THE ANNIHILATING-SUBMODULE GRAPH OF MODULES OVER COMMUTATIVE RINGS

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Abstract. Let M be a module over a commutative ring R. In this paper, we continue our study of annihilating-submodule graph $AG(M)$ which was introduced in (The Zariski topology-graph of modules over commutative rings, Comm. Algebra., 42 (2014), 3283–3296). $AG(M)$ is a (undirected) graph in which a nonzero submodule N of M is a vertex if and only if there exists a nonzero proper submodule K of M such that $NK = (0)$, where NK, the product of N and K, is defined by $(N : M)(K : M)M$ and two distinct vertices N and K are adjacent if and only if $NK = (0)$. We obtain useful characterizations for those modules M for which either $AG(M)$ is a complete (or star) graph or every vertex of $AG(M)$ is a prime (or maximal) submodule of M. Moreover, we study coloring of annihilating-submodule graphs.

1. Introduction

Throughout this paper R is a commutative ring with a non-zero identity and M is a unital R-module. By $N \leq M$ (resp. $N < M$) we mean that N is a submodule (resp. proper submodule) of M. Let $\Lambda(M)$ and $\Lambda(M)^*$ be the set of proper submodules of M and nonzero proper submodules of M, respectively.

Define $(N :_R M)$ or simply $(N : M) = \{r \in R | \ rM \subseteq N\}$ for any $N \leq M$. We denote $((0) : M)$ by $Ann_R(M)$ or simply $Ann(M)$. M is said to be faithful if $Ann(M) = (0)$.

Let $N, K \leq M$. Then the product of N and K, denoted by NK, is defined by $(N : M)(K : M)M$ (see [3]).

There are many papers on assigning graphs to rings or modules (see, for example, [1, 4, 7, 8]). The annihilating-ideal graph $AG(R)$, was introduced and studied in [8]. $AG(R)$ is a graph whose vertices are ideals of R with nonzero annihilators and in which two vertices I and J are adjacent if and only if $IJ = (0)$.

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In [4], we generalized the above idea to submodules of M and defined the (undirected) graph $AG(M)$, called *the annihilating-submodule graph*, with vertices $V(AG(M))= \{N \leq M|$ there exists $(0) \neq K < M$ with $NK = (0)\}$. In this graph, distinct vertices $N, L \in V(AG(M))$ are adjacent if and only if $NL = (0)$. Let $AG(M)^*$ be the subgraph of $AG(M)$ with vertices $V(AG(M)^*) = \{N < M$ with $(N : M) \neq Ann(M)|$ there exists a submodule $K < M$ with $(K : M) \neq Ann(M)$ and $NK = (0)\}$. Note that M is a vertex of $AG(M)$ if and only if there exists a nonzero proper submodule N of M with $(N : M) = Ann(M)$ if and only if every nonzero submodule of M is a vertex of $AG(M)$.

A prime submodule of M is a submodule $P \neq M$ such that whenever $re \in P$ for some $r \in R$ and $e \in M$, we have $r \in (P : M)$ or $e \in P$ [12, 13].

The prime spectrum (or simply, the spectrum) of M is the set of all prime submodules of M and denoted by $Spec(M)$. Also, $Max(M)$ will denote the set of all maximal submodules of M.

The prime radical $rad_M(N)$ is defined to be the intersection of all prime submodules of M containing N, and in case N is not contained in any prime submodule, $rad_M(N)$ is defined to be M [12].

Let $Z(R)$ and $Nil(R)$ be the set of zero-divisors and nilpotent elements of R, respectively. Let $Z_R(M)$ or simply $Z(M)$ be the set $\{r \in R|\ rm = 0$ for some $0 \neq m \in M\}$.

Let N and K be submodules of M. Then the product of N and K is defined by $(N : M)(K : M)M$ and denoted by NK (see [3]).

A clique of a graph is a maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $cl(G)$, is called the clique number of G. Let $\chi(G)$ denote the chromatic number of the graph G, that is, the minimal number of colors needed to color the vertices of G so that no two adjacent vertices have the same color. Obviously $\chi(G) \geq cl(G)$.

In section 2, we continue all modules M for which $AG(M)$ is a complete (resp. star) graph or every vertex of $AG(M)$ is a prime (or maximal) submodule (see Theorems 2.14, 2.15, and 2.17). In section 3, we study the coloring of the annihilating-submodule graph of modules. At first, among other results, we give a characterization of $\chi(AG(M)^*) = 2$ (see Theorem 3.2). It is shown that for a semiprime module M, the following conditions are equivalent. (1) $\chi(AG(M)^*)$ is finite. (2) $cl(AG(M)^*)$ is finite. (3) $AG(M)^*$ does not have an infinite clique (see Corollary 3.8). Also, it is shown that for a faithful module M with $rad_M(0) = (0)$, the following conditions are equivalent. (1) $\chi(AG(M)^*)$ is finite. (2) $cl(AG(M)^*)$ is finite. (3) $AG(M)^*$ does not have an infinite clique. (4) R has a finite number of prime ideals (see Proposition 3.11).

Let us introduce some graphical notions and denotations that are used in what follows: A graph G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set of vertices, $V(G)$, a set $E(G)$ of edges, and an incident function ψ_G that associates an unordered pair of distinct vertices with each edge. The edge e joins x and y if $\psi_G(e) = \{x, y\}$, and we say x and y are adjacent. A path in graph G is a finite sequence of vertices $\{x_0, x_1, \ldots, x_n\}$, where x_{i-1} and x_i are adjacent for each $1 \leq i \leq n$ and we denote $x_{i-1} - x_i$ for existing an edge between x_{i-1} and x_i.

A graph H is a subgraph of G if $V(H) \subset V(G)$, $E(H) \subseteq E(G)$ and ψ_H is the restriction of ψ_G to $E(H)$. A bipartite graph is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in

U to one in V; that is, U and V are each independent sets and complete bipartite graph on n and m vertices, denoted by $K_{n,m}$, where V and U are of size n and m, respectively, and $E(G)$ connects every vertex in V with all vertices in U. Note that a graph $K_{1,m}$ is called a star graph and the vertex in the singleton partition is called the center of the graph. (see [14]).

2. The Annihilating-submodule graph

An ideal $I \leq R$ is said to be nil if I consist of nilpotent elements; I is said to be nilpotent if $I^n = (0)$ for some natural number n.

Proposition 2.1. Suppose that e is an idempotent element of R. We have the following statements.

(a) $R = R_1 \oplus R_2$, where $R_1 = eR$ and $R_2 = (1-e)R$.
(b) $M = M_1 \oplus M_2$, where $M_1 = eM$ and $M_2 = (1-e)M$.
(c) For every submodule N of M, $N = N_1 \times N_2$ such that N_1 is an R_1-submodule M_1, N_2 is an R_2-submodule M_2, and $(N :_R M) = (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)$.
(d) For submodules N and K of M, $NK = N_1K_1 \times N_2K_2$ such that $N = N_1 \times N_2$ and $K = K_1 \times K_2$.

Proof. This is clear. □

We need the following lemmas.

Lemma 2.2. (See [2, Proposition 7.6].) Let $R_1, R_2, \dots, R_n$ be non-zero ideals of R. Then the following statements are equivalent:

(a) ${}_RR = R_1 \oplus \dots \oplus R_n$;
(b) As an abelian group R is the direct sum of $R_1, \dots, R_n$;
(c) There exist pairwise orthogonal central idempotents $e_1, \dots, e_n$ with $1 = e_1 + \dots + e_n$, and $R_i = Re_i$, $i = 1, \dots, n$.

Lemma 2.3. (See [11, Theorem 21.28].) Let I be a nil ideal in R and $u \in R$ be such that $u + I$ is an idempotent in R/I. Then there exists an idempotent e in uR such that $e - u \in I$.

Lemma 2.4. Let N be a minimal submodule of M and let $Ann(M)$ be a nil ideal. Then we have $N^2 = (0)$ or $N = eM$ for some idempotent $e \in R$.

Proof. Assume $N^2 \neq (0)$. Since $N^2 \neq (0)$ and N is a minimal submodule of M, we have $(N : M)m \neq (0)$ for some $m \in N$ so that $(N : M)m = N$. Choose $u \in (N : M)$ such that $m = um$. So $N = uM$. Since $m \in ((0) :_N u - 1)$, $N = ((0) :_N u - 1)$ and hence $u(u-1)M = (0)$. Thus $u(u-1) \in Ann(M)$. By Lemma 2.3, there is an idempotent e in R with $e - u \in Ann(M)$. So $(e-u)M = (0)$. It is clear that $eM = uM$. Hence $N = eM$. □

Theorem 2.5. *Let $Ann(M)$ be a nil ideal. There exists a vertex of $AG(M)$ which is adjacent to every other vertex if and only if $M = eM \oplus (1-e)M$, where eM is a simple module and $(1-e)M$ is a prime module for some idempotent $e \in R$ or $Z(M) = Ann((N : M)M)$, where N is a nonzero proper submodule of M or M is a vertex of $AG(M)$.*

Proof. Suppose that N is adjacent to every other vertex of $AG(M)$, $Z(M) \neq Ann((K : M)M)$ for every nonzero proper submodule K of M and M is not a vertex of $AG(M)$. If $N^2 = (0)$, then $Z(M) = Ann((N : M)M)$, a contradiction (note that if $r \in Z(M)$, then there exists a nonzero element $m \in M$ such that $rm = 0$. If $rM = (0)$, then $r \in Ann((N : M)M)$. Otherwise, since $(rM : M)(mR : M)M = (0)$, we have $(rM : M)(N : M)M = (0)$). Thus $N^2 \neq (0)$. Again by the above arguments, $N = (N : M)M$. By Lemma 2.4, $N = eM$ for some idempotent $e \in R$. We may assume that $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Also, by Proposition 2.1, we may assume that $M_1 \times (0)$ is adjacent to every other vertex of $AG(M)$. Now we show that M_2 is a prime module. Otherwise, there exist $0 \neq m \in M_2$ and $r \in R$ such that $rm = 0$ and $r \notin Ann(M_2)$. It follows that $(M_1 \times mR_2)((0) \times rM_2) = (0)$. So $M_1 \times (0)$ is adjacent to $M_1 \times mR_2$. This implies that $R_1 = (0)$, a contradiction. Therefore M_2 is a prime module. Conversely, assume that $M = eM \oplus (1-e)M$, where eM is a simple module and $(1-e)M$ is a prime module such that e is an idempotent. One can see that $eM \times (0)$ is adjacent to every other vertex of $AG(M)$. If M is a vertex of $AG(M)$, then there exists a nonzero proper submodule N of M such that $(N : M) = Ann(M)$ and hence N is adjacent to every other vertex. Now suppose that $Z(M) = Ann((N : M)M)$, where N is a nonzero proper submodule of M. Then it is easy to see that N is a vertex of $AG(M)$ which is adjacent to every other vertex or M is a vertex of $AG(M)$. □

Example 2.6. Let $M := \mathbb{Z}_2 \oplus \mathbb{Z}_3$ as a $\mathbb{Z}_{12}-$module. Clearly, $Ann(M) = \{\bar{0}, \bar{6}\}$ is a nil ideal and $AG(\mathbb{Z}_2 \oplus \mathbb{Z}_3)$ is a star graph with the only edge $\mathbb{Z}_2 \oplus (0) - (0) \oplus \mathbb{Z}_3$.

Theorem 2.7. *Let M be a faithful module. There exists a vertex of $AG(M)^*$ which is adjacent to every other vertex of $AG(M)^*$ if and only if $M = M_1 \oplus M_2$, where M_1 is a simple module and M_2 is a prime module or $Z(R)$ is an annihilator ideal.*

Proof. ($\Longrightarrow$). Suppose that $Z(R)$ ia not an annihilator ideal and N is adjacent to every other vertex. If $N^2 = (0)$, then $Z(R) = Ann((N : M))$, a contradiction (note that if $r \in Z(R)$, then there exists a nonzero element $s \in R$ such that $rs = 0$. So we have $(rM : M)(sM : M) = (0)$. Hence $(rM : M)(N : M) = (0)$). Thus $N^2 \neq (0)$. Now the claim follows by using similar arguments as in the proof of theorem 2.5.
($\Longleftarrow$). Assume that $M = M_1 \oplus M_2$, where M_1 is a simple R-module and M_2 is a prime R-module. Since M_1 is a simple R-module, $Ann(M_1)$ is a maximal ideal of R and since $Ann(M) = (0)$, we have $Ann(M_2) + Ann(M_1) = R$ and so we may assume that $R = R_1 \oplus R_2$. Then Lemma 2.2 and Theorem 2.5 imply that there is a vertex of $AG(M)^*$ which is adjacent to every other vertex of $AG(M)^*$. Now let $Z(R) = Ann(I)$ for some nonzero proper ideal I of R. In this case, clearly, IM is a vertex of $AG(M)^*$ which is adjacent to every other vertex of $AG(M)^*$. □

Example 2.8. It is easy to see that $\mathbb{Q} \oplus \mathbb{Q}$ as $\mathbb{Q} \oplus \mathbb{Z}-$module is faithful and $AG(\mathbb{Q} \oplus \mathbb{Q})^*$ is a star graph with the only edge $\mathbb{Q} \oplus (0) - (0) \oplus \mathbb{Q}$.

Corollary 2.9. Let R be a reduced ring and let $Ann(M)$ be a nil ideal. Then the following statements are equivalent.

(a) There is a vertex of $AG(M)^*$ which is adjacent to every other vertex of $AG(M)^*$.
(b) $AG(M)^*$ is a star graph.
(c) $M = M_1 \oplus M_2$, where M_1 is a simple module and M_2 is a prime module.

Proof. $(a) \Rightarrow (b)$ Suppose that there is a vertex of $AG(M)^*$ which is adjacent to every other vertex. If $Z(R) = Ann(x)$ for some $0 \neq x \in R$, then we have $x^2 = 0$, a contradiction. Therefore by Theorem 2.7, $M = M_1 \oplus M_2$, where M_1 is a simple module and M_2 is a prime module. Then every nonzero submodule of M is of the form $M_1 \times N_2$ and $(0) \times N_2$, where N_2 is a nonzero submodule of M_2. By our hypothesis, we can not have any vertex of the form $M_1 \times N_2$, where N_2 is a nonzero proper submodule of M_2. Also $M_1 \times (0)$ is adjacent to every other vertex, and non of the submodules of the form $(0) \times N_2$ can be adjacent to each other. So $AG(M)^*$ is a star graph.
$(b) \Rightarrow (c)$ This follows by Theorem 2.7.
$(c) \Rightarrow (a)$ This follows by Theorem 2.7. □

Let M be an R-module. The set of associated prime ideals of M, denoted by $Ass_R(M)$ (or simply $Ass(M)$), is defined as $Ass(M) = \{p \in Spec(R) | \ p = (0 :_R m)$ for some $0 \neq m \in M\}$.

Corollary 2.10. Let R be an Artinian ring and let $Ann(M)$ be a nil ideal. Then there is a vertex of $AG(M)$ which is adjacent to every other vertex if and only if either $M = M_1 \oplus M_2$, where M_1 is a simple module and M_2 is a prime semisimple module or R is a local ring with maximal ideal $p \in Ass(M)$ or M is a vertex of $AG(M)$.

Proof. $(\Longrightarrow)$ Let N be a vertex of $AG(M)$ which is adjacent to every other vertex and suppose M is not a vertex of $AG(M)$. As we have seen in Theorem 2.5, either $M = M_1 \oplus M_2$, where M_1 is a simple module and M_2 is a prime module or $Z(M) = Ann((K : M)M)$, where K is a nonzero proper submodule of M. Let $M = M_1 \oplus M_2$, where M_1 is a simple module and M_2 is a prime module. It is easy to see that M_2 is a vector space over $R/Ann(M_2)$ and so is a semisimple R-module. If $Z(M)$ is an ideal of R , since R is an Artinian ring, then $Z(M) = p \in Ass(M)$.
$(\Longleftarrow)$ First suppose that R is not a local ring. Hence by [5, Theorem 8.7], $R = R_1 \times \ldots \times R_n$, where R_i is an Artinian local ring for $i = 1, \ldots, n$. By Lemma 2.2 and Theorem 2.5, we may assume that $eM \times (0)$ is adjacent to every other vertex of $AG(M)$. If R is a local ring with maximal ideal $p \in Ass(M)$, then there exists $0 \neq m \in M$ such that $p = Ann(m)$. We claim that Rm is adjacent to every other vertex. Suppose N is a vertex. We have $(N : M) \subseteq p$. Hence $(N : M)(mR) = (0)$. So we have $N(mR) = (0)$. □

Example 2.11. Let $M := \mathbb{Z}_3 \oplus \mathbb{Z}_8$ as a $\mathbb{Z}_{48}-$module. Clearly, $Ann(M) = \{\bar{0}, \bar{24}\}$ is a nil ideal and $AG(\mathbb{Z}_3 \oplus \mathbb{Z}_8)$ is a star graph with the center $\mathbb{Z}_3 \oplus (0)$ and $V(AG(\mathbb{Z}_3 \oplus \mathbb{Z}_8)) = \{\mathbb{Z}_3 \oplus (0), (0) \oplus \mathbb{Z}_8, (0) \oplus N, (0) \oplus K\}$, where $N = (\bar{0}, \bar{2})\mathbb{Z}_{48}$ and $K = (\bar{0}, \bar{4})\mathbb{Z}_{48}$.

Corollary 2.12. Let R be an Artinian ring and let M be a faithful R-module. Then there is a vertex of $AG(M)^*$ which is adjacent to every other vertex if and only if either $M = M_1 \oplus M_2$, where M_1 and M_2 are simple modules or R is a local ring with maximal ideal $p \in Ass(M)$.

Proof. $(\Longrightarrow)$ Let N be a vertex of $AG(M)^*$ which is adjacent to every other vertex. So there is a vertex of $AG(R)$ which is adjacent to every other vertex of $AG(R)$. By [8, Corollary 2.4], we may assume that $R = F_1 \oplus F_2$, where F_1 and F_2 are fields or $Z(R)$ is an annihilator ideal. If $R = F_1 \oplus F_2$, then $AG(R)$ is a complete graph. It follows that $AG(M)^*$ is a complete graph. Hence $AG(M)^*$ have exactly two vertices $M_1 \times (0)$ and $(0) \times M_2$. So M_2 is a simple module. If $Z(R)$ is an annihilator ideal,

since R is an Artinian ring, then $Z(R)$ is the unique maximal ideal of R. Since R is a Noetherian ring, $Z(M) = p \in Ass(M)$ and hence $Z(R) = Z(M)$.
($\Longleftarrow$) This is clear by Corollary 2.10. □

Lemma 2.13. Let R be an Artinian ring and assume that $Ann(M)$ is a nil ideal and $AG(M)$ is a star graph. Then either $M = M_1 \oplus M_2$, where M_1 is a simple module and M_2 is a prime semisimple module or R is a local ring with maximal ideal $p = Ann(m)$, $(mR)^2 = (0)$ and $p^4M = (0)$ or M is a vertex of $AG(M)$.

Proof. Let R be an Artinian ring and assume that $AG(M)$ is a star graph and M is not a vertex of $AG(M)$. Then by Corollary 2.10, either $M = M_1 \oplus M_2$, where M_1 is a simple module and M_2 is a prime semisimple module or R is a local ring with maximal ideal $p \in Ass(M)$. If $M = M_1 \oplus M_2$, then there is nothing to prove. If R is a local ring with maximal ideal $p = Ann(m)$, where $0 \neq m \in M$, then as we showed in Corollary 2.10, mR is adjacent to every other vertex. Since R is an Artinian ring and M is not a vertex, there exists an integer $n > 1$ such that $p^nM = (0)$ and $p^{n-1}M \neq (0)$. As $AG(M)$ is a star graph (resp. $(mR : M) \subseteq p$), we have $p^4M = (0)$ and $mR = p^3M$ (resp. $(mR)^2 = (0)$). □

Theorem 2.14. *Let R be an Artinian ring and assume that M is not a vertex of $AG(M)$ and $Ann(M)$ is a nil ideal. Then $AG(M)$ is a star graph if and only if either $M = M_1 \oplus M_2$, where M_1 and M_2 are simple modules or R is a local ring with maximal ideal $p = (0 : m) \in Ass(M)$ and one of the following cases holds.*

(a) $p^2M = (0)$ *and* $pM = mR$ *is the only nonzero proper submodule of* M*.*
(b) $p^3M = (0)$ *and* $p^2M = mR$ *is the only minimal submodule of* M *and for every distinct proper submodules* N_1, N_2 *of* M *such that* $mR \neq N_i$ $(i = 1, 2)$*,* $N_1N_2 = Rm$*.*
(c) $p^4M = (0)$*,* $p^3M \neq (0)$*, and* $\Lambda(M)^* = \{N < M | \ (N : M) = (pM : M)\} \cup \{p^2M, p^3M = mR\}$*.*

Proof. ($\Longrightarrow$) Suppose that $AG(M)$ is a star graph. By Lemma 2.13, we may have $p^4M = (0)$. We proceed by the following cases:

Case 1. $p^2M = (0)$. Hence every nonzero proper submodule N of M is a vertex and $N^2 = (0)$. It is clear that pM is a p-prime submodule of M and is adjacent to every other vertex. Thus $pM = Rm$. Since for every nonzero proper submodule N and K of M, $NK = (0)$ and $AG(M)$ is a star graph, M has at most two nonzero proper submodules. So M is a Noetherian module and Rm is a subset of every nonzero submodule of M. It is easy to see that M is cyclic and hence a multiplication module. It follows that $pM = mR$ is the only nonzero proper submodule of M.

Case 2. $p^3M = (0)$ and $p^2M \neq (0)$. It is clear that p^2M is adjacent to every other vertex. So $p^2M = Rm$. We claim that Rm is the only minimal submodule of M. Suppose N is another minimal submodule of M. It is easy to see that $Ann(N) = p$. Let K be a nonzero proper submodule of M. Thus $(K : M) \subseteq Ann(N) = p$ so that $NK = (0)$. Hence $N = Rm$. Finally, suppose $N_1, N_2 \neq Rm$ are distinct nonzero proper submodules of M. We have $(N_1 : M), (N_2 : M) \subseteq p$. Since $AG(M)$ is a star graph, we have $N_1N_2 \neq (0)$. Hence by minimality of $p^2M = mR$, $N_1N_2 = mR$.

Case 3. $p^4M = (0)$ and $p^3M \neq (0)$. Since $AG(M)$ is a star graph and R is a

local ring, then the center of the star graph must be a nonzero cyclic submodule $p^3M = mR$. Since $p^3 \subsetneq p^2$, there are elements $a, b \in p \setminus p^2$ such that $ab \in p^2 \setminus p^3$. Then $abp^2M = (0)$, so $p^2M = abM$ because $AG(M)$ is a star graph. If $a^2 \in p^3M$, then $(aM)(abM) = (0)$, a contradiction with the star shaped assumption. Hence $p^2M = a^2M$. Another application of the star shaped assumption yields $a^3 \neq 0$. So $p^3M = a^3M$. For $c \in p \setminus p^2$, $ca^2 \in a^3R \setminus (0)$. We conclude that $c \in aR$. Hence $p = aR$ and hence every non-zero ideal of R is a power of p. So $(aM : M) = aR$, $(a^2M : M) = a^2R$, and $(a^3M : M) = a^3R$. It is easy to see that $AG(M)$ is a star graph with the center $a^3M = mR$, and the other vertices are aM, a^2M, and every nonzero proper submodule N of M with $(N : M) = aR$. (Note that $Spec(M) = \{(0) \neq N < M |\ N \neq a^2M, a^3M\}$.)
($\Longleftarrow$) This is clear. □

Theorem 2.15. *Assume that M is a faithful module and is not a vertex of $AG(M)$. Then $AG(M)$ is a complete graph if and only if M is one of the three types of modules.*

(a) *$M = M_1 \oplus M_2$, where M_1 and M_2 are simple modules,*
(b) *$Z(R)$ is an ideal with $(Z(R))^2 = (0)$, or*
(c) *Every nonzero proper submodule of M is a vertex, $Spec(M) = Max(M) = \{aM\}$, where R is a local ring with exactly two nonzero proper ideals $Z(R) = aR$, $Z(R)^2$ such that $a^3 = 0$, and for every nonzero proper submodule $N \neq aM$, $(N : M) = a^2R$.*

Proof. ($\Longrightarrow$) Assume that $AG(M)$ is a complete graph. So $AG(R)$ is a complete graph. By Theorem 2.5, $M = M_1 \oplus M_2$, where M_1 is a simple module and M_2 is a prime module or $Z(R)$ is an ideal. Suppose that we have the first case. If M_2 has a nonzero proper submodule, say N_2, then $(0) \times M_2$ and $(0) \times N_2$ are vertices of $AG(M)$ which are not adjacent, a contradiction. Thus M_2 can not have any nonzero proper submodule, and hence it is a simple module. Now assume that $Z(R)$ is an ideal of R. So (b) holds if $Z(R)^2 = (0)$. Otherwise, then by [8, Theorem 2.7], R is a local ring with exactly two nonzero proper ideals $Z(R) = aR$ and $Z(R)^2$ (note that $a^3 = 0$). Hence every nonzero proper submodule M is a vertex. Since $AG(M)$ is a complete graph, for every nonzero proper submodule $N \neq aM$, we have $(N : M) = a^2R$. It follows that $Spec(M) = Max(M) = \{aM\}$.
($\Longleftarrow$) This is clear. □

Corollary 2.16. Assume that M is a faithful module and is not a vertex of $AG(M)$. Then we have the following.

(a) $AG(M)$ is a complete graph with one vertex if and only if M has only one nonzero proper submodule.
(b) $AG(M)$ is a graph with two vertices if and only if $M = M_1 \oplus M_2$, where M_1 and M_2 are simple modules or M is a module with exactly two nonzero proper submodules $Z(R)M$ and $Z(R)^2M$.
(c) $AG(M)$ is a graph with three vertices if and only if M has exactly three nonzero proper submodules m_1R, m_2R, m_3R such that $m_3R = m_1R \cap m_2R, Z(R) = Ann(m_3), (m_1R)^2 = (m_2R)^2 = (m_3R)^2 = (0)$, where $0 \neq m_1, m_2, m_3 \in M$, or $\Lambda(M)^* = \{Z(R)M, Z(R)^2M, Z(R)^3M\}$.

Proof. (a) ($\Longrightarrow$) This follows by [4, Theorem 3.6 and Proposition 3.5].
($\Longleftarrow$) Suppose M has only one nonzero proper submodule. It follows that $M \cong R$

and hence [8, Corollary 2.9(a)] completes the proof.
(b) ($\Longrightarrow$) Suppose $AG(M)$ is a graph with two vertices. By [4, Theorem 3.6 and Proposition 3.5], M has exactly two nonzero proper submodules. Since $AG(M)$ is connected, then $AG(M)$ is a complete (or star) graph. Thus by Theorem 2.15 and Theorem 2.14, $M = M_1 \oplus M_2$, where M_1 and M_2 are simple modules or M is a module with exactly two nonzero proper submodules $Z(R)M$ and $Z(R)^2M$.
($\Longleftarrow$) This is clear.
(c) ($\Longrightarrow$) Suppose $AG(M)$ is a graph with three vertices. By [4, Theorem 3.6 and Proposition 3.5], M has exactly three nonzero proper submodules. Since $AG(M)$ is connected, either it is a complete (or star) graph. If $AG(M)$ is a complete graph, then we may have the cases (b) and (c) in Theorem 2.15. First we assume that the case (b) in this theorem is true. Then $Z(R)$ is an ideal of R with $Z(R)^2 = (0)$. It follows that R is an Artinian ring and $Z(R) = Nil(R) = Ann(m)$ is the only prime ideal of R, where Rm is a minimal submodule of M. Let N_1, N_2, and N_3 be the only nonzero proper submodules of M. We proceed by the following cases:

Case 1. $N_1 \subset N_2 \subset N_3$. Then we have M and R are isomorphic which is a contradiction by [8, Corollary 2.9].

Case 2. N_1, N_2, and N_3 are minimal submodules of M. In this case, M is a multiplication cyclic module. But this yields a contradiction.

Case 3. $N_1 \subset N_2$, and N_3 is not comparable with N_i, $i = 1, 2$. Then since $(N_2 : M) = (N_3 : M) = Z(R)$, it follows that $N_1 = N_3$ or $N_2 = N_3$, a contradiction.

Case 4. $N_1 \subset N_2$, and $N_3 \subset N_2$. Then we have $(N_2 : M) = Z(R)$. If M is a multiplication module, then it is cyclic and hence similar to the case (1), we get a contradiction. Otherwise, $N_1 \subset N_3$ or $N_3 \subset N_1$, which is again a contradiction.

Case 5. $N_3 \subset N_1$ and $N_3 \subset N_2$. It follows that $N_1 = Rm_1$, $N_2 = Rm_2$, $N_3 = Rm_3 = N_1 \cap N_2$, $Z(R) = Ann(m_3)$, and $N_1^2 = N_2^2 = N_3^2 = (0)$, as desired.

Now suppose that the case (c) in Theorem 2.15 is true. Then we have $\Lambda(M)^* = \{aM, a^2M, N\}$ such that $Spec(M) = Max(M) = \{aM\}$, where $Z(R) = aR$ and $(N : M) = a^2R$. It follows that $N \subseteq aM$ and hence similar to case (4), we get again a contradiction. Finally suppose that $AG(M)$ is a star graph with three vertices. Then we may have the cases (b) and (c) in Theorem 2.15. We prove that the case (b) in this theorem is not true. Otherwise, $Z(R)^3M = (0)$ and $Z(R)^2M$ is the only minimal submodule of M. Let N be a nonzero proper submodule of M and $N \neq pM, p^2M$. If N is a maximal submodule of M, then $p^2M \subset pM \subset N$ and hence similar to case (1), we have a contradiction. Otherwise, $p^2M \subset N \subset pM$, which is again a contradiction. Thus M has exactly three nonzero proper submodules $Z(R)M$, $Z(R)^2M$, and $Z(R)^3M$.
($\Longleftarrow$) This is clear.

□

Theorem 2.17. *Suppose that R is not a domain and M is a faithful module. If every vertex of $AG(M)$ is a prime submodule of M, then either $M = M_1 \oplus M_2$, where M_1 and M_2 are simple modules or M has only one nonzero proper submodules.*

Proof. Case 1. Let $x \in Z(R)$ and $x^2 \neq 0$. It follows that xM and x^2M are vertices of $AG(M)$. We have $(xM : M)(xM) \subseteq x^2M$. This implies that $xM \subseteq x^2M$ or $(xM : M) \subseteq (x^2M : M)$ so that $xM = x^2M$. Let N be a nonzero proper submodule of M with $N \leq xM$ and let $m \in M$. Since $xm \in x^2M$, there exists $m' \in M$ such that $xm = x^2m'$. Since $M \neq xM$, hence $m - xm' \neq 0$. Thus we have $x(m - xm') = 0 \in N$. Again, since $M \neq xM$, $x \in (N : M)$. It follows that $N = xM$ and hence xM is a minimal submodule of M. By Lemma 2.4, we have $N = eM$ for some idempotent $e \in R$. Now we show that $(1 - e)M$ is a minimal submodule of M. Let $0 \neq K \subset (1 - e)M$. Then there exists $m \in M$ such that $m(1-e) \notin K$. We have $e(m(1-e)) \in K$. So $e \in (K : M)$. It follows that $e^2M = 0$, a contradiction. This implies that $M = M_1 \oplus M_2$, where M_1 and M_2 are simple modules.

Case 2. Assume that $x^2 = 0$ for every $0 \neq x \in Z(R)$. At first, we show that for every $x, y \in Z(R) \setminus \{0\}$, $xM = yM$. Otherwise, there exists $m, m' \in M$ such that $xm \notin yM$ and $ym' \notin xM$. We have $x(xm) = 0 \in yM$ and $y(ym') = 0 \in xM$. It follows that $xM \subseteq yM$ and $yM \subseteq xM$, a contradiction. Hence $xy = 0$. It implies that for every vertex N and K of $AG(M)$, $NK = (0)$. Therefore $AG(M)$ is a complete graph. One can see that for every $0 \neq x \in Z(R)$, xM is a minimal submodule of M and hence there exists $0 \neq m \in M$ such that $xM = Rm$. This case and $Z(R) = Nil(R)$ yield that $Z(R) = Ann(m)$ is a unique prime ideal of R. So every nonzero proper submodule of M is a vertex. Now, if xM is the only nonzero proper submodule of M, then there is nothing to prove. Otherwise, let N be a nonzero proper submodule of M such that $xM \subset N$. Thus there exists $m \in N$ such that $m \notin xM$. If $Ann(m) = (0)$, then $R \cong Rm$. So every nonzero proper ideal of R is a prime ideal and hence R has only one nonzero proper ideal. Now the result follows from Theorem 2.14. If $Ann(m) \neq (0)$, then Rm is a minimal submodule of M. Since $xM \subseteq Rm$ ($xM = (xM : M)M = Z(R)M = (Rm : M)M$), we have $xM = Rm$, a contradiction, as desired. □

Corollary 2.18. Assume that R is not a domain and M is a faithful module. Then we have the following.

(a) $V(AG(M)) \subseteq Max(M)$, i.e., every vertex of $AG(M)$ is a maximal submodule of M.
(b) $V(AG(M)) = Max(M)$
(c) $V(AG(M)) = Spec(M)$.
(d) $V(AG(M)) \subseteq Spec(M)$.
(e) Either $M = M_1 \oplus M_2$, where M_1 and M_2 are simple modules or M has only one nonzero proper submodule.

Proof. This is clear. □

3. Coloring of the annihilating-submodule graphs

We recall that M is an R-module.
The purpose of this section is to study of coloring of the annihilating-submodule graphs of modules and investigate the interplay between $\chi(AG(M))$ and $cl(AG(M))$.

Proposition 3.1. Let M be a faithful module. Then $\chi(AG(M)) = 1$ if and only if M has only one nonzero proper submodule.

Proof. Let $\chi(AG(M)) = 1$. Since $AG(M)$ is a connected graph, it can not have more than one vertex. If M is a faithful module, then by Corollary 2.16(a), $AG(M)$ is a graph with one vertex if and only if M has only one nonzero proper submodule. □

Theorem 3.2. *Let M be a faithful module. Then the following statements are equivalent.*

(a) $\chi(AG(M)^*) = 2$.
(b) $AG(M)^*$ *is a bipartite graph with two nonempty parts.*
(c) $AG(M)^*$ *is a complete bipartite graph with two nonempty parts.*
(d) *Either R is a reduced ring with exactly two minimal prime ideals or $AG(M)^*$ is a star graph with more than one vertex.*

Proof. $(a) \Longleftrightarrow (b)$ and $(c) \Longrightarrow (b)$ are clear.
$(b) \Longrightarrow (d)$ Suppose that $AG(M)^*$ is a bipartite graph with two nonempty parts V_1 and V_2. One can see that $AG(M)^*$ is a bipartite graph with two nonempty parts V_1 and V_2 if and only if $AG(R)$ is a bipartite graph with two nonempty parts U_1 and U_2 such that if $N \in V_i$, then $(N : M) \in U_i$ and if $I \in U_i$, then $IM \in V_i$, for $i = 1, 2$. Hence by [9, Theorem 2.3], R is a reduced ring with exactly two minimal prime ideals p_1 and p_2 or $AG(R)$ is a star graph with more than one vertex. If R is a reduced ring with exactly two minimal prime ideals p_1 and p_2, then there is nothing to prove. If $AG(R)$ is a star graph with more than one vertex, then $AG(M)^*$ is a star graph with more than one vertex.
$(d) \Longrightarrow (c)$ Assume that R is a reduced ring with exactly two minimal prime ideals p_1 and p_2. Then by [9, Theorem 2.3], $AG(R)$ is a complete bipartite graph with two nonempty parts so that $AG(M)^*$ is a complete bipartite graph with two nonempty parts. If $AG(M)^*$ is a star graph with more than one vertex, then $AG(M)^*$ is a complete bipartite graph. □

Corollary 3.3. Let R be an Artinian ring and assume that M is a faithful module. Then the following statements are equivalent.

(a) $\chi(AG(M)^*) = 2$.
(b) $AG(M)^*$ is a bipartite graph with two nonempty parts.
(c) $AG(M)^*$ is a complete bipartite graph with two nonempty parts.
(d) Either $M = M_1 \oplus M_2$, where M_1 and M_2 are simple modules or $AG(M)^*$ is a star graph with more than one vertex such that R is a local ring.

Proof. By Theorem 3.2, $(a) \Longleftrightarrow (b) \Longleftrightarrow (c)$.
$(b) \Longrightarrow (d)$ Assume that $AG(M)^*$ is a bipartite graph with two nonempty parts. Hence $AG(R)$ is a bipartite graph with two nonempty parts. By [9, Corollary 2.4], if $R \cong F_1 \oplus F_2$, then $AG(R)$ is a star graph. So $AG(M)^*$ is a star graph. Hence by Corollary 2.12, either $M = M_1 \oplus M_2$, where M_1 and M_2 are simple modules or R is a local ring with maximal ideal $p = (0 : m) \in Ass(M)$. In the first case, as desired. In the second case, $AG(M)^*$ is a star graph with the center Rm. On the other hand, if R is a local ring such that $AG(R)$ is a star graph, then we are done. □

Corollary 3.4. Let R be a reduced ring and assume that M is a faithful module. Then the following statements are equivalent.

(a) $\chi(AG(M)^*) = 2$.

(b) $AG(M)^*$ is a bipartite graph with two nonempty parts.
(c) $AG(M)^*$ is a complete bipartite graph with two nonempty parts.
(d) R has exactly two minimal prime ideals.

Proof. Use Theorem 3.2. □

Recall that $N < M$ is said to be a semiprime submodule of M if for every ideal I of R and every submodule K of M, $I^2K \subseteq N$ implies that $IK \subseteq N$. Further M is called a semiprime module if $(0) \subseteq M$ is a semiprime submodule. Every intersection of prime submodules is a semiprime submodule (see [15]). A prime submodule N of M will be called extraordinary if whenever K and L are an intersection of prime submodules of M with $K \cap L \subseteq N$, then $K \subseteq N$ or $L \subseteq N$ (see [13]).

Lemma 3.5. Let M be a semiprime R-module such that $AG(M)^*$ does not have an infinite clique. Then M has a.c.c. on submodules of the form $Ann_M(I)$, where I is an ideal of R.

Proof. Suppose that $Ann_M(I_1) \subset Ann_M(I_2) \subset Ann_M(I_3) \subset ...$ (strict inclusions) so that M does not satisfy the a.c.c. on submodules of the form $Ann_M(I)$, where I is an ideal of R. Clearly, $I_iAnn_M(I_{i+1}) \neq 0$, for each $i \geq 1$. Thus for each $i \geq 1$, there exists $x_i \in I_i$ such that $x_iAnn_M(I_{i+1}) \neq (0)$. Let $J_i = x_iAnn_M(I_{i+1})$, $i = 1, 2, 3, \ldots$. Then if $i \neq j$ (we may assume that $i < j$), $J_i \neq J_j$ because if $x_iAnn_M(I_{i+1}) = x_jAnn_M(I_{j+1})$, then we have $Ann_M(I_{i+1}) \subseteq Ann_M(I_j)$. So $x_jAnn_M(I_{i+1}) = (0)$. Hence $x_j^2Ann_M(I_{j+1}) = (0)$. This yields a contradiction because M is a semiprime module and $x_jAnn_M(I_{j+1}) \neq 0$. On the other hand, one can see that $J_iJ_j \subseteq x_ix_jAnn_M(I_{i+1})$ and $J_iJ_j \subseteq x_ix_jAnn_M(I_{j+1})$. Hence we have $J_iJ_j = (0)$. □

Lemma 3.6. Let $P_1 = Ann_M(x_1)$ and $P_2 = Ann_M(x_2)$ be two distinct elements of $Spec(M)$. Then $(x_1M)(x_2M) = (0)$.

Proof. The proof is straightforward. □

Theorem 3.7. *M is a faithful module if one of the following holds.*
(a) *R is a reduced ring and $Z(M) = p_1 \cup p_2 \cup ... \cup p_k$, where $Min(R) = \{p_1, p_2, ..., p_k\}$.*
(b) *M is a semiprime module and $AG(M)^*$ does not have an infinite clique.*

Proof. (a). Let $(0) = p_1 \cap p_2 \cap ... \cap p_k$, where p_1, p_2,..., p_k are minimal prime ideals of R. We have $Ann(M) \subseteq (p_1M : M) \cap ... \cap (p_kM : M)$. It is enough to show that $(p_iM : M) = p_i$, $i = 1, ..., k$. For $1 \leq i \leq n$, $p_1p_2...p_{i-1}p_{i+1}...p_n \neq (0)$ because if $p_1p_2...p_{i-1}p_{i+1}...p_n = (0)$, then $p_j \subseteq p_i$, where $j \neq i$, a contradiction. Also, if $(p_1p_2...p_{i-1}p_{i+1}...p_n)M = (0)$, then $p_1p_2...p_{i-1}p_{i+1}...p_n \subseteq Ann(M)$. So for every nonzero element $m \in M$, we have $p_1p_2...p_{i-1}p_{i+1}...p_n \subseteq Ann(m) \subseteq Z(M)$. It follows that there exists $j \neq i$ such that $Ann(m) \subseteq p_j$. Hence $Z(M) = p_1 \cup p_2 \cup ... \cup p_{i-1} \cup p_{i+1} \cup ... \cup p_n$, a contradiction. So $(p_1p_2...p_{i-1}p_{i+1}...p_n)M \neq (0)$. We have $(p_iM)((p_1p_2...p_{i-1}p_{i+1}...p_n)M) = 0$ and so $(p_iM : M) \subseteq Z(M)$. It follows that $(p_iM : M) = p_i$, as desired.
(b) Suppose that M is a semiprime module and $AG(M)^*$ does not have an infinite clique. Then by Lemma 3.5, M has a.c.c. on submodules of the form $Ann_M(I)$, where I is an ideal of R. Therefore the set $\{Ann_M(x)| \ x \notin Ann(M)\}$ has maximal submodules so that they are prime submodules of M. Let $Ann_M(x_\lambda)$, where $\lambda \in \Lambda$, be the different maximal members of the family $\{Ann_M(x)| \ x \notin Ann(M)\}$. By Lemma 3.6, the index set Λ is finite. Let $x \in R$ such that $x \notin Ann(M)$. Then

$Ann_M(x) \subseteq Ann_M(x_{\lambda_1})$ for some $\lambda_1 \in \Lambda$. We claim that $\cap_{\lambda\in\Lambda}(Ann_M(x_\lambda) : M) = (0)$. Let $0 \neq x \in \cap_{\lambda\in\Lambda}(Ann_M(x_\lambda) : M)$. So $xM \subseteq Ann_M(x_\lambda)$ for every $\lambda \in \Lambda$. We have $Ann_M(x) \subseteq Ann_M(x_{\lambda_1})$. Since $xM \subseteq Ann_M(x_{\lambda_1})$, $x_{\lambda_1}M \subseteq Ann_M(x)$. Thus $x^2_{\lambda_1}M = (0)$, a contradiction. Now the proof is completed because $Ann(M) \subseteq (Ann_M(x_\lambda) : M)$ for every $\lambda \in \Lambda$.

□

Corollary 3.8. Assume that M is a semiprime module. Then the following statements are equivalent.

(a) $\chi(AG(M)^*)$ is finite.
(b) $cl(AG(M)^*)$ is finite.
(c) $AG(M)^*$ does not have an infinite clique.

Proof. $(a) \Longrightarrow (b) \Longrightarrow (c)$ is clear.
$(c) \Longrightarrow (d)$ Suppose $AG(M)^*$ does not have an infinite clique. It follows directly from the proof of Theorem 3.7(b), there exists a finite number of prime submodules $P_1, ..., P_k$ of M such that $(0) = P_1 \cap P_2 \cap ... \cap P_k$. Define a coloring $f(N) = min\{n \in N| \ (N : M)M \nsubseteq P_n\}$ such that N is a vertex of $AG(M)^*$. We have $\chi(AG(M)^*) \leq k$. □

Corollary 3.9. Assume that $rad_M(0) = (0)$ and every prime submodule of M is extraordinary. Then the following statements are equivalent.

(a) $\chi(AG(M)^*)$ is finite.
(b) $cl(AG(M)^*)$ is finite.
(c) $AG(M)^*$ does not have an infinite clique.
(d) M has a finite number of minimal prime submodules.

Proof. $(a) \Longrightarrow (b) \Longrightarrow (c)$ is clear.
$(c) \Longrightarrow (d)$ Suppose $AG(M)^*$ does not have an infinite clique. Once again, it follows directly from the proof of Theorem 3.7(b), there exists a finite number of prime submodules $P_1, ..., P_k$ of M such that $(0) = P_1 \cap P_2 \cap ... \cap P_k$. Since every prime submodule of M is extraordinary, M has a finite number of minimal prime submodules.
$(d) \Longrightarrow (a)$ Assume that M has a finite number of minimal prime submodules so that $(0) = P_1 \cap P_2 \cap ... \cap P_k$, where $P_1, ..., P_k$ are minimal prime submodules of M. Define a coloring $f(N) = min\{n \in N| \ (N : M)M \nsubseteq P_n\}$ such that N is a vertex of $AG(M)^*$. We have $\chi(AG(M)^*) \leq k$. □

Lemma 3.10. Let R be a reduced ring and M a faithful R-module. Then $AG(R)$ has an infinite clique if and only if $AG(M)^*$ has an infinite clique.

Proof. This is clear. □

Proposition 3.11. Assume that $rad_M(0) = (0)$ and M is a faithful R-module. Then the following statements are equivalent.

(a) $\chi(AG(M)^*)$ is finite.
(b) $cl(AG(M)^*)$ is finite.
(c) $AG(M)^*$ does not have an infinite clique.
(d) R has a finite number of minimal prime ideals.

Proof. $(a) \Longrightarrow (b) \Longrightarrow (c)$ is clear.
$(c) \Longrightarrow (d)$ Suppose $AG(M)^*$ does not have an infinite clique. Then by Theorem

3.7(b), M is a faithful module. Since $rad_M(0) = (0)$, it follows that R is a reduced ring. So by Lemma 3.5, $AG(R)$ does not have an infinite clique. Then by [9, Corollary 2.10], R has a finite number of minimal prime ideals so that $(0) = p_1 \cap p_2 \cap ... \cap p_k$, where $p_1, ..., p_k$ are prime ideals.
$(d) \Longrightarrow (a)$ Assume that R has a finite number of minimal prime ideals. Since M is a faithful module and $rad_M(0) = (0)$, then R is a reduced ring. So R has a finite number of minimal prime ideals $p_1, ..., p_k$ such that $(0) = p_1 \cap p_2 \cap ... \cap p_k$. Define a coloring $f(N) = min\{n \in N | \ (N : M) \nsubseteq p_n\}$ such that N is a vertex of $AG(M)^*$. We have $\chi(AG(M)^*) \leq k$. □

Corollary 3.12. Assume that $rad_M(0) = (0)$ and M is a faithful module. Then $\chi(AG(M)^*) = cl(AG(M)^*)$. Moreover, if $\chi(AG(M)^*)$ is finite, then R has a finite number of minimal prime ideals, and if k is this number, then $\chi(AG(M)^*) = cl(AG(M)^*) = k$.

Proof. Suppose $\chi(AG(M)^*)$ is finite. Then by Proposition 3.11, R has a finite number of minimal prime ideals $p_1, ..., p_k$. One can see that R is a reduced ring. So $cl(AG(M)^*) \leq \chi(AG(M)^*) \leq k$. By [6, Theorem 6], $cl(AG(R)) \geq k$, and so $cl(AG(M)^*) \geq k$, as desired. □

Lemma 3.13. If $cl(AG(M)^*)$ is finite, then for every nonzero submodule N of M with $N^2 = (0)$ and $(N : M) \neq Ann(M)$, N has a finite number of R-submodules K such that $(K : M) \neq Ann(M)$.

Proof. This is clear. □

Theorem 3.14. *Let M be a Noetherian module and $\Upsilon = \{N \in V(AG(M)^*)| \ N^2 = (0)\}$. Assume that every $N \in \Upsilon$ has a finite number of R-submodules in Υ. If one of the following statements holds, then $cl(AG(M)^*)$ is finite.*

(a) *We have $(\Sigma_{N\in\Upsilon} N : M)M = (\Sigma_{N\in\Upsilon}(N : M)M : M)M$*
(b) *For every $N \in \Upsilon$, the subset $\{K < M| \ (N : M)M = (K : M)M\}$ is finite.*

Proof. Suppose that every $N \in \Upsilon$ has a finite number of R-submodules in Υ and we have (a). Let C be a largest clique in $AG(M)^*$ and let Υ_1 be the set of all vertices N of C with $N^2 = (0)$. If $\Upsilon_1 \neq \emptyset$, then $K = \Sigma_{N\in\Upsilon_1} N$ is again a vertex of C and $K^2 = (0)$ because for every $L \in C$, we have

$$(L : M)(\Sigma_{N\in\Upsilon_1} N : M)M = (L : M)(\Sigma_{N\in\Upsilon_1}(N : M)M : M)M \subseteq$$
$$(L : M)(\Sigma_{N\in\Upsilon_1}(N : M)M) \subseteq \Sigma_{N\in\Upsilon_1}(L : M)(N : M)M = (0).$$

Hence $K \in C$. We have

$$K^2 = (\Sigma_{N\in\Upsilon_1} N : M)^2 M = (\Sigma_{N\in\Upsilon_1}(N : M)M : M)^2 M \subseteq ...$$
$$\subseteq \Sigma_{N,N'\in\Upsilon_1}(N : M)(N' : M)M = (0).$$

So by our hypothesis, K has a finite number of R-submodules in Υ. But if $N \in \Upsilon_1$, every R-submodule of N is an R-submodule of K. Thus for every $N \in \Upsilon_1$, N has a finite number of R-submodules in Υ and hence Υ_1 has a finite elements. We claim that $C\setminus\Upsilon_1$ has also a finite elements. Suppose that $\{N_1, N_2, ...\}$ is an infinite subset of $C\setminus\Upsilon_1$. Consider the chain $N_1 \subseteq N_1 + N_2 \subseteq N_1 + N_2 + N_3 \subseteq \dots$. Since M is a Noetherian module, there exists $n \in N$ such that $N_1+...+N_n = N_1+...+N_n+N_{n+1}$, i.e., $N_{n+1} \subseteq N_1 + ... + N_n$. So

$$N_{n+1}^2 \subseteq N_{n+1}(N_1 + ... + N_n) \subseteq (N_{n+1} : M)((N_1 : M)M + ... + (N_n : M)M)$$

$$\subseteq (N_{n+1} : M)(N_1 : M)M + ... + (N_{n+1} : M)(N_n : M)M = (0).$$

It follows that $N_{n+1}^2 = (0)$, a contradiction. Thus C has a finite number of vertices and from there, $cl(AG(M)^*)$ is finite. Now assume that we have the hypothesis in case (b). Let $K = \Sigma_{N \in \Upsilon_1}(N : M)M$. By using similar arguments as in case (a), we have $K^2 = (0)$. But by hypotheses, K has a finite number of submodules in Υ. We claim that Υ_1 has a finite number of elements. Suppose not. Then there exists $N \in \Upsilon_1$ such that the subset $\{L \in \Upsilon | (N : M)M = (L : M)M\}$ is infinite, a contradiction. Thus C has a finite number of vertices and from there, $cl(AG(M)^*)$ is a finite set.

□

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