

Hamiltonization of Elementary Nonholonomic Systems

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Abstract. In this paper, we develop the Chaplygin reducing multiplier method; using this method, we obtain a conformally Hamiltonian representation for three nonholonomic systems, namely, for the nonholonomic oscillator, for the Heisenberg system, and for the Chaplygin sleigh. Furthermore, in the case of an oscillator and the nonholonomic Chaplygin sleigh, we show that the problem reduces to the study of motion of a mass point (in a potential field) on a plane and, in the case of the Heisenberg system, on the sphere. Moreover, we consider an example of a nonholonomic system (suggested by Blackall) to which one cannot apply the reducing multiplier method.

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1 Introduction

In the present paper, a few fairly simple (model) dynamical systems of nonholonomic mechanics are considered in connection with the Hamiltonization problem, that is, the problem of reducing these systems to Hamiltonian form. For meaningful problems of the theory of nonholonomic systems (describing, as a rule, the dynamics of systems with rolling), see the surveys [13, 17], and for the general Hamiltonization problem (of general dynamical systems), see [7].

The paper discusses the problem of motion of a mass point on a three-dimensional Riemannian manifold in the presence of a nonholonomic (nonintegrable) constraint and a potential field. If the constraint is integrable and the potential is absent, then the problem is reduced to the well-studied case of motion along geodesics on a two-dimensional manifold. As is well known, in this case, the behavior of geodesics is related to a variational problem, and its global aspect is related to methods of the calculus of variations in general. The case of nonzero potential can also be reduced to the problem of geodesics, however, with a different metric, namely, the Maupertuis metric [42]. In this connection we mention the work [31], in which an example is given of the geodesic flow with integrals rational in the velocities.

In general, the nonintegrability of a constraint is incompatible with the variational principle, which was already clear to Hertz, Poincaré and Hamel [24]. Nevertheless, in some cases, by a suitable reparameterization of time (depending on the configuration variables only), the trajectories of the system can again be obtained using the Hamilton variational principle, and the equations of motion are represented in a conformally Hamiltonian form. The most natural methods of Hamiltonization can be developed for Chaplygin systems by using his reducing multiplier theory.

In this paper, we discuss three model problems (the nonholonomic oscillator, the Heisenberg system, and the Chaplygin sleigh) for which the Hamiltonization can be carried out explicitly and, moreover, we consider another system (which was introduced by Blackall) for which there are essential obstacles to the Hamiltonization (in the entire phase space) and, as a result of these obstacles, the behavior of the system differs significantly from that for the variational problem of geodesics.

2 Equations of motion

Consider a (mechanical) system with three degrees of freedom and generalized coordinates q_1, q_2, q_3 .

Suppose that the coordinate q_3 is cyclic, i.e., that it is not included explicitly in the Lagrangian of the system L . Moreover, we assume that one can always carry out a Legendre transform of the Lagrangian L .

Assume that the motion is subject to a nonholonomic constraint which is linear, homogeneous in the velocities, and can be represented as

$$f = \dot{q}_3 - a_1(\mathbf{q})\dot{q}_1 - a_2(\mathbf{q})\dot{q}_2 = 0, \quad \mathbf{q} = (q_1, q_2). \quad (1)$$

Let us write the equations of motion in the form of the Euler–Lagrange equations with the undetermined multiplier λ ,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \lambda \frac{\partial f}{\partial \dot{q}_i}, \quad i = 1, 2, 3. \quad (2)$$

It follows from the last equation that

$$\lambda = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_3} \right).$$

Denote by $L^*(\mathbf{q}, \dot{\mathbf{q}})$ the Lagrangian of the system after substituting into it the expression for \dot{q}_3 from the constraint equation. Using the standard rule of indirect differentiation, we see that

$$\frac{\partial L^*}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial L}{\partial \dot{q}_3} a_i, \quad \frac{\partial L^*}{\partial q_i} = \frac{\partial L}{\partial q_i} + \sum_{k=1}^2 \frac{\partial L}{\partial \dot{q}_3} \frac{\partial a_k}{\partial q_i} \dot{q}_k, \quad i = 1, 2.$$

Substituting the last relations into equations (2) and reducing similar terms, we obtain a close system for the variables $(\mathbf{q}, \dot{\mathbf{q}})$,

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L^*}{\partial \dot{q}_1} \right) - \frac{\partial L^*}{\partial q_1} &= S \dot{q}_2, & \frac{d}{dt} \left(\frac{\partial L^*}{\partial \dot{q}_2} \right) - \frac{\partial L^*}{\partial q_2} &= -S \dot{q}_1, \\ S &= \left(\frac{\partial L}{\partial \dot{q}_3} \right)^* \left(\frac{\partial a_1}{\partial q_2} - \frac{\partial a_2}{\partial q_1} \right), \end{aligned} \quad (3)$$

where the expression $\left(\frac{\partial L}{\partial \dot{q}_3} \right)^*$ means that the substitution \dot{q}_3 is made after the differentiation.

Thus, the problem reduces to the study of the system (3) with two degrees of freedom; according to [15], we refer to this system as the generalized Chaplygin system. For known solutions $q_1(t)$ and $q_2(t)$, the law of modification of the remaining variable q_3 is obtained, according to (1), by a quadrature. Further, we concentrate on the investigation of system (3). To study the integrability of systems of this kind, one can use the results of [29, 30, 38].

Remark 1. *The nonholonomicity of a constraint means that this constraint cannot be represented in the form*

$$F(q_1, q_2, q_3) = 0, \quad \text{where} \quad \frac{dF}{dt} = f,$$

which implies the condition

$$\frac{\partial a_1}{\partial q_2} \neq \frac{\partial a_2}{\partial q_1}, \quad (4)$$

which is assumed to be valid everywhere below.

3 Invariant measure and the reducing multiplier method

As is well known (see, e.g., [9, 28]), the equations of motion in nonholonomic mechanics generally cannot be represented in Hamiltonian form. Nevertheless, there are problems in which such a representation can be obtained only after rescaling time, i.e., the equations of motion are represented in conformally Hamiltonian form.

The *reducing multiplier* method is the most efficient to reduce the generalized Chaplygin systems to conformally Hamiltonian form, and we proceed with a presentation of the method (see also [8, 15]).

First of all, note that the homogeneity in the generalized velocities of the constraint (1) results in the fact that the system (3) preserves the energy integral

$$E = \sum_{i=1}^2 \frac{\partial L^*}{\partial \dot{q}_i} \dot{q}_i - L^*. \quad (5)$$

Making the Legendre transform

$$P_i = \frac{\partial L^*}{\partial \dot{q}_i}, \quad H = \sum_{i=1}^2 P_i \dot{q}_i - L^* \Big|_{\dot{q}_i \rightarrow P_i}, \quad i = (1, 2),$$

we obtain the following system of equations:

$$\dot{q}_i = \frac{\partial H}{\partial P_i}, \quad \dot{P}_1 = -\frac{\partial H}{\partial q_1} + \frac{\partial H}{\partial P_2} S, \quad \dot{P}_2 = -\frac{\partial H}{\partial q_2} - \frac{\partial H}{\partial P_1} S. \quad (6)$$

Here H stands for the integral (5), which, together with S , is expressed in terms of the new variables.

Invariant measure. Let us find cases in which system (6) has an invariant measure that can be represented as

$$\mathcal{N}(\mathbf{q}) d\mathbf{q} d\mathbf{P}. \quad (7)$$

Recall that the function $\mathcal{N}(\mathbf{q})$ is called the density of the invariant measure and satisfies the Liouville equation [44, 22]

$$\operatorname{div}(\mathcal{N}\mathbf{v}) = 0, \quad (8)$$

where \mathbf{v} is the vector field determined by (6).

Remark 2. *As a rule, it is assumed that the density of the invariant measure is a smooth and positive function on the entire phase space. Nevertheless, in applications, one can face a situation in which $\mathcal{N}(\mathbf{q})$ has singularities in some domain of the phase space. In this case, the system is said to admit a singular invariant measure.*

Consider in more detail the Liouville equation (8), which in this case can be represented as

$$\left(\frac{\partial}{\partial \mathbf{q}} \ln \mathcal{N}(\mathbf{q}) + \xi, \dot{\mathbf{q}} \right) = 0, \quad \xi = \left(-\frac{\partial S}{\partial P_2}, \frac{\partial S}{\partial P_1} \right). \quad (9)$$

Since the previous relation must hold for arbitrary $\dot{\mathbf{q}}$, it follows that

$$\frac{\partial}{\partial \mathbf{q}} \ln \mathcal{N}(\mathbf{q}) + \xi = 0.$$

In this case, for the solution $\mathcal{N}(\mathbf{q})$ of (9) to exist, it is necessary that the vector field ξ be potential. This condition leads to the relation

$$\frac{\partial^2 S}{\partial q_1 \partial P_1} + \frac{\partial^2 S}{\partial q_2 \partial P_2} = 0. \quad (10)$$

Thus, the following proposition holds.

Proposition 1. *If system (6) has an invariant measure (7), then (10) holds.*

Note that, for systems for which the nonholonomic model admits an invariant measure, one should take into account the friction forces to describe the asymptotic behavior [37].

Reducing multiplier method. Suppose we are given an invariant measure $\mathcal{N}(\mathbf{q})$. Then we make the following change of variables

$$P_i = \frac{p_i}{\mathcal{N}(\mathbf{q})}, \quad i = 1, 2.$$

Denote the functions in the new variables by $\overline{H}(\mathbf{q}, \mathbf{p}) = H(\mathbf{q}, \mathbf{P}(\mathbf{q}, \mathbf{p}))$ and $\overline{S}(\mathbf{q}, \mathbf{p}) = S(\mathbf{q}, \mathbf{P}(\mathbf{q}, \mathbf{p}))$, respectively. Then the following relations hold for the derivatives:

$$\begin{aligned} \frac{\partial H}{\partial P_i} &= \mathcal{N} \frac{\partial \overline{H}}{\partial p_i}, & \frac{\partial S}{\partial P_i} &= \mathcal{N} \frac{\partial \overline{S}}{\partial p_i}, \\ \frac{\partial H}{\partial q_i} &= \frac{\partial \overline{H}}{\partial q_i} + \frac{1}{\mathcal{N}} \frac{\partial \mathcal{N}}{\partial q_i} \left(\frac{\partial \overline{H}}{\partial p_1} p_1 + \frac{\partial \overline{H}}{\partial p_2} p_2 \right). \end{aligned}$$

Further, substituting these relations into (6) and using (9), we obtain

$$\begin{aligned} \dot{q}_i &= \mathcal{N} \frac{\partial \overline{H}}{\partial p_i}, & \dot{p}_1 &= \mathcal{N} \left(-\frac{\partial \overline{H}}{\partial q_1} + K \frac{\partial \overline{H}}{\partial p_2} \right), & \dot{p}_2 &= \mathcal{N} \left(-\frac{\partial \overline{H}}{\partial q_2} - K \frac{\partial \overline{H}}{\partial p_1} \right), \\ & & K &= \mathcal{N} \left(\overline{S} - \frac{\partial \overline{S}}{\partial p_1} p_1 - \frac{\partial \overline{S}}{\partial p_2} p_2 \right). \end{aligned} \quad (11)$$

Write $\mathbf{x} = (\mathbf{p}, \mathbf{q})$ and represent the system (11) as

$$\dot{\mathbf{x}} = \mathcal{N} J \frac{\partial \overline{H}}{\partial \mathbf{x}}, \quad J = \begin{pmatrix} 0 & K & 1 & 0 \\ -K & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

For the skew-symmetric matrix J to define the Poisson bracket

$$\{q_i, q_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}, \quad \{p_1, p_2\} = K, \quad i = 1, 2,$$

and, thus, for the invariant measure \mathcal{N} to be a reducing multiplier, it is necessary that the Jacobi identity be valid, which in this case is reduced to the following two equations for \bar{S} :

$$p_1 \frac{\partial^2 \bar{S}}{\partial p_1^2} + p_2 \frac{\partial^2 \bar{S}}{\partial p_1 \partial p_2} = 0, \quad p_2 \frac{\partial^2 \bar{S}}{\partial p_2^2} + p_1 \frac{\partial^2 \bar{S}}{\partial p_1 \partial p_2} = 0. \quad (12)$$

If \bar{S} is a linear function in p_1 and p_2 , then the previous relation holds, and thus the equations of motion (11) can be represented in conformally Hamiltonian form.

Obviously, the linearity of \bar{S} means that \dot{q}_1 and \dot{q}_2 enter the Lagrangian L^* linearly and quadratically, which, as a rule, occurs in practice. We formulate the result thus obtained more clearly in the form of the following theorem.

Theorem 1. (reducing multiplier method) *If a system with a constraint of the form (1) and the Lagrangian*

$$L = \frac{1}{2} \sum_{i,j=1}^2 g_{ij}(\mathbf{q}) \dot{q}_i \dot{q}_j + \sum_{i=1}^2 c_i(\mathbf{q}) \dot{q}_i - U(\mathbf{q}), \quad (13)$$

where $\mathbf{q} = (q_1, q_2)$ and $\mathbf{g} = \|g_{ij}(\mathbf{q})\|$ is a symmetric matrix, admits a smooth invariant measure with density $\mathcal{N}(\mathbf{q})$, then the equations of motion (on the entire phase space) can be represented in conformally Hamiltonian form.

Proof. Indeed, if (10) holds, then, using the solution of the Liouville equation, one can represent the equations of motion in the form (11), and, in this case, $K = K(\mathbf{q})$ (i.e., K is a function depending on \mathbf{q} only).

In that case, it follows from (12) that J defines a Poisson bracket, and the equations of motion are represented in conformally Hamiltonian form. \square

Remark 3. *If a generalized Chaplygin system has a singular invariant measure with density $\mathcal{N}(\mathbf{q})$ (depending on the configuration variables only), then the equations of motion are represented in conformally Hamiltonian form, except for the domain in the phase space in which the density of the invariant measure either has a singularity or vanishes.*

Below we illustrate Theorem 1 and the above arguments by considering several problems of nonholonomic mechanics.

4 Motion of a mass point

Consider the motion of a mass point in Euclidean space \mathbb{R}^3 . In this case, (q_1, q_2, q_3) are the Cartesian coordinates of the point, and the Lagrangian function

is of the form

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - U(\mathbf{q}), \quad \mathbf{q} = (q_1, q_2), \quad (14)$$

where m is the mass of the particle and $U(\mathbf{q})$ is the potential of the external forces. In this case, condition (9) can be represented as

$$\frac{\partial}{\partial q_1} \left(\frac{a_1}{1 + a_1^2 + a_2^2} \left(\frac{\partial a_1}{\partial q_2} - \frac{\partial a_2}{\partial q_1} \right) \right) + \frac{\partial}{\partial q_2} \left(\frac{a_2}{1 + a_1^2 + a_2^2} \left(\frac{\partial a_1}{\partial q_2} - \frac{\partial a_2}{\partial q_1} \right) \right) = 0. \quad (15)$$

Let us consider three examples in more detail.

4.1 Nonholonomic Oscillator

Let the nonholonomic constraint be of the form

$$\dot{q}_3 - q_2 \dot{q}_1 = 0. \quad (16)$$

In accordance with [2], we refer to the system thus obtained as a *nonholonomic oscillator*. It is usually associated with the book of Rosenberg [35], although it was considered much earlier by Bottema [18] from the viewpoint of equilibrium positions and their stability and by Hamel [25] (the potential field of this system is usually assumed to be quadratic in \mathbf{q}).

Remark 4. Note that in [45] an implementation of the constraint (16) using a plate sliding on a knife edge was suggested. Here (q_2, q_3) are the coordinates of the center of mass of the plate and q_1 is the rotation angle of the plate (i.e., the configuration space in this case is $\mathbb{R}^2 \times S^1$).

In the case under consideration, $a_1(\mathbf{q}) = q_2$ and $a_2(\mathbf{q}) = 0$; then condition (15) holds, and the density of the invariant measure is

$$\mathcal{N}(\mathbf{q}) = (1 + q_2^2)^{\frac{1}{2}}.$$

Hence, after rescaling time $d\tau = \mathcal{N}dt$ (as follows from Theorem 1), the equations of motion of the nonholonomic oscillator are represented in Hamiltonian form with the canonical Poisson bracket and the Hamiltonian

$$\overline{H} = \frac{p_1^2}{2m} + \frac{(1 + q_2^2)p_2^2}{2m} + U(\mathbf{q}). \quad (17)$$

For the case where there is no potential, this result was obtained in [36] and for the case $U(\mathbf{q}) = q_2^2/2$ it was presented in [23].

Moreover, it turns out that, after the canonical change of variables

$$q_1 = x, \quad p_1 = p_x, \quad q_2 = \ln((1 + y^2)^{\frac{1}{2}} + y), \quad p_2 = (1 + y^2)^{\frac{1}{2}} p_y,$$

the Hamiltonian \overline{H} becomes

$$H^{(p)} = \frac{p_x^2 + p_y^2}{2m} + V(x, y), \quad (18)$$

where $V(x, y) = U(q_1, q_2(y))$.

Thus, the problem reduces to the investigation of the motion of a mass point with (Cartesian) coordinates (x, y) on a plane, in the potential field $V(x, y)$. The isomorphisms found (for a nonholonomic oscillator and the Heisenberg system) provide a natural explanation of the possibility of adding integrable potentials presented in [39].

4.2 Heisenberg System

Consider another example, namely, the *Heisenberg system*, for which the nonholonomic constraint is represented as

$$\dot{q}_3 - q_2 \dot{q}_1 + q_1 \dot{q}_2 = 0, \quad (19)$$

and which apparently first appeared in the book [20] in connection with control problems. Other nonholonomic systems (involving rolling motion) were considered in [11, 27, 41] in connection with control problems.

The authors of [33] consider the motion of a point in a potential field of the form

$$U(\mathbf{q}) = \frac{1}{2}(\alpha_1 q_1^2 + \alpha_2 q_2^2)$$

and prove, using the Poincaré mapping, that in this case the system exhibits chaotic behavior. It can be seen from the Poincaré map that the behavior of the trajectories of this system is similar to that of the trajectories of nonintegrable two-degree-of-freedom Hamiltonian systems.

It turns out that this similarity is not accidental and is due to the fact that in this case Theorem 1 applies for an (arbitrary) potential field $U(\mathbf{q})$.

Indeed, in this case, condition (15) holds identically, and the density of the invariant measure is of the form

$$\mathcal{N}(\mathbf{q}) = (q_1^2 + q_2^2 + 1)^{-1}.$$

Thus, the equations of motion are represented in conformally Hamiltonian form with Hamiltonian

$$\overline{H} = \frac{1 + q_1^2 + q_2^2}{2m} ((q_1 p_1 + q_2 p_2)^2 + p_1^2 + p_2^2) + U(\mathbf{q}) \quad (20)$$

and a canonical Poisson bracket.

It turns out that, in this case, the system (20) reduces to the investigation of the motion of a mass point on the sphere S^2 in a potential field. Indeed, let us carry out the central projection (for details, see, e.g., [4]),

$$q_1 = \tan \theta \cos \varphi, \quad q_2 = \tan \theta \sin \varphi,$$

where $\theta \in (0, \pi)$ and $\varphi \in [0, 2\pi)$, and pass to the (canonically conjugate) momenta

$$p_\theta = -\frac{1 + q_1^2 + q_2^2}{\sqrt{q_1^2 + q_2^2}}(q_1 p_1 + q_2 p_2), \quad p_\varphi = q_2 p_1 - q_1 p_2.$$

As a result, the Hamiltonian (20) becomes

$$H^{(s)} = \frac{p_\theta^2}{2m} + \frac{p_\varphi^2}{2m \sin^2 \theta} + V(\theta, \varphi),$$

where $V(\theta, \varphi) = U(q_1(\theta, \varphi), q_2(\theta, \varphi))$.

Because of the ambiguity of the central projection (taking two points on a sphere into one on a plane), the above isomorphism is defined only on half the hemisphere, and all the trajectories crossing the equator ($\theta = \frac{\pi}{2}$) are transformed into infinite trajectories on the plane.

4.3 Blackall Nonholonomic Constraint

In conclusion of the present section, consider the constraint $\dot{q}_3 - q_1 q_2 \dot{q}_1 = 0$ (suggested in [5]), for which condition (15) does not hold. In other words, in this case there is no invariant measure with density $\mathcal{N}(\mathbf{q})$ (depending on the configuration variables only), and hence, Theorem 1 does not apply.

In this case, the equations of motion (6) become

$$\begin{aligned} \dot{P}_1 &= \frac{q_1 q_2 P_1}{m(1 + q_1^2 q_2^2)} \left(\frac{q_2 P_1}{1 + q_1^2 q_2^2} - q_1 P_2 \right), & \dot{P}_2 &= \frac{2q_1^2 q_2 P_1^2}{m(1 + q_1^2 q_2^2)^2} \\ \dot{q}_1 &= \frac{P_1}{m(1 + q_1^2 q_2^2)}, & \dot{q}_2 &= \frac{P_2}{m}. \end{aligned} \quad (21)$$

It turns out that system (21) has a singular invariant measure (depending on phase variables)

$$(1 + q_1^2 q_2^2)^{\frac{1}{4}} |P_1|^{\frac{1}{2}} d\mathbf{q} d\mathbf{P}.$$

Note that system (21) has the following family of particular solutions:

$$P_1 = 0, \quad P_2 = \text{const}, \quad q_1 = \text{const}, \quad q_2 = \frac{P_2}{m} t, \quad (22)$$

in which the density of the invariant measure has a singularity.

Numerical experiments show that system (21) exhibits asymptotic behavior on the time interval $t \in (-\infty, +\infty)$, i.e., as $t \rightarrow -\infty$, the motion begins with an unstable solution (22) and, as $t \rightarrow +\infty$, tends to a stable solution. Thus, system (21) exhibits a behavior that differs substantially from that in the Hamiltonian case; see also [32] and [34].

5 Chaplygin sleigh on a plane

In this section, we consider the motion of a Chaplygin sleigh on a horizontal fixed plane. As a rule, by the Chaplygin sleigh one means a rigid body in the plane supported at two (or more) absolutely smooth legs and a sharp weightless wheel (disk or knife edge), which prevents its contact point P from slipping in the direction perpendicular to the plane of the wheel (see Fig. 1).

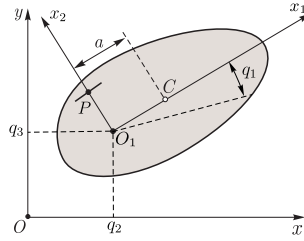


Fig. 1: Chaplygin sleigh on a plane

Note that, as an example illustrating the reducing multiplier method, Chaplygin [48] considered the motion of the Chaplygin sleigh. However, his considerations use substantially the quasi-coordinate introduced by him (which gave rise to a debate concerning the correctness of the method [46]).

Remark 5. *It is of interest that the reducing multiplier method has allowed the equations of motion to be represented in conformally Hamiltonian form in another well-known problem, due to Chaplygin [47], on the rolling motion of a dynamically asymmetric ball on a horizontal plane (for various generalizations of this problem, see [3, 6, 12, 14]). A qualitative and topological analysis of the motion of the contact point of the Chaplygin ball has been made recently in [10].*

In what follows we prove that the reducing multiplier method is applicable only in the case $a = 0$, i.e., if the center of mass, C , is placed on the perpendicular to the plane of the knife edge passing through the contact point P .

As a historical remark, we note that, although the Chaplygin sleigh is customarily associated with the works of Chaplygin [48] and Carathéodory [21], it was considered somewhat earlier by Brill in the book [19], as an example of the mechanism of a nonholonomic planimeter.

Remark 6. *Diverse generalizations (variations) of the problem of motion of the Chaplygin sleigh were considered in [43, 16]. For example, in [43], the motion of the Chaplygin sleigh with torque and on an inclined plane in a gravitational field was considered, and in [16] the equations of motion were obtained and the equilibrium positions were studied for the Chaplygin sleigh on a rotating plane.*

Introduce two coordinate systems: an inertial (fixed) one, Oxy , and a noninertial coordinate system $O_1x_1x_2$ attached to the Chaplygin sleigh (see Fig. 1).

The configuration space in this case coincides with the motion group of the plane $SE(2)$. To parameterize this space, we choose the angle q_1 of rotation of the axes of Oxy with respect to Ox_1x_2 and the Cartesian coordinates (q_2, q_3) of point O_1 in the coordinate system Oxy . Then the constraint equation in the chosen variables can be represented as

$$\dot{q}_3 - \frac{\cos q_1}{\sin q_1} \dot{q}_2 = 0.$$

The Lagrangian function is

$$L = \frac{1}{2}m(\dot{q}_2^2 + \dot{q}_3^2) + \frac{1}{2}I\dot{q}_1^2 + ma\dot{q}_1(-\dot{q}_2 \sin q_1 + \dot{q}_3 \cos q_1) - U(\mathbf{q}),$$

where $U(\mathbf{q})$ is the potential, and m and I are, respectively, the mass of the body and its moment of inertia relative to point O_1 . As a result, we obtain

$$S = \frac{\cos q_1}{\sin q_1} \left(\frac{2maP_1 \sin q_1 - ma^2P_2 \cos 2q_1 + IP_2}{I - ma^2 \cos^2 2q_1} \right).$$

A straightforward verification shows that relation (10) holds only for $a = 0$. In this case, we find a singular invariant measure (a reducing multiplier) in the form $\mathcal{N} = \sin q_1$.

Note that, for $a \neq 0$, the dynamics of the system is of asymptotic nature, which is why there is no (smooth) invariant measure with density of the form $\mathcal{N}(\mathbf{q})$.

Remark 7. *Nevertheless, for $U(\mathbf{q}) = 0$, but $a \neq 0$, one can represent the equations of motion in Hamiltonian form [43] using the method developed in [26].*

Applying Theorem 1 (for $a = 0$), we find the Hamiltonian

$$\overline{H} = \frac{p_1^2}{2I \sin^2 q_1} + \frac{p_2^2}{2m} + U(\mathbf{q}),$$

which, after the change of variables

$$x = \sqrt{\frac{I}{m}} \cos q_1, \quad p_x = \sqrt{\frac{m}{I}} \frac{p_1}{\sin q_1}, \quad y = q_2, \quad p_y = p_2,$$

reduces to the Hamiltonian (18), i.e., in this case, the problem reduces to investigating the motion of a mass point on a plane.

Remark 8. *In this example, in the constraint equation the singularity at $q_1 = 0, \pi$ is related to the choice of local coordinates. For this reason, in some cases, it is more convenient to study the equations of motion in quasi-coordinates.*

6 Conclusion

In conclusion, we discuss some open problems.

Above we have obtained equation (15) for Euclidean metric \mathbb{R}^3 ; this equation describes nonholonomic constraints of Chaplygin type for which the system admits a conformally Hamiltonian representation. Moreover, for the two examples of constraints, (16) and (19), in the absence of an external field, the system turns out to be integrable. In this connection, it would be of interest to find a general parameterization of constraints satisfying (15) and to find out in the general case whether the system is integrable without potential. We also note that, for holonomic systems, there are projective transformations [1] preserving the

trajectories of the metrics on the plane and on the sphere. The existence of such transformations for the systems considered remains an open problem.

As was shown above, by combining different constraints and potentials, one can obtain diverse systems that exhibit various effects typical of nonholonomic mechanics. It would be of interest to choose the simplest of these systems, which (for different parameter values) would possess all unusual properties of nonholonomic systems, such as limit cycles, strange attractors, etc. (a similar problem statement was considered in [40]).

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