion (iv) holds true, as desired.

(5) To show that G contains a bipartite spanning graph H with $\deg_H(u) \geq \frac{1}{2}\deg_G(u)$ for every $u \in V(H) = V(G)$, we make a partition $V(G) = S_1 \cup S_2$ for $S_1 \cap S_2 = \emptyset$ such that the cardinality of subset $E' = \{uv : u \in S_1, v \in S_2\}$ is as large as possible, and obtain an edge induced graph $G[E']$ over E'. If it is not that $2\deg_H(u) \geq \deg_G(u)$ for every vertex $u \in V(H) = V(G)$. Hence, there is a vertex $v_0 \in S_1$ such that $2\deg_H(v_0) < \deg_G(v_0)$, so we can take $S_1^* = S_1 \setminus \{v_0\}$ and $S_2^* = S_2 \cup \{v_0\}$, and get $E^* = \{uv : u \in S_1^*, v \in S_2^*\}$ such that $|E^*| > |E'|$; a contradiction with the choice of E'. This bipartite graph H was discovered first by Erdös. Since $\deg_H(u) \geq \frac{1}{2}\deg_G(u) \geq \frac{n}{3}$ and $\chi(G) \leq \Delta(G)$ according to $G \notin \{C_{2m+1}, K_n\}$, the assertion (v) follows from the assertion (iii), Lemma 5 and a result of the article [4] stated in $(P2)$ above.

(6) Let T be a spanning tree of G. Then

$$\chi''_{2s}(G) \leq \chi''_{2s}(T) + \chi'(G - E(T)), \tag{1}$$

and

$$\Delta(G - E(T)) \leq \Delta(G) - 1, \quad \chi'(G - E(T)) \leq \Delta(G) \tag{2}$$

by Vizing's theorem. If $G = K_n$, we have a star $T = K_{1,n-1}$. Furthermore, $\chi''_{2s}(K_{1,n-1}) = n$, $\chi'(G - E(T)) = n - 1$. The assertion (vi) follows from (1).

(7) Notice that G contains a Hamilton path P_n which is a spanning tree of G, and $\chi'(G - E(P_n)) \leq \Delta(G)$ by (2). The assertion (vii) follows from (1).

(8) If G contains a spanning tree T with $n_2(T) = 0$, then $n_1(T) \leq \chi''_{2s}(T) \leq n_1(T) + 1$. The last assertion follows from both (1) and (2).

The proof of the theorem is complete. $\qquad\square$

Theorem 8. *For a graph $G \in \mathcal{F}_{3s}(n)$ we have*
(i) $\chi''_{2as}(G) \leq 8$ if $\Delta(G) \leq 3$ and $G \notin \{C_{2m+1}, K_3\}$.
(ii) $\chi''_{2as}(G) \leq \Delta(G) + 4$ if G is bipartite.
(iii) $\chi''_{2as}(G) \leq 2\Delta(G) + 3$ if G is a 3-colorable, Hamilton graph, and $G \notin \{C_{2m+1}, K_n\}$.
(iv) $\chi''_{2as}(G) \leq 2\Delta(G) + 2$ if G is a planar graph G with girth $g \geq 6$ and $\Delta(G) \geq 3$.
(v) G has a spanning subgraph G^* with $|E(G^*)| < \frac{1}{2}|E(G)|$ such that

$$\chi''_{2as}(G) \leq \chi''(G^*) + \Delta(G) - \delta(G^*) + 2.$$

Proof. In the article [3], the authors shown that $\chi'_{as}(G) \leq 5$ if $\Delta(G) \leq 3$, and $\chi'_{as}(G) \leq \Delta(G) + 2$ if G is bipartite. By Lemma 4 we obtain the assertions (i) and (ii).

The assertion (iii) follows from Lemma 4 and a result of the article [11]: every connected 3-colourable Hamiltonian graph G holds $\chi'_{as}(G) \leq \Delta(G) + 3$.

Wang and Wang [12] distribute that a planar graph G with girth $g \geq 6$ and $\Delta(G) \geq 3$ holds $\chi'_{as}(G) \leq \Delta(G) + 2$. This result and Lemma 4 induce the assertion (iv).

To show that assertion (v), we apply that a certain bipartite spanning subgraph H of G with $\deg_H(u) \geq \frac{1}{2}\deg_G(u)$ for every $u \in V(H)$ exists according to the proof of the assertion (v) of Theorem 7. So, $|E(H)| \geq \frac{1}{2}|E(G)|$, which means $|E(G^*)| < \frac{1}{2}|E(G)|$, where $G^* = G - E(H)$. Let f be a proper total coloring of the graph $G^* = G - E(H)$ with the color set $[1, \chi''(G^*)]$, and let h be a adjacent vertex distinguishing edge coloring of H with the color set $\{\chi''(G^*) + 1, \chi''(G^*) + 2, \ldots, \chi''(G^*) + \chi'_{as}(H)\}$. Both colorings give $\chi''_{2as}(G) \leq \chi''(G^*) + \chi'_{as}(H) \leq \chi''(G^*) + \Delta(H) + 2$

8

since H is bipartite and $\chi'_{as}(H) \le \Delta(H) + 2$ (cf. [3]). Notice that $\Delta(H) = \Delta(G) - \delta(G^*)$. We obtain the assertion (v).

This theorem is covered. $\qquad\square$

2.3 Construction of graphs having total colorings with additional constrained conditions

Theorem 9. *Suppose that a graph $H \in \mathcal{F}_{3s}(n)$ is not a complete graph, then the graph G obtained by adding an edge of the complement of H to H holds $\chi''_{2s}(G) \le \chi''_{2s}(H) + 1$.*

Proof. Let f be a $\mu(k)$-coloring of a non-complete graph $H \in \mathcal{F}_{3s}(n)$ with $k = \chi''_{2s}(H)$. Suppose that u is not adjacent to v in H. We have a graph $G = H + uv$, and define a coloring g of G in the following cases.

Case 1. $f(u) \ne f(v)$. We set $g(uv) = k + 1$, $g(z) = f(z)$ for $z \in V(G) \cup (E(G) \setminus \{uv\})$. Notice that $N_2[g, u] \setminus \{k + 1\} = N_2[f, u] \ne N_2[f, v] = N_2[g, v] \setminus \{k + 1\}$. We can confirm $N_2[f, x] \ne N_2[f, y]$ for distinct $x, y \in V(G)$. Thereby, g is a $\mu(k)$-coloring of G, and furthermore $\chi''_{2s}(G) \le \chi''_{2s}(H) + 1$.

Case 2. $f(u) = f(v)$, $N_2[f, u] \not\subset N_2[f, v]$ and $N_2[f, v] \not\subset N_2[f, u]$. Notice that there is a color $\alpha \in N_2[f, u]$, but $\alpha \notin N_2[f, v]$; and there is a color $\beta \in N_2[f, v]$, but $\beta \notin N_2[f, u]$.

Case 2.1. $f(uu_i) \ne \alpha$ for $u_i \in N(u)$. We set $g(uv) = \alpha$, $g(v) = k + 1$, and $g(z) = f(z)$ for $z \in (V(G) \setminus \{v\}) \cup (E(G) \setminus \{uv\})$. Notice that $k + 1 \in N_2[g, u]$ and $k + 1 \in N_2[g, v]$. Since $\beta \notin N_2[f, u]$, so $\beta \notin N_2[g, u]$, it follows that g is a $\mu(k)$-coloring of G.

Case 2.2. $f(uu_i) = a$ for some $u_i \in N(u)$. Define f' of H as: $f'(z) = k + 1$ if $f(z) = \alpha$ and $f'(z) = f(z)$ if $f(z) \ne a$, $z \in V(H) \cup E(H)$. Notice that f' is a $\mu(k)$-coloring of H. Next, we set $g(uv) = a$, $g(v) = k + 1$, and $g(z) = f'(z)$ for $z \in (V(G) \setminus \{v\}) \cup (E(G) \setminus \{uv\})$. We can see $a \notin N_2[g, z]$ for $z \in V(G) \setminus \{u, v\}$, and $\beta \notin N_2[f, u]$. This coloring g gives $\chi''_{2s}(G) \le \chi''_{2s}(H) + 1$.

Case 3. $f(u) = f(v)$ and $N_2[f, u] \subset N_2[f, v]$. Notice that $N_2[f, v] \setminus N_2[f, u] \ne \emptyset$, so we can obtain the desired $\mu(k)$-coloring g of G by the methods in Case 2.1 and Case 2.2.

We complete the proof of the theorem. $\qquad\square$

Theorem 10. (i) *Adding a leaf to a connected graph H produces a graph G holding $\chi''_{2s}(G) \le \chi''_{2s}(H) + 1$.*

(ii) *If a graph G obtained by deleting a vertex of degree m from a connected graph H is connected. Then $\chi''_{2s}(G) \le \chi''_{2s}(H) + m$.*

(iii) *Adding a leaf $v_i \notin V(H)$ to a vertex u_i of a connected graph H for $i \in [1, m]$ produces a connected graph G holding $\chi''_{2as}(G) \le \chi''_{2as}(H) + 1$.*

Proof. (1) Let f be a $\mu(k)$-coloring of a connected graph H with $k = \chi''_{2s}(H)$. We add a leaf v to H by joining v and a vertex $u \in V(G)$, and the resulting graph is denoted as G. We define a coloring g of G as: $g(w) = f(w)$ for $w \in (V(G) \cup E(G)) \setminus \{v, uv\}$; $g(uv) = k + 1$, and $g(v) = f(u')$, where $u' \in N(u)$. Notice that $N_2[g, v] = \{f(u'), f(u), k + 1\}$ is a proper subset of $N_2[g, u]$ since $f(uu') \notin N_2[g, v]$. We see that g is a $\mu(k + 1)$-coloring of G, which induces the assertion (i).

(2) Let H be a connected graph, and let $G = H - w$ be connected with $\deg_H(w) = m$ and $N(w) = \{w_1, w_2, \ldots, w_m\}$. By the assertion (i) a connected graph G_1 by adding a leaf w to G

9

through joining w and a vertex $w_1 \in V(G)$ holds $\chi''_{2s}(G_1) \leq \chi''_{2s}(H) + 1$. Applying Theorem 9 repeatedly $(m-1)$ times by joining w and each $w_i \in N(w) \setminus \{w_1\}$, we get $\chi''_{2s}(G) \leq \chi''_{2s}(H) + m$, as desired.

(3) Let h be a $\mu_e(k)$-coloring of a connected graph H with $k = \chi''_{2as}(H)$. Take distinct vertices $v_i \notin V(H)$, and select arbitrarily distinct vertices $u_i \in V(H)$, $i \in [1, m]$. We have a graph G obtained by joining v_i and u_i by an edge for $i \in [1, m]$, and define a coloring β of G in the way that $\beta(z) = h(z)$ for $z \in (V(G) \cup E(G)) \setminus \{v_i, u_i v_i : i \in [1, m]\}$; $\beta(u_i v_i) = k + 1$ and $\beta(v_i) = h(u_i u'_i)$ for $i \in [1, m]$, where $u'_i \in N(u_i)$. It follows that the coloring β is a $\mu_e(k)$-coloring of G, since $N_2[\beta, x] \neq N_2[\beta, y]$ for every edge $xy \in E(G)$. $\square$

3 Problems for further works

As further works, we propose the following problems:

Problem 1. (1) If $G_1 \subseteq H \subseteq G_2$ and $\chi''_\lambda(G_1) = \chi''_\lambda(G_2) = k$, do we have $\chi''_\lambda(H) = k$ for $\lambda = 2s, 2as$?

(2) Let $D(G) = 2$. If the smallest number k satisfies that $\binom{k}{\Delta+1} \geq n$, then $|\chi''_{2s}(G) - k| \leq 1$?

(3) Suppose that $D(G) \geq 3$ and $\delta(G) \neq \Delta(G)$. If $\binom{k}{\Delta+1} \geq n$ and $\binom{m}{\delta+1} \geq n$ and $m \leq k$, then $m \leq \chi''_{2s}(G) \leq k$?

(4) Characterize simple graphs G such that $N(u) \cup \{u\} \neq N(v) \cup \{v\}$ for distinct $u, v \in V(G)$ (resp. for every edge $uv \in E(G)$).

Conjecture 3. *Every connected, simple graph G holds $\chi'_s(G) \leq \chi''_{(s)}(G) \leq \chi'_s(G) + 1$ and $\chi'_{as}(G) \leq \chi''_{(as)}(G) \leq \chi'_{as}(G) + 1$.*

It has been discovered that there are infinite simple graphs G and some their proper subgraphs H such that $\chi'_\lambda(H) > \chi'_\lambda(G)$ and $\chi''_\lambda(H) > \chi''_\lambda(G)$ for $\lambda = s, as$ (cf. [13]). But, we have

Conjecture 4. *No simple graph G and its proper subgraphs H hold $\chi''_{(8)}(G) < \chi''_{(8)}(H)$ true.*

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