A GREEDY ALGORITHM FOR $B_h[g]$ SEQUENCES

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ABSTRACT. For any positive integers $h \geq 2$ and $g \geq 1$, we present a greedy algorithm that provides an infinite $B_h[g]$ sequence with $a_n \leq g n^{h+(h-1)/g}$.

1. INTRODUCTION

Given positive integers $h \geq 2$ and $g \geq 1$, we say that a sequence of integers A is $B_h[g]$ sequence if the number of representations of any integer n in the form

$$n = a_1 + \cdots + a_h, \quad a_1 \leq \cdots \leq a_h, \quad a_i \in A$$

is bounded by g. The $B_h[1]$ sequences are simply called B_h sequences.

A trivial counting argument shows that if $A = \{a_n\}$ is a $B_h[g]$ sequence then $a_n \gg n^h$. On the other hand, the greedy algorithm introduced by Erdős [1] provides an infinite B_h sequence with $a_n \leq n^{2h-1}$.

Classic greedy algorithm: *Let $a_1 = 1$ and for $n \geq 2$, define a_n as the smallest positive integer, greater than a_{n-1}, such that $a_1, \dots, a_n$ is a $B_h[g]$ sequence.*

When $g = 1$, the greedy algorithm defines a_n as the smallest positive integer that is not of the form $a_{i_1} + \cdots + a_{i_h} - (a_{i'_1} \cdots + a_{i'_{h-1}})$ with $1 \leq i_1, \dots, i_h, i'_1, \dots, i'_{h-1}$ $\leq n-1$. Since there are at most $(n-1)^{2h-1}$ forbidden elements for a_n, then $a_n \leq (n-1)^{2h-1} + 1 \leq n^{2h-1}$.

It is possible that the classic greedy algorithm may provide a denser sequence when $g > 1$, but it is not clear how to take advantage of g. For this reason other methods have been used to obtain dense infinite $B_h[g]$ sequences:

Theorem A. *Given $h \geq 2$ and $g \geq 1$, there exists an infinite $B_h[g]$ sequence with $a_n \ll n^{h+\epsilon}$ with $\epsilon = \epsilon(h, g) \to 0$ when $g \to \infty$.*

Erdős and Renyi [8] proved Theorem A for $h = 2$ using the probabilistic method. Ruzsa gave the first proof for any $h \geq 3$ (a sketch of that proof, which is not probabilistic, appeared in [7] and a detailed proof in [5]). Other probabilistic proofs of Theorem A, with better dependence of $\epsilon(h, g)$, have appeared in [2], [5] and [9].

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[1]This algorithm has been atributed to Mian and Chowla, but it seems (see [6]) that was Erdős who found this algorithm to give a first answer to a question of Sidon

The aim of this paper is to present a new greedy algorithm that provides a $B_h[g]$ sequence that grows slower than all the previous known constructions for $g > 1$.

We must mention that for $g = 1$ there are special constructions of B_h sequences with slower growth (see [1],[3],[4] and [10]).

2. A new greedy algorithm

We need to introduce the notion of *strong* $B_h[g]$ set.

Definition 1. *We say that* $A_n = \{a_1, \dots, a_n\}$ *is a strong* $B_h[g]$ *set if the following conditions are satisfied:*

i) A_n *is a* $B_h[g]$ *set.*
ii) $R_s(A_n) \le n^{h+(1-s)(h-1)/g}$, *for* $s = 1, \dots, g$, *where*

$$R_s(A_n) = |\{x : r_{A_n}(x) \ge s\}|$$

and

$$r_{A_n}(x) = |\{(a_{i_1}, \dots a_{i_h}) : \quad 1 \le i_1 \le \cdots \le i_h \le n, \quad a_{i_1} + \cdots + a_{i_h} = x\}|.$$

Theorem 2.1. *Let* $a_1 = 1$ *and for* $n \ge 1$ *define* a_{n+1} *as the smallest positive integer, distinct to* $a_1, \dots, a_n$, *such that* $a_1, \dots, a_{n+1}$ *is a strong* $B_h[g]$ *set. The infinite sequence* $A = \{a_n\}$ *given by this greedy algorithm is a* $B_h[g]$ *sequence with* $a_n \le g n^{h+(h-1)/g}$.

Proof. Assume that $A_n = \{a_1, \dots, a_n\}$ is a strong $B_h[g]$ set. We will find an upper bound for the number of forbiden positive integers for a_{n+1}. We classify the forbidden elements m in the following sets:

i) $F_n = \{m : \ m \in A_n\}$.
ii) $F_{0,n} = \{m : \ A_n \cup m \text{ is not a } B_h[g] \text{ set}\}$
iii) $F_{s,n} = \{m : \ R_s(A_n \cup m) > (n+1)^{h+(1-s)(h-1)/g}\}, \quad s = 1, \dots, g.$

Hence a_{n+1} is the smallest positive integer not belonging to $(\bigcup_{s=0}^{g} F_{s,n}) \cup F_n$ and then the proof will be completed if we prove that

$$\left| \left(\bigcup_{s=0}^{g} F_{s,n} \right) \cup F_n \right| \le g(n+1)^{h+(h-1)/g} - 1. \tag{2.1}$$

It is clear that $|F_n| = n$. Next, we find an upper bound for the cardinality of $F_{s,n}, \ s = 0, \dots, g$.

The elements of $F_{0,n}$ are of the form $x - (a_{i_1} + \cdots + a_{i_{h-1}})$ for some $1 \le i_1, \dots, i_{h-1} \le n$ and for some x with $r_{A_n}(x) \ge g$. Thus

$$|F_{0,n}| \le n^{h-1} |\{x : \ r_{A_n}(x) \ge g\}| = n^{h-1} R_g(A_n) \le n^{h+(h-1)/g}. \tag{2.2}$$

For $s = 1$, note that $R_1(A_n \cup m) \le (n+1)^h$ for any m. Thus $|F_{1,n}| = 0$.

For $s = 2, \dots, g$, and for any m we have

$$R_s(A_n \cup m) \le R_s(A_n) + T_{s,n}(m), \tag{2.3}$$

where

$$T_{s,n}(m) = |\{x: \ r_{A_n}(x) \geq s-1, \ x-m \in A_n + \overset{h-1}{\cdots} + A_n\}|.$$

We observe that if $T_s(m) \leq n^{h-1+(1-s)(h-1)/g}$, using (2.3) and that A_n is a strong $B_h[g]$ set, we have

$$R_s(A_n \cup m) \leq n^{h+(1-s)(h-1)/g} + n^{h-1+(1-s)(h-1)/g} \leq (n+1)^{h+(1-s)(h-1)/g}$$

and then $m \notin F_{s,n}$. Thus,

$$\sum_m T_{s,n}(m) \geq \sum_{m \in F_{s,n}} T_{s,n}(m) \geq n^{h-1+(1-s)(h-1)/g}|F_{s,n}|. \tag{2.4}$$

On the other hand, when we sum $T_{s,n}(m)$ over all m, each x with $r_{A_n}(x) \geq s-1$ is counted no more than n^{h-1} times. Then

$$\sum_m T_{s,n}(m) \leq n^{h-1} R_{s-1}(A_n) \leq n^{2h-1+(2-s)(h-1)/g}. \tag{2.5}$$

Inequalities (2.4) and (2.5) impply

$$|F_{s,n}| \leq n^{h+(h-1)/g}. \tag{2.6}$$

Taking into account (2.2), the inequalities (2.6) for $s = 2, \dots, g$ and the estimate $|F_n| = n$, we get

$$\begin{aligned} \left| \left(\bigcup_{s=0}^{g} F_{s,n} \right) \cup F_n \right| &\leq n^{h+(h-1)/g} + (g-1)n^{h+(h-1)/g} + n \\ &\leq g(n+1)^{h+(h-1)/g} - 1, \end{aligned}$$

which completes the proof. □

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