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WEYL-EINSTEIN STRUCTURES ON K-CONTACT MANIFOLDS

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Abstract. We show that a compact K-contact manifold (M, g, ξ) has a closed Weyl-Einstein connection compatible with the conformal structure $[g]$ if and only if it is Sasaki-Einstein.

1. Introduction

K-contact structures — see the definition in Section 3 — can be viewed as the odd-dimensional counterparts of *almost Kähler* structure, in the same way as *Sasakian* structures are the odd-dimensional counterparts of *Kähler structures*. It has been shown in [4], cf. also [1], that *compact* Einstein K-contact structures are actually Sasakian, hence Sasaki-Einstein. In this note, we consider the more general situation of a compact K-contact manifold (M, g, ξ) carrying in addition a *Weyl-Einstein* connection D compatible with the conformal class $[g]$, already considered by a number of authors, in particular in [8] and [10]. We show — Theorem 3.2 and Corollary 3.1 below — that g is then Einstein and D is the Levi-Civita connection of g, in particular that the K-contact structure is Sasaki-Einstein.

2. Conformal vector fields

Let (M, c) be a (positive definite) conformal manifold of dimension n. A vector field ξ on M is called *conformal Killing* with respect to c if it preserves c, meaning that for any metric g in c, the trace-free part $(\mathcal{L}_\xi g)_0$ of the Lie derivative $\mathcal{L}_\xi g$ of g along ξ is identically zero, hence that $\mathcal{L}_\xi g = f\, g$, for some function f, depending on ξ and g, and it is then easily checked that $f = -\frac{\delta^g \eta_g}{n}$, where η_g denotes the 1-form dual to ξ and $\delta^g \eta_g$ the co-differential of ξ with respect to g. In particular, a conformal Killing vector field ξ on M is Killing with respect to some metric g in c if and only if $\delta^g \eta_g = 0$. In this section, we present a number of facts concerning conformal Killing vector fields for further use in this note.

Proposition 2.1. *Let (M, g) be a connected compact oriented Riemannian manifold of dimension n, $n \geq 2$, carrying a non-trivial parallel vector field T. Let ξ be any conformal Killing vector field on M with respect to the conformal class $[g]$ of g. Then, ξ is Killing with respect to g; moreover, it commutes with T and the inner product $a := g(\xi, T)$ is constant.*

Proof. Denote by $\eta = \xi^\flat$ the 1-form dual to ξ and by $\delta\eta$ the co-differential of η with respect to g; then, ξ is Killing if and only if $\delta\eta = 0$. Denote by ∇ the Levi-Civita connection of g and by $\mathcal{L}_T$ the Lie derivative along T; then, $\nabla_T \xi = [T, \xi] = \mathcal{L}_T \xi$ is conformal Killing, and we have:

$$\delta\big(\nabla_T \eta\big) = \delta(\mathcal{L}_T \eta) = \mathcal{L}_T(\delta\eta) = T(\delta\eta). \tag{2.1}$$

Since T is non-trivial, we may assume $|T| \equiv 1$. Denote $a = g(\xi, T) = \eta(T)$. Since ξ is conformal Killing, $\nabla\xi = A - \frac{\delta\eta}{n}\,\mathrm{Id}$, where A is skew-symmetric and Id denotes the identity;

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for any vector field X we then have: $\mathrm{d}a(X) = g(\nabla_X \xi, T) = -g(\nabla_T \xi, X) - \frac{2\delta\eta}{n}(X, T)$. We thus get:

$$\nabla_T \eta = -\mathrm{d}a - \frac{2\delta\eta}{n}\,\theta, \tag{2.2}$$

where $\theta = T^\flat$ denotes the 1-form dual to T. By evaluating both members of (2.2) at T, we get:

$$\delta\eta = -n\,\mathrm{d}a(T), \tag{2.3}$$

whereas, by considering their co-differential and by using (2.1), we get:

$$\Delta a = -\frac{(n-2)}{n}\,T(\delta\eta), \tag{2.4}$$

where $\Delta a = \delta \mathrm{d}a$ denotes the Laplacian of a. Denote by v_g the volume form determined by g and the chosen orientation; from (2.3) and (2.4), we then infer:

$$\begin{aligned}\int_M a\,\Delta a\, v_g = \int_M |\mathrm{d}a|^2\, v_g &= -\frac{(n-2)}{n}\int_M \delta(a\theta)\,\delta(\eta)\, v_g\\ &= \frac{(n-2)}{n}\int_M \mathrm{d}a(T)\,\delta\eta\, v_g = -(n-2)\int_M (\delta\eta)^2\, v_g,\end{aligned}$$

hence

$$\int_M |\mathrm{d}a|^2\, v_g = -(n-2)\int_M (\delta\eta)^2\, v_g. \tag{2.5}$$

This readily implies that $\mathrm{d}a = 0$ and, either by (2.5) if $n > 2$ or by (2.3) if $n = 2$, that $\delta\eta = 0$, i.e. that ξ is Killing. Finally, by (2.2) we infer that $\nabla_T \xi = [T, \xi] = 0$. ☐

Remark 2.1. Proposition 2.1 can be viewed as a particular case of a more general statement (Theorem 2.1 in [11]) concerning conformal Killing forms on Riemannian products.

Proposition 2.2. *Assume that (M^n, g) is a compact oriented Einstein manifold carrying a conformal vector field which is not Killing. Then (M, g) is, up to constant rescaling, isometric to the round sphere $\mathbb{S}^n$.*

Proof. We first recall the following lemma, due to A. Lichnerowicz, cf. [9, Paragarph 85]:

Lemma 2.1. *Let (M, g) be a connected compact Einstein manifold of dimension $n \geq 2$. of positive (constant) scalar curvature* Scal. *Denote by λ_1 the smallest positive eigenvalue of the Riemannian Laplacian acting on functions. Then,*

$$\lambda_1 \geq \frac{\mathrm{Scal}}{(n-1)}, \tag{2.6}$$

with equality if and only if $\mathrm{grad}_g f$ *is conformally Killing for each function f in the eigenspace of λ_1.*

Proof. As before denote by ∇ the Levi-Civita connection of the metric g and denote by Ric the Ricci tensor of g. For any vector field ξ on M, denote by $\eta = \xi^\flat$ the 1-form dual to ξ with respect to g. The covariant derivative $\nabla\eta$ of η then splits as follows:

$$\nabla\eta = \frac{1}{2}\left(\mathcal{L}_\xi g\right)_0 + \frac{1}{2}\mathrm{d}\eta - \frac{\delta\eta}{n}\, g, \tag{2.7}$$

where $(\mathcal{L}_\xi g)_0$ denotes the trace-free part of $\mathcal{L}_\xi g$. By using (2.7) the *Bochner identity*

$$\Delta\eta = \delta\nabla\eta + \mathrm{Ric}(\xi) \tag{2.8}$$

can be rewritten as

$$\mathrm{Ric}(\xi) = -\frac{1}{2}\delta\left(\mathcal{L}_\xi g\right)_0 + \frac{(n-1)}{n}\,\mathrm{d}\delta\xi + \frac{1}{2}\delta\mathrm{d}\xi. \tag{2.9}$$

Let λ be any positive eigenvalue of Δ and f any non-zero element of the corresponding eigenspace, so that $\Delta f = \lambda\, f$. By substituting $\eta = \mathrm{d}f$ and $\mathrm{Ric} = \frac{\mathrm{Scal}}{n}\, g$ in (2.9), we get

$$\Delta \mathrm{d}f = \lambda\,\mathrm{d}f = \frac{\mathrm{Scal}}{(n-1)}\,\mathrm{d}f + \frac{n}{2(n-1)}\,\delta\left(\mathcal{L}_{\mathrm{grad}_g f}\, g\right)_0, \tag{2.10}$$

where $\mathrm{grad}_g f = (\mathrm{d}f)^\sharp$ denotes the gradient of f with respect to g. By contracting with $\mathrm{d}f$ and integrating over M, we obtain

$$\left(\lambda - \frac{\mathrm{Scal}}{(n-1)}\right)\int_M |\mathrm{d}f|^2\, v_g = \frac{n}{2(n-1)}\int_M |\left(\mathcal{L}_{\mathrm{grad}_g f}\, g\right)_0|^2\, v_g \geq 0, \tag{2.11}$$

so that $\lambda \geq \frac{\mathrm{Scal}}{(n-1)}$, with equality if and only if $\left(\mathcal{L}_{\mathrm{grad}_g f}\, g\right)_0 = 0$, hence if and only if $\mathrm{grad}_g f$ is conformal Killing. ☐

In view of Lemma 2.1, the proof of Proposition 2.2 goes as follows. Let ξ be any conformal Killing vector field on M, with dual 1-form η. From (2.9), we get:

$$\mathrm{Ric}(\xi)\frac{\mathrm{Scal}}{n}\,\eta = \frac{(n-1)}{n}\,\mathrm{d}\delta\eta + \frac{1}{2}\delta\mathrm{d}\eta, \tag{2.12}$$

hence

$$\Delta(\delta\eta) = \frac{\mathrm{Scal}}{(n-1)}\,\delta\eta. \tag{2.13}$$

We then infer from Lemma 2.1 that $\mathrm{grad}_g(\delta\eta)$ is conformal Killing. It is a well-known theorem of M. Obata [14], previously proved in [12], [13] in the Einstein setting, that the only compact Riemannian manifold admitting a non trivial gradient as a conformal Killing vector field is the standard sphere $\mathbb{S}^n$. If $(M, g) \neq \mathbb{S}^n$, we then infer that $\delta\eta = 0$. ☐

3. Weyl-Einstein connections on K-contact manifolds

Definition 3.1. *A K-contact manifold is an oriented Riemannian manifold (M, g) of odd dimension $n = 2m+1$, endowed with a unit Killing vector field ξ whose covariant derivative $\varphi := \nabla\xi$ satisfies*

$$\varphi^2 = -\mathrm{Id} + \eta\otimes\xi, \tag{3.1}$$

where η is the metric dual 1-form of ξ.

Since ξ is Killing, we have $\mathrm{d}\eta(X, Y) = 2g(\varphi(X), Y)$ for all vector fields X and Y. The kernel of the 2-form $\mathrm{d}\eta$, equal to that of φ, is then spanned by ξ:

$$\ker(\mathrm{d}\eta) = \ker(\varphi) = \mathbb{R}\xi. \tag{3.2}$$

It follows that the restriction of $\mathrm{d}\eta$ to $\mathcal{D} := \ker(\eta)$ is non-degenerate, hence that $\mathcal{D}$ is a contact distribution on M. Moreover, since $\eta(\xi) = 1$ and $\xi\lrcorner\mathrm{d}\eta = 0$, ξ is the Reeb vector field of the contact 1-form η.

Proposition 3.1 (cf. [3])**.** *The Ricci tensor of a K-contact manifold satisfies* $\mathrm{Ric}(\xi,\cdot) = 2m\, g(\xi,\cdot) = 2m\,\eta$.

Proof. Since ξ is Killing and $\nabla\xi = \varphi$, we have

$$\nabla_X\varphi = \mathrm{R}_{\xi,X}, \tag{3.3}$$

from which we get:

$$\nabla_\xi\varphi = 0, \tag{3.4}$$

and

$$\delta\varphi = \mathrm{Ric}(\xi) \tag{3.5}$$

— here $\delta\varphi$ denotes the co-differential of φ and Ric is regarded as a field of endomorphisms of $\mathrm{T}M$ — whereas, from (3.1) we readily infer

$$\nabla_X\varphi\circ\varphi + \varphi\circ\nabla_X\varphi = \frac{1}{2}X\lrcorner \mathrm{d}\eta\otimes\xi + \eta\otimes\varphi(X), \tag{3.6}$$

hence

$$(\nabla_X\varphi)(\xi) = \mathrm{R}_{\xi,X}\xi = X - \eta(X)\xi, \tag{3.7}$$

for any vector field X, from which we get

$$\mathrm{Ric}(\xi,\xi) = n-1 = 2m. \tag{3.8}$$

In view of (3.8) and (3.2), to prove Proposition 3.1 it is sufficient to check that $\varphi\big(\mathrm{Ric}(\xi)\big) = 0$, or else, by (3.5), that $\varphi(\delta\varphi) = 0$. In view of (3.4), we have

$$\delta\varphi = -\sum_{i=1}^{2m}(\nabla_{e_i}\varphi)(e_i), \tag{3.9}$$

for any auxiliary (local) orthonormal frame of $\mathcal{D}$; from (3.6) we thus get

$$\varphi(\delta\varphi) = \sum_{i=1}^{2m}(\nabla_{e_i}\varphi)\big(\varphi(e_i)\big). \tag{3.10}$$

Since φ is associated to the *closed* 2-form $\mathrm{d}\eta$, for any vector field X we have:

$$g\left(\sum_{i=1}^{2m}(\nabla_{e_i}\varphi)\big(\varphi(e_i)\big), X\right) = -\frac{1}{2}\sum_{i=1}^{2m} g\big((\nabla_X\varphi)(e_i), \varphi(e_i)\big) = -g(\nabla_X\varphi,\varphi),$$

which is equal to zero since the norm of φ is constant. □

Definition 3.2. *A Weyl connection on a conformal manifold* (M,c) *is a torsion-free linear connection* D *which preserves the conformal class* c*.*

The latter condition means that for any metric g in the conformal class c, there exists a real 1-form, θ^g, called the *Lee form* of D with respect to g, such that $Dg = -2\theta^g\otimes g$, and D is then related to the Levi-Civita connection, ∇^g, of g by

$$D_XY = \nabla^g_XY + \theta^g(X)Y + \theta^g(Y)X - g(X,Y)\ (\theta^g)^{\sharp_g}\,, \tag{3.11}$$

cf. e.g. [5]. A Weyl connection D is said to be *closed* if it is locally the Levi-Civita connection of a (local) metric in c, *exact* if it is the Levi-Civita connection of a (globally defined) metric in c; equivalently, D is closed, resp. exact, if its Lee form is closed, resp. exact, with respect to one, hence any, metric in c.

If M is compact, for any Weyl connection on (M, c) there exists a distinguished metric, say g_0, in c, usually called the *Gauduchon metric* of D, unique up to scaling, whose Lee form θ^{g_0} is co-closed with respect to g_0, [6]. If D is closed, θ^{g_0} is then g_0-harmonic, identically zero if D is exact.

The *Ricci tensor*, Ric^D, of a Weyl connection D is the bilinear defined by $\mathrm{Ric}(X, Y) = \mathrm{trace}\{Z \mapsto \mathrm{R^D}_{X,Z}Y\} = \sum_{i=1}^n g(\mathrm{R}^D_{X,e_i}Y, e_i)$, for any metric g in c and any g-orthonormal basis $\{e_i\}_{i=1}^n$. The Ricci tensor Ric^D defined that way is symmetric if and only if D is closed.

A Weyl connection D is called *Weyl-Einstein* is the trace-free component of the symmetric part of Ric^D is identically zero. A closed Weyl-Einstein connection is locally the Levi-Civita connection of a (local) Einstein metric in c; an exact Weyl-Einstein connection is the Levi-Civita connection of a (globally defined) Einstein metric.

We here recall the following well-known result:

Theorem 3.1 ([15], [7])**.** *Let D be a Weyl-Einstein connection defined on a compact connected oriented conformal manifold (M, c) and denote by g_0 denote its Gauduchon metric. Then the vector field T on M dual to the Lee form θ^{g_0} is Killing with respect to g_0. If D is closed, T is parallel with respect to g_0, identically zero if and only if D is exact.*

The aim of this section is to prove the following:

Theorem 3.2. *Let (M^{2m+1}, g, ξ) be a compact K-contact manifold carrying a closed Weyl-Einstein structure D compatible with the conformal class $c = [g]$. Then D is the Levi-Civita connection of g, which is then Einstein.*

Proof. Let $g_0 := e^{2f} g$ denote the Gauduchon metric of D and let T denote the g_0-dual of the Lee form of D with respect to g_0. According to Theorem 3.1, T is ∇^{g_0}-parallel. We first show that $T \equiv 0$, i.e. that the closed Weyl-Einstein connection D is actually exact.

Assume, for a contradiction, that T is non-zero. By rescaling the Gauduchon metric g_0 if necessary, we may assume that $g_0(T, T) = 1$. Denote by η, resp. η_0, the 1-form dual to ξ with respect to g, resp. g_0. Both η and η_0 are contact 1-forms for the contact distribution $\mathcal{D}$, and, as already noticed, ξ is the Reeb vector field of η According to Proposition 2.1, ξ, which is Killing with respect to g, hence conformal Killing with respect to $[g] = [g_0]$, is Killing with respect to g_0 as well, commutes with T, and the inner product $a := g_0(\xi, T) = \eta_0(T)$ is constant; we then have: $\mathcal{L}_T\eta_0 = 0$, hence that $T \lrcorner \mathrm{d}\eta_0 = \mathcal{L}_T\eta_0 - \mathrm{d}\big(\eta_0(T)\big) = -\mathrm{d}a = 0$. Moreover, since $\eta_0(T) = a$ and $T \lrcorner \mathrm{d}\eta_0$, a cannot be zero — otherwise, η_0 would not be a contact 1-form — and $\xi_0 := a^{-1} T$ is then the Reeb vector field of η_0. Since $\eta_0 = e^{2f}\, \eta_0$, the Reeb vector fields ξ_0 and ξ are related by

$$\xi_0 = e^{-2f}\, \xi + Z_f, \tag{3.12}$$

where Z_f is the section of $\mathcal{D}$ defined by

$$Z_f \lrcorner \mathrm{d}\eta = 2e^{-2f}\, \mathrm{d}f_{|\mathcal{D}}. \tag{3.13}$$

Since M is compact, f has critical points and for each of them, say x, it follows from (3.13) that $Z_f(x) = 0$, hence $\xi_0(x) = a^{-1}\, T(x) = e^{-2f(x)}\, \xi(x)$. Since, $g_0(T(x), T(x)) = 1$ and $g_0(\xi(x), T(x)) = a$ for any x, we infer that $e^{2f(x)} = a^2$ for *any* critical point x of f, in particular for points where f takes its minimal or its maximal value. It follows that f is *constant*, with $e^{2f} \equiv a^2$, that $g_0 = a^2 g$ and $\xi = a T$. In particular, η and $\eta_0 = a^2 \eta$ are parallel, with respect to g and g_0, hence closed. This contradicts the fact that they are contact 1-forms.

In view of the above, T must be identically zero. This means that D is the Levi-Civita connection of the Gauduchon metric g_0, which is thus Einstein. Since ξ is conformal with respect to g_0, from Proposition 2.2, either ξ is Killing with respect to g_0, or (M, g_0) is homothetic to $\mathbb{S}^{2m+1}$ (in which case one can rescale g_0 in order to have $(M, g_0) = \mathbb{S}^{2m+1}$).

Case 1. ξ is Killing with respect to g_0. Since ξ is Killing with respect to $g = e^{-2f} g_0$, this means that $\xi(f) = 0$, or equivalently

$$g(\xi, \mathrm{grad}_g f) = 0. \tag{3.14}$$

Let λ denote the Einstein constant of (M, g_0), so that $\mathrm{Ric}^0 = \lambda g_0 = e^{2f} \lambda g$. The classical formula relating the Ricci tensors Ric and Ric^0 of g and g_0 reads (cf. [2], p. 59):

$$\mathrm{Ric}^0 = \mathrm{Ric} - (2m-1)(\nabla^g \mathrm{d}f - \mathrm{d}f \otimes \mathrm{d}f) + (\Delta^g f - (2m-1)|\mathrm{d}f|^2_g) g. \tag{3.15}$$

Contracting (3.15) with ξ and using Proposition 3.1 we get

$$\lambda e^{2f} \eta = 2m\eta - (2m-1)\nabla^g_\xi \mathrm{d}f + (\Delta^g f - (2m-1)|\mathrm{d}f|^2_g)\eta.$$

Taking the metric duals with respect to g this equation reads

$$\nabla^g_\xi(\mathrm{grad}_g f) = h\xi, \qquad \text{with } h := \frac{1}{2m-1}\left(\Delta^g f - (2m-1)|\mathrm{d}f|^2_g + 2m - \lambda e^{2f}\right). \tag{3.16}$$

On the other hand, we have $0 = \mathrm{d}\mathcal{L}_\xi f = \mathcal{L}_\xi \mathrm{d}f$, thus $\mathcal{L}_\xi(\mathrm{grad}_g f) = 0$ and therefore

$$\nabla^g_\xi(\mathrm{grad}_g f) = \nabla^g_{\mathrm{grad}_g f}\xi = \varphi(\mathrm{grad}_g f).$$

Since the image of φ is orthogonal to ξ, (3.16) implies that $\varphi(\mathrm{grad}_g f) = 0$, thus by (3.2), $\mathrm{grad}_g f$ is proportional to ξ. From (3.14) we thus get $\mathrm{grad}_g f = 0$, so f is constant and D is the Levi-Civita connection of g, and hence g is Einstein.

Case 2. (M, g_0) is isometric to $\mathbb{S}^{2m+1}$. The vector field ξ is a conformal vector field of the round sphere, thus it is a sum $\xi = K + G$ where K is Killing and G is a gradient conformal vector field, i.e. $G = \mathrm{grad}_{g_0} h$ where h is a first spherical harmonic. More precisely, there exists a skew-symmetric matrix $A \in \mathfrak{so}_{2m+2}$ and a vector $v \in \mathbb{R}^{2m+2}$ such that $K_x = Ax$ and $h(x) = \langle x, v\rangle$ for every $x \in \mathbb{S}^{2m+1} \subset \mathbb{R}^{2m+2}$.

Since ξ has unit length with respect to $g = e^{-2f} g_0$, the conformal factor satisfies $e^{2f} = g_0(\xi, \xi)$. Consider as before the metric duals η and $\eta_0 = e^{2f}\eta$ of ξ with respect to g and g_0. Since $\eta_0 = K^\flat + G^\flat = K^\flat + \mathrm{d}h$, we have

$$\mathrm{d}\eta_0 = \mathrm{d}K^\flat = A, \tag{3.17}$$

where A is the restriction to $\mathbb{S}^{2m+1}$ of the constant 2-form $A \in \mathfrak{so}_{2m+2} = \Lambda^2\mathbb{R}^{2m+2}$. Since $\mathrm{d}\eta = e^{-2f}\big(-2df \wedge \eta_0 + \mathrm{d}\eta_0\big)$, by (3.17), at any critical point x of f the 2-form $\mathrm{d}\eta$ reads

$$\mathrm{d}\eta_x = e^{-2f(x)}\mathrm{d}\eta_0 = e^{-2f(x)} A.$$

From (3.1) it follows that the symmetric endomorphism $e^{-4f(x)}A^2$ of $\mathbb{R}^{2m+2}$ has two eigenvalues, 0 and -1, (the 0-eigenspace being generated by x and ξ_x). Using this at a minimum and maximum points x_m and x_M of f, we get $e^{-4f(x_m)} = e^{-4f(x_M)}$, thus f is constant, and like before, g is homothetic to g_0 and thus Einstein. This concludes the proof of the theorem. □

As a direct corollary of Theorem 3.2 above together with Theorem 1.1 in [1] (see also [4]), we obtain the following result:

Corollary 3.1. *If (M^{2m+1}, g, ξ) is a compact K-contact manifold carrying a closed Weyl-Einstein structure compatible with g, then M is Sasaki-Einstein.*

Remark 3.1. *In* [10] *it is claimed that if (M^{2m+1}, g, ξ) is a compact K-contact manifold carrying a compatible closed Weyl-Einstein structure, then M is Sasakian if and only if it is η-Einstein. Our above result show that the hypotheses in* [10] *already imply both conditions.*

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