QUANTUM SUBGROUPS OF SIMPLE TWISTED QUANTUM GROUPS AT ROOTS OF ONE

GASTÓN A. GARCÍA AND JAVIER A. GUTIÉRREZ

Abstract. Let G be a connected, simply connected simple complex algebraic group and let ϵ be a primitive ℓth root of unity with ℓ odd and coprime with 3 if G is of type G_2. We determine all Hopf algebra quotients of the twisted multiparameter quantum function algebra $\mathcal{O}_\epsilon^\varphi(G)$ introduced in [CV1]. This extends the results of [AG], where the untwisted case is treated.

1. Introduction

Let G be a connected, simply connected complex algebraic group. In these notes we determine all Hopf algebra quotients of the twisted multiparameter quantum function algebra $\mathcal{O}_\epsilon^\varphi(G)$ introduced by Costantini and Varagnolo in [CV1], where ϵ is a primitive ℓth root of unity with ℓ odd and coprime with 3 if G is of type G_2. The dual notion of this deformation was introduced by Reshetikhin [R] to produce multiparameter enveloping algebras of $\mathfrak{g} = \mathrm{Lie}(G)$, see also [Su]. It is constructed as a twist deformation of the topological Hopf algebra $U_\hbar(\mathfrak{g})$ over $\mathbb{C}[[\hbar]]$, where the twist only invlolves elements of a fixed Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. In the dual function algebra, this deformation corresponds to a skew endomorphism φ on the weight lattice of G. When $\varphi = 0$, one recovers the standard quantum function algebra on G and the resuIts on this paper reproduce the classification obtained in [AG].

It turns out that $\mathcal{O}_\epsilon^\varphi(G)$ is a 2-cocycle deformation of $\mathcal{O}_\varepsilon(G)$, see Lemma 2.14. For this reason, we call $\mathcal{O}_\epsilon^\varphi(G)$ a *twisted quantum groups*. This is not an isolated example. The relation between multiparameter quantum groups and 2-cocycle deformations has been explained for particular instances of quantum groups; see for example [Ma], [Tk2], [AST], [HLT]. In general, multiparameter quantum groups were intensively studied. They appeared first in the work of Manin [Ma] and were subsequently treated by different authors, among them [AE, BW, CM, DPW, H, HLT, HPR, LS, OY, R, Tk].

An important problem in the theory of quantum groups is the determination of the general properties that a quantum group should have, since up to date there is no axiomatic definition of an *algebraic* quantum group. In this sense, the description of all possible Hopf algebra quotients, *i.e.* the quantum subgroups, of the known examples would give some insight on the structure of the quantum group. This can be viewed as the quantum

2010 Mathematics Subject Classification: 81R50, 17B37, 20G42, 16W30, 16W35.
Keywords: twisted quantum function algebras, quantum subgroups, Hopf algebra quotients
This work was partially supported by ANPCyT-Foncyt, CONICET, Secyt (UNC).

version of the classical problem of studying subgroups of a simple algebraic group. This is an actual area of research since the description by P. Podleś [P] of the compact quantum subgroups of Woronowicz's quantum groups $SU_q(2)$ and $SO_q(3)$ for $q \in [-1,1]\backslash 0$. Besides the result of Podleś, the main contributions are

- ▷ the description of the finite quantum subgroups of $GL_q(n)$ and $SL_q(n)$ for q an odd root of unity by Müller [Mu];
- ▷ the classification in [AG] of the quantum subgroups of G_q, for G a connected, simply connected, complex simple algebraic group G, with q a primitive ℓth root of unity with ℓ odd and coprime with 3 if G is of type G_2.
- ▷ the description in [G] of the quantum subgroups of the two-parameter deformation $GL_{\alpha,\beta}(n)$ for $\alpha^{-1}\beta$ a primitive root of unity of odd order;
- ▷ the compact quantum subgroups of $SO_{-1}(3)$ were determined by Banica and Bichon [BB];
- ▷ the study of the quantum subgroups of $SU_q(2)$ for $q=-1$ in [BN] and for $q \neq -1$ in [FST].
- ▷ the description by Bichon and Dubois-Violette [BD] of the compact quantum subgroups of the half-liberated orthogonal quantum groups O_n^* from [BS].
- ▷ the classification of the quantum subgroups of $SU_{-1}(3)$ by Bichon and Yuncken [BY].

As the reader might have noticed, the problem splits into the algebraic case and the compact case. The latter is reduced mainly to the case when $q=-1$. In this paper, we study the algebraic case, hence we will assume that q is a primitive root of unity. The main result reads

Theorem 1. *There is a bijection between*

(a) *Hopf algebra quotients* $q : \mathcal{O}_\epsilon^\varphi(G) \to A$.
(b) *Twisted subgroup data up to equivalence.*

For the definition of the twisted subgroup data see Definition 4.8. We prove Theorem 1 in Section 4 through Theorems 4.9, 3.4 and 4.15. We use the strategy developed in [AG] for the untwisted case, where the quantum subgroups are constructed using commutative diagrams whose rows are central extension of Hopf algebras. Since $\mathcal{O}_\epsilon^\varphi(G)$ is a 2-cocycle deformation of $\mathcal{O}_\varepsilon(G)$, several steps of the construction can be carried out without much effort. On the other hand, special attention has to be paid for certain constructions. To describe them we use the study of $\mathcal{O}_\epsilon^\varphi(G)$ carried out by Costantini and Varagnolo in [CV1] and [CV2], which is in turn a generalization of [DL].

A consequence of Theorem 1 is the construction of new examples of finite-dimensional Hopf algebras with different properties which might not be necessarily 2-cocycle deformation of quantum subgroups of $\mathcal{O}_\varepsilon(G)$. These examples are given by central exact sequences of Hopf algebras. Due to a result of Ştefan [Ş], which characterize Hopf algebras with certain properties as quantum subgroups of $SL_q(2)$, these examples might help to understand better the classification problem of finite-dimensional Hopf algebras over the complex numbers.

The paper is organized as follows. In Section 2 we recall the definition and general properties of the twisted quantum groups $U_q^{\varphi}(\mathfrak{g})$, its divided power algebra, the twisted quantum function algebra $\mathcal{O}_q^{\varphi}(G)$ and their specializations at roots of unity, and we show that $\mathcal{O}_\epsilon^{\varphi}(G)$ is a 2-cocycle deformation of the one-parameter quantum group $\mathcal{O}_\varepsilon(G)$. In Section 3 we describe the twisted Frobenius-Lusztig kernels $\mathbf{u}_\epsilon^{\varphi}(\mathfrak{g})$ and all the Hopf algebra quotients of $\mathbf{u}_\epsilon^{\varphi}(\mathfrak{g})^*$. We also prove that $\mathbf{u}_\epsilon^{\varphi}(\mathfrak{g})$ is a twist deformation of $\mathbf{u}_\varepsilon(\mathfrak{g})$. Finally, in Section 4 we prove the main theorem.

Conventions and Preliminaries. Our references for the theory of Hopf algebras are [Mo], [Ra]. We use standard notation for Hopf algebras; the comultiplication, counit and antipode are denoted by Δ, ε and $\mathcal{S}$, respectively. Let $\Bbbk$ be a field. The set of group-like elements of a coalgebra C is denoted by $G(C)$. We also denote by $C^+ = \operatorname{Ker}\varepsilon$ the augmentation ideal of C. Let H be a Hopf algebra. H^{op} denotes the Hopf algebra with the same coalgebra structure but opposite multiplication and H^{cop} denotes the Hopf algebra with the same algebra structure but opposite comultiplication. Let $g, h \in G(H)$, the set of (g,h)-primitive elements is given by $P_{g,h}(H) = \{x \in H : \ \Delta(x) = x \otimes g + h \otimes x\}$. We call $P_{1,1}(H) = P(H)$ the set of primitive elements.

Recall that a convolution invertible linear map σ in $\operatorname{Hom}_{\Bbbk}(H \otimes H, \Bbbk)$ is a *normalized multiplicative 2-cocycle* if

$$\sigma(b_{(1)}, c_{(1)})\sigma(a, b_{(2)}c_{(2)}) = \sigma(a_{(1)}, b_{(1)})\sigma(a_{(2)}b_{(2)}, c) \tag{1}$$

and $\sigma(a, 1) = \varepsilon(a) = \sigma(1, a)$ for all $a, b, c \in H$, see [Mo, Sec. 7.1]. In particular, the inverse of σ is given by $\sigma^{-1}(a, b) = \sigma(\mathcal{S}(a), b)$ for all $a, b \in H$. Using a 2-cocycle σ it is possible to define a new algebra structure on H by deforming the multiplication, which we denote by H_σ. Moreover, H_σ is indeed a Hopf algebra with $H = H_\sigma$ as coalgebras, deformed multiplication $m_\sigma = \sigma * m * \sigma^{-1} : H \otimes H \to H$ given by

$$m_\sigma(a, b) = a \cdot_\sigma b = \sigma(a_{(1)}, b_{(1)})a_{(2)}b_{(2)}\sigma^{-1}(a_{(3)}, b_{(3)}) \qquad \text{for all } a, b \in H,$$

and antipode $\mathcal{S}_\sigma = \sigma * \mathcal{S} * \sigma^{-1} : H \to H$ given by (see [Do] for details)

$$\mathcal{S}_\sigma(a) = \sigma(a_{(1)}, \mathcal{S}(a_{(2)}))\mathcal{S}(a_{(3)})\sigma^{-1}(\mathcal{S}(a_{(4)}), a_{(5)}) \qquad \text{for all } a \in H.$$

Remark 1.1. Let H be a Hopf algebra, I a Hopf ideal, $A = H/I$ and $\pi : H \to A$ the canonical map. Clearly, any 2-cocycle on A can be lifted through π to a 2-cocycle on H. Let $\sigma : H \otimes H \to \Bbbk$ a normalized multiplicative 2-cocycle on H such that $\sigma|_{I \otimes H + H \otimes I} = 0$. Then the map $\hat{\sigma} : A \otimes A \to \Bbbk$ given by $\hat{\sigma}(\pi(h), \pi(k)) = \sigma(h, k)$ for all $h, k \in H$ defines a normalized multiplicative 2-cocycle on A and the induced map $\pi_\sigma : H_\sigma \to A_{\hat{\sigma}}$ is a Hopf algebra map. In particular, if B is a central Hopf subalgebra of H such that $\sigma|_{B \otimes B} = \varepsilon \otimes \varepsilon$, then the formula above defines a 2-cocycle on $A = H/B^+H$.

Let H be a Hopf algebra and $J \in H \otimes H$ an invertible element. We say that J is a *normalized twist* if

$$(\Delta \otimes \mathrm{id})(J)(J \otimes 1) = (\mathrm{id} \otimes \Delta)(J)(1 \otimes J) \quad \text{and} \quad (\varepsilon \otimes \mathrm{id})(J) = 1 = (\mathrm{id} \otimes \varepsilon)(J).$$

Given a twist J for H, one can define a new Hopf algebra H^J with the same algebra structure and counit as H, but different comultiplication and antipode

$$\Delta^J(h) = J\Delta(h)J, \qquad \mathcal{S}^J(h) = Q_J^{-1}\mathcal{S}(h)Q_J,$$

for all $h \in H$, where we denote $J = J^{(1)} \otimes J^{(2)}$ and $Q_J = \mathcal{S}(J^{(1)})J^{(2)}$. We say that H^J is a twist deformation of H.

The notion of 2-cocycle and twist are dual of each other. If H is finite-dimensional, then J is a twist for H if and only if J^* is a 2-cocycle on H^*.

Definition 1.2. A *Hopf pairing* between two Hopf algebras U and H over a ring R is a bilinear form $b : H \times U \to R$ such that, for all $u,\ v \in U$ and $f,\ h \in H$,

$$\begin{array}{llll} (i) & b(h, uv) = b(h_{(1)}, u)b(h_{(2)}, v); & (iii) & b(1, u) = \varepsilon(u); \\ (ii) & b(fh, u) = b(f, u_{(1)})b(h, u_{(2)}); & (iv) & b(h, 1) = \varepsilon(h). \end{array}$$

It follows that $b(h, \mathcal{S}(u)) = b(\mathcal{S}(h), u)$ for all $u \in U,\ h \in H$. Given a Hopf pairing, one has Hopf algebra maps $U \to H^\circ$ and $H \to U^\circ$, where H° and U° are the Sweedler duals. The pairing is called *perfect* if these maps are injections.

Let G be a connected, simply connected simple complex algebraic group and $\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra of G. We fix $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra and Φ the root system associated to $\mathfrak{h}$ with simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$, where $n = \mathrm{rk}(\mathfrak{g}) := \dim(\mathfrak{h})$. Let $(-,-)$ be the symmetric bilinear form over $\mathfrak{h}^*$ induced by the Killing form. Then, the Cartan matrix associated $A = (a_{ij})_{1\le i,j\le n} \in \mathbb{Z}^{n\times n}$ to $\mathfrak{g}$ is given by $a_{i,j} = \frac{2(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)}$. If we write $d_i = \frac{(\alpha_i,\alpha_i)}{2}$ and $D = \mathrm{diag}(d_1, \dots, d_n)$, then $(\alpha_i, \alpha_j) = d_i a_{ij}$ and DA is symmetric. The fundamental weights $\omega_1, \dots, \omega_n$ are given by the property $(\omega_i, \alpha_j) = d_i\delta_{ij}$ for all $1 \le i \le n$. Then, $\alpha_i = \sum_{j=1}^n a_{ji}\omega_j$ for all $1 \le i \le n$. We denote by $P = \sum_{i=1}^n \mathbb{Z}\omega_i$ the weight lattice, P_+ the positive weights, $Q = \sum_{i=1}^n \mathbb{Z}\alpha_i$ the root lattice, Q_+ the positive roots and $\mathcal{W}$ the Weyl group associated to Φ. The bilinear form $(-,-)$ defines a $\mathbb{Z}$-pairing over $P \times Q$.

Let q be an indeterminate, $R = \mathbb{Q}[q, q^{-1}]$ and $\mathbb{Q}(q)$ its field of fractions. Let ϵ be an ℓth root of unity with $\operatorname{ord}\epsilon = \ell$ odd and $3 \nmid \ell$ if G is of type G_2. If $\chi_\ell(q)$ denotes the ℓth cyclotomic polynomial, then $R/[\chi_\ell(q)R] = \mathbb{Q}(\epsilon)$. We denote $q_i = q^{d_i}$ for all $1 \le i \le n$.

For $n > 0$ define

$$(n)_q = \frac{q^n - 1}{q - 1} = q^{n-1} + \cdots + q + 1, \quad (n)_q! = (n)_q(n-1)_q \cdots (2)_q(1)_q \text{ and } (0)_q = 1,$$

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = [n]_q[n-1]_q \cdots [1]_q \quad \text{and} \quad [0]_q = 1,$$

$$\binom{n}{k}_q = \frac{(n)_q}{(k)_q(n-k)_q}, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

2. Twisted quantum groups

In this section we recall the definition of the twisted (multiparameter simply connected) quantum enveloping algebra $\check{U}_q^\varphi(\mathfrak{g})$, its divided power algebra and the twisted quantum function algebra $\mathcal{O}_q^\varphi(G)$. The former is isomorphic to the multiparameter quantum group defined in [R] and [CKP], see [CV1], and the latter is introduced by Constantini and Varagnolo in [CV2]. We follow mainly [CV2] for the description.

These algebras depend on a $\mathbb{Q}$-linear map on the weight lattice that induces a deformation on the coproduct of $\check{U}_q(\mathfrak{g})$, and on the product of $\mathcal{O}_q(G)$. This deformation is given by a multiplicative 2-cocycle on $\mathcal{O}_q(G)$ and *resembles* a twist deformation on $\check{U}_q(\mathfrak{g})$. For this reason, we call them *twisted* quantum groups. We describe also the corresponding objects at roots of unity and some basic properties such as PBW basis, a Hopf algebra pairing and the quantum Frobenius map. In particular, twisted quantum function algebras at roots of unity fit into an exact sequence of Hopf algebras.

Throughout we omit the supraindex φ when $\varphi = 0$ on the quantum groups and on the corresponding maps if no possible confusion arise.

2.1. **The twisting map φ.** Consider the $\mathbb{Q}$-linear space $\mathbb{Q}P = \sum_{i=1}^n \mathbb{Q}\omega_i$ spanned by the weigths and define a $\mathbb{Q}$-linear map satisfying:

$$\begin{cases} (\varphi x, y) = -(x, \varphi y), & \forall x, y \in \mathbb{Q}P, \\ \varphi\alpha_i = \delta_i = 2\tau_i, & \tau_i \in P, i = 1, ..., n, \\ \frac{1}{2}(\varphi\lambda, \mu) \in \mathbb{Z}, & \forall \lambda, \mu \in P, \end{cases} \tag{2}$$

where $(-,-)$ is consider as a linear extension of the $\mathbb{Z}$-pairing over $P \times Q$ to a symmetric bilinear form $\mathbb{Q}P \times \mathbb{Q}P \to \mathbb{Q}$. In particular, φ is antisymmetric with respect to this form.

According to the first two conditions we have that $(2\tau_i, \alpha_j) = -(2\tau_j, \alpha_i)$. Writing $\tau_i = \sum_{j=1}^n x_{ji}\omega_j = \sum_{j=1}^n y_{ji}\alpha_j$ with $x_{ji}, y_{ij} \in \mathbb{Z}$ for all $1 \leq i, j \leq n$, it follows that

$$d_j x_{ji} = \left(\sum_{k=1}^n x_{ki}\omega_k, \alpha_j\right) = -\left(\sum_{l=1}^n x_{lj}\omega_l, \alpha_i\right) = -\sum_{l=1}^n x_{lj}(\omega_l, \alpha_i) = -d_i x_{ij}.$$

If we denote $X = (x_{ij})_{1\leq i,j\leq n}$, $Y = (y_{ij})_{1\leq i,j\leq n} \in \mathbb{Z}^{n\times n}$, then $AY = X$ and $DX = (d_i x_{ij})_{1\leq i,j\leq n}$ is antisymmetric. In particular, $x_{ii} = 0$ for all $1 \leq i \leq n$ and φ depends on at most $\frac{n(n+1)}{2}$ integer parameters. Moreover, by [CV1, Lemma 2.1] the matrix $A + 2X$ is invertible and the maps $1 \pm \varphi : \mathbb{Q}P \to \mathbb{Q}P$ are $\mathbb{Q}$ isomorphisms that satisfy that

$$((1 + \varphi)^{\pm 1}\lambda, \mu) = (\lambda, (1 - \varphi)^{\pm 1}\mu) \qquad \text{for all } \lambda,\ \mu \in P.$$

Write $r = (1 + \varphi)^{-1}$, $\bar{r} = (1 - \varphi)^{-1}$. Note that if $\lambda \in r(P)$ and $\mu \in P$, then $(\lambda, \mu) \in \frac{1}{\det(A+2X)}\mathbb{Z}$. Let u be an element contained in the algebraic clausure of $\mathbb{Q}(q)$ such that $q = u^{\det(A+2X)}$. If $z \in \frac{1}{\det(A+2X)}$, then we write q^z for $u^{z \det(A+2X)}$.

2.2. **Twisted quantum enveloping algebras.** Let $Q \subseteq M \subseteq P$ be a lattice. For convenience, we recall the definition of the one-parameter quantum enveloping algebras $U_q(\mathfrak{g}, M)$, see [BG, I.6.5].

Definition 2.1. $U_q(\mathfrak{g}, M)$ is the $\mathbb{Q}(q)$-algebra generated by the elements $\{E_i, F_i\}_{i=1}^n$, $\{K_\lambda : \lambda \in M\}$ satisfying the relations

$$\begin{aligned}
K_0 = 1, \qquad K_\lambda K_\mu &= K_{\lambda+\mu} = K_\mu K_\lambda \quad \text{for all} \quad \lambda, \mu \in M,\\
K_\lambda E_j K_{-\lambda} &= q^{(\lambda,\alpha_j)} E_j,\\
K_\lambda F_j K_{-\lambda} &= q^{-(\lambda,\alpha_j)} F_j,\\
E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},\\
\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} E_i^{1-a_{ij}-l} E_j E_i^l &= 0 \quad (i \neq j),\\
\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_{q_i} F_i^{1-a_{ij}-l} F_j F_i^l &= 0 \quad (i \neq j).
\end{aligned}$$

It is well-known that $U_q(\mathfrak{g}, M)$ is a Hopf algebra with its comultiplication defined by setting E_i to be $(1, K_{\alpha_i})$-primitive and F_i to be $(K_{-\alpha_i}, 1)$-primitive, for all $1 \leq i \leq n$. Using the map φ, one may define a different coproduct, counit and antipode on $U_q(\mathfrak{g}, M)$ as follows (see [CV2, §1.3])

$$\begin{cases} \Delta_\varphi(E_i) = E_i \otimes K_{\tau_i} + K_{\alpha_i - \tau_i} \otimes E_i, \\ \Delta_\varphi(F_i) = F_i \otimes K_{-\alpha_i - \tau_i} + K_{\tau_i} \otimes F_i, \\ \Delta_\varphi(K_\lambda) = K_\lambda \otimes K_\lambda, \end{cases} \quad \begin{cases} \varepsilon_\varphi(E_i) = 0, \\ \varepsilon_\varphi(F_i) = 0, \\ \varepsilon_\varphi(K_\alpha) = 1, \end{cases} \quad \begin{cases} \mathcal{S}_\varphi(E_i) = -K_{-\alpha_i} E_i, \\ \mathcal{S}_\varphi(F_i) = -F_i K_{\alpha_i}, \\ \mathcal{S}_\varphi(K_\lambda) = K_{-\lambda}. \end{cases}$$

Note that the coproduct is well-defined by (2). With this new structure, $U_q(\mathfrak{g}, M)$ is again a Hopf algebra which is denoted by $U_q^\varphi(\mathfrak{g}, M)$. Clearly, $U_q^\varphi(\mathfrak{g}, M) = U_q(\mathfrak{g}, M)$ if $\varphi = 0$. We write $U_q^\varphi(\mathfrak{g}, P) = \check{U}_q^\varphi(\mathfrak{g})$ and $U_q^\varphi(\mathfrak{g}, Q) = U_q^\varphi(\mathfrak{g})$.

Remark 2.2. From the defining relations we have that $K_{\tau_i} E_i = E_i K_{\tau_i}$ and $K_{\tau_i} F_i = F_i K_{\tau_i}$ for all $1 \leq i \leq n$. Indeed,

$$K_{\tau_i} E_i = \prod_{j=1}^n K_{\omega_j}^{x_{ji}} E_i = \left(\prod_{j=1}^n q_j^{x_{ji}(\omega_j, \alpha_i)} \right) E_i \left(\prod_{j=1}^n K_{\omega_j}^{x_{ji}} \right) = \left(\prod_{j=1}^n q_j^{d_j x_{ji} \delta_{ij}} \right) E_i K_{\tau_i} = E_i K_{\tau_i}.$$

The second assertion follows analogously.

Definition 2.3. [CV2, §1.4] Let $\check{U}_q^\varphi(\mathfrak{b}_+)$ and $U_q^\varphi(\mathfrak{b}_+)$ be the Hopf subalgebras of $\check{U}_q^\varphi(\mathfrak{g})$ generated by the elements K_λ, E_i with $\lambda \in P$ and $\lambda \in Q$, respectively. Similarly, let $\check{U}_q^\varphi(\mathfrak{b}_-)$ and $U_q^\varphi(\mathfrak{b}_-)$ be the Hopf subalgebras of generated by the elements K_λ, F_i, with $\lambda \in P$, and $\lambda \in Q$, respectively. The algebra $U_q^\varphi(\mathfrak{g})$ is the Hopf subalgebra generated by K_λ, E_i, F_i with $\lambda \in Q$ and $1 \leq i \leq n$.

2.3. **Pairings, Borel subalgebras and integer forms.** By [CV1, § 2], see also [CV2, §1.4], [DL, §3], there exist perfect Hopf pairings $\pi_\varphi : \check{U}_q^\varphi(\mathfrak{b}_-)^{\mathrm{cop}} \times \check{U}_q^\varphi(\mathfrak{b}_+) \to \mathbb{Q}(u)$ and $\bar{\pi}_\varphi : \check{U}_q^\varphi(\mathfrak{b}_+)^{\mathrm{cop}} \times \check{U}_q^\varphi(\mathfrak{b}_-) \to \mathbb{Q}(u)$. These are given by

$$\begin{cases} \pi_\varphi(K_\lambda, K_\mu) = q^{(r(\lambda),\mu)}, \\ \pi_\varphi(K_\lambda, E_i) = \pi_\varphi(F_i, K_\lambda) = 0, \\ \pi_\varphi(F_i, E_j) = \dfrac{\delta_{ij}}{q_i - q_i^{-1}} q^{(r(\tau_i),\tau_i)}, \end{cases} \qquad \begin{cases} \overline{\pi}_\varphi(K_\lambda, K_\mu) = q^{-(\overline{r}(\lambda),\mu))}, \\ \overline{\pi}_\varphi(E_i, K_\lambda) = \overline{\pi}_\varphi(K_\lambda, F_i) = 0, \\ \overline{\pi}_\varphi(E_i, F_j) = \dfrac{\delta_{ij}}{q_i^{-1} - q_i} q^{-(\overline{r}(\tau_i),\tau_i)}, \end{cases}$$

for $\lambda, \mu \in P$ and $1 \leq i, j \leq n$. The pairing $\overline{\pi}_\varphi$ can be obtained from $\pi_{-\varphi}$ by the conjugation of the Hopf algebra $\mathbb{Q}$-anti-isomorphism $\zeta_\varphi : \check{U}_q^\varphi(\mathfrak{g}) \to \check{U}_q^{-\varphi}(\mathfrak{g})$ given by $E_i \mapsto F_i$, $F_i \mapsto E_i$, $K_\lambda \mapsto K_{-\lambda}$ and $q \mapsto q^{-1}$. Clearly, ζ_φ maps $\check{U}_q^\varphi(\mathfrak{b}_+)$ into $\check{U}_q^{-\varphi}(\mathfrak{b}_-)$ and $\overline{\pi}_\varphi = \zeta_\varphi \circ \pi_{-\varphi} \circ (\zeta_\varphi \otimes \zeta_\varphi)$. If $\varphi = 0$, denote by π_0 the corresponding bilinear form. Using these pairings we will define four R-Hopf algebras that will be needed later.

Fix a reduced expresion of the longest element $w_0 = s_{i_1} \cdots s_{i_N}$ in the Weyl group $\mathcal{W}$ and consider the total ordering on Φ_+ given by

$$\beta_1 = \alpha_{i_1}, \qquad \beta_2 = \alpha_{i_1}\alpha_{i_2}, \qquad \ldots \qquad \beta_N = \alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_N}.$$

The braid group $B_{\mathcal{W}}$ associated to $\mathcal{W}$ acts on $\check{U}_q^\varphi(\mathfrak{g})$ via the Lusztig automorphisms T_{i_j} for $1 \leq j \leq N$, and one may define the root vectors

$$E_{\beta_k} = T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(E_k) \qquad \text{and} \qquad F_{\beta_k} = T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(F_k).$$

For $s \in \mathbb{N}$, $1 \leq i \leq n$, $1 \leq k \leq N$ and $G_i = E_i$ or F_i define

$$G_i^{(s)} = \frac{G_i^{(s)}}{[s]_{q_i}!}, \qquad \text{and} \qquad G_{\beta_k}^{(s)} = T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(G_k^{(s)}).$$

For $\alpha \in \Phi_+$, let

$$q_\alpha = q^{\frac{(\alpha,\alpha)}{2}}, \qquad \tau_\alpha = \frac{1}{2}\varphi(\alpha), \qquad e_\alpha^\varphi = (q_\alpha^{-1} - q_\alpha) E_\alpha K_{-\tau_\alpha},$$
$$e_i^\varphi = e_{\alpha_i}^\varphi, \qquad f_i^\varphi = f_{\alpha_i}^\varphi, \qquad f_\alpha^\varphi = (q_\alpha - q_\alpha^{-1}) F_\alpha K_{-\tau_\alpha}.$$

Definition 2.4. Denote by $R_q^\varphi[B_-]'$ and $R_q^\varphi[B_-]''$ the R-subalgebras of $\check{U}_q^\varphi(\mathfrak{b}_+)^{\mathrm{op}}$ and $\check{U}_q^\varphi(\mathfrak{b}_+)^{\mathrm{cop}}$, respectively, generated by the elements e_α^φ and $K_{(1-\varphi)\omega_i}$ for $1 \leq i \leq n$ and $\alpha \in \Phi_+$. Similarly, let $R_q^\varphi[B_+]'$ and $R_q^\varphi[B_+]''$ be the R-subalgebras of $\check{U}_q^\varphi(\mathfrak{b}_-)^{\mathrm{op}}$ and $\check{U}_q^\varphi(\mathfrak{b}_-)^{\mathrm{cop}}$, generated by the elements f_α^φ and $K_{(1+\varphi)\omega_i}$ for $1 \leq i \leq n$ and $\alpha \in \Phi_+$.

By restriction, we get the following pairings

$$\pi'_\varphi : \check{U}_q^\varphi(\mathfrak{b}_-) \otimes_R R_q^\varphi[B_-]' \to \mathbb{Q}(q), \qquad \pi''_\varphi : R_q^\varphi[B_+]'' \otimes_R \check{U}_q^\varphi(\mathfrak{b}_+) \to \mathbb{Q}(q),$$
$$\bar{\pi}'_\varphi : \check{U}_q^\varphi(\mathfrak{b}_+) \otimes_R R_q^\varphi[B_+]' \to \mathbb{Q}(q), \qquad \bar{\pi}''_\varphi : R_q^\varphi[B_-]'' \otimes_R \check{U}_q^\varphi(\mathfrak{b}_-) \to \mathbb{Q}(q).$$

They are given by

$$\begin{aligned} \pi'_\varphi(K_\lambda, K_{(1-\varphi)\mu}) &= q^{(\lambda,\mu)}, & \pi'_\varphi(F_j, e_i^\varphi) &= -\delta_{ij},\\ \pi''_\varphi(K_{(1+\varphi)\mu}, K_\lambda) &= q^{(\mu,\lambda)}, & \pi''_\varphi(f_i^\varphi, E_j) &= \delta_{ij},\\ \overline{\pi}'_\varphi(K_\lambda, K_{(1+\varphi)\mu}) &= q^{-(\lambda,\mu)}, & \overline{\pi}'_\varphi(E_j, f_i^\varphi) &= -\delta_{ij},\\ \overline{\pi}''_\varphi(K_{(1-\varphi)\mu}, K_\lambda) &= q^{-(\mu,\lambda)}, & \overline{\pi}''_\varphi(e_i^\varphi, F_j) &= \delta_{ij}. \end{aligned} \tag{3}$$

By [L], one may take as bases of $U_q^\varphi(\mathfrak{b}_+)$ and $U_q^\varphi(\mathfrak{b}_-)$, respectively, the elements[1],

$$\xi_{m,t} = \prod_{j=N}^{1} E_{\beta_j}^{(m_j)} \prod_{i=1}^{n} \binom{K_{\alpha_i};0}{t_i} K_{\alpha_i}^{-\left[\frac{t_i}{2}\right]}, \qquad \eta_{m,t} = \prod_{j=N}^{1} F_{\beta_j}^{(m_j)} \prod_{i=1}^{n} \binom{K_{\alpha_i};0}{t_i} K_{\alpha_i}^{-\left[\frac{t_i}{2}\right]}.$$

Divided power algebras. We describe now an integer form of $U_q^\varphi(\mathfrak{g})$, which is used to define the algebra $U_\epsilon^\varphi(\mathfrak{g})$ at the root of unity ϵ.

Definition 2.5. Let $\Gamma^\varphi(\mathfrak{b}_+)$ and $\Gamma^\varphi(\mathfrak{b}_-)$ be the R-submodules of $U_q^\varphi(\mathfrak{g})$ given by

$$\begin{aligned} \Gamma^\varphi(\mathfrak{b}_+) &= \{u \in U_q^\varphi(\mathfrak{b}_+) \mid \pi''_\varphi(R_q^\varphi[B_+]''^{\mathrm{cop}} \otimes u) \subset R\},\\ \Gamma^\varphi(\mathfrak{b}_-) &= \{u \in U_q^\varphi(\mathfrak{b}_-) \mid \pi'_\varphi(u \otimes R_q^\varphi[B_-]'^{\mathrm{op}}) \subset R\}. \end{aligned}$$

It is known that the sets $\{\xi_{m,t}\}$ and $\{\eta_{m,t}\}$ are R-bases of $\Gamma^\varphi(\mathfrak{b}_+)$ and $\Gamma^\varphi(\mathfrak{b}_-)$, respectively. This implies that both are algebras isomorphic to $\Gamma(\mathfrak{b}_+)$ and $\Gamma(\mathfrak{b}_-)$. Moreover, they are also subcoalgebras with the coproduct given by

$$\begin{cases} \Delta_\varphi E_i^{(p)} = \sum\limits_{r+s=p} q_i^{-rs} E_i^{(r)} K_{s(\alpha_i-\tau_i)} \otimes E_i^{(s)} K_{r\tau_i},\\ \Delta_\varphi F_i^{(p)} = \sum\limits_{r+s=p} q_i^{-rs} F_i^{(r)} K_{s\tau_i} \otimes F_i^{(s)} K_{-r(\alpha_i+\tau_i)},\\ \Delta_\varphi \binom{K_i;0}{t} = \sum\limits_{r+s=t} K_i^s q_i^{-rs} \binom{K_i;0}{r} \otimes \binom{K_i;0}{s}. \end{cases} \tag{4}$$

Again by restriction, we get the Hopf pairings

$$\begin{aligned} \pi'_\varphi &: \Gamma^\varphi(\mathfrak{b}_-) \otimes_R R_q^\varphi[B_-]' \to R, & \pi''_\varphi &: R_q^\varphi[B_+]'' \otimes_R \Gamma^\varphi(\mathfrak{b}_+) \to R,\\ \bar{\pi}'_\varphi &: \Gamma^\varphi(\mathfrak{b}_+) \otimes_R R_q^\varphi[B_+]' \to R, & \bar{\pi}''_\varphi &: R_q^\varphi[B_-]'' \otimes_R \Gamma^\varphi(\mathfrak{b}_-) \to R. \end{aligned}$$

By [CV2, Lemma 1.12], the algebras $R_q^\varphi[B_\pm]'$, $R_q^\varphi[B_\pm]''$ admit a Hopf algebra structure such that the pairings above become perfect Hopf algebra pairings. Moreover, we have that $R_q^\varphi[B_\pm]' \simeq R_q^\varphi[B_\pm]''$ as Hopf algebras.

[1] [] represents the integer part function

Definition 2.6. [DL, §3.4] The algebra $\Gamma^\varphi(\mathfrak{g})$ is the R-subalgebra of $U_q^\varphi(\mathfrak{g})$ generated by $\Gamma^\varphi(\mathfrak{b}_+)$ and $\Gamma^\varphi(\mathfrak{b}_-)$. In particular, it is generated by the elements

$$K_{\alpha_i}^{-1} \qquad (1 \leq i \leq n),$$

$$\binom{K_{\alpha_i};0}{t} := \prod_{s=1}^{t} \left(\frac{K_{\alpha_i} q_i^{-s+1} - 1}{q_i^s - 1} \right) \qquad (t \geq 1, 1 \leq i \leq n),$$

$$E_i^{(t)} := \frac{E_i^t}{[t]_{q_i}!} \qquad (t \geq 1, 1 \leq i \leq n),$$

$$F_i^{(t)} := \frac{F_i^t}{[t]_{q_i}!} \qquad (t \geq 1, 1 \leq i \leq n).$$

2.4. **Twisted quantum function algebras.** In this subsection, we introduce the dual algebras $\mathcal{O}_q^\varphi(G)$ of $U_q^\varphi(\mathfrak{g})$ and $R_q^\varphi[G]$ of $\Gamma^\varphi(\mathfrak{g})$. They are obtained as the submodules generated by the matrix coefficients of representations of type one.

Let $\mathcal{C}_\varphi$ be the full faithfull subcategory in $U_q^\varphi(\mathfrak{g})$-mod consisting of finite-dimensional modules on which the elements K_{α_i} act diagonally by powers of q. Then $\mathcal{C}_\varphi$ is a tensor category which is strict. Denote by $\mathcal{O}_q^\varphi(G)$ the $\mathbb{Q}(q)$-submodule of $\mathrm{Hom}_{\mathbb{Q}(q)}(U_q^\varphi(\mathfrak{g}), \mathbb{Q}(q))$ spanned by all the matrix coefficients of objects in $\mathcal{C}_\varphi$. Then $\mathcal{O}_q^\varphi(G)$ is a $\mathbb{Q}(q)$-Hopf algebra with the usual structure. Given $V \in \mathcal{C}_\varphi$, $v \in V$ and $f \in V^*$, then the matrix coefficient $c_{f,v} : U_q^\varphi(\mathfrak{g}) \to \mathbb{Q}(q)$ is defined by $c_{f,v}(x) = f(x \cdot v)$ for all $x \in U_q^\varphi(\mathfrak{g})$. Then we have

$$\Delta(c_{f,v})(x \otimes y) = c_{f,v}(xy) \qquad \text{and} \qquad m_\varphi(c_{f,v} \otimes c_{g,w}) = c_{f\otimes g, v \otimes w},$$

for $V, W \in \mathcal{C}_\varphi$, $v \in V$, $f \in V^*$, $w \in W$, $g, \in W^*$ and $x, y \in U_q^\varphi(\mathfrak{g})$.

For $\Lambda \in P_+$, let $L(\Lambda)$ be a simple highest weight module of $U_q^\varphi(\mathfrak{g})$. Then, $L(\Lambda) = \bigoplus L(\Lambda)_\lambda$ is a graded module and by the Peter-Weyl Theorem we have that $\mathcal{O}_q(G) = \bigoplus_{\Lambda \in P_+} L(\Lambda) \otimes L(\Lambda)^*$, where $L(\Lambda)^* \simeq L(-\omega_0 \Lambda)$. If $v \in L(\Lambda)_\mu$, $f \in L(\Lambda)_{-\lambda}$, then write $\Delta(c_{f,v}) = \sum_i c_{f,v}^{-\lambda,\nu} \otimes c_{f,v}^{-\nu,\mu} \in \bigoplus_\nu \mathcal{O}_q(G)_{-\lambda,\nu} \otimes \mathcal{O}_q(G)_{-\nu,\mu}$. Since $\mathcal{O}_q^\varphi(G)$ equals $\mathcal{O}_q(G)$ as coalgebra, we keep this notation for the coproduct on $\mathcal{O}_q^\varphi(G)$.

Lemma 2.7. [LS] *For $i = 1, 2$ and $\Lambda_i \in P_+$, $v_i \in L(\Lambda_i)_{\mu_i}$, $f_i \in L(\Lambda_i)_{-\lambda_i}$ we have*

$$m_\varphi(c_{f_1,v_1} \otimes c_{f_2,v_2}) = q^{\frac{1}{2}((\varphi(\mu_1),\mu_2) - (\varphi(\lambda_1),\lambda_2))} m(c_{f_1,v_1} \otimes c_{f_2,v_2}).$$

□

Remark 2.8. Following [HLT, §2], the quantum coordinate algebra $\mathcal{O}_q(G)$ is a P-bigraded Hopf algebra. In particular, if $f \in L(\Lambda)_\lambda^*$, $v \in L(\Lambda)_\mu$ and $f(v) \neq 0$ we have that $f(v) = f(1 \cdot v) = f(K_{\alpha_i}^{-1} K_{\alpha_i} \cdot v) = f(\mathcal{S}(K_{\alpha_i}) K_{\alpha_i} \cdot v) = (K_{\alpha_i} \cdot f)(K_{\alpha_i} \cdot v) = q_i^{(\lambda,\alpha_i)+(\mu,\alpha_i)} f(v)$ for all $1 \leq i \leq n$, which implies that $\lambda = -\mu$ if $f(v) \neq 0$.

If we define the anti-symmetric bicharacter $p : P \times P \to \mathbb{Q}(q)$ by $p(\lambda_1, \lambda_2) = q^{-\frac{1}{2}(\varphi(\lambda_1),\lambda_2)}$, then it induces a group 2-cocycle $\tilde{p}$ on $P \times P$ given by

$$\tilde{p}((\lambda_1, \mu_1), (\lambda_2, \mu_2)) = p(\lambda_1, \lambda_2) p(\mu_1, \mu_2)^{-1} = q^{\frac{1}{2}((\varphi(\mu_1),\mu_2) - (\varphi(\lambda_1),\lambda_2))},$$

and by [HLT, Theorem 2.1], $\mathcal{O}_q^{\varphi}(G)$ is isomorphic to the deformed P-bigraded Hopf algebra $\mathcal{O}_q(G)_p$ where the product is given

$$m_{\varphi}(c_{f_1,v_1}\otimes c_{f_2,v_2}) = p(\lambda_1,\lambda_2)p(\mu_1,\mu_2)^{-1}m(c_{f_1,v_1}\otimes c_{f_2,v_2}),$$

for $\Lambda_i \in P_+$, $v_i \in L(\Lambda_i)_{\mu_i}$, $f_i \in L(\Lambda_i)_{-\lambda_i}$.

Corollary 2.9. *$\mathcal{O}_q^{\varphi}(G)$ is a 2-cocycle deformation of $\mathcal{O}_q(G)$. The 2-cocycle $\sigma : \mathcal{O}_q(G)\otimes \mathcal{O}_q(G) \to \mathbb{Q}(q)$ is given by the formula*

$$\sigma(c_{f_1,v_1}, c_{f_2,v_2}) = \varepsilon(c_{f_1,v_1})\varepsilon(c_{f_2,v_2})q^{-\frac{1}{2}(\varphi(\lambda_1),\lambda_2)}$$

for $\Lambda_i \in P_+$, $v_i \in L(\Lambda_i)_{\mu_i}$, $f_i \in L(\Lambda_i)_{-\lambda_i}$, and $i = 1, 2$.

Proof. Denote $\chi(\lambda_1,\lambda_2) = q^{-\frac{1}{2}(\varphi(\lambda_1),\lambda_2)}$. Clearly, $\sigma(x,1) = \sigma(1,x) = \varepsilon(x)$ for all $x \in \mathcal{O}_q(G)$. We first prove condition (1). For $1 \le i \le 3$, let $c_{f_i,v_i} \in \mathcal{O}_q(G)$ with $f_i \in L(\Lambda_i)_{\lambda_i}$ and $v_i \in L(\Lambda_i)_{\mu_i}$. On one hand, we have

$$\begin{aligned}
&\sigma((c_{f_2,v_2})_{(1)},(c_{f_3,v_3})_{(1)})\sigma(c_{f_1,v_1},(c_{f_2,v_2})_{(2)}(c_{f_3,v_3})_{(2)})\\
&= \sum_{\nu_1,\nu_2} \sigma(c_{f_2,v_2}^{-\lambda_2,\nu_1}, c_{f_3,v_3}^{-\lambda_3,\nu_2})\sigma(c_{f_1,v_1}, c_{f_2,v_2}^{-\nu_1,\mu_2}c_{f_3,v_3}^{-\nu_2,\mu_3})\\
&= \sum_{\nu_1,\nu_2} \varepsilon(c_{f_2,v_2}^{-\lambda_2,\nu_1})\varepsilon(c_{f_3,v_3}^{-\lambda_3,\nu_2})\chi(\lambda_1,\lambda_3)\varepsilon(c_{f_1,v_1})\varepsilon(c_{f_2,v_2}^{-\nu_1,\mu_2})\varepsilon(c_{f_3,v_3}^{-\nu_2,\mu_3})\chi(\lambda_1,\nu_2+\nu_3)\\
&= \varepsilon(c_{f_1,v_1})\varepsilon(c_{f_2,v_2})\varepsilon(c_{f_3,v_3})\chi(\lambda_1,\lambda_3)\chi(\lambda_1,\lambda_2+\lambda_3).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sigma((c_{f_1,v_1})_{(1)},(c_{f_2,v_2})_{(1)})\sigma((c_{f_1,v_1})_{(2)})(c_{f_2,v_2})_{(2)}, c_{f_3,v_3})\\
&= \sum_{\nu_1,\nu_2} \sigma(c_{f_1,v_1}^{-\lambda_1,\nu_1}, c_{f_2,v_2}^{-\lambda_2,\nu_2})\sigma(c_{f_1,v_1}^{-\nu_1,\mu_1}c_{f_2,v_2}^{-\nu_2,\mu_2}, c_{f_3,v_3})\\
&= \sum_{\nu_1,\nu_2} \varepsilon(c_{f_1,v_1}^{-\lambda_1,\nu_1})\varepsilon(c_{f_1,v_1}^{-\lambda_2,\nu_2})\chi(\lambda_1,\lambda_2)\varepsilon(c_{f_1,v_1}^{-\nu_1,\mu_1})\varepsilon(c_{f_2,v_2}^{-\nu_2,\mu_2})\varepsilon(c_{f_3,v_3})\chi(\nu_1+\nu_2,\lambda_3)\\
&= \varepsilon(c_{f_1,v_1})\varepsilon(c_{f_2,v_2})\varepsilon(c_{f_3,v_3})\chi(\lambda_1,\lambda_2)\chi(\lambda_1+\lambda_2,\lambda_3).
\end{aligned}$$

Thus, σ is a 2-cocycle on $\mathcal{O}_q(G)$. We prove now that it satisfies the equation given in Lemma 2.7. It actually follows by a direct computation using that $\chi(0,0) = 1$, $\chi(\lambda,0) = \chi(0,\lambda) = 1$, $\sigma^{-1}(c_{f_1,v_1}, c_{f_2,v_2}) = \sigma(\mathcal{S}(c_{f_1,v_1}), c_{f_2,v_2}) = \varepsilon(c_{f_1,v_1})\varepsilon(c_{f_2,v_2})\chi(\mu_1,\lambda_2)$ for $\Lambda_i \in P_+$,

$v_i \in L(\Lambda_i)_{\mu_i}$, $f_i \in L(\Lambda_i)_{-\lambda_i}$ and $i = 1, 2$, and $\varepsilon(\mathcal{O}_q(G)_{\lambda,\mu}) = 0$ if $-\lambda \neq \mu$:

$$\begin{aligned}
m_\sigma(c_{f_1,v_1}, c_{f_2,v_2}) &= \sum_{\nu_1,\nu_2,\eta_1,\eta_2} \sigma(c^{-\lambda_1,\nu_1}_{f_1,v_1}, c^{-\lambda_2,\nu_2}_{f_2,v_2}) m(c^{-\nu_1,\eta_1}_{f_1,v_1} \otimes c^{\nu_2,\eta_2}_{f_2,v_2}) \sigma^{-1}(c^{-\eta_1,\mu_1}_{f_1,v_1} c^{-\eta_2,\mu_2}_{f_2,v_2}) \\
&= \sum_{\nu_1,\nu_2,\eta_1,\eta_2} \varepsilon(c^{-\lambda_1,\nu_1}_{f_1,v_1}) \varepsilon(c^{-\lambda_2,\nu_2}_{f_2,v_2}) \chi(\lambda_1,\lambda_2) m(c^{-\nu_1,\eta_1}_{f_1,v_1} \otimes c^{\nu_2,\eta_2}_{f_2,v_2}) \varepsilon(c^{-\eta_1,\mu_1}_{f_1,v_1}) \varepsilon(c^{-\eta_2,\mu_2}_{f_2,v_2}) \chi(\mu_1,\eta_2) \\
&= \chi(\lambda_1,\lambda_2) m(c_{f_1,v_1} \otimes c_{f_2,v_2}) \chi(\mu_1, -\mu_2) \\
&= \chi(\lambda_1,\lambda_2) \chi(\mu_1,\mu_2)^{-1} m(c_{f_1,v_1} \otimes c_{f_2,v_2}) = q^{\frac{1}{2}((\varphi(\mu_1),\mu_2)-(\varphi(\lambda_1),\lambda_2))} m(c_{f_1,v_1} \otimes c_{f_2,v_2}).
\end{aligned}$$

□

For more details on twisting, deformation and r-matrices, see [HLT], [CV2, §2.2].

Definition 2.10. Let $\mathcal{E}_\varphi$ be the full faithfull subcategory in $\Gamma^\varphi(\mathfrak{g})$-mod whose objects are the free R-modules of finite rank such that the elements K_i and $\binom{K_i;0}{t}$ act by diagonal matrices with eigenvalues q_i^m and $\binom{m}{t}_{q_i}$ respectively. Define $R_q^\varphi[G]$ as the R-submodule of $\mathrm{Hom}_R(\Gamma^\varphi(\mathfrak{g}), R)$ generated by the matrix coefficients of elements in $\mathcal{E}_\varphi$. Analogously, we define $R_q^\varphi[B_\pm]$ as the R-module generated by the matrix coefficients of elements of the full subcategories of $\Gamma^\varphi(\mathfrak{b}_+)$-mod and $\Gamma^\varphi(\mathfrak{b}_-)$-mod, respectively.

Since the categories are strict and tensorial, $R_q^\varphi[G]$ and $R_q^\varphi[B_\pm]$ are R-Hopf algebras. Moreover, by [CV2, §2.3], we have the isomorphims

$$R_q^\varphi[B_\pm]' \simeq R_q^\varphi[B_\pm] \simeq R_q^\varphi[B_\pm]''$$

Consider the linear map $\Gamma^\varphi(\mathfrak{b}_+) \otimes_R \Gamma^\varphi(\mathfrak{b}_-) \to \Gamma^\varphi(\mathfrak{g})$ given by the multiplication. The dual map composed with the isomorphism above give the injection

$$\mu''_\varphi : R_q^\varphi[G] \to R_q^\varphi[B_+]'' \otimes_R R_q^\varphi[B_-]''. \tag{5}$$

Lemma 2.11. [CV2, Lemma 2.5] *The image of μ''_φ is contained in the R-subalgebra $\mathbb{A}''_\varphi$ generated by elements the $1 \otimes e^\varphi_\alpha$, $f^\varphi_\alpha \otimes 1$ and $K_{-(1+\varphi)\lambda} \otimes K_{(1-\varphi)\lambda}$ for $\lambda \in P$, $\alpha \in \Phi_+$.* □

Let $\lambda \in P_+$ and $v_{\pm\lambda}$ be a highest (resp. lowest) weight vector of $L(\lambda)$ (resp. $L(-\lambda)$). Let $\phi_{\pm\lambda}$ be the unique element in $L(\pm\lambda)^*$, such that $\phi_{\pm\lambda}(v_{\pm\lambda}) = 1$ and vanish over the complement $\Gamma(\mathfrak{h})$-invariant of $\mathbb{Q}(q)v_{\pm\lambda} \subset L(\pm\lambda)$. Denote by $\psi_{\pm\lambda} = c_{\phi_{\pm\lambda},v_{\pm\lambda}}$ the corresponding matrix coefficient.

As in [DL], we define for all $\alpha \in \Phi_+$, the matrix coefficient $\psi^{\pm\alpha}_{\pm\lambda}$ by

$$\psi^\alpha_\lambda(x) = \phi_\lambda((E_\alpha x) \cdot v_\lambda), \qquad \psi^\alpha_{-\lambda}(x) = \phi_{-\lambda}((x E_\alpha) \cdot v_{-\lambda}),$$
$$\psi^{-\alpha}_\lambda(x) = \phi_\lambda((x F_\alpha) \cdot v_\lambda), \qquad \psi^{-\alpha}_{-\lambda}(x) = \phi_{-\lambda}((F_\alpha x) \cdot v_{-\lambda}).$$

Remark 2.12. (a) Let $\lambda \in P_+$, then $\mu''_\varphi(\psi_{-\lambda}) = K_{-(1+\varphi)\lambda} \otimes K_{(1-\varphi)\lambda}$.

Indeed, evaluating both expressions in $EM \otimes FN$ where $EM = \xi_{m_1,0}\eta_{0,t_2}$ and $FN = \eta_{m_2,0}\xi_{0,t_1}$ for suitable m_1, t_2, m_2, t_1 of the basis of $\Gamma^\varphi(\mathfrak{b}_+)$ and $\Gamma^\varphi(\mathfrak{b}_-)$ (c.f. Definition 2.5) respectively, and using [DL, Lemma 4.4 (iv)] we have that

$$\langle \mu''_\varphi(\psi_{-\lambda}), EM \otimes NF \rangle = \psi_{-\lambda}(EMNF) = \delta_{1,E}\delta_{1,F} MN(-\lambda),$$

where $M(\lambda) = \pi_0(K_\lambda, M)$ and $N(\lambda) = \bar{\pi}_0(K_{-\lambda}, N)$. Then $MN(-\lambda) = \pi_0(K_{-\lambda}, MN) = \pi_0(K_{-\lambda}, M)\pi_0(K_{-\lambda}, N) = \pi_0(K_{-\lambda}, M)\bar{\pi}_0(K_\lambda, N) = M(-\lambda)N(-\lambda)$. Moreover, using (3) we have

$$\langle \mu''_\varphi(\psi_\lambda), EM \otimes NF\rangle = \delta_{1,E}\delta_{1,F}M(-\lambda)N(-\lambda) = \delta_{1,E}\delta_{1,F}\pi''_\varphi(K_{-(1+\varphi)\lambda}, M)\bar{\pi}''_\varphi(K_{(1-\varphi)\lambda}, N).$$

On the other hand, using the pairings π''_φ and $\bar{\pi}''_\varphi$ we have that

$$\langle K_{-(1+\varphi)\lambda} \otimes K_{(1-\varphi)\lambda}, EM \otimes NF\rangle = \delta_{1,E}\delta_{1,F}\pi''_\varphi(K_{-(1+\varphi)\lambda}, M)\bar{\pi}''_\varphi(K_{(1-\varphi)(\lambda)}, N)$$

and the claim follows.

(b) By [CV2, Propositions 1.9 & 2.7], for all $1 \leq i \leq n$ we have that

$$\begin{aligned} \mu''_\varphi(\psi^{-\alpha_i}_{-\omega_i}) &= q^{-(\tau_i,\omega_i)} f^\varphi_{\alpha_i} K_{-(1+\varphi)\omega_i} \otimes K_{(1-\varphi)\omega_i}, \\ \mu''_\varphi(\psi^{\alpha_i}_{-\omega_i}) &= q^{-(\tau_i,\omega_i)} K_{-(1+\varphi)\omega_i} \otimes K_{(1-\varphi)\omega_i} e^\varphi_{\alpha_i}. \end{aligned} \tag{6}$$

We check the first formula, the second follows similarly. Since $\mu''_\varphi(\psi^{-\alpha_i}_{-\omega_i}) = \mu''_0(\psi^{-\alpha_i}_{-\omega_i})$, and by [DL, Lemma 4.5 (vi)], it holds that $\mu''_0(\psi^{-\alpha_i}_{-\omega_i}) = f_{\alpha_i}K_{-\omega_i} \otimes K_{\omega_i}$, we have

$$\begin{aligned} \langle \mu''_\varphi(\psi^{-\alpha_i}_{-\omega_i}), EM \otimes NF\rangle &= \langle f_{\alpha_i}K_{-\omega_i} \otimes K_{\omega_i}, EM \otimes NF\rangle = \pi''_0(f_{\alpha_i}K_{-\omega_i}, EM)\bar{\pi}''_0(K_{\omega_i}, NF) \\ &= \pi''_0(f_{\alpha_i}K_{-\omega_i}, EM)\bar{\pi}''_0(K_{\omega_i}, N)\bar{\pi}''_0(K_{\omega_i}, F) \\ &= \pi''_0(f_{\alpha_i}K_{-\omega_i}, EM)N(-\omega_i)\delta_{1,F}. \end{aligned}$$

On the other hand, since $\pi''_\varphi(f^\varphi_{\alpha_i}K_{-(1+\varphi)\omega_i}, EM) = q^{(\tau_i,\omega_i)}\pi''_0(f^0_{\alpha_i}K_{-\omega_i}, EM)$ by [CV2, Proposition 1.9] and [DL, (3.3)], using the definitions in (3), we obtain

$$\begin{aligned} \langle f^\varphi_{\alpha_i}K_{-(1+\varphi)\omega_i} \otimes K_{(1-\varphi)\omega_i}, EM \otimes NF\rangle &= \pi''_\varphi(f^\varphi_{\alpha_i}K_{-(1+\varphi)\omega_i}, EM)\pi''_\varphi(K_{(1-\varphi)\omega_i}, NF) \\ &= \pi''_\varphi(f^\varphi_{\alpha_i}K_{-(1+\varphi)\omega_i}, EM)N(-\omega_i)\delta_{1,F} \\ &= q^{(\tau_i,\omega_i)}\pi''_0(f^0_{\alpha_i}K_{-\omega_i}, EM)N(-\omega_i)\delta_{1,F}, \end{aligned}$$

and the assertion is proved.

The following lemma is a twisted version of [DL, Lemma 4.1].

Lemma 2.13. *$R^\varphi_q[G]$ coincides with the R-Hopf subalgebra of $U^\varphi_q(\mathfrak{g})^\circ$ given by the set of all linear functions $f : \Gamma^\varphi(\mathfrak{g}) \to R$ such that there exists a cofinite ideal $I \subset \Gamma^\varphi(\mathfrak{g})$ and $N \in \mathbb{N}$ which satisfy that $f(I) = 0$ and $\prod_{p=-N}^{N}(K_i - q_i^p) \in I$ for all $1 \leq i \leq n$. Further, the induced Hopf pairing ρ between $R^\varphi_q[G]$ and $\Gamma^\varphi(\mathfrak{g})$ is non-degenerate.*

Proof. Since $\Gamma^\varphi(\mathfrak{g}) = \Gamma(\mathfrak{g})$ as algebras, $R^\varphi_q[G]$ coincides with the set above by [DL, Lemma 4.1]. The Hopf algebra structure is the one induced from $\Gamma^\varphi(\mathfrak{g})^\circ$. The last claim follows from the fact that $\Gamma^\varphi(\mathfrak{g})$ has a PBW-basis and its dual basis lie in $R^\varphi_q[G]$. □

2.5. **Specializations at roots of one.** In this subsection we recall the definition at roots of unity of the twisted quantum groups and state some results that will be needed later. For all $Q \leq M \leq P$, we define

$$U_\epsilon^\varphi(\mathfrak{g}; M) = U_q^\varphi(\mathfrak{g})(\mathfrak{g}; M) \otimes_R \mathbb{Q}(\epsilon),\ \Gamma_\epsilon^\varphi(\mathfrak{g}) := \Gamma^\varphi(\mathfrak{g}) \otimes_R \mathbb{Q}(\epsilon),\ \mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)} := R_q^\varphi[G] \otimes_R \mathbb{Q}(\epsilon).$$

Note that $\Gamma_\epsilon^\varphi(\mathfrak{g}) \simeq \Gamma^\varphi(\mathfrak{g})/[\chi_l(q)\Gamma^\varphi(\mathfrak{g})(\mathfrak{g})]$ and $\mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)} \simeq R_q^\varphi[G]/[\chi_\ell(q)R_q^\varphi[G]]$, where $R/[\chi_\ell(q)R] \simeq \mathbb{Q}(\epsilon)$. We denote $U_\epsilon^\varphi(\mathfrak{g}; P) := \check{U}_\epsilon^\varphi(\mathfrak{g})$ and $U_\epsilon^\varphi(\mathfrak{g}; Q) := U_\epsilon^\varphi(\mathfrak{g})$. For $r \in R$, denote by $\bar{r}$ the image of the canonical projection $R \twoheadrightarrow \mathbb{Q}(\epsilon)$.

Lemma 2.14. *$\mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)}$ is a 2-cocycle deformation of $\mathcal{O}_\epsilon(G)_{\mathbb{Q}(\epsilon)}$.*

Proof. Let $\sigma : \mathcal{O}_q(G) \otimes \mathcal{O}_q(G) \to \mathbb{Q}(q)$ denote the 2-cocycle defined in Corollary 2.9. Then, the map $\bar{\sigma} : \mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)} \otimes \mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)} \to \mathbb{Q}(\epsilon)$ given by

$$\bar{\sigma}(\bar{x}, \bar{y}) = \overline{\sigma(x, y)} \qquad \text{for all } x, y \in \mathcal{O}_q(G),$$

is a well-defined 2-cocycle for $\mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)}$, where $\bar{x}$ denotes the image of $x \in \mathcal{O}_q(G)$ under the canonical projection $\mathcal{O}_q^\varphi(G) \twoheadrightarrow \mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)}$. □

Remark 2.15. The relations $E_i^\ell = 0$, $F_i^\ell = 0$, $K_{\alpha_i}^\ell = 1$ hold in $\Gamma_\epsilon^\varphi(\mathfrak{g})$ for all $1 \leq i \leq n$. Indeed, we have that $\prod_{s=1}^{\ell}(K_{\alpha_i}q^{(-s+1)} - 1) = \prod_{s=1}^{\ell}(q^s - 1)\binom{K_{\alpha_i}; 0}{\ell}$ in $\Gamma^\varphi(\mathfrak{g})$. If we specialize q at ϵ, then we have $\prod_{s=1}^{\ell}(K_{\alpha_i}\epsilon^{(-s+1)} - 1) = 0$. Since $K_{\alpha_i}^\ell - 1 = \prod_{s=0}^{\ell-1}(K_{\alpha_i} - \epsilon^s) = \epsilon^{\frac{(\ell-1)\ell}{2}} \prod_{s=0}^{\ell-1}(K_{\alpha_i}\epsilon^{-s+1} - 1)$, we have that $K_{\alpha_i}^\ell = 1$ as desired. The other two relations follow from the fact $(\ell)_\epsilon = 0$.

The following lemma is analogue to [DL, Lemma 6.1].

Lemma 2.16. *There exists a perfect Hopf pairing $\bar{\rho} : \mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)} \otimes_{\mathbb{Q}(\epsilon)} \Gamma_\epsilon^\varphi(\mathfrak{g}) \to \mathbb{Q}(\epsilon)$.*

Proof. Let $\rho : R_q^\varphi[G] \otimes_R \Gamma^\varphi(\mathfrak{g}) \to R$ denote the pairing defined in Lemma 2.13. Then, we may define the pairing $\bar{\rho} : \mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)} \otimes_{\mathbb{Q}(\epsilon)} \Gamma_\epsilon^\varphi(\mathfrak{g}) \to \mathbb{Q}(\epsilon)$ via $\bar{\rho}(\bar{x}, \bar{u}) = \overline{(\rho(x, u))}$ for all $x \in R_q^\varphi[G]$ and $u \in \Gamma^\varphi(\mathfrak{g})$, where $\bar{x}$ and $\bar{u}$ denote the images of x and u under the canonical projections $R_q^\varphi[G] \to \mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)}$ and $\Gamma^\varphi(\mathfrak{g}) \to \Gamma_\epsilon^\varphi(\mathfrak{g})$, respectively. A direct computation shows that $\bar{\rho}$ is a well-defined map and it is a non-degenerate Hopf pairing. □

Now we introduce the twisted quantum Frobenius map. For details, see [CV2, §3]. For $1 \leq i \leq n$, let e_i, f_i and h_i denote the Chevalley generators of $\mathfrak{g}$ and write $e_i^{(m)} := e_i/m!$, $f_i^{(m)} := f_i/m!$, $\binom{h_i}{m} := \frac{h_i(h_i-1)\cdots(h_i-m+1)}{m!}$ for all $m \geq 0$.

Lemma 2.17. [CV2, §3.2 (i)] *There is a unique Hopf algebra epimorphism* $\mathrm{Fr} : \Gamma_\epsilon^\varphi(\mathfrak{g}) \longrightarrow U(\mathfrak{g})_{\mathbb{Q}(\epsilon)}$ *given for all* $1 \leq i \leq n$ *and* $m > 0$*, by*

$$\mathrm{Fr}(E_i^{(m)}) = e_i^{(m/\ell)}, \quad \mathrm{Fr}(F_i^{(m)}) = f_i(m/\ell), \quad \mathrm{Fr}\begin{pmatrix} K_{\alpha_i};0 \\ m \end{pmatrix} = \begin{pmatrix} h_i \\ m/\ell \end{pmatrix}, \quad \mathrm{Fr}(K_{\alpha_i}) = 1,$$

if $\ell \mid p$ *or* 0 *otherwise. Its kernel is the ideal generated by the elements* $K_{\alpha_i} - 1$*,* E_i *and* F_i*. In particular, there is a Hopf algebra monomorphism* ${}^t\mathrm{Fr} : \mathcal{O}(G)_{\mathbb{Q}(\epsilon)} \to \Gamma_\epsilon^\varphi(\mathfrak{g})^\circ$. □

Let $\Bbbk$ be a field extension of $\mathbb{Q}(\epsilon)$. We call $\mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}_{(\epsilon)}} \otimes_{\mathbb{Q}(\epsilon)} \Bbbk$ the $\Bbbk$-form of $\mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}_{(\epsilon)}}$. When $\Bbbk = \mathbb{C}$ we simply write $\mathcal{O}_\epsilon^\varphi(G)$.

Lemma 2.18. [CV2, §3.3] $\mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}_{(\epsilon)}}$ *contains a central Hopf subalgebra* F_0 *isomorphic to* $\mathcal{O}(G)_{\mathbb{Q}(\epsilon)}$*. Moreover, an element of* $\mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}_{(\epsilon)}}$ *belongs to* F_0 *if only if it vanishes on* I *and*

$$F_0 = \mathbb{Q}(\epsilon)\langle \overline{c}_{f,v} \in \mathcal{O}_q^\varphi(G)_{\mathbb{Q}(\epsilon)} | f \in \overline{L}(\ell\Lambda)^*_{-\ell v}, v \in \overline{L}(\ell\Lambda)_{\ell\mu}; v, \mu \in P_+\rangle,$$

where $\overline{L}(e\Lambda)$ *is the* $\Gamma^\varphi(\mathfrak{g})$*-module* $\Gamma^\varphi(\mathfrak{g})v_{e\Lambda}$ *with* $v_{e\Lambda}$ *the highest weight vector of* $L(e\Lambda)$. □

Proposition 2.19. $\mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}_{(\epsilon)}}$ *is a free* $\mathcal{O}(G)_{\mathbb{Q}(\epsilon)}$*-module of rank* $\ell^{\dim \mathfrak{g}}$.

Proof. Follows from [CV2, Proposition 3.5], [DL] and [BGS]. □

Let $\overline{\mathcal{O}_\epsilon^\varphi(G)}$ be the quotient $\mathcal{O}_\epsilon^\varphi(G)/[\mathcal{O}(G)^+\mathcal{O}_\epsilon^\varphi(G)]$ and $\pi : \mathcal{O}_\epsilon^\varphi(G) \longrightarrow \overline{\mathcal{O}_\epsilon^\varphi(G))}$ the canonical projection. By Proposition 2.19, $\overline{\mathcal{O}_\epsilon^\varphi(G)}$ is a Hopf algebra of dimension $\ell^{\dim \mathfrak{g}}$. Moreover, since $\mathcal{O}_\epsilon^\varphi(G)$ is a free $\mathcal{O}(G)$-module, it is faithfully flat. Then, by [Mo, Proposition 3.4.3] $\mathcal{O}_\epsilon^\varphi(G)$ fits into the short exact sequence of Hopf algebras.

$$1 \longrightarrow \mathcal{O}(G) \longrightarrow \mathcal{O}_\epsilon^\varphi(G) \longrightarrow \overline{\mathcal{O}_\epsilon^\varphi(G)} \longrightarrow 1.$$

3. Twisted Frobenius-Luzstig kernels

In this section we define and study the twisted Frobenius-Lusztig kernels and the quotients of their duals. They are finite-dimensional pointed Hopf algebras which are twist deformations of the usual kernels.

Let Z_0^φ be the smaller $B_{\mathcal{W}}$-invariant subalgebra of $U_\epsilon^\varphi(\mathfrak{g})$ that contains the elements $K_{\ell\alpha} = K_\alpha^\ell$, E_i^ℓ, F_i^ℓ for all $\alpha \in Q$ and $1 \leq i \leq n$.

Theorem 3.1. *(i)* Z_0^φ *is a central Hopf subalgebra of* $U_\epsilon^\varphi(\mathfrak{g})$.
(ii) Z_0^φ *is a polynomial ring in* $\dim \mathfrak{g}$ *generators, with* n *generators inverted.*
(iii) $U_\epsilon^\varphi(\mathfrak{g})$ *is a free* Z_0^φ*-module of rank* $\ell^{dim \mathfrak{g}}$.

Proof. The proof follows the same lines as [BG, Theorem III.6.2], using that the algebra W spanned by the elements $K_{\ell\alpha} = K_\alpha^\ell$, E_i^ℓ, F_i^ℓ for all $\alpha \in Q$ and $1 \leq i \leq n$ is a Hopf subalgebra, and this follows from a simple computation using the q-binomial formula. For example, $\Delta_\varphi(E_i^\ell) = (E_i \otimes K_{\tau_i} + K_{\alpha_i - \tau_i} \otimes E_i)^\ell = E_i^\ell \otimes K_{\ell\tau_i} + K_{\ell(\alpha_i - \tau_i)} \otimes E_i^\ell$, since $(E_i \otimes K_{\tau_i})(K_{\alpha_i - \tau_i} \otimes E_i) = \epsilon^{-2d_i}(K_{\alpha_i - \tau_i} \otimes E_i)(E_i \otimes K_{\tau_i})$. □

Definition 3.2. The *twisted Frobenius-Luzstig kernel* is defined as the quotient

$$\mathbf{u}_\epsilon^\varphi(\mathfrak{g}) = U_\epsilon^\varphi(\mathfrak{g})/[Z_0^+ U_\epsilon^\varphi(\mathfrak{g})].$$

By the theorem above, $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ is a finite-dimensional pointed Hopf algebra of dimension $\ell^{\dim \mathfrak{g}}$ and $G(\mathbf{u}_\epsilon^\varphi(\mathfrak{g})) = \langle K_{\alpha_i} |\ 1 \le i \le n\rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^n$. We denote $G(\mathbf{u}_\epsilon^\varphi(\mathfrak{g})) = \mathbb{T}^\varphi$.

Lemma 3.3. *Let $\widehat{U}_\epsilon^\varphi(\mathfrak{g})$ be the Hopf subalgebra of $\Gamma_\epsilon^\varphi(\mathfrak{g})$ generated by the elements E_i, F_i, K_{α_i} with $1 \le i \le n$. Then $\widehat{U}_\epsilon^\varphi(\mathfrak{g})$ and $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ are isomorphic as Hopf algebras.*

Proof. By definition, there exists a Hopf epimorphism $\widehat{U}_\epsilon^\varphi(\mathfrak{g}) \twoheadrightarrow \mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ given by $E_i \mapsto E_i$, $F_i \mapsto F_i$ and $K_{\alpha_i} \mapsto K_{\alpha_i}$ for all $1 \le i \le n$. Since by Remark 2.15, $\dim \widehat{U}_\epsilon^\varphi(\mathfrak{g}) \le \ell^{\dim \mathfrak{g}}$, the claim follows. □

Adapting the proof of [BG, Theorem III.7.10], we have the following.

Theorem 3.4. *The Hopf algebras $\overline{\mathcal{O}_\epsilon^\varphi(G)}$ and $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})^*$ are isomorphic.*

Proof. The pairing defined in Lemma 2.16 induces a perfect Hopf pairing $\overline{\mathcal{O}_\epsilon^\varphi(G)} \otimes_{\mathbb{Q}(\epsilon)} \widehat{U}_\epsilon^\varphi(\mathfrak{g}) \to \mathbb{Q}(\epsilon)$. In particular, we have a Hopf algebra monomorphism $\overline{\mathcal{O}_\epsilon^\varphi(G)} \hookrightarrow \widehat{U}_\epsilon^\varphi(\mathfrak{g})^*$. Since both algebras have the same dimension, the assertion follows by Lemma 3.3. □

As a consequence of the theorem above, the following sequence of Hopf algebras is exact

$$1 \longrightarrow \mathcal{O}(G)_{\mathbb{Q}(\epsilon)} \xrightarrow{\iota} \mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)} \xrightarrow{\pi} \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^* \longrightarrow 1.$$

Proposition 3.5. $\mathbf{u}_\epsilon^\varphi(\mathfrak{g}) \simeq \mathbf{u}_\varepsilon(\mathfrak{g})^J$ *for a twist $J \in \mathbb{Q}(\epsilon)[\mathbb{T}^\varphi \times \mathbb{T}^\varphi]$.*

Proof. By Lemma 2.14, $\mathcal{O}_\epsilon^\varphi(G)_{\mathbb{Q}(\epsilon)}$ is a 2-cocycle deformation of $\mathcal{O}_\varepsilon(G)$. Denote this cocycle by $\bar{\sigma}$. Then, it holds that $\bar{\sigma}|_{\mathcal{O}(G)\otimes\mathcal{O}(G)} = \varepsilon \otimes \varepsilon$ and by Remark 1.1, we have that $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})^*$ is a 2-cocycle deformation of $\mathbf{u}_\varepsilon(\mathfrak{g})^*$, where the cocycle is given by the formula $\hat{\sigma}(\pi(x), \pi(y)) = \bar{\sigma}(x, y)$ for all $x, y \in \mathcal{O}_\varepsilon(G)$. We may consider $\hat{\sigma} : \mathbf{u}_\varepsilon(\mathfrak{g})^* \otimes \mathbf{u}_\varepsilon(\mathfrak{g})^* \to \mathbb{Q}(\epsilon)$ as an element in $\mathbf{u}_\varepsilon(\mathfrak{g}) \otimes \mathbf{u}_\varepsilon(\mathfrak{g})$, say $J = \sum_i u_i \otimes u^i$. Then,

$$\begin{aligned}\hat{\sigma}(\pi(c_{f_1,v_1}) \otimes \pi(c_{f_2,v_2})) = \langle J, \pi(c_{f_1,v_1}) \otimes \pi(c_{f_2,v_2})\rangle = \sum_i f_1(u_i \cdot v_1) f_2(u^i \cdot v_2) \\ = \varepsilon(c_{f_1,v_1})\varepsilon(c_{f_2,v_2})\epsilon^{\frac{1}{2}(\varphi(\lambda_1),\lambda_2)} = f_1(v_1)f_2(v_2)\epsilon^{\frac{1}{2}(\varphi(\lambda_1),\lambda_2)},\end{aligned}$$

for all $\Lambda_i \in P_+,\ v_i \in L(\Lambda_i)_{\mu_i},\ f_i \in L(\Lambda_i)_{-\lambda_i}$, and $i = 1, 2$, where $\langle \cdot, \cdot\rangle$ is the perfect pairing given by the evaluation. Thus, the components of J must act diagonally and consequently, $J \in \mathbb{Q}(\epsilon)[\mathbb{T}^\varphi \times \mathbb{T}^\varphi]$. □

3.1. **Subalgebras of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$.** In this subsection we discuss a parametrization of the Hopf subalgebras of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$. Since $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ is a pointed Hopf algebra, any Hopf subalgebra is also pointed, and in this case, it is generated by a subgroup of the group of group-like elements and a subset of skew-primitive elements.

Lemma 3.6. *The Hopf subalgebras of* $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ *are parametrized by triples* $(I_+, I_-, \Sigma^\varphi)$ *where* $I_\pm \subset \pm\Pi$ *and* Σ^φ *is a subgroup of* $G(\mathbf{u}_\epsilon^\varphi(\mathfrak{g}))$ *such that* $K_{(1\mp\varphi)(\alpha_i)} \in \Sigma^\varphi$ *if* $\alpha_i \in I_\pm$*. Denote* $\widetilde{E}_i := E_i K_{-\tau_i}$ *and* $\widetilde{F}_j := K_{(\alpha_j+\tau_j)} F_j$*. Then the Hopf subalgebra of* $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ *corresponding to the triple* $(I_+, I_-, \Sigma^\varphi)$ *is the subalgebra generated by the set* $\{g, \widetilde{E}_i, \widetilde{F}_j |\ g \in \Sigma^\varphi, \alpha_i \in I_+$ *and* $\alpha_j \in I_-\}$*.*

Proof. The proof follows from [AG, Corollary 1.12], since $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ is generated by group-like and skew-primitive elements. In particular, $\Delta_\varphi(\widetilde{E}_i) = \widetilde{E}_i \otimes 1 + K_{(1-\varphi)(\alpha_i)} \otimes \widetilde{E}_i$, $\Delta_\varphi(\widetilde{F}_j) = \widetilde{F}_j \otimes 1 + K_{(1+\varphi)(\alpha_j)} \otimes \widetilde{F}_j$. □

Each pair (I_+, I_-) determines a regular parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing the fixed Cartan subalgebra $\mathfrak{h}$. Next we define the corresponding twisted quantum group.

Definition 3.7. For every pair (I_+, I_-) with $I_\pm \subset \pm\Pi$, we define $\Gamma^\varphi(\mathfrak{p})$ as the subalgebra of $\Gamma^\varphi(\mathfrak{g})$ generated by the elements

$$\begin{aligned}
&K_{\alpha_i}^{-1} && (1 \le i \le n),\\
&\binom{K_{\alpha_i};0}{m} := \prod_{s=1}^{m}\left(\frac{K_{\alpha_i} q_i^{-s+1} - 1}{q_i^s - 1}\right) && (m \ge 1, 1 \le i \le n),\\
&E_j^{(m)} := \frac{E_j^m}{[m]_{q_j}!} && (m \ge 1, \alpha_j \in I_+),\\
&F_k^{(m)} := \frac{F_k^m}{[m]_{q_k}!} && (m \ge 1, \alpha_k \in I_-).
\end{aligned}$$

Proposition 3.8. [AG, Proposition 2.3 (a)] *Let* $\Gamma_\epsilon^\varphi(\mathfrak{p}) := \Gamma^\varphi(\mathfrak{p})/[\chi_\ell(q)\Gamma^\varphi(\mathfrak{p})] \simeq \Gamma^\varphi(\mathfrak{p}) \otimes_R R/[\chi_\ell(q)R]$ *denote the* $\mathbb{Q}(\epsilon)$*-algebra given by the specialization. Then* $\Gamma_\epsilon^\varphi(\mathfrak{p})$ *is a Hopf subalgebra of* $\Gamma_\epsilon^\varphi(\mathfrak{g})$*.* □

Next we define a family of parabolic twisted Frobenius-Lusztig kernels.

Definition 3.9. For every pair (I_+, I_-) with $I_\pm \subset \pm\Pi$, we define the *twisted regular (parabolic) Frobenius-Lusztig kerner* $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})$ as the subalgebra of $\Gamma_\epsilon^\varphi(\mathfrak{p})$ generated by the elements $\{K_{\alpha_i}, E_j, F_k : 1 \le i \le n, \alpha_j \in I_+, \alpha_k \in I_-\}$.

In the following propositions we collect some properties.

Proposition 3.10. $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})$ *is the Hopf subalgebra of* $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ *given by* $\Gamma_\epsilon^\varphi(\mathfrak{p}) \cap \mathbf{u}_\epsilon^\varphi(\mathfrak{g}) = \mathbf{u}_\epsilon^\varphi(\mathfrak{p})$*. It corresponds to the triple* $(I_+, I_-, \mathbb{T}^\varphi)$*.*

Proof. Follows from Lemmata 3.3 and 3.6. □

Proposition 3.11. (*i*) *Let* $U(\mathfrak{p})_{\mathbb{Q}(\epsilon)} := \mathrm{Fr}(\Gamma_\epsilon^\varphi(\mathfrak{p}))$ *and denote* $\mathrm{Fr}_{res} = \mathrm{Fr}|_{\Gamma_\epsilon^\varphi(\mathfrak{p})}$*. Then the following diagram is commutative and all rows are exact sequences of Hopf* *algebras*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{u}_\epsilon^\varphi(\mathfrak{g}) & \longrightarrow & \Gamma_\epsilon^\varphi(\mathfrak{g}) & \xrightarrow{\mathrm{Fr}} & U(\mathfrak{g})_{\mathbb{Q}(\epsilon)} & \longrightarrow & 1 \\ & & \big\uparrow & & \big\uparrow & & \big\uparrow & & \\ 1 & \longrightarrow & \mathbf{u}_\epsilon^\varphi(\mathfrak{p}) & \longrightarrow & \Gamma_\epsilon^\varphi(\mathfrak{p}) & \xrightarrow{\mathrm{Fr}_{res}} & U(\mathfrak{p})_{\mathbb{Q}(\epsilon)} & \longrightarrow & 1. \end{array} \tag{7}$$

(*ii*) *There is a surjective algebra map* $\theta : \Gamma_\epsilon^\varphi(\mathfrak{p}) \to \mathbf{u}_\epsilon^\varphi(\mathfrak{p})$ *such that* $\theta|_{\mathbf{u}_\epsilon^\varphi(\mathfrak{p})} = \mathrm{id}$.

Proof. (*i*) It follows from [CV2, DL] and Proposition 2.17 that $\operatorname{Ker}\mathrm{Fr} = \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^+\Gamma_\epsilon^\varphi(\mathfrak{g})$. The proof that $\Gamma_\epsilon^\varphi(\mathfrak{g})^{\mathrm{co\,Fr}} = \mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ follows from [A, Lemma 3.4.2] but using the formula (4) instead of the formula (1.1.3) in [A]. So the first row is exact. To prove that the second row is exact, note that $\mathbf{u}_\epsilon^\varphi(\mathfrak{p}) = \mathbf{u}_\epsilon^\varphi(\mathfrak{g}) \cap \Gamma_\epsilon^\varphi(\mathfrak{p}) = \Gamma_\epsilon^\varphi(\mathfrak{g})^{\mathrm{co\,Fr}} \cap \Gamma_\epsilon^\varphi(\mathfrak{p}) = \Gamma_\epsilon^\varphi(\mathfrak{p})^{\mathrm{co\,Fr}_{res}}$ and $\operatorname{Ker}\mathrm{Fr}_{res} = \operatorname{Ker}\mathrm{Fr} \cap \Gamma_\epsilon^\varphi(\mathfrak{p}) = \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^+\Gamma^\varphi(\mathfrak{g}) \cap \Gamma_\epsilon^\varphi(\mathfrak{p}) = \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^+\Gamma_\epsilon^\varphi(\mathfrak{p})$.

(*ii*) Follows from [AG, Lemma 1.10 & Proposition 2.6]. □

Remark 3.12. Let $\mathfrak{p}$ be the set of primitive elements in $U(\mathfrak{p})_{\mathbb{Q}(\epsilon)}$. Then, $\mathfrak{p}$ is a regular parabolic Lie subalgebra of $\mathfrak{g}$, and $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})$ is the Frobenius-Lusztig kernel associated to it.

Proposition 3.13. $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})$ *is a twist deformation of* $\mathbf{u}_\epsilon(\mathfrak{p})$.

Proof. We know that $\mathbf{u}_\epsilon^\varphi(\mathfrak{g}) \simeq \mathbf{u}_\varepsilon(\mathfrak{g})^J$ for a twist $J \in \mathbb{Q}(\epsilon)[\mathbb{T}^\varphi \times \mathbb{T}^\varphi]$. Thus, $J \in \mathbf{u}_\varepsilon(\mathfrak{p}) \otimes \mathbf{u}_\varepsilon(\mathfrak{p})$ and $\mathbf{u}_\varepsilon(\mathfrak{p})^J$ is the subalgebra of $\mathbf{u}_\varepsilon(\mathfrak{g})^J$ that is isomorphic to the Hopf subalgebra of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ which corresponds to the triple $(I_+, I_-, \mathbb{T}^\varphi)$. Hence, $\mathbf{u}_\varepsilon(\mathfrak{p})^J \simeq \mathbf{u}_\epsilon^\varphi(\mathfrak{p})$. □

3.2. **Quotients of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})^*$.** Denote the $\mathbb{C}$-form of the twisted Frobenius-Lusztig kernel just by $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$. Let H be a Hopf algebra quotient of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})^*$. Then, H^* is a Hopf subalgebra of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ and whence, by Lemma 3.6, it is determined by a triple $(I_+, I_-, \Sigma^\varphi)$. Let $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})$ be the regular parabolic Frobenius-Lusztig kernel associated to the pair (I_+, I_-). Then $H^* \hookrightarrow \mathbf{u}_\epsilon^\varphi(\mathfrak{p}) \hookrightarrow \mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ as Hopf algebras, and consequently we have a sequence of Hopf algebra epimorphisms

$$\mathbf{u}_\epsilon^\varphi(\mathfrak{g})^* \twoheadrightarrow \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* \xrightarrow{\nu} H.$$

Let $I = I_+ \cup I_-$, $I' = I_+ \cap I_-$ and $I^c = (I_+ \cup I_-)^c = I_+^c \cap I_-^c$. We define the abelian subgroups $\mathbb{T}_I^\varphi$ and $\mathbb{T}_{I'}^\varphi$ of Σ^φ as follows:

$$\begin{aligned} \mathbb{T}_I^\varphi &= \langle \overline{K}_i := K_{(1-\varphi)(\alpha_i)}, \widetilde{K}_j := K_{(1+\varphi)(\alpha_j)} : \quad \text{if } \alpha_i \in I_+, \alpha_j \in I_- \rangle, \\ \mathbb{T}_{I'}^\varphi &= \langle K_{\alpha_i} : \quad \text{if } \alpha_i \in I_+ \cap I_- \rangle. \end{aligned}$$

Note that if $\alpha_i \in I_+ \cap I_-$, then $K_{\alpha_i} \in \mathbb{T}_I^\varphi$. Hence, $\mathbb{T}_{I'}^\varphi \subseteq \mathbb{T}_I^\varphi \subseteq \Sigma^\varphi \subseteq \mathbb{T}^\varphi$. Denote $\mathbb{T}_{I^c}^\varphi = \mathbb{T}^\varphi / \mathbb{T}_I^\varphi$ and $\Omega^\varphi = \Sigma^\varphi / \mathbb{T}_I^\varphi$; so $\Omega^\varphi \subseteq \mathbb{T}_{I^c}^\varphi$.

Definition 3.14. For all $i \in \{1, \dots, n\}$ such that $\alpha_i \in (I_+ \cap I_-)^c$, we define the algebra homomorphism $D_i : \mathbf{u}_\epsilon^\varphi(\mathfrak{p}) \to \mathbb{C}$ by

$$D_i(E_j) = 0 = D_i(F_k), \qquad D_i(K_{\alpha_t}) = \epsilon^{\delta_{it}} \quad \text{for all } \alpha_j \in I_+,\ \alpha_k \in I_-,\ t \in \{1 \dots n\}.$$

Remark 3.15. (*a*) For $1 \leq i \leq n$, let $\hat{D}_i \in \widehat{\mathbb{T}^\varphi}$ given by $\hat{D}_i(K_{\alpha_t}) = \epsilon^{\delta_{it}}$ for all $1 \leq t \leq n$. Then $\langle \hat{D}_i : 1 \leq i \leq n \rangle = \widehat{\mathbb{T}^\varphi}$ and we may identify $(\mathbb{Z}/\ell\mathbb{Z})^n \simeq \widehat{\mathbb{T}^\varphi}$ by $z \mapsto \hat{D}^z = \hat{D}_1^{z_1} \cdots \hat{D}_n^{z_n}$. In particular, one has that $\hat{D}_i = D_i|_{\mathbb{T}^\varphi}$ for all $i \in (I_+ \cap I_-)^c$.

(*b*) Assume $(I_+ \cap I_-)^c = \{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$. For all $z \in (\mathbb{Z}/\ell\mathbb{Z})^m$, denote

$$D^z = D_{i_1}^{z_1} \cdots D_{i_m}^{z_m} \in G(\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*).$$

If $f \in G(\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*)$ then $f = D^z$ for some $z \in (\mathbb{Z}/\ell\mathbb{Z})^m$. In particular, we may identify

$$G(\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*) \simeq \widehat{\mathbb{T}^\varphi/\mathbb{T}^\varphi_{I'}} \simeq (\mathbb{Z}/\ell\mathbb{Z})^m.$$

(*c*) Since $\mathbb{T}^\varphi_{I'} \subseteq \mathbb{T}^\varphi_I$, there is a group monomorphism $\widehat{\mathbb{T}^\varphi_{I^c}} \hookrightarrow \widehat{\mathbb{T}^\varphi/\mathbb{T}^\varphi_{I'}} = G(\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*)$ given for any $f \in \widehat{\mathbb{T}^\varphi_{I^c}}$ by the composition $\mathbb{T}^\varphi/\mathbb{T}^\varphi_{I'} \twoheadrightarrow \mathbb{T}^\varphi_{I^c} \xrightarrow{f} \mathbb{C}$.

(*d*) The inclusions $\mathbb{T}^\varphi_I \overset{\iota}{\hookrightarrow} \Sigma^\varphi \overset{j}{\hookrightarrow} \mathbb{T}^\varphi$ induce the surjective maps $\widehat{\mathbb{T}^\varphi} \overset{{}^t j}{\twoheadrightarrow} \widehat{\Sigma^\varphi}$ with $\operatorname{Ker} {}^t j = \{f \in \widehat{\mathbb{T}^\varphi} : f(\Sigma^\varphi) = 1\}$ and $\widehat{\mathbb{T}^\varphi_{I^c}} \overset{{}^t j_s}{\twoheadrightarrow} \widehat{\Omega^\varphi}$ with $N^\varphi = \operatorname{Ker} {}^t j_s = \{f \in \widehat{\mathbb{T}^\varphi_{I^c}} : f(\Omega^\varphi) = 1\}$. In particular, we have

$$|\Sigma^\varphi| = |\mathbb{T}^\varphi_I||\Omega^\varphi| = |\mathbb{T}^\varphi_I|\frac{|\mathbb{T}^\varphi_{I^c}|}{|N^\varphi|} = |\mathbb{T}^\varphi_I|\frac{|\mathbb{T}^\varphi|}{|\mathbb{T}^\varphi_I||N^\varphi|} = \frac{\ell^n}{|N^\varphi|}. \tag{8}$$

Moreover, one has that $\operatorname{Ker} {}^t(j\iota) \simeq \widehat{\mathbb{T}^\varphi_{I^c}}$, since there is a group monomorphism $\operatorname{Ker} {}^t(j\iota) \to \widehat{\mathbb{T}^\varphi_{I^c}}$ and $|\operatorname{Ker} {}^t(j\iota)| = |\mathbb{T}^\varphi_{I^c}|$. Hence, in what follows we identify the elements of $\widehat{\mathbb{T}^\varphi_{I^c}}$ and $\operatorname{Ker} {}^t(j\iota)$. On the other hand, if we denote $\hat{D}^z = \hat{D}_1^{z_1} \cdots \hat{D}_n^{z_n}$ for all $z = (z_1, \ldots, z_n) \in (\mathbb{Z}/\ell\mathbb{Z})^n$, then

$$\operatorname{Ker} {}^t(j\iota) = \{\hat{D}^z |\ \hat{D}^z(\overline{K}_i) = 1 = \hat{D}^z(\widetilde{K}_j),\ \text{for } i \in I_+,\ j \in I_-,\ z \in (\mathbb{Z}/\ell\mathbb{Z})^n\} \simeq \widehat{\mathbb{T}^\varphi_{I^c}}.$$

Therefore, if $\hat{D}^z \in \operatorname{Ker} {}^t(j\iota)$, then

$$1 = \hat{D}^z(\overline{K}_i) = \hat{D}^z(K_{(1-\varphi)(\alpha_i)}) = \hat{D}^z(K_{\alpha_i}\prod_{j=1}^n K_{\alpha_j}^{-2y_{ji}}) = \epsilon^{z_i}\prod_{j=1}^n \epsilon^{-2y_{ji}z_j}, \tag{9}$$

$$1 = \hat{D}^z(\widetilde{K}_j) = \hat{D}^z(K_{(1+\varphi)(\alpha_j)}) = \hat{D}^z(K_{\alpha_j}\prod_{k=1}^n K_{\alpha_j}^{2y_{kj}}) = \epsilon^{z_j}\prod_{k=1}^n \epsilon^{2y_{kj}z_j}, \tag{10}$$

for all $i \in I_+$ and $j \in I_-$. Thus, to find the generators of $\operatorname{Ker} {}^t(j\iota)$ it suffices to solve a linear system over $\mathbb{Z}/\ell\mathbb{Z}$. Indeed, if $I_+ = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$ and $I_- = \{\alpha_{j_1}, \ldots, \alpha_{j_r}\}$, by (9) and (10) we have a system of linear equations over $\mathbb{Z}/\ell\mathbb{Z}$ whose matrix S_ℓ^φ is given by

$$\begin{pmatrix} 2y_{i_1 1} & \dots & 2y_{i_1 i_s} & \dots & 2y_{i_1 j_1} & \dots & 2y_{i_1 j_l} & \dots & 2y_{i_1 i_1} - 1 & \dots & 2y_{i_1 n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 2y_{i_s 1} & \dots & 2y_{i_s i_s} + 1 & \dots & 2y_{i_s j_1} & \dots & 2y_{i_s j_l} & \dots & 2y_{i_s i_1} & \dots & 2y_{i_s n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 2y_{j_1 1} & \dots & 2y_{j_1 i_s} & \dots & 2y_{j_1 j_1} - 1 & \dots & 2y_{j_1 j_l} & \dots & 2y_{j_1 i_1} & \dots & 2y_{j_1 n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 2y_{j_l 1} & \dots & 2y_{j_l i_s} & \dots & 2y_{j_l j_1} & \dots & 2y_{j_l j_l} + 1 & \dots & 2y_{j_l i_1} & \dots & 2y_{j_l n} \end{pmatrix}$$

In particular, $|\operatorname{Ker}\ {}^t(j\iota)| = |\widehat{\mathbb{T}^\varphi_{I^c}}| = \ell^{n - \operatorname{rk} S_\ell^\varphi}$. Analogously, it is possible to characterize in the same way the kernel N^φ. In this case we have to consider the system of linear equations determined by the conditions $\hat{D}^z(\Omega^\varphi) = 1$ for all $\hat{D}^z \in \widehat{\mathbb{T}^\varphi_{I^c}}$.

Example 3.16. Assume $\mathfrak{g}$ is of type C_3 with associated Cartan matrix $A = \left(\begin{smallmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{smallmatrix}\right)$. Then the multiparametric matrix Y is given by

$$Y = \begin{pmatrix} a + b/2 & -a + c/2 & -b/2 - c/2 \\ 2a + b & -a + c & -b/2 - c \\ 2a + 3b/2 & -a + 3c/2 & -b/2 - c \end{pmatrix},$$

where $a \in \mathbb{Z}$, and $b, c \in 2\mathbb{Z}$. Set $a = 1$, $b = 2$, $c = 0$ and $\ell = 11$. Then, $\varphi(\alpha_1) = 4\alpha_1 + 8\alpha_2 + 10\alpha_3$, $\varphi(\alpha_2) = -2\alpha_1 - 2\alpha_2 - 2\alpha_3$ and $\varphi(\alpha_3) = -2\alpha_1 - 2\alpha_2 - 2\alpha_3$.

(a) If we choose $I_+ = \{\alpha_2\}$ and $I_- = \{\alpha_1\}$, then $S_{11}^\varphi = \left(\begin{smallmatrix} 5 & 8 & 10 \\ 2 & 3 & 2 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & 0 & 8 \\ 0 & 1 & 10 \end{smallmatrix}\right)$ and $\widehat{\mathbb{T}^\varphi_{I^c}} = \langle \hat{D}_1^3 \hat{D}_2 \hat{D}_3 \rangle \simeq \mathbb{Z}/11\mathbb{Z}$. If we take $\Sigma^\varphi = \langle K_{(1-\varphi)(\alpha_2)}, K_{(1+\varphi)(\alpha_1)}, K_{\tau_3}, K_{\tau_2} \rangle$, then we have that $\Sigma^\varphi = \mathbb{T}^\varphi \simeq (\mathbb{Z}/11\mathbb{Z})^3$ and N^φ is trivial.

(b) If we choose $I_+ = \{\alpha_2\}$, $I_- = \emptyset$ and $\Sigma^\varphi = \langle K_{(1+\varphi)(\alpha_1)}, K_{(1-\varphi)(\alpha_2)} \rangle$, then we have that $\widehat{\mathbb{T}^\varphi_{I^c}} = \langle \hat{D}^{(1,0,10)}, \hat{D}^{(0,1,2)} \rangle \simeq (\mathbb{Z}/11\mathbb{Z})^2$, $\Omega^\varphi \simeq \langle K_{(1+\varphi)(\alpha_1)} \rangle$ and $N^\varphi = \langle \hat{D}^{(1,10,8)} \rangle$.

The following proposition states that the elements in $\widehat{\mathbb{T}^\varphi_{I^c}}$ are central in $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$.

Proposition 3.17. *The subgroup $\widehat{\mathbb{T}^\varphi_{I^c}}$ of $G(\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*)$ consists of central group-like elements.*

Proof. Let $z \in (\mathbb{Z}/\ell\mathbb{Z})^m$ and $D^z \in G(\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*)$ such that $D^z \in \widehat{\mathbb{T}^\varphi_{I^c}}$. Then $D^z(\overline{K}_i) = 1 = D^z(\widetilde{K}_j)$ for all $i \in I_+$ and $j \in I_-$. We show that D^z is central in $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$.

By [L, Theorem 6.7] and [AG, Lemma 2.14], $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})$ has a basis

$$\left\{ \prod_{\beta \geq 0} F_\beta^{n_\beta} \prod_{i=1}^{n} K_{\alpha_i}^{t_i} \prod_{\alpha \geq 0} E_\alpha^{m_\alpha} : 0 \leq n_\beta, t_i, m_\alpha \leq \ell,\ 1 \leq i \leq n,\ \beta \in Q_{I_-},\ \alpha \in Q_{I_+} \right\}.$$

The hypotesis on D^z ensures that $D^z f(E_i) = f D^z(E_i)$ and $D^z f(F_j) = f D^z(F_j)$ for all $f \in \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$, $z \in (\mathbb{Z}/\ell\mathbb{Z})^m$, $i \in I_+$ and $j \in I_-$. Moreover, since the elements $K_{\alpha_t} \in$

$\mathbf{u}_\epsilon^\varphi(\mathfrak{p})$ are group-like for all $1 \leq t \leq n$, $D^z f(K_{\alpha_t}) = fD^z(K_{\alpha_t})$. As D^z is a group-like element in $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$, we have that $D^z f(MN) = fD^z(MN)$ for $M, N \in \{K_{\alpha_t}, E_i, F_j : i \in I_+, j \in I_-\}$, since $D^z f(MN) = (D^z f)_{(1)}(M)(D^z f)_{(2)}(N) = D^z f_{(1)}(M) D^z f_{(2)}(N) = f_{(1)} D^z(M) f_{(2)} D^z(N) = fD^z(MN)$. Analogously, using an inductive argument one may prove that D^z and f commute when evaluated on every element of the basis. □

The following proposition gives a characterization of all quotients of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})^*$.

Proposition 3.18. *Let H be a Hopf algebra quotient of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})^*$ such that H^* is determined by the triple $(I_+, I_-, \Sigma^\varphi)$ and $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})$ the twisted regular Frobenius-Lusztig kernel associated to (I_+, I_-). Then $H = \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*/\langle D^z - 1 :\ D^z \in N^\varphi\rangle$.*

Proof. If $(I_+ \cap I_-)^c = \{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$ and we write $D^z = D_{i_1}^{z_1} \cdots D_{i_m}^{z_m}$, then Remark 3.15 (b), $G(\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*) = \{D^z |\ z \in (\mathbb{Z}/\ell\mathbb{Z})^m\}$. By Proposition 3.17, we know that the elements of $\widehat{\mathbb{T}_{I^c}^\varphi}$ are central in $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$. Since $N^\varphi \subseteq \widehat{\mathbb{T}_{I^c}^\varphi}$, the two-sided ideal $\mathcal{I}$ of $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$ generated by the elements $D^z - 1 :\ D^z \in N^\varphi$ is a Hopf ideal and whence $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*/\mathcal{I}$ is a Hopf algebra.

On the other hand, we know that H^* is determined by the triple $(I_+, I_-, \Sigma^\varphi)$, and consequently, H^* is included in $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})$. If we denote by $\nu : \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* \to H$ the epimorphism induced by this inclusion, we have that $\operatorname{Ker}\nu = \{f \in \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* :\ f(h) = 0 \text{ for all } h \in H^*\}$. Since by Remark 3.15 (c), $D^z(g) = 1$ for all $g \in \Sigma^\varphi$ and $D^z \in N^\varphi$, we have that $D^z - 1 \in \operatorname{Ker}\nu$ and whence there is a Hopf algebra epimorphism $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*/\mathcal{I} \twoheadrightarrow H$. But by (8) we have that

$$\dim H = |\Sigma^\varphi| \ell^{|I_+|+|I_-|} = \frac{\ell^n}{|N^\varphi|} \ell^{|I_+|+|I_-|} = \dim \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*/\mathcal{I},$$

which implies that the epimorphism is indeed an isomorphism. □

Example 3.19. Let φ be the twisting map defined in Example 3.16 over $\mathfrak{g} = \mathfrak{sp}_6$. If we take $I_+ = \{\alpha_2\}$, $I_- = \{\alpha_1\}$ and $\Sigma^\varphi = \langle K_{(1-\varphi)(\alpha_2)}, K_{(1+\varphi)(\alpha_1)}, K_{\tau_3}, K_{\tau_2}\rangle$, then $\Sigma^\varphi = \mathbb{T}^\varphi \simeq (\mathbb{Z}/11\mathbb{Z})^3$ and N^φ is trivial. On the other hand, if we set $\varphi = 0$, then $\Sigma = \langle K_{\alpha_1}, K_{\alpha_2}\rangle$ and N is not trivial. This implies that the quotient $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*/\langle D^z - 1 :\ D^z \in N^\varphi\rangle$ cannot be a 2-cocycle deformation of $\mathbf{u}_\varepsilon(\mathfrak{p})^*/\langle D^z - 1 :\ D^z \in N\rangle$, since they have different dimension.

4. Quantum subgroups

In this section we determine all quantum subgroups of the twisted quantum group $\mathcal{O}_\epsilon^\varphi(G)$. We first construct a family of quantum subgroups using the root datum associated to $\mathfrak{g} = \operatorname{Lie}(G)$ and an algebraic subgroup Γ of G. Then we prove that any quantum subgroup of $\mathcal{O}_\epsilon^\varphi(G)$ is isomorphic to one constructed in this way. We end the section with a parametrization of the isomorphism classes.

From now on, we work with the complex form of all quantum groups introduced above.

4.1. **Twisted quantum parabolic subgroups.** Let $I_\pm \subseteq \pm\Pi$. Let $\Gamma_\epsilon^\varphi(\mathfrak{p})$ be the Hopf algebra associated to the pair (I_+, I_-) as in Definition 3.7, and $\mathfrak{p}$ the parabolic Lie subalgebra of $\mathfrak{g}$ given by Remark 3.12. In this subsection we construct a quantum subgroup related to the pair (I_+, I_-).

Denote by $\mathrm{Res} : \Gamma_\epsilon^\varphi(\mathfrak{g})^\circ \to \Gamma_\epsilon^\varphi(\mathfrak{p})^\circ$ the Hopf algebra map induced by the inclusion $\Gamma_\epsilon^\varphi(\mathfrak{p}) \hookrightarrow \Gamma_\epsilon^\varphi(\mathfrak{g})$. Using Lemma 2.16, we know that $\mathcal{O}_\epsilon^\varphi(G) \subseteq \Gamma_\epsilon^\varphi(\mathfrak{g})^\circ$.

Definition 4.1. We define the twisted quantum algebra associated to the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ as the Hopf algebra given by

$$\mathcal{O}_\epsilon^\varphi(P) := \mathrm{Res}(\mathcal{O}_\epsilon^\varphi(G)).$$

If $\varphi = 0$, we have that $\mathcal{O}_\epsilon^0(P) = \mathcal{O}_\epsilon(P)$, see [AG, §2.3.1]. Since $\mathcal{O}(G)$ is a central Hopf subalgebra of $\mathcal{O}_\epsilon^\varphi(G)$, $\mathrm{Res}(\mathcal{O}(G))$ is a central Hopf subalgebra of $\mathcal{O}_\epsilon^\varphi(P)$. Thus, there exists P an algebraic subgroup of G such that $\mathrm{Res}(\mathcal{O}(G)) = \mathcal{O}(P)$. Since $\mathcal{O}(P)$ is a central Hopf subalgebra of $\mathcal{O}_\epsilon^\varphi(P)$, the quotient

$$\overline{\mathcal{O}_\epsilon^\varphi(P)} := \mathcal{O}_\epsilon^\varphi(P)/[\mathcal{O}(P)^+\mathcal{O}_\epsilon^\varphi(P)],$$

is a Hopf algebra, which is in fact isomorphic to $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$.

Proposition 4.2. *(i) P is a connected algebraic group and* $\mathrm{Lie}(P) = \mathfrak{p}$.

(ii) The following sequence of Hopf algebras is exact

$$1 \longrightarrow \mathcal{O}(P) \longrightarrow \mathcal{O}_\epsilon^\varphi(P) \longrightarrow \overline{\mathcal{O}_\epsilon^\varphi(P)} \longrightarrow 1.$$

(iii) There exists a Hopf algebra epimorphism $\overline{\mathrm{Res}} : \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^* \to \overline{\mathcal{O}_\epsilon^\varphi(P)}$ making the following diagram commutative

$$\begin{CD} 1 @>>> \mathcal{O}(G) @>\iota>> \mathcal{O}_\epsilon^\varphi(G) @>\pi>> \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^* @>>> 1 \\ @. @VV\mathrm{res}V @VV\mathrm{Res}V @VV\overline{\mathrm{Res}}V \\ 1 @>>> \mathcal{O}(P) @>\iota_P>> \mathcal{O}_\epsilon^\varphi(P) @>\pi_P>> \overline{\mathcal{O}_\epsilon^\varphi(P)} @>>> 1. \end{CD} \tag{11}$$

(iv) $\mathcal{O}_\epsilon^\varphi(P)$ and $\overline{\mathcal{O}_\epsilon^\varphi(P)}$ are 2-cocycle deformations of $\mathcal{O}_\epsilon(P)$ and $\overline{\mathcal{O}_\epsilon(P)}$, respectively.

(v) $\overline{\mathcal{O}_\epsilon^\varphi(P)} \simeq \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$ as Hopf algebras.

Proof. (i), (ii), (iii) follow *mutadis mutantis* from [AG, Propositions 2.7 & 2.8].

(iv) By Lemma 2.14, we know that $\mathcal{O}_\epsilon^\varphi(G)$ is a 2-cocycle deformation of $\mathcal{O}_\varepsilon(G)$, say by the cocycle $\bar{\sigma}$. Since the kernel $\mathcal{I}$ of the Hopf algebra map $\mathrm{Res} : \mathcal{O}_\varepsilon(G) \to \mathcal{O}_\epsilon(P)$ is spanned by matrix coefficients that vanish when restricted to $\Gamma_\epsilon(\mathfrak{p})$, using the definition of $\bar{\sigma}$ we see that $\bar{\sigma}|_{\mathcal{I}\otimes\mathcal{O}_\varepsilon(G)+\mathcal{O}_\varepsilon(G)\otimes\mathcal{I}} = 0$. Thus by Remark 1.1, Res induces a 2-cocycle $\hat{\sigma}$ on $\mathcal{O}_\varepsilon(G)/\mathcal{I}$ and we have that $\mathcal{O}_\epsilon^\varphi(P) = \mathrm{Res}((\mathcal{O}_\varepsilon(G))_{\bar{\sigma}}) = (\mathcal{O}_\varepsilon(G)/\mathcal{I})_{\hat{\sigma}} = (\mathcal{O}_\epsilon(P))_{\hat{\sigma}}$. The same argument applies for $\overline{\mathcal{O}_\epsilon^\varphi(P)}$ and $\overline{\mathcal{O}_\epsilon(P)}$, since $\mathcal{O}(P)$ is a central Hopf subalgebra of $\mathcal{O}_\epsilon(P)$ and the cocyle $\hat{\sigma}$ is trivial on it.

(v) Dualizing the diagram (7) we get

$$\begin{CD} 1 @>>> U(\mathfrak{g})^\circ @>{}^t\,\mathrm{Fr}>> \Gamma_\epsilon^\varphi(\mathfrak{g})^\circ @>\alpha>> \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^* @>>> 1 \\ @. @VVV @VV\mathrm{Res}V @VVV \\ 1 @>>> U(\mathfrak{p})^\circ @>{}^t\,\mathrm{Fr}_{res}>> \Gamma_\epsilon^\varphi(\mathfrak{p})^\circ @>\beta>> \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* @>>> 1. \end{CD}$$

Since $\mathfrak{g}$ is simple, we have that $\mathcal{O}(G) \simeq U(\mathfrak{g})^\circ$. Thus, as $\mathcal{O}(P) = \mathrm{Res}(\mathcal{O}(G))$ and $\mathcal{O}_\epsilon^\varphi(P) = \mathrm{Res}(\mathcal{O}_\epsilon^\varphi(G))$, we have that ${}^t\,\mathrm{Fr}_{res}(\mathcal{O}(P)) \subseteq U(\mathfrak{p})^\circ$ and consequently $\mathcal{O}(P)^+ \subseteq \mathrm{Ker}\,\beta$. Moreover, since $\alpha(\mathcal{O}_\epsilon^\varphi(G)) = \pi(\mathcal{O}_\epsilon^\varphi(G)) = \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^*$, we have that $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* = \beta\,\mathrm{Res}(\mathcal{O}_\epsilon^\varphi(G)) = \beta(\mathcal{O}_\epsilon^\varphi(P))$. Hence, there exists a surjective Hopf algebra map $\gamma : \overline{\mathcal{O}_\epsilon(P)} \to \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$. But by (iv), [AG, Proposition 2.8 (c)] and Proposition 3.13, we have that $\dim \overline{\mathcal{O}_\epsilon^\varphi(P)} = \dim \overline{\mathcal{O}_\epsilon(P)} = \dim \mathbf{u}_\varepsilon(\mathfrak{p}) = \dim \mathbf{u}_\epsilon^\varphi(\mathfrak{p})$ and the epimorphism is in fact an isomorphism. □

Remark 4.3. By the proposition above, we know that the quantum group $\mathcal{O}_\epsilon^\varphi(P)$ fits into the central exact sequence of Hopf algebras $\mathcal{O}(P) \xhookrightarrow{\iota_P} \mathcal{O}_\epsilon^\varphi(P) \xrightarrow{\pi_P} \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$ and that $\mathcal{O}_\epsilon^\varphi(P)$ is a 2-cocycle deformation of $\mathcal{O}_\epsilon(P)$, where the 2-cocycle $\hat{\sigma}$ is given by the formula $\hat{\sigma}(\mathrm{Res}(x), \mathrm{Res}(y)) = \bar{\sigma}(x, y)$ for all $x, y \in \mathcal{O}_\varepsilon(G)$. On the other hand, by Propositions 3.5 and 3.13 we know that $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* = (\mathbf{u}_\varepsilon(\mathfrak{p})^*)_\tau$ for the 2-cocycle τ given by $\tau(\overline{\mathrm{Res}}(\pi(x)), \overline{\mathrm{Res}}(\pi(y)) = \bar{\sigma}(x, y)$. Since the diagram (11) for $\varphi = 0$ is commutative, the pullback of the cocycle τ coincides with the cocycle $\hat{\sigma}$.

4.2. **Quantum subgroups from classical subgroups.** In this subsection we construct a quantum subgroup of $\mathcal{O}_\epsilon^\varphi(G)$ associated to the pair (I_+, I_-) and an algebraic subgroup of G included in P. This is based in the *pushout construction*, which is a general method for constructing Hopf algebras from central exact sequences.

The following proposition follows from the arguments in [AG, §2.2]. If $\gamma : \Gamma \to G$ is a homomorphism of algebraic groups, then ${}^t\gamma : \mathcal{O}(G) \to \mathcal{O}(\Gamma)$ denotes the corresponding algebra map between the coordinate algebras.

Proposition 4.4. *Let Γ be an algebraic group and $\gamma : \Gamma \to G$ an injective homomorphism of algebraic groups such that $\sigma(\Gamma) \subseteq P$. Let $\mathcal{J}$ denote the two-sided ideal of $\mathcal{O}_\epsilon^\varphi(P)$ generated by $\iota(\mathrm{Ker}\ {}^t\gamma)$. Then $A_{\epsilon,\mathfrak{p},\gamma}^\varphi = \mathcal{O}_\epsilon^\varphi(P)/\mathcal{J}$ is a Hopf algebra and there exist a Hopf algebra monomorphism $j : \mathcal{O}(\Gamma) \hookrightarrow A_{\epsilon,\mathfrak{p},\gamma}^\varphi$, and Hopf algebra epimorphism $\bar{\pi} : A_{\epsilon,\mathfrak{p},\gamma}^\varphi \twoheadrightarrow \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*$ such that $A_{\epsilon,\mathfrak{p},\gamma}^\varphi$ fits into the exact sequence of Hopf algebras*

$$1 \longrightarrow \mathcal{O}(\Gamma) \xrightarrow{j} A_{\epsilon,\mathfrak{p},\gamma}^\varphi \xrightarrow{\bar{\pi}} \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* \longrightarrow 1.$$

If in addition $|\Gamma|$ is finite, then $\dim A_{\epsilon,\mathfrak{p},\gamma}^\varphi = |\Gamma| \dim \mathbf{u}_\epsilon^\varphi(\mathfrak{p})$. *Moreover, the following diagram is commutative*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_\epsilon^\varphi(G) & \xrightarrow{\pi} & \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^* & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \overline{\mathrm{Res}}} & & \\
1 & \longrightarrow & \mathcal{O}(P) & \xrightarrow{\iota_P} & \mathcal{O}_\epsilon^\varphi(P) & \xrightarrow{\pi_P} & \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle {}^t\gamma} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{j} & A_{\epsilon,\mathfrak{p},\gamma}^\varphi & \xrightarrow{\bar{\pi}} & \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* & \longrightarrow & 1.
\end{array} \tag{12}$$

□

Proposition 4.5. *$A^{\varphi}_{\epsilon,\mathfrak{p},\gamma}$ is a 2-cocycle deformation of $A_{\epsilon,\mathfrak{p},\gamma}$.*

Proof. By Proposition 4.2 (iv), we know that $\mathcal{O}^{\varphi}_{\epsilon}(P)$ is a 2-cocycle deformation of $\mathcal{O}_{\epsilon}(P)$, say by the cocycle $\hat{\sigma}$, see Remark 4.3 above. Then, by Remark 1.1 it is enough to check that $\hat{\sigma}|_{\mathcal{O}^{\varphi}_{\epsilon}(P)\otimes\mathcal{J}+\mathcal{J}\otimes\mathcal{O}^{\varphi}_{\epsilon}(P)} = 0$. Since $\mathcal{J} = \mathcal{O}^{\varphi}_{\epsilon}(P)\iota_P(\mathrm{Ker}\ {}^t\gamma)$ and $\mathrm{Ker}\ {}^t\gamma$ is generated by matrix coefficients $c_{f,v}$ in $\mathcal{O}(P)$, of degree $(\ell\lambda,\ell\mu)$ for some $\lambda,\mu\in P$, we have that $\hat{\sigma}|_{\iota_p(\mathrm{Ker}\ {}^t\gamma)\otimes\iota_p(\mathrm{Ker}\ {}^t\gamma)} = \varepsilon\otimes\varepsilon = 0$ and whence $\hat{\sigma}|_{\mathcal{O}^{\varphi}_{\epsilon}(P)\otimes\mathcal{J}+\mathcal{J}\otimes\mathcal{O}^{\varphi}_{\epsilon}(P)} = 0$. Thus, we may define a 2-cocycle $\tilde{\sigma} : A_{\epsilon,\mathfrak{p},\gamma}\otimes A_{\epsilon,\mathfrak{p},\gamma}\to\mathbb{C}$ by $\tilde{\sigma}(\psi(x),\psi(y)) = \hat{\sigma}(x,y)$ for all $x,y\in\mathcal{O}_{\epsilon}(P)$ and $(A_{\epsilon,\mathfrak{p},\gamma})_{\tilde{\sigma}} = A^{\varphi}_{\epsilon,\mathfrak{p},\gamma}$. Note that $\tilde{\sigma}$ coincides with the pullback through $\bar{\pi}$ of the 2-cocycle τ on $\mathbf{u}_{\varepsilon}(\mathfrak{p})^*$. □

By Proposition 3.11 (ii), we know that there exists an injective coalgebra map ${}^t\theta$: $\mathbf{u}^{\varphi}_{\epsilon}(\mathfrak{g})^*\to\Gamma^{\varphi}_{\epsilon}(\mathfrak{g})^{\circ}$, and since $\mathbf{u}^{\varphi}_{\epsilon}(\mathfrak{g})^*\simeq\overline{\mathcal{O}^{\varphi}_{\epsilon}(P)}$ by Proposition 4.2, we have that $\mathrm{Im}\ {}^t\theta\subseteq\mathcal{O}^{\varphi}_{\epsilon}(P)$. Thus, the image of the central subgroup $\widehat{\mathbb{T}^{\varphi}_{I^c}}$ of $G(\mathbf{u}^{\varphi}_{\epsilon}(\mathfrak{p})^*)$ is a subgroup of $G(\mathcal{O}^{\varphi}_{\epsilon}(P))$. Denote $d^z = {}^t\theta(D^z)$ for $z\in(\mathbb{Z}/\ell\mathbb{Z})^p$, $D^z\in\widehat{\mathbb{T}^{\varphi}_{I^c}}$.

Lemma 4.6. *There exists a subgroup $\mathcal{A} = \{\partial^z = \psi({}^t\theta(D^z)) :\ D^z\in\widehat{\mathbb{T}^{\varphi}_{I^c}}\}$ of $G(A^{\varphi}_{\epsilon,\mathfrak{p},\gamma})$ isomorphic to $\widehat{\mathbb{T}^{\varphi}_{I^c}}$ consisting of central elements. In particular, $|\mathcal{A}| = \ell^{n-\mathrm{rk}\, S^{\varphi}_{\ell}}$.*

Proof. Using the same argument as in the proof of Proposition 3.17, one sees that the elements d^z are central in $\mathcal{O}^{\varphi}_{\epsilon}(P)$. Indeed, if $f\in\mathcal{O}^{\varphi}_{\epsilon}(P)$, then $d^zf(M) = fd^z(M)$ for every generator M of $\Gamma^{\varphi}_{\epsilon}(\mathfrak{p})$ from Definition 3.7. For example, let $i\in I_+$ and $m\geq 0$, then by (4) we have

$$\begin{aligned} d^zf(E_i^{(m)}) &= \sum_{r+s=m} q_i^{-rs} d^z(E_i^{(r)}K_{s(\alpha_i-\tau_i)})f(E_i^{(s)}K_{r\tau_i}) \\ &= \sum_{r+s=m} q_i^{-rs} d^z(E_i^{(r)})d^z(K_{s(\alpha_i-\tau_i)})f(E_i^{(s)}K_{r\tau_i}) = d^z(K_{m(\alpha_i-\tau_i)})f(E_i^{(m)}), \text{ and} \\ d^zf(E_i^{(m)}) &= \sum_{r+s=m} q_i^{-rs} f(E_i^{(r)}K_{s(\alpha_i-\tau_i)})d^z(E_i^{(s)}K_{r\tau_i}) \\ &= \sum_{r+s=m} q_i^{-rs} f(E_i^{(r)}K_{s(\alpha_i-\tau_i)})d^z(E_i^{(s)})d^z(K_{r\tau_i}) = f(E_i^{(m)})d^z(K_{m\tau_i}). \end{aligned}$$

Since $d^z(\overline{K_i}) = d^z(K_{\alpha_i-2\tau_i}) = D^z(\theta(K_{\alpha_i-2\tau_i})) = 1$, we have that $d^z(K_{m(\alpha_i-\tau_i)}) = d^z(K_{m\tau_i})$ for all $m\geq 0$, and then $d^zf(E_i^{(m)}) = fd^z(E_i^{(m)})$. Analogously, using that $1 = d^z(\widetilde{K_j})$ for all $j\in I_-$, we have that $d^zf(F_j^{(m)}) = fd^z(F_j^{(m)})$ for all $m\geq 0$. The equality on the generators $K_{\alpha_i}^{-1}$ and $\binom{K_{\alpha_i};0}{m}$ follows easily since the coproduct is cocommutative on them. Applying an inductive argument on monomials on the generators we have that d^z is central in $\mathcal{O}^{\varphi}_{\epsilon}(P)$. Since $\psi : \mathcal{O}^{\varphi}_{\epsilon}(P)\to A^{\varphi}_{\epsilon,\mathfrak{p},\gamma}$ is surjective, the group-like elements ∂^z are also central in $A^{\varphi}_{\epsilon,\mathfrak{p},\gamma}$.

Now we show that $\mathcal{A} \simeq \widehat{\mathbb{T}^{\varphi}_{I^c}}$ as groups. By construction, we have that $\psi \circ {}^t\theta : \widehat{\mathbb{T}^{\varphi}_{I^c}} \to \mathcal{A}$ is a group epimorphism. As the diagram

$$\begin{array}{ccc} \mathcal{O}^{\varphi}_{\epsilon}(P) & \xrightarrow{\pi_P} & \mathbf{u}^{\varphi}_{\epsilon}(\mathfrak{p})^* \\ {\scriptstyle\psi}\downarrow & \nearrow{\scriptstyle\overline{\pi}} & \\ A^{\varphi}_{\epsilon,\mathfrak{p},\sigma} & & \end{array}$$

is commutative by (12), we have that $\overline{\pi}(\mathcal{A}) = \overline{\pi}(\psi({}^t\theta(\widehat{\mathbb{T}^{\varphi}_{I^c}})) = \pi_P({}^t\theta(\widehat{\mathbb{T}^{\varphi}_{I^c}}) = \widehat{\mathbb{T}^{\varphi}_{I^c}}$, which implies that $\psi \circ {}^t\theta$ is indeed an isomorphism. □

4.3. **Quantum subgroups from subalgebras of the twisted Frobenius-Lusztig kernels.** In this subsection we construct quantum subgroups from a Hopf subalgebra of $\mathbf{u}^{\varphi}_{\epsilon}(\mathfrak{g})$ and an algebraic subgroup of G.

Let L be a Hopf subalgebra of $\mathbf{u}^{\varphi}_{\epsilon}(\mathfrak{g})$. By Lemma 3.6, it is determined by a triple $(I_+, I_-, \Sigma^{\varphi})$ with Σ^{φ} a subgroup of $G(\mathbf{u}^{\varphi}_{\epsilon}(\mathfrak{g}))$ and $I_{\pm} \subset \pm\Pi$ are such that $K_{(1\mp\varphi)(\alpha_i)} \in \Sigma^{\varphi}$ if $\alpha_i \in I_{\pm}$. If $H = L^*$, then by Proposition 3.18, $H = \mathbf{u}^{\varphi}_{\epsilon}(\mathfrak{p})^*/\langle D^z - 1 : \ D^z \in N^{\varphi}\rangle$, where N^{φ} is determined by Σ^{φ} as in Remark 3.15 (d). Let P be the parabolic regular subgroup of G determined by the pair (I_+, I_-) with $\mathfrak{p} = \mathrm{Lie}(P)$ and $\mathcal{O}^{\varphi}_{\epsilon}(P)$, $\mathbf{u}^{\varphi}_{\epsilon}(\mathfrak{p})$ the corresponding twisted quantum groups.

Proposition 4.7. *Let Γ be an algebraic group and $\gamma : \Gamma \to G$ an injective morphism of algebraic groups such that $\gamma(\Gamma) \subseteq P$. For every group homomorphism $\delta : N^{\varphi} \to \widehat{\Gamma}$, the two-sided ideal J_{δ} of $A^{\varphi}_{\epsilon,\mathfrak{p},\gamma}$ generated by the elemens $\delta(D^z) - \partial^z$ for $\partial^z \in \mathcal{A}$ and D^z in N^{φ}, is a Hopf ideal and the Hopf algebra $A_{\epsilon,\mathfrak{p},\gamma}/J_{\delta}$ fits into the central exact sequence*

$$1 \longrightarrow \mathcal{O}(\Gamma) \xrightarrow{\hat{\iota}} A^{\varphi}_{\epsilon,\mathfrak{p},\gamma}/J_{\delta} \xrightarrow{\hat{\pi}} H \longrightarrow 1.$$

If in addition $|\Gamma|$ is finite, then $\dim A^{\varphi}_{\epsilon,\mathfrak{p},\gamma}/J_{\delta} = |\Gamma| \dim H$*. Moreover, the following diagram is commutative*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}^{\varphi}_{\epsilon}(G) & \xrightarrow{\pi} & \mathbf{u}^{\varphi}_{\epsilon}(\mathfrak{g})^* & \longrightarrow & 1 \\ & & {\scriptstyle\mathrm{res}}\downarrow & & \downarrow{\scriptstyle\mathrm{Res}} & & \downarrow{\scriptstyle\overline{\mathrm{Res}}} & & \\ 1 & \longrightarrow & \mathcal{O}(P) & \xrightarrow{\iota_P} & \mathcal{O}^{\varphi}_{\epsilon}(P) & \xrightarrow{\pi_P} & \mathbf{u}^{\varphi}_{\epsilon}(\mathfrak{p})^* & \longrightarrow & 1 \\ & & {\scriptstyle{}^t\gamma}\downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\mathrm{id}} & & \\ 1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{j} & A^{\varphi}_{\epsilon,\mathfrak{p},\gamma} & \xrightarrow{\overline{\pi}} & \mathbf{u}^{\varphi}_{\epsilon}(\mathfrak{p})^* & \longrightarrow & 1 \\ & & {\scriptstyle\mathrm{id}}\downarrow & & \downarrow & & \downarrow{\scriptstyle\nu} & & \\ 1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{\hat{\iota}} & A^{\varphi}_{\epsilon,\mathfrak{p},\gamma}/J_{\delta} & \xrightarrow{\hat{\pi}} & H & \longrightarrow & 1. \end{array} \tag{13}$$

Proof. Follows by the proof [AG, Theorem 2.17]. We reproduce the first part here to give an idea. By Lemma 4.6, we know that the group-like elements $\partial^z \in \mathcal{A}$ are central in $A^{\varphi}_{\epsilon,\mathfrak{p},\gamma}$. Since $\delta(D^z) \in \mathcal{O}(\Gamma)$ for all $D^z \in N^{\varphi}$, the ideal J_δ in $A^{\varphi}_{\epsilon,\mathfrak{p},\gamma}$ generated by the elements $\delta(D^z) - \partial^z$ is a Hopf ideal, and whence $A^{\varphi}_{\epsilon,\mathfrak{p},\gamma}/J_\delta$ is a Hopf algebra. If we write $\mathcal{J}_\delta = J_\delta \cap \mathcal{O}(\Gamma)$, then $A^{\varphi}_{\epsilon,\mathfrak{p},\gamma}/J_\delta$ fits into the central exact sequence

$$1 \longrightarrow \mathcal{O}(\Gamma)/\mathcal{J}_\delta \longrightarrow A^{\varphi}_{\epsilon,\mathfrak{p},\gamma}/J_\delta \longrightarrow \mathbf{u}^{\varphi}_{\epsilon}(\mathfrak{p})^*/\overline{\pi}(J_\delta) \longrightarrow 1.$$

Since $\overline{\pi}(\delta(D^z)) = 1$ and $\overline{\pi}(\partial^z) = D^z$ by the proof of Lemma 4.6, it follows that $\overline{\pi}(J_\delta) = \langle D^z - 1 : \ D^z \in N^{\varphi}\rangle$. Thus, by Proposition 3.18 (ii), we have that $\mathbf{u}^{\varphi}_{\epsilon}(\mathfrak{p})^*/\overline{\pi}(J_\delta) = H$. The proof that $\mathcal{J}_\delta = 0$ and that $A^{\varphi}_{\epsilon,\mathfrak{p},\gamma}/J_\delta$ fits into the commutative diagram follow the same arguments used in *loc. cit.* □

4.4. **Parametrization of quantum subgroups.** In this subsection we parametrize the quantum subgroups of $\mathcal{O}^{\varphi}_{\epsilon}(G)$ by a 6-tuple called *twisted subgroup datum*. We show first that there is a 1-1 correspondance between twisted quantum subgroup and twisted subgroup data, and then we classify the quantum subgroups up to isomorphism.

Definition 4.8. A *twisted subgroup datum* is a collection $\mathcal{D}^{\varphi} := (I_+, I_-, N^{\varphi}, \Gamma, \gamma, \delta)$ where

- ▷ $I_\pm \subset \pm\prod$. Let $\Psi_\pm = \{\alpha \in \Phi : \operatorname{Supp}\alpha \in I_\pm\}$, $\mathfrak{p} = \sum_{\alpha\in\Psi_\pm} \mathfrak{g}_\alpha$ and $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{h} \oplus \mathfrak{p}_-$. Let P be the connected Lie subgroup of G with $\operatorname{Lie}(P) = \mathfrak{p}$.
- ▷ N^{φ} a subgroup of $\widetilde{\mathbb{T}^{\varphi}_{I^c}}$, see Remark 3.15 (d).
- ▷ Γ an algebraic group.
- ▷ $\gamma : \Gamma \to P$ is a injective algebraic group homomorphism.
- ▷ $\delta : N^{\varphi} \to \hat{\Gamma}$ is a group homomorphism.

If Γ is finite, we call D^{φ} a *finite twisted subgroup datum.*

Summarizing the previous results we obtain the first main result of the paper.

Theorem 4.9. *Let $\mathcal{D}^{\varphi} = (I_+, I_-, N^{\varphi}, \Gamma, \gamma, \delta)$ be a twisted subgroup datum. Then there exists a quantum subgroup $A_{\mathcal{D}^{\varphi}} = A_{\epsilon,\mathfrak{p},\gamma}/J_\delta$ of $\mathcal{O}^{\varphi}_{\epsilon}(G)$ that fits into the central exact sequence*

$$1 \longrightarrow \mathcal{O}(\Gamma) \xrightarrow{\hat{\iota}} A_{\mathcal{D}^{\varphi}} \xrightarrow{\hat{\pi}} H \longrightarrow 1.$$

In particular, if $|\Gamma|$ is finite, then $\dim A_{\mathcal{D}^{\varphi}} = |\Gamma| \dim H$.

Proof. By Lemma 3.6 and Remark 3.15 (d), the triple (I_+, I_-, N^{φ}) determines a quotient H of $\mathbf{u}^{\varphi}_{\epsilon}(\mathfrak{g})^*$. Besides, by Proposition 4.2, the pair (I_+, I_-) determines a parabolic subgroup P of G, a parabolic Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ and the quantum groups $\mathcal{O}^{\varphi}_{\epsilon}(P)$ and $\mathbf{u}^{\varphi}_{\epsilon}(\mathfrak{p})$, which makes the upper part of the diagram (13) commutative. Then by Proposition 4.4, the morphism $\gamma : \Gamma \to P \subset G$ give rise to the quantum subgroup $A^{\varphi}_{\epsilon,\mathfrak{p},\gamma}$ through the pushout construction. Finally, by Proposition 4.7 the group homomorphism $\delta : N^{\varphi} \to \widehat{\Gamma}$ defines the Hopf ideal J_δ of $A^{\varphi}_{\epsilon,\mathfrak{p},\gamma}$ and the Hopf algebra $A_{D^{\varphi}} = A^{\varphi}_{\epsilon,\mathfrak{p},\sigma}/J_\delta$ fits into the commutative diagram (13). □

The next theorem establishes the converse of Theorem 4.9. We give its proof in several lemmata.

Theorem 4.10. *Let $\kappa : \mathcal{O}_\epsilon^\varphi(G) \to A$ be a surjective Hopf algebra morphism, then there exists a twisted subgroup datum $\mathcal{D}^\varphi$ such that $A \simeq A_{\mathcal{D}^\varphi}$ as Hopf algebras.* □

Lemma 4.11. *There exists an algebraic group Γ and an injective homomorphism of algebraic groups $\gamma : \Gamma \to G$ such that $\mathcal{O}(\Gamma)$ is a Hopf subalgebra of A and A fits into the central exact sequence of Hopf algebras $1 \longrightarrow \mathcal{O}(\Gamma) \xrightarrow{\hat{\iota}} A \xrightarrow{\hat{\pi}} H \longrightarrow 1$, where $H = A/\mathcal{O}(\Gamma)^+A$. Moreover, the following diagram is commutative*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_\epsilon^\varphi(G) & \xrightarrow{\pi} & \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^* & \longrightarrow & 1 \\
 & & \downarrow {\scriptstyle {}^t\gamma} & & \downarrow {\scriptstyle \kappa} & & \downarrow & & \\
1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{\hat{\iota}} & A & \xrightarrow{\hat{\pi}} & H & \longrightarrow & 1.
\end{array} \tag{14}$$

Proof. Let $K = \kappa(\iota(\mathcal{O}(G)))$. Since $\mathcal{O}(G)$ is central in $\mathcal{O}_\epsilon^\varphi(G)$, K is central in A and there exists an algebraic group Γ and an algebraic group homomorphism $\gamma : \Gamma \to G$ such that $K = \mathcal{O}(\Gamma)$ and ${}^t\gamma : \mathcal{O}(G) \to \mathcal{O}(\Gamma)$ is the Hopf algebra epimorphism $\kappa \circ \iota|_{\mathcal{O}(G)}$. Moreover, if we set $H = A/K^+A$, then the sequence $1 \longrightarrow \mathcal{O}(\Gamma) \xrightarrow{\hat{\iota}} A \xrightarrow{\hat{\pi}} H \longrightarrow 1$ is exact and the diagram (14) is commutative. □

By the lemma above, H^* is a Hopf subalgebra of $\mathbf{u}_\epsilon^\varphi(\mathfrak{g})$. Thus, by Lemma 3.6 it is determined by a triple $(I_+, I_-, \Sigma^\varphi)$. Let P be the subgroup of G determined by the pair (I_+, I_-), $\mathfrak{p} = \mathrm{Lie}(P)$ and $\mathcal{O}_\epsilon^\varphi(P)$, $\mathbf{u}_\epsilon^\varphi(\mathfrak{p})$ the quantum groups given by Proposition 4.2. In particular, we have that $H^* \subseteq \mathbf{u}_\epsilon^\varphi(\mathfrak{p}) \subseteq \mathbf{u}_\epsilon^\varphi(\mathfrak{g})$ and by Proposition 3.18, $H \simeq \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*/\langle D^z - 1 : \ D^z \in N^\varphi\rangle$, where N^φ is determined by Σ^φ as in Remark 3.15 (d). Denote by $\nu : \mathbf{u}_\epsilon^\varphi(\mathfrak{p}) \to H$ the corresponding epimorphism.

The next lemma follows from [AG, Lemma 3.1], but adapted to the twisted case.

Lemma 4.12. *The diagram* (14) *factorizes through the central exact sequence*

$$1 \longrightarrow \mathcal{O}(P) \xrightarrow{\iota_P} \mathcal{O}_\epsilon^\varphi(P) \xrightarrow{\pi_P} \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* \longrightarrow 1.$$

Proof. We want to show that A fits into the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_\epsilon^\varphi(G) & \xrightarrow{\pi} & \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^* & \longrightarrow & 1 \\
 & & \downarrow {\scriptstyle \mathrm{res}} & & \downarrow {\scriptstyle \mathrm{Res}} & & \downarrow {\scriptstyle \overline{\mathrm{Res}}} & & \\
1 & \longrightarrow & \mathcal{O}(P) & \xrightarrow{\iota_P} & \mathcal{O}_\epsilon^\varphi(P) & \xrightarrow{\pi_P} & \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* & \longrightarrow & 1 \\
 & & \downarrow {\scriptstyle {}^t\zeta} & & \downarrow {\scriptstyle \psi} & & \downarrow {\scriptstyle \nu} & & \\
1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{\hat{\iota}} & A & \xrightarrow{\hat{\pi}} & H & \longrightarrow & 1.
\end{array} \tag{15}$$

To prove it, it suffices to show that $\mathrm{Ker}\,\mathrm{Res} \subseteq \mathrm{Ker}\,\kappa$. In order to do so, we realize $\mathcal{O}_\epsilon^\varphi(G)$ as a subalgebra of $\mathbb{A}_\epsilon^\varphi = \mathbb{A}_\varphi'' \otimes_R \mathbb{Q}(\epsilon)$, see [CV2, §3.6], Lemma 2.11.

Let $\mu_\epsilon^\varphi : \mathcal{O}_\epsilon^\varphi(G) \to \check{U}_\epsilon^\varphi(\mathfrak{b}_-)^{\mathrm{cop}} \otimes \check{U}_\epsilon^\varphi(\mathfrak{b}_+)^{\mathrm{cop}}$ be the complexification of the injective algebra map μ''_φ given by (5). Then by Lemma 2.11, $\mu_\epsilon^\varphi(\mathcal{O}_\epsilon^\varphi(G)) \subseteq \mathbb{A}_\epsilon^\varphi$, which is the algebra generated by $f_\alpha^\varphi \otimes 1$, $1 \otimes e_\alpha^\varphi$ and $K_{-(1+\varphi)\lambda} \otimes K_{(1-\varphi)\lambda}$ for $\lambda \in P$ and $\alpha \in \Phi_+$.

The proof follows by showing $\mu_\epsilon^\varphi(\mathrm{Ker\,Res}) \subseteq \mu_\epsilon^\varphi(\mathrm{Ker}\,\kappa)$. First note that $\mu_\epsilon^\varphi(\mathrm{Ker\,Res})$ is the two-sided ideal $\mathcal{I}$ generated by $\{1 \otimes e_k^\varphi, f_i^\varphi \otimes 1 |\ \alpha_k \notin I_-,\ \alpha_j \notin I_+\}$. Indeed, by [CV2, Proposition 2.7], Remark 2.12 (a) and (6) we have

$$\begin{aligned}\mu_\epsilon^\varphi(\psi_{-\omega_i}^{-\alpha_i}\psi_{\omega_i}) &= \left(\left(\epsilon^{-(\tau_i,\omega_i)} f_{\alpha_i}^\varphi\right) K_{-(1+\varphi)(\omega_i)} \otimes K_{(1-\varphi)(\omega_i)}\right)\left(K_{(1+\varphi)(\omega_i)} \otimes K_{-(1-\varphi)(\omega_i)}\right) \\ &= \epsilon^{-(\tau_i,\omega_i)} f_{\alpha_i}^\varphi \otimes 1.\end{aligned}$$

Analogously, we have $\mu_\epsilon^\varphi(\psi_{\omega_i}\psi_{-\omega_i}^{\alpha_k}) = \epsilon^{-(\tau_i,\omega_i)} 1 \otimes e_\alpha^\varphi$. Since by definition $\psi_{-\omega_i}^{\alpha_k}$, $\psi_{-\omega_i}^{-\alpha_k} \in \mathrm{Ker\,Res}$ when $\alpha_k \notin I_-$, $\alpha_j \notin I_+$, we obtain that $1 \otimes e_k^\varphi$, $f_i^\varphi \otimes 1 \in \mu_\epsilon^\varphi(\mathrm{Ker\,Res})$ for $\alpha_k \notin I_-$ and $\alpha_j \notin I_+$. Conversely, assume $f \in \mathrm{Ker\,Res}$. Then $f_{|\Gamma_\epsilon^\varphi(\mathfrak{p})} = 0$ and $\langle \mu_\epsilon^\varphi(f), FM \otimes NE\rangle = f(FMNE) = 0$ for all elements $FMNE$ in a basis of $\Gamma_\epsilon^\varphi(\mathfrak{p})$. Using the perfect pairing (3) on ϵ, it follows that $\mu_\epsilon^\varphi(f) \subseteq \mathcal{I}$.

The proof that $\mathcal{I} \subseteq \mu_\epsilon^\varphi(\mathrm{Ker}\,\kappa)$ is analogous to the proof of [AG, Lemma 3.1]. □

Note that the map ${}^t\zeta : \mathcal{O}(P) \to \mathcal{O}(\Gamma)$ is given by the restriction $\psi|_{\mathcal{O}(P)}$. Hence, ${}^t\zeta\,\mathrm{res} = {}^t\gamma$ and $\mathrm{Im}\,\gamma \subseteq P$.

We end the proof of Theorem 4.10 with the following lemma. Its proof is analogous to the case $\varphi = 0$ and will be given without any detail.

Lemma 4.13. [AG, Lemmata 3.2 & 3.3] *There exists a group homomorphism $\delta : N^\varphi \to \widehat{\Gamma}$ such that the two-sided ideal J_δ of $A_{\epsilon,\mathfrak{p},\gamma}^\varphi$ generated by the elemens $\delta(D^z) - \partial^z$ for D^z in N^φ is a Hopf ideal, $A \simeq A_{\mathcal{D}^\varphi} = A_{\epsilon,\mathfrak{p},\gamma}^\varphi / J_\delta$ as Hopf algebras and A fits into the commutative diagram*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}(G) & \xrightarrow{\iota} & \mathcal{O}_\epsilon^\varphi(G) & \xrightarrow{\pi} & \mathbf{u}_\epsilon^\varphi(\mathfrak{g})^* & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \overline{\mathrm{Res}}} & & \\
1 & \longrightarrow & \mathcal{O}(P) & \xrightarrow{\iota_P} & \mathcal{O}_\epsilon^\varphi(P) & \xrightarrow{\pi_P} & \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle {}^t\gamma} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{j} & A_{\epsilon,\mathfrak{p},\gamma}^\varphi & \xrightarrow{\overline{\pi}} & \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^* & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \nu} & & \\
1 & \longrightarrow & \mathcal{O}(\Gamma) & \xrightarrow{\hat{\iota}} & A & \xrightarrow{\hat{\pi}} & H & \longrightarrow & 1.
\end{array}$$

Proof. (*Sketch*) Using that $A_{\epsilon,\mathfrak{p},\gamma}^\varphi$ is given by a pushout, one first shows that A fits into the commutative diagram above. Then, using the commutativity of the diagram, one proves that there exists a group homomorphism $\delta : N^\varphi \to \widehat{\Gamma}$ and a Hopf ideal J_δ such that $A \simeq A_{\epsilon,\mathfrak{p},\gamma}^\varphi / J_\delta$. □

4.4.1. *Isomorphism classes of quantum subgroups.* In this subsection we parametrize the quantum subgroups of $\mathcal{O}_\epsilon^\varphi(G)$ up to isomorphism. To do so, we first define a partial order on the isomorphism classes of quotients of $\mathcal{O}_\epsilon^\varphi(G)$ and on the set of twisted subgroup data.

Let $\mathcal{Q}(\mathcal{O}_\epsilon^\varphi(G))$ be the category whose objects are surjective Hopf algebra maps $\kappa : \mathcal{O}_\epsilon^\varphi(G) \to A$. If $\kappa : \mathcal{O}_\epsilon^\varphi(G) \to A$ and $\kappa' : \mathcal{O}_\epsilon^\varphi(G) \to A'$ are such maps, then an arrow $\kappa \xrightarrow{\alpha} \kappa'$ in $\mathcal{Q}(\mathcal{O}_\epsilon^\varphi(G))$ is a Hopf algebra map $\alpha : A \to A'$ such that $\alpha\kappa = \kappa'$. In this language, a *quotient* of $\mathcal{O}_\epsilon^\varphi(G)$ is just an isomorphism class of objects in $\mathcal{Q}(\mathcal{O}_\epsilon^\varphi(G))$; let $[\kappa]$ denote the class of the map κ. There is a partial order in the set of quotients of $\mathcal{O}_\epsilon^\varphi(G)$, given by $[\kappa] \leq [\kappa']$ iff there exists an arrow $\kappa \xrightarrow{\alpha} \kappa'$ in $\mathcal{Q}(\mathcal{O}_\epsilon^\varphi(G))$. Note that $[\kappa] \leq [\kappa']$ and $[\kappa'] \leq [\kappa]$ implies $[\kappa] = [\kappa']$. Our goal is to describe the partial order in $\mathcal{Q}(\mathcal{O}_\epsilon^\varphi(G))$.

Let $I_\pm, I'_\pm \subseteq \pm\Pi$. If $I'_+ \subseteq I_+$ and $I'_- \subseteq I_-$, then $I' \subseteq I$ and $\mathbb{T}^\varphi_{I'} \subseteq \mathbb{T}^\varphi_I$. Thus, there exists an epimorphism $\mathbb{T}^\varphi_{I'^c} \twoheadrightarrow \mathbb{T}^\varphi_{I^c}$ which induces a monomorphism $\eta : \widehat{\mathbb{T}^\varphi_{I^c}} \hookrightarrow \widehat{\mathbb{T}^\varphi_{I'^c}}$.

Definition 4.14. Let $\mathcal{D}^\varphi = (I_+, I_-, N^\varphi, \Gamma, \sigma, \delta)$ and $\mathcal{D}^{\varphi\prime} = (I'_+, I'_-, N^{\varphi\prime}, \Gamma', \gamma', \delta')$ be twisted subgroup data with respect to $\mathcal{O}_\epsilon^\varphi(G)$. We say that $\mathcal{D}^\varphi \leq \mathcal{D}^{\varphi\prime}$ if and only if:

▷ $I'_+ \subseteq I_+$, $I'_- \subseteq I_-$.
▷ $\eta(N^\varphi) \subseteq N^{\varphi\prime}$.
▷ there exists an algebraic group homomorphism $\tau : \Gamma' \to \Gamma$ such that $\gamma\tau = \gamma'$.
▷ $\delta'\eta = \tau^t\delta$.

Moreover, we say that $\mathcal{D}^\varphi \sim \mathcal{D}^{\varphi\prime}$ if and only if $\mathcal{D}^\varphi \leq \mathcal{D}^{\varphi\prime}$ and $\mathcal{D}^\varphi \leq \mathcal{D}^{\varphi\prime}$. In particular, this implies that $I'_+ = I_+$, $I'_- = I_-$, $N^\varphi = N^{\varphi\prime}$, the morphism τ is an isomorphism and $\delta' = \tau^t\delta$.

Our last theorem yields the parametrization of the quotients of $\mathcal{O}_\epsilon^\varphi(G)$ up to isomorphism. The proof is analogous to the case $\varphi = 0$ since it relies on the commutativity of the diagram (12) and general constructions of the sucesive quotients. For these reasons, it will be omitted.

Theorem 4.15. [AG, Theorem 2.20] *Let $\mathcal{D}^\varphi$ and $\mathcal{D}^{\varphi\prime}$ be twisted subgroup data and $\kappa : \mathcal{O}_\epsilon^\varphi(G) \to A_{\mathcal{D}^\varphi}$, $\kappa' : \mathcal{O}_\epsilon^\varphi(G) \to A_{\mathcal{D}^{\varphi\prime}}$ the corresponding quotients. Then $[\kappa] \leq [\kappa']$ if only if $\mathcal{D}^\varphi \leq \mathcal{D}^{\varphi\prime}$.* □

4.4.2. *Properties of the quotients.* We end the paper with a list of some properties of the quotients. Apart from item (v), the proof is analogous to [AG2, Proposition 3.8].

Proposition 4.16. *Let $D^\varphi = (I_+, I_-, N^\varphi, \Gamma, \gamma, \delta)$ be a twisted subgroup datum.*

(i) *If A_{D^φ} is pointed, then $I_+ \cap I_- = \emptyset$ and Γ is a subgroup of the group of upper triangular matrices of some size. In particular, if Γ is finite, then it is abelian.*
(ii) *A_{D^φ} is semisimple if and only if $I_+ \cup I_- = \emptyset$ and Γ is finite.*
(iii) *If $\dim A_{D^\varphi} < \infty$ and $A^*_{D^\varphi}$ is pointed, then $\gamma(\Gamma)$ is included in the fixed torus of G.*
(iv) *If A_{D^φ} is co-Frobenius then Γ is reductive.*
(v) *If φ and $(I_+, I_-, \Sigma^\varphi)$ are such that $\Sigma^\varphi = \mathbb{T}^\varphi$ but $\Sigma \neq \mathbb{T}$, then A_{D^φ} is not a 2-cocycle deformation of A_D.*

Proof. We prove only (v). If φ and $(I_+, I_-, \Sigma^\varphi)$ are such that $\Sigma^\varphi = \mathbb{T}^\varphi$ but $\Sigma \neq \mathbb{T}$, then $N^\varphi = 1$ and $N \neq 1$. Then, the quotient $H^\varphi = \mathbf{u}_\epsilon^\varphi(\mathfrak{p})^*/\langle D^z - 1 : \ D^z \in N^\varphi \rangle$ cannot be a 2-cocycle deformation of $H = \mathbf{u}_\varepsilon(\mathfrak{p})^*/\langle D^z - 1 : \ D^z \in N \rangle$ since they have different dimension. If A_{D^φ} were a 2-cocycle deformation of A_D, then by a chasing diagram argument we would have that H^φ is a 2-cocycle deformation of H, a contradiction, see Example 3.19. □

References

[A] N. Andruskiewitsch, Notes on Extensions of Hopf Algebras. *Can. J. Math.* **48** (1996), no. 1, 3–42.

[AE] N. Andruskiewitsch and B. Enriquez, Examples of compact matrix pseudogroups arising from the twisting operation. *Comm. Math. Phys.* **149** (1) (1992), 195-207.

[AG] N. Andruskiewitsch and G. García, Quantum Subgroups of a Simple Quantum Group at Roots of One. *Compos. Math.* **145** (2009), 476–500.

[AG2] ———, Extensions of finite quantum groups by finite groups. *Transform. Groups* **14** (1) (2009), 1–27.

[AST] M. Artin, W. Schelter and J. Tate, Quantum deformations of GL_n. *Comm. Pure Appl. Math.* **44** (8-9) (1991), 879–895.

[BW] G. Benkart and S. Witherspoon, Two-parameter quantum groups and Drinfel'd doubles. *Algebr. Represent. Theory* **7** (3) (2004), 261–286.

[BB] T. Banica and J. Bichon, Quantum groups acting on 4 points. *J. Reine Angew. Math.* **626** (2009) 75–114.

[BS] T. Banica and R. Speicher, Liberation of orthogonal Lie groups. *Adv. Math.* **222** (2009) 1461–1501.

[BD] J. Bichon and M. Dubois-Violette, Half-commutative orthogonal Hopf algebras. *Pacific J. Math.* **263** (2013), 13–28.

[BN] J. Bichon and S. Natale, Hopf algebra deformations of binary polyhedral groups. *Transform. Groups* **16** (2011), 339–374.

[BY] J. Bichon and R. Yuncken, Quantum Subgroups of the Compact Quantum Group $SU_{-1}(3)$. *Bull. London Math. Soc* **46** (2014), 315–328.

[BG] K. Brown and K. Goodearl, Lectures on Algebraic Quantum Groups. Birkhäuser Verlag, Basel, 2002.

[BGS] K. Brown, I. Gordon and J. Stafford. $\mathcal{O}_\varepsilon(G)$ is a free module over $\mathcal{O}(G)$. Preprint: `arXiv:math/0007179v1`.

[CM] W. Chin and I. Musson, Multiparameter quantum enveloping algebras, in: Contact Franco-Belge en Algèbre, Diepenbeek, 1993. *J. Pure Appl. Algebra* **107** (2ï£¡3) (1996), 171–191.

[CK] C. de Concini and V. Kac. Representations of Quantum Groups at Roots of 1, in Operator Algebras Unitary Representations, Enveloping Algebras, and Invariant Theory, Birkhäuser, Boston, 1990 471–506.

[CKP] C. de Concini, V. Kac and C. Procesi. Quantum Coadjoint Action. *Journal AMS* **5** (1992), no. 1 151–189.

[CV1] M. Costantini and M. Varagnolo. Quantum Double and Multiparameter Quantum Groups. *Comm. Algebra* **5** (1994), no. 5, 6305–63213.

[CV2] M. Costantini and M. Varagnolo. Multiparameter quantum function algebra at roots of 1. *Math. Ann.* **306** (1996), no. 4, 759–780.

[DL] C. De Concini and V. Lyubashenko. Quantum Function Algebra at Root of 1. *Adv. Math.* **108** (1994), 205–262.

[Do] Y. Doi. Braided Bialgebras and Quadratic Bialgebras. *Comm. Alg.* **21:5** (1993), 1731–1749.

[DPW] J. Du, B. Parshall and J.P. Wang, Two-parameter quantum linear groups and the hyperbolic invariance of q-Schur algebras. *J. Lond. Math. Soc.* (2) **44** (3) (1991), 420–436.

[FST] U. Franz, A. Skalski and R. Tomatsu, Idempotent states on compact quantum groups and their classification on $U_q(2)$, $SU_q(2)$, and $SO_q(3)$. *J. Noncommut. Geom.* **7** (2013) 221–254.

[G] G. A. García, Quantum subgroups of $GL_{\alpha,\beta}(n)$. *J. Algebra* **324** (2010), 1392–1428.

[H] T. Hayashi, Quantum groups and quantum determinants. *J. Algebra* **152** (1) (1992), 146–165.

[HLT] T.J. Hodges, T. Levasseur and M. Toro, Algebraic strcuture of Multi-Parameter quantum groups. *Adv. Math.* **126**(1997), 52–92.

[HPR] N. Hu, Y. Pei and M. Rosso, Multi-parameter quantum groups and quantum shuffles. I. Quantum affine algebras, extended affine Lie algebras, and their applications, 145–171, *Contemp. Math.*, **506**, Amer. Math. Soc., Providence, RI, 2010.

[LS] S. Levendorskiĭ and Y. Soibelman, Algebras of Functions on Compact Quantum Groups, Schubert Cells and Quantum Tori. *Comm. Math. Phys.* **139**(1991), 139–141.

[L] G. Lusztig, Quantum groups at root of 1. *Geometriae Dedicata* **35** (1990), 89–114.

[Ma] Y.I. Manin, Multiparametric quantum deformation of the general linear supergroup. *Comm. Math. Phys.* **123** (1) (1989), 16–175.

[Mo] S. Montgomery, Hopf Algebras and their Actions on Rings. AMS-NSF, CBMS-No. 82 Chicago, 1992.

[Mu] E. Müller, Finite Subgroups of the Quantum General Linear Group. *Proc. London Math. Soc.* **81** (2000), no. 1, 190–210 .

[OY] M. Okado and H. Yamane, R-matrices with gauge parameters and multi-parameter quantized enveloping algebras, in: Special Functions, ICM-90 Satell. Conf. Proc., Okayama, 1990, Springer-Verlag, Tokyo, 1991, pp. 289–293.

[P] P. Podleś, Symmetries of quantum spaces. Subgroups and quotient spaces of quantum $SU(2)$ and $SO(3)$ groups. *Comm. Math. Phys.* **170** (1995), 1–20.

[Ra] D. E. Radford, Hopf algebras. Series on Knots and Everything, 49. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[R] N. Reshetikhin, Multiparameter quantum groups and twisted quasitriangular Hopf algebras. *Lett. Math. Phys.* **20** (1990), no. 4, 331–335.

[Ş] D. Ştefan, Hopf algebras of low dimension, *J. Algebra* **211** (1999), 343–361.

[Su] H.-J. Sudbery, Consistent multiparameter quantisation of $GL(n)$. *J. Phys. A* **23** (15) (1990), L697–L704.

[Tk] M. Takeuchi, A two-parameter quantization of $GL(n)$, *Proc. Japan Acad. Ser. A Math. Sci.* **66** (5) (1990), 11–114.

[Tk2] ———, Cocycle deformations of coordinate rings of quantum matrices. *J. Algebra* **189** (1) (1997) 23–33.

[W] S.L. Woronowicz, Compact Quantum Groups, *Symétries Quantiques*, North Holland, Amsterdan (1998), 845–884.

G. A. G. : Departamento de Matemática, Facultad de Ciencias Exactas, Universidad Nacional de La Plata. CONICET. Casilla de Correo 172, (1900) La Plata, Argentina.

J. A. G. : FaMAF-CIEM (CONICET), Universidad Nacional de Córdoba. Medina Allende s/n, Ciudad Universitaria, 5000 Córdoba. Argentina.

E-mail address: ggarcia@mate.unlp.edu.ar

E-mail address: puiguti@gmail.com