The Mumford–Tate conjecture for the product of an abelian surface and a K3 surface

by J.M. Commelin Wednesday, the 6th of January, 2016

1 Introduction

The main result of this paper is the following theorem. In the next paragraph we recall the Mumford–Tate conjecture; and in §1.3 we give an outline of the proof. The ambitious reader may skip to section 7 and dive head first into the proof.

1.1 Theorem. — *Let K be a finitely generated subfield of $\mathbb{C}$. If A is an abelian surface over K and X is a K3 surface over K, then the Mumford–Tate conjecture is true for* $\mathrm{H}^2(A \times X)(1)$.

1.2 The Mumford–Tate conjecture. — Let K be a finitely generated field of characteristic 0; and let $K \hookrightarrow \mathbb{C}$ be an embedding of K into the complex numbers. Let $\overline{K}$ be the algebraic closure of K in $\mathbb{C}$. Let X/K be a smooth projective variety. One may attach several cohomology groups to X. For the purpose of this article we are interested in two cohomology theories: Betti cohomology and ℓ-adic étale cohomology (for a prime number ℓ). We will write $\mathrm{H}^i_{\mathrm{B}}(X)$ for the $\mathbb{Q}$-Hodge structure $\mathrm{H}^i_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q})$. Similarly, we write $\mathrm{H}^i_\ell(X)$ for the $\mathrm{Gal}(\overline{K}/K)$-representation $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$.

The Mumford–Tate conjecture is a precise way of saying that the cohomology groups $\mathrm{H}^i_{\mathrm{B}}(X)$ and $\mathrm{H}^i_\ell(X)$ contain the same information about X. To make this precise, let $\mathrm{G}_{\mathrm{B}}(\mathrm{H}^i_{\mathrm{B}}(X))$ be the Mumford–Tate group of the Hodge structure $\mathrm{H}^i_{\mathrm{B}}(X)$, and let $\mathrm{G}^\circ_\ell(\mathrm{H}^i_\ell(X))$ be the connected component of the Zariski closure of $\mathrm{Gal}(\overline{K}/K)$ in $\mathrm{GL}(\mathrm{H}^i_\ell(X))$. The comparison theorem by Artin, comparing singular cohomology with étale cohomology, canonically identifies $\mathrm{GL}(\mathrm{H}^i_{\mathrm{B}}(X)) \otimes \mathbb{Q}_\ell$ with $\mathrm{GL}(\mathrm{H}^i_\ell(X))$. The Mumford–Tate conjecture (for the prime ℓ, and the embedding $K \hookrightarrow \mathbb{C}$) states that under this identification

$$\mathrm{G}_{\mathrm{B}}(\mathrm{H}^i_{\mathrm{B}}(X)) \otimes \mathbb{Q}_\ell \cong \mathrm{G}^\circ_\ell(\mathrm{H}^i_\ell(X)).$$

1.3 Outline of the proof. — Let A/K be an abelian surface, and let X/K be a K3 surface. Observe that, by Künneth's theorem, $\mathrm{H}^2_{\mathrm{B}}(A \times X) \cong \mathrm{H}^2_{\mathrm{B}}(A) \oplus \mathrm{H}^2_{\mathrm{B}}(X)$. Similarly $\mathrm{H}^2_\ell(A \times X) \cong \mathrm{H}^2_\ell(A) \oplus \mathrm{H}^2_\ell(X)$. Recall that the Mumford–Tate conjecture for A is known in degree 1, and hence in all degrees. (This is classical, but see corollary 4.4 of [14] for a reference.) By [24, 23, 1], the Mumford–Tate conjecture for X (in degree 2) is true as well. Still, it is not a formal consequence that the the Mumford–Tate conjecture for $A \times X$ is true in degree 2.

The proof of theorem 1.1 falls apart into four cases, that use very different techniques. All cases build on the Hodge theory of K3 surfaces and abelian varieties, of which we provide an overview in section 6.

Let V be the transcendental part of $\mathrm{H}^2_{\mathrm{B}}(X)$. The first case (lemma 7.4) inspects $\mathrm{End}(V)$, and exploits Chebotaryov's density theorem, which we recall in section 2. The second case (lemma 7.6) looks at the Lie type of $\mathrm{G}_{\mathrm{B}}(V)$, and uses results about semisimple groups over number fields, which we assemble in section 3.

The third case (lemma 7.7) deals with Kummer varieties, and other K3 surfaces for which $\dim(V)$ is small. We use the theory of Kuga–Satake varieties, and apply techniques of Lombardo, developed in [14]. The preliminaries of this part of the proof are gathered in section 5.

The final case (lemma 7.9) is the only case where we use that $\mathrm{H}^2(X)$ is a motive coming from a K3 surface. We use information about the reduction of X modulo a place of K, and combine this with a result about non-split groups and results about compatible systems of ℓ-adic representations.

1.4 Notation and terminology. — Let K be a finitely generated field of characteristic 0; and fix an embedding $K \hookrightarrow \mathbb{C}$. In this article we use the language of motives à la André, [2]. To be precise, our category of base pieces is the category of smooth projective varieties over K, and our reference cohomology is Betti cohomology, $\mathrm{H}_\mathrm{B}(\_)$; which, we stress, depends on the chosen embedding $K \hookrightarrow \mathbb{C}$. We write $\mathrm{H}^i(X)$ for the motive of weight i associated with a smooth projective variety X/K.

The Mumford–Tate conjecture naturally generalises to motives. Let M be a motive. We will write $\mathrm{H}_\mathrm{B}(M)$ for its Hodge realisation; $\mathrm{H}_\ell(M)$ for its ℓ-adic realisation; $\mathrm{G}_\mathrm{B}(M)$ for its Mumford–Tate group (*i.e.,* the Mumford–Tate group of $\mathrm{H}_\mathrm{B}(M)$); and $\mathrm{G}^\circ_\ell(M)$ for $\mathrm{G}^\circ_\ell(\mathrm{H}_\ell(M))$. We will use the notation $\mathrm{MTC}_\ell(M)$ for the conjectural statement

$$\mathrm{G}_\mathrm{B}(M) \otimes \mathbb{Q}_\ell \cong \mathrm{G}^\circ_\ell(M),$$

and $\mathrm{MTC}(M)$ for the assertion $\mathrm{MTC}_\ell(M)$ for all prime numbers ℓ. In this paper, we never use specific properties of the chosen embedding $K \hookrightarrow \mathbb{C}$, and all statements are valid for every such embedding. In particular, we will speak about subfields of $\mathbb{C}$, where the embedding is implicit.

In this paper, we will use compatible systems of ℓ-adic representations. We refer to the letters of Serre to Ribet (see [20]) or the work of Larsen and Pink [11, 12] for more information.

Throughout this paper, A is an abelian variety, over some base field. (Outside section 5, it is even an abelian surface.) Assume A is absolutely simple; and choose a polarisation of A. Let $(D, \dagger)$ be its endomorphism ring $\mathrm{End}^0(A)$ together with the Rosati involution associated with the polarisation. The simple algebra D together with the positive involution $\dagger$ has a certain type in the Albert classification that does not depend on the chosen polarisation. We say that A is of type $\textsc{x}$ if $(D, \dagger)$ is of type $\textsc{x}$, where $\textsc{x}$ runs through $\{\textsc{i}, \ldots, \textsc{iv}\}$. If E denotes the center of D, with degree $e = [E : \mathbb{Q}]$, we also say that A is of type $\textsc{x}(e)$.

Whenever we speak of (semi)simple groups or (semi)simple Lie algebras, we mean *non-commutative* (semi)simple groups, and *non-abelian* (semi)simple Lie-algebras.

Let T be a type of Dynkin diagram (*e.g.,* A_n, B_n, C_n or D_n). Let $\mathfrak{g}$ be a semisimple Lie algebra over K. We say that T does not occur in the Lie type of $\mathfrak{g}$, if the Dynkin diagram of $\mathfrak{g}_{\overline{K}}$ does not have a component of type T. For a semisimple group G over K, we say that T does not occur in the Lie type of G, if T does not occur in the Lie type of $\mathrm{Lie}(G)$.

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2 Some remarks on Chebotaryov's density theorem and transitive group actions

2.1 Theorem (Chebotaryov's density theorem). — *Let $K \subset E$ be an extension of number fields. Let $E \subset L$ be a Galois closure of E, and let $G = \mathrm{Gal}(L/K)$ be the Galois group of L over K. Let $\Sigma = \mathrm{Hom}_K(E, L)$ be the set of field embeddings over K of E in L.*

- » *Let $\mathfrak{p}$ be a prime of K that is unramified in L, and let $C_\mathfrak{p} \subset G$ be the conjugacy class of the Frobenius elements associated with $\mathfrak{p}$. The decomposition type of $\mathfrak{p}$ in $\mathcal{O}_E$ is equal to the cycle type of $C_\mathfrak{p}$ acting on Σ.*
- » *Let $C \subset G$ be a union of conjugacy classes of G. The set*

$$\{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_E) \mid \mathfrak{p} \text{ is unramified, and } C_\mathfrak{p} \subset C\}$$

has density $\frac{|C|}{|G|}$ as subset of $\mathrm{Spec}(\mathcal{O}_E)$.

Proof. See fact 2.1 and theorem 3.1 of [13]. See Theorem 13.4 of [18] for the case where E/K is Galois. □

2.2 Lemma. — *Let G be a finite group acting transitively on a finite set Σ. Let $n \in \mathbb{Z}_{\geq 0}$ be a non-negative integer, and let $C \subset G$ be the set of elements $g \in G$ that have at least n fixed points:*

$$C = \left\{ g \in G \;\middle|\; |\Sigma^g| \geq n \right\}$$

If $n \cdot |C| \geq |G|$, then $|\Sigma| = n$. If furthermore the action of G on Σ is faithful, then $|G| = n$, and Σ is principal homogeneous under G.

Proof. Burnside's lemma gives

$$1 = |G \backslash \Sigma| = \frac{1}{|G|} \sum_{g \in G} |\Sigma^g| \geq \frac{n \cdot |C|}{|G|} \geq 1.$$

Hence $n \cdot |C| = |G|$ and all elements in C have exactly n fixed points. In particular the identity element has n fixed points, which implies $|\Sigma| = n$. If G acts faithfully on Σ, then $|\Sigma| = n$ implies $C = \{e\}$, and thus $|G| = n = |\Sigma|$. So Σ is principal homogeneous under G. □

[1] A preliminary version of lemma 2.3 arose from a question on MathOverflow titled “How simple does a $\mathbb{Q}$-simple group remain after base change to $\mathbb{Q}_\ell$?” (`http://mathoverflow.net/q/214603/78087`). The answers also inspired lemma 2.2.

2.3 Lemma. — *Let F_1 be a Galois extension of $\mathbb{Q}$. Let F_2 be a number field. If for all prime numbers ℓ, the product of local fields $F_1 \otimes \mathbb{Q}_\ell$ is a factor of $F_2 \otimes \mathbb{Q}_\ell$, then $F_1 \cong F_2$.*
Proof. Let L be a Galois closure of F_2, and let G be the Galois group $\mathrm{Gal}(L/\mathbb{Q})$, which acts naturally on the set of field embeddings $\Sigma = \mathrm{Hom}(F_2, L)$. Let n be the degree of F_1, and let C be the set $\{g \in G \mid |\Sigma^g| \geq n\}$ of elements in G that have at least n fixed points in Σ.

By Chebotaryov's density theorem (2.1), the set of primes that split completely in F_1 has density $1/n$. Another application of theorem 2.1 shows that the set of primes ℓ for which $F_2 \otimes \mathbb{Q}_\ell$ has a semisimple factor isomorphic to $(\mathbb{Q}_\ell)^n$ must have density $\geq 1/n$. Our assumption therefore implies that $n \cdot |C| \geq |G|$. By lemma 2.2, this implies $|\Sigma| = n$, and since G acts faithfully on Σ, we find that $F_2/\mathbb{Q}$ is Galois of degree n. Because Galois extensions of number fields can be recovered from their set of splitting primes (Satz VII.13.9 of [18]), we conclude that $F_2 \cong F_1$. □

2.4 Lemma. — *Let F_1 be a quadratic extension of $\mathbb{Q}$. Let F_2 be a number field of degree ≤ 5 over $\mathbb{Q}$. If for all prime numbers ℓ, the products of local fields $F_1 \otimes \mathbb{Q}_\ell$ and $F_2 \otimes \mathbb{Q}_\ell$ have an isomorphic factor, then $F_1 \cong F_2$.*
Proof. Let L be a Galois closure of F_2, and let G be the Galois group $\mathrm{Gal}(L/\mathbb{Q})$, which acts naturally on the set of field embeddings $\Sigma = \mathrm{Hom}(F_2, L)$. Observe that G acts transitively on Σ, and we identify G with its image in $\mathfrak{S}(\Sigma)$. Write n for the degree of F_2 over $\mathbb{Q}$, which also equals $|\Sigma|$. The order of G is divisible by n. Hence, if n is prime, then G must contain an n-cycle.

Suppose that G contains an n-cycle. By Chebotaryov's density theorem (2.1) there must be a prime number ℓ that is inert in F_2. By our assumption $F_2 \otimes \mathbb{Q}_\ell$ also contains a factor of at most degree 2 over $\mathbb{Q}_\ell$. This shows that $n = 2$.

If $n = 4$, then G does not contain an n-cycle if and only if it is isomorphic to V_4 or A_4. If $G \cong V_4$, only the identity element has fixed points, and by Chebotaryov's density theorem this means that the set of primes ℓ for which $F_2 \otimes \mathbb{Q}_\ell$ has a factor $\mathbb{Q}_\ell$ has density $1/4$, whereas the set of primes splitting in F_1 has density $1/2$. On the other hand, if $G \cong A_4$, only 3 of the 12 elements have a 2-cycle in the cycle decomposition, and by Chebotaryov's density theorem this means that the set of primes ℓ for which $F_2 \otimes \mathbb{Q}_\ell$ has a factor isomorphic to a quadratic extension of $\mathbb{Q}_\ell$ has density $1/4$, whereas the set of primes inert in F_1 has density $1/2$. This gives a contradiction. We conclude that n must be 2; and therefore $F_1 \cong F_2$, by lemma 2.3. □

3 Several results on semisimple groups over number fields

Throughout this section K is a field of characteristic 0.

3.1 Lemma. — *Let G be a connected algebraic group over K, and let $H \subset G$ be a subgroup. If $\mathrm{Lie}(H) = \mathrm{Lie}(G)$, then $H = G$.*
Proof. This is immediate, since H is a subgroup of G of the same dimension as G. □

3.2 Lemma (Goursat's lemma for Lie algebras). — *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras over K, and let $\mathfrak{h} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a sub-Lie algebra such that the projections $\pi_i \colon \mathfrak{h} \to \mathfrak{g}_i$ are surjective. Let $\mathfrak{n}_1$ be the kernel of π_2, and $\mathfrak{n}_2$ the kernel of π_1. The projection π_i identifies $\mathfrak{n}_i$ with an ideal of $\mathfrak{g}_i$,* *and the image of the canonical map*

$$\mathfrak{h} \longrightarrow (\mathfrak{g}_1/\pi_1(\mathfrak{n}_1)) \oplus (\mathfrak{g}_2/\pi_2(\mathfrak{n}_2))$$

is the graph of an isomorphism $\mathfrak{g}_1/\pi_1(\mathfrak{n}_1) \to \mathfrak{g}_2/\pi_2(\mathfrak{n}_2)$.

Proof. Observe that π_i is injective on $\mathfrak{n}_i$. If $x \in \pi_i(\mathfrak{n}_i)$ and $y \in \mathfrak{g}_i$, then $[x,y] \in \pi_i(\mathfrak{n}_i)$, because π_i is surjective, and $\mathfrak{n}_i$ is an ideal of $\mathfrak{h}$. Let $\bar{\mathfrak{h}}$ be the image of the canonical map

$$\mathfrak{h} \longrightarrow (\mathfrak{g}_1/\pi_1(\mathfrak{n}_1)) \oplus (\mathfrak{g}_2/\pi_2(\mathfrak{n}_2))$$

By construction, the projections $\bar{\mathfrak{h}} \to \mathfrak{g}_i/\pi_i(\mathfrak{n}_i)$ are injective; and they are surjective by assumption. This proves the lemma. □

3.3 Remark. — Let $\mathfrak{h} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be Lie algebras over K satisfying the conditions of lemma 3.2. Assume that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are finite-dimensional and semisimple. It follows from the proof of lemma 3.2 that there exist semisimple Lie algebras $\mathfrak{s}_1$, $\mathfrak{t}$, and $\mathfrak{s}_2$ such that $\mathfrak{g}_1 \cong \mathfrak{s}_1 \oplus \mathfrak{t}$, $\mathfrak{g}_2 \cong \mathfrak{t} \oplus \mathfrak{s}_2$, and $\mathfrak{h} \cong \mathfrak{s}_1 \oplus \mathfrak{t} \oplus \mathfrak{s}_2$.

3.4 Corollary. — *Let $K \subset L$ be a field extension. Let G_1 and G_2 be connected semisimple groups over K. Let $\iota\colon G \hookrightarrow G_1 \times G_2$ be a subgroup, with surjective projections onto both factors. If $\mathrm{Lie}(G_1)_L$ and $\mathrm{Lie}(G_2)_L$ have no isomorphic factor over L, then ι is an isomorphism.*

3.5 Lemma. — *Let $K \subset F$ be a finite field extension. Let G be an algebraic group over F. The Lie algebra $\mathrm{Lie}(\mathrm{Res}_{F/K} G)$ is isomorphic to the Lie algebra $\mathrm{Lie}(G)$, viewed as Lie algebra over K.*

Proof. This follows from the following diagram, the rows of which are exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Lie}(\mathrm{Res}_{F/K} G) & \longrightarrow & (\mathrm{Res}_{F/K} G)(K[\varepsilon]) & \longrightarrow & (\mathrm{Res}_{F/K} G)(K) & \longrightarrow & 0 \\ & & & & \downarrow\wr & & \downarrow\wr & & \\ 0 & \longrightarrow & \mathrm{Lie}(G) & \longrightarrow & G(K[\varepsilon]) & \longrightarrow & G(K) & \longrightarrow & 0 \end{array}$$

□

3.6 Lemma. — *Let F_1/K and F_2/K be finite field extensions. Let $\mathfrak{g}_i/F_i$ $(i = 1,2)$ be a finite product of absolutely simple Lie algebras (cf. our conventions in §1.4). Write $(\mathfrak{g}_i)_K$ for the Lie algebra $\mathfrak{g}_i$ viewed as Lie algebra over K. If $(\mathfrak{g}_1)_K$ and $(\mathfrak{g}_2)_K$ have an isomorphic factor, then $F_1 \cong_K F_2$.*

Proof. The K-simple factors of $(\mathfrak{g}_i)_K$ are all of the form $(\mathfrak{t}_i)_K$, where $\mathfrak{t}_i$ is an F_i-simple factor of $\mathfrak{g}_i$. So if $(\mathfrak{g}_1)_K$ and $(\mathfrak{g}_2)_K$ have an isomorphic factor, there exist F_i-simple factors $\mathfrak{t}_i$ of $\mathfrak{g}_i$ for which there exists an isomorphism $f\colon (\mathfrak{t}_1)_K \to (\mathfrak{t}_2)_K$. Let $\overline{K}$ be an algebraic closure of K. Observe that

$$(\mathfrak{t}_i)_K \otimes_K \overline{K} \cong \bigoplus_{\sigma \in \mathrm{Hom}_K(F_i, \overline{K})} \mathfrak{t}_i \otimes_{F_i,\sigma} \overline{K},$$

and note that $\mathrm{Gal}(\overline{K}/K)$ acts transitively on $\mathrm{Hom}_K(F_i, \overline{K})$. By assumption, the $\mathfrak{t}_i$ are F_i-simple, and therefore the $\mathfrak{t}_i \otimes_{F_i,\sigma} \overline{K}$ are precisely the simple ideals of $(\mathfrak{t}_i)_K \otimes_K \overline{K}$. Thus the isomorphism f gives a $\mathrm{Gal}(\overline{K}/K)$-equivariant bijection between the simple ideals of $(\mathfrak{t}_1)_K \otimes_K \overline{K}$ and $(\mathfrak{t}_2)_K \otimes_K \overline{K}$; and therefore $\mathrm{Hom}_K(F_1, \overline{K})$ and $\mathrm{Hom}_K(F_2, \overline{K})$ are isomorphic as $\mathrm{Gal}(\overline{K}/K)$-sets. This proves the result. □

3.7 Lemma. — *Let F_1 and F_2 be number fields. Let G_i/F_i $(i = 1, 2)$ be an almost direct product of connected absolutely simple F_i-groups. Let ℓ be a prime number, and let $\iota_\ell \colon G \hookrightarrow (\mathrm{Res}_{F_1/\mathbb{Q}} G_1)_{\mathbb{Q}_\ell} \times (\mathrm{Res}_{F_2/\mathbb{Q}} G_2)_{\mathbb{Q}_\ell}$ be a subgroup over $\mathbb{Q}_\ell$, with surjective projections onto both factors. If ι_ℓ is not an isomorphism, then $F_1 \otimes \mathbb{Q}_\ell$ and $F_2 \otimes \mathbb{Q}_\ell$ have an isomorphic simple factor.*
Proof. Observe that $(\mathrm{Res}_{F_i/\mathbb{Q}} G_i) \otimes \mathbb{Q}_\ell \cong \prod_{\lambda|\ell} \mathrm{Res}_{F_{i,\lambda}/\mathbb{Q}_\ell}(G_i \otimes_{F_i} F_\lambda)$. If ι_ℓ is not an isomorphism, then by corollary 3.4, there exist places λ_i of F_i over ℓ such that $\mathrm{Lie}(\mathrm{Res}_{F_{1,\lambda_1}/\mathbb{Q}_\ell}(G_1 \otimes_{F_1} F_{1,\lambda_1}))$ and $\mathrm{Lie}(\mathrm{Res}_{F_{2,\lambda_2}/\mathbb{Q}_\ell}(G_2 \otimes_{F_2} F_{2,\lambda_2}))$ have an isomorphic factor. By lemmas 3.5 and 3.6, this implies that $F_{1,\lambda_1} \cong_{\mathbb{Q}_\ell} F_{2,\lambda_2}$, which proves the lemma. □

4 Several results on abelian motives

4.1 Lemma. — *The Mumford–Tate conjecture on centres is true for abelian motives. In other words, let M be an abelian motive. Let $Z_\mathrm{B}(M)$ be the centre of the Mumford–Tate group $\mathrm{G}_\mathrm{B}(M)$, and let $Z_\ell(M)$ be the centre of $\mathrm{G}_\ell^\circ(M)$. Then $Z_\ell(M) \cong Z_\mathrm{B}(M) \otimes \mathbb{Q}_\ell$.*
Proof. The result is true for abelian varieties (see theorem 1.3.1 of [26] or corollary 2.11 of [25]).

By definition of abelian motive, there is an abelian variety A such that M is contained in the Tannakian subcategory of motives generated by $\mathrm{H}(A)$. This yields a surjection $\mathrm{G}_\mathrm{B}(A) \twoheadrightarrow \mathrm{G}_\mathrm{B}(M)$, and therefore $Z_\mathrm{B}(M)$ is the image of $Z_\mathrm{B}(A)$ under this map. The same is true on the ℓ-adic side. Thus we obtain a commutative diagram with solid arrows

$$\begin{array}{ccccc} Z_\ell(A) & \twoheadrightarrow & Z_\ell(M) & \hookrightarrow & \mathrm{G}_\ell^\circ(M) \\ \downarrow \wr & & \vdots\, \wr & & \downarrow \\ Z_\mathrm{B}(A) \otimes \mathbb{Q}_\ell & \twoheadrightarrow & Z_\mathrm{B}(M) \otimes \mathbb{Q}_\ell & \hookrightarrow & \mathrm{G}_\mathrm{B}(M) \otimes \mathbb{Q}_\ell \end{array}$$

which shows that the dotted arrow exists and is an isomorphism. □

4.2 Lemma. — *Let $\mathbb{1}$ denote the trivial motive. If M is a motive, then the Mumford–Tate conjecture for M is equivalent to the Mumford–Tate conjecture for $M \oplus \mathbb{1}$.*
Proof. Indeed, M and $M \oplus \mathbb{1}$ generate the same Tannakian subcategory of motives. □

4.3 Lemma. — *Let $K \subset L$ be an extension of finitely generated subfields of $\mathbb{C}$. If M is a motive over K, then $\mathrm{MTC}(M) \iff \mathrm{MTC}(M_L)$.*
Proof. See proposition 1.3 of [15]. □

4.4 Lemma. — *Let M be an abelian motive. Assume that the ℓ-adic realisations of M form a compatible system of ℓ-adic representations. If the Mumford–Tate conjecture for M is true for one prime ℓ', then it is true for all primes ℓ.*
Proof. Since M is an abelian motive, we have $\mathrm{G}_\ell^\circ(M) \subset \mathrm{G}_\mathrm{B}(M) \otimes \mathbb{Q}_\ell$. By our assumption on the ℓ-adic realisations of M, the proofs of theorem 4.3 and lemma 4.4 of [12] apply verbatim to our situation. □

4.5 — Let K be a finitely generated subfield of $\mathbb{C}$. A pair (A, X), consisting of an abelian surface A and a K3 surface X over K, is said to satisfy condition 4.5 for ℓ if

$$\mathrm{G}_\ell^\circ\big(\mathrm{H}^2(A \times X)(1)\big)^{\mathrm{der}} \hookrightarrow \mathrm{G}_\ell^\circ\big(\mathrm{H}^2(A)(1)\big)^{\mathrm{der}} \times \mathrm{G}_\ell^\circ\big(\mathrm{H}^2(X)(1)\big)^{\mathrm{der}}$$

is an isomorphism.

4.6 Lemma. — *Let K be a finitely generated subfield of $\mathbb{C}$, and let C/K be a smooth (not necessarily proper) curve over K, with generic point η. Let A/C be an abelian scheme, and let X/C be a K3 surface. There exists a closed point $c \in C$ and a prime number ℓ such that $\mathrm{G}_\ell^\circ(\mathrm{H}^2(A_\eta)(1)) \cong \mathrm{G}_\ell^\circ(\mathrm{H}^2(A_c)(1))$ and $\mathrm{G}_\ell^\circ(\mathrm{H}^2(X_\eta)(1)) \cong \mathrm{G}_\ell^\circ(\mathrm{H}^2(X_c)(1))$. Furthermore, if (A_c, X_c) satisfies condition 4.5 for ℓ, then so does (A_η, X_η).*

Proof. The existence of the point c follows immediately from theorem 1.1 of [4]. The diagram

$$\begin{array}{ccc} \mathrm{G}_\ell^\circ(\mathrm{H}^2(A_c \times X_c)(1))^{\mathrm{der}} & \hookrightarrow & \mathrm{G}_\ell^\circ(\mathrm{H}^2(A_\eta \times X_\eta)(1))^{\mathrm{der}} \\ \downarrow & & \downarrow \\ \mathrm{G}_\ell^\circ(\mathrm{H}^2(A_c)(1))^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(\mathrm{H}^2(X_c)(1))^{\mathrm{der}} & \xrightarrow{\simeq} & \mathrm{G}_\ell^\circ(\mathrm{H}^2(A_\eta)(1))^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(\mathrm{H}^2(X_\eta)(1))^{\mathrm{der}} \end{array}$$

shows that (A_η, X_η) satisfies condition 4.5 for ℓ if (A_c, X_c) satisfies it. □

4.7 — Let ℓ be a prime number. Let G_1 and G_2 be connected reductive groups over $\mathbb{Q}_\ell$. By a (G_1, G_2)*-tuple* over K we shall mean a pair (A, X), where A is an abelian surface over K, and X is a K3 surface over K such that $\mathrm{G}_\ell^\circ(\mathrm{H}^2(A)(1)) \cong G_1$ and $\mathrm{G}_\ell^\circ(\mathrm{H}^2(X)(1)) \cong G_2$. We will show in section 7 that there exist groups G_1 and G_2 that satisfy the hypothesis of the following lemma, namely that condition 4.5 for ℓ is satisfied for all (G_1, G_2)-tuples over number fields.

4.8 Lemma. — *Let ℓ be a prime number. Let G_1 and G_2 be connected reductive groups over $\mathbb{Q}_\ell$. If for all number fields K, all (G_1, G_2)-tuples (A, X) over K satisfy condition 4.5 for ℓ, then for all finitely generated subfields L of $\mathbb{C}$, all (G_1, G_2)-tuples (A, X) over L satisfy condition 4.5 for ℓ.*

Proof. The proof goes by induction on the transcendence degree n of L. If $n = 0$, the result is true by assumption. Suppose that $n > 0$, and assume as induction hypothesis that condition 4.5 for ℓ is satisfied for all (G_1, G_2)-tuples over all finitely generated subfields of $\mathbb{C}$ with transcendence degree $< n$.

There exists a field $K \subset L$, and a smooth curve C/K such that L is the function field of C. Observe that $\mathrm{trdeg}(K) = n - 1$. By the induction hypothesis and lemma 4.6, the claim of the lemma is true for L. The result follows by induction. □

5 Some remarks on the Mumford–Tate conjecture for abelian varieties

5.1 — For the convenience of the reader, we copy some results from [14]. Before we do that, let us recall the notion of the *Hodge group*, $\mathrm{Hdg}_{\mathrm{B}}(A)$, of an abelian variety. Let A be an abelian variety over a finitely generated field $K \subset \mathbb{C}$. By definition, the Mumford–Tate group of an abelian variety is $\mathrm{G}_{\mathrm{B}}(A) = \mathrm{G}_{\mathrm{B}}(\mathrm{H}^1_{\mathrm{B}}(A)) \subset \mathrm{GL}(\mathrm{H}^1_{\mathrm{B}}(A))$, and we put

$$\mathrm{Hdg}_{\mathrm{B}}(A) = (\mathrm{G}_{\mathrm{B}}(A) \cap \mathrm{SL}(\mathrm{H}^1_{\mathrm{B}}(A)))^{\circ} \quad \text{and} \quad \mathrm{Hdg}_{\ell}(A) = (\mathrm{G}_{\ell}(A) \cap \mathrm{SL}(\mathrm{H}^1_{\ell}(A)))^{\circ}.$$

We leave it as an easy exercise to the reader to verify that

$$\mathrm{MTC}_{\ell}(A) \quad \iff \quad \mathrm{Hdg}_{\mathrm{B}}(A) \otimes \mathbb{Q}_{\ell} \cong \mathrm{Hdg}_{\ell}(A).$$

5.2 Definition (1.1 in [14]). — Let A be an absolutely simple abelian variety of dimension g over K. The endomorphism ring $D = \mathrm{End}^0(A)$ is a division algebra. Write E for the centre of D. The ring E is a field, either TR (totally real) or CM. Write e for $[E : \mathbb{Q}]$. The degree of D over E is a perfect square d^2.

The *relative dimension* of A is

$$\mathrm{reldim}(A) = \begin{cases} \frac{g}{de}, & \text{if } A \text{ is of type I, II, or III,} \\ \frac{2g}{de}, & \text{if } A \text{ is of type IV.} \end{cases}$$

Note that $d = 1$ if A is of type I, and $d = 2$ if A is of type II or III.

In definition 2.22 of [14], Lombardo defines when an abelian variety is of *general Lefschetz type*. This definition is a bit unwieldy, and its details do not matter too much for our purposes. What matters are the following results, that prove that certain abelian varieties are of general Lefschetz type, and that show why this notion is relevant for us.

5.3 Lemma. — *Let A be an absolutely simple abelian variety over a finitely generated subfield of $\mathbb{C}$. Assume that A is of type* I *or* II. *If* $\mathrm{reldim}(A)$ *is odd, or equal to* 2*, then A is of general Lefschetz type.*

Proof. If $\mathrm{reldim}(A)$ is odd, then this follows from theorems 6.9 and 7.12 of [3]. Lombardo notes (remark 2.25 in [14]) that the proof of [3] also works if $\mathrm{reldim}(A) = 2$, and also refers to theorem 8.5 of [6] for a proof of that fact. □

5.4 Lemma. — *Let K be a finitely generated subfield of $\mathbb{C}$. Let A_1 and A_2 be two abelian varieties over K that are isogenous to products of abelian varieties of general Lefschetz type. If D_4 does not occur in the Lie type of* $\mathrm{Hdg}_{\ell}(A_1)$ *and* $\mathrm{Hdg}_{\ell}(A_2)$*, then either*

$$\mathrm{Hom}_K(A_1, A_2) \neq 0, \quad \textit{or} \quad \mathrm{Hdg}_{\ell}(A_1 \times A_2) \cong \mathrm{Hdg}_{\ell}(A_1) \times \mathrm{Hdg}_{\ell}(A_2).$$

Proof. This is remark 4.3 of [14], where Lombardo observes that, under the assumption of the lemma, theorem 4.1 of [14] can be applied to products of abelian varieties of general Lefschetz type. □

5.5 Lemma. — *Let A be an abelian variety over a finitely generated field $K \subset \mathbb{C}$. Let $L \subset \mathbb{C}$ be a finite extension of K for which A_L is isogenous over L to a product of absolutely simple abelian varieties $\prod A_i^{k_i}$. Assume that for all i the following conditions are valid:*

(a) either A_i is of general Lefschetz type or A_i is of CM *type;*
(b) the Lie type of $\mathrm{Hdg}_\ell(A_i)$ *does not contain* D_4*;*
(c) the Mumford–Tate conjecture is true for A_i*.*

Under these conditions the Mumford–Tate conjecture is true for A.

Proof. By lemma 4.3 we know that $\mathrm{MTC}(A) \iff \mathrm{MTC}(A_L)$. Furthermore, note that $\mathrm{MTC}(A_L)$ is equivalent to $\mathrm{MTC}(\prod A_i)$. Observe that

$$\mathrm{Hdg}_\ell(A) \subset \mathrm{Hdg}_\mathrm{B}(A) \otimes \mathbb{Q}_\ell \subset \prod \mathrm{Hdg}_\mathrm{B}(A_i) \otimes \mathbb{Q}_\ell = \prod \mathrm{Hdg}_\ell(A_i),$$

where the first inclusion is Deligne's "Hodge = absolute Hodge" theorem; the second inclusion is a generality; and the last equality is condition (c).

If we ignore the factors that are CM, then an inductive application of the previous lemma yields $\mathrm{Hdg}_\ell(A) = \prod \mathrm{Hdg}_\ell(A_i)$. If we do not ignore the factors that are CM, then we actually get $\mathrm{Hdg}_\ell(A)^{\mathrm{der}} = \prod \mathrm{Hdg}_\ell(A_i)^{\mathrm{der}}$. Together with lemma 4.1, this proves $\mathrm{Hdg}_\ell(A) = \mathrm{Hdg}_\mathrm{B}(A) \otimes \mathbb{Q}_\ell$.□

As an illustrative application of this result, Lombardo observes in corollary 4.5 of [14] that the Mumford–Tate conjecture is true for arbitrary products of elliptic curves and abelian surfaces.

6 Hodge theory of K3 surfaces and abelian surfaces

6.1 — In this section we recall some results of Zarhin that describe all possible Mumford–Tate groups of Hodge structures of K3 type, *i.e.,* Hodge structures of weight 0 with Hodge numbers of the form $(1, n, 1)$.

The canonical example of a Hodge structure of K3 type is the cohomology in degree 2 of a complex K3 surface X. Namely the Hodge structure $\mathrm{H}^2_\mathrm{B}(X)(1)$ has Hodge numbers $(1, 20, 1)$. Another example is provided by abelian surfaces, which is the content of remark 6.6 below.

6.2 Lemma. — *Let V be an irreducible Hodge structure of K3 type, and let ψ be a polarisation on V.*

1. *The endomorphism algebra E of V is a field.*
2. *The field E is* TR *(totally real) or* CM*.*
3. *If E is* TR*, then* $\dim_E(V) \geq 3$.
4. *Let $\tilde{\psi}$ be the unique E-bilinear (resp. hermitian) form such that $\psi = \mathrm{tr}_{E/\mathbb{Q}} \circ \tilde{\psi}$ if E is* TR *(resp.* CM*). Let E_0 be the maximal totally real subfield of E. The Mumford–Tate group of V is*

$$\mathrm{G}_\mathrm{B}(V) \cong \begin{cases} \mathrm{Res}_{E/\mathbb{Q}} \mathrm{SO}(\tilde{\psi}), & \text{if } E \text{ is TR;} \\ \mathrm{Res}_{E_0/\mathbb{Q}} \mathrm{U}_{E/E_0}(\tilde{\psi}), & \text{if } E \text{ is CM.} \end{cases}$$

Proof. The first (resp. second) claim is theorem 1.6.a (resp. theorem 1.5) of [28]; the third claim is observed by Van Geemen, in lemma 3.2 of [9]; and the final claim is a combination of theorems 2.2 and 2.3 of [28]. (We note that [28] deals with Hodge groups, but because our Hodge structure has weight 0, the Mumford–Tate group and the Hodge group coincide.) □

6.3 Remark. — Let V, E and $\tilde{\psi}$ be as in lemma 6.2. If E is CM, then $\mathrm{U}(\tilde{\psi})^{\mathrm{der}} = \mathrm{SU}(\tilde{\psi})$ is absolutely simple over E. If E is TR and $\dim_E(V) \neq 4$, then $\mathrm{SO}(\tilde{\psi})$ is absolutely simple over E. Assume E is TR and $\dim_E(V) = 4$. In this case $\mathrm{SO}(\tilde{\psi})$ is not absolutely simple over E; it has Lie type $D_2 = A_1 \oplus A_1$. In this remark we will take a close look at this special case, because a good understanding of it will play a crucial rôle in the proof of lemma 7.9.

Geometrically we find $\mathrm{SO}(\tilde{\psi})_{\overline{E}} \cong (\mathrm{SL}_{2,\overline{E}} \times \mathrm{SL}_{2,\overline{E}})/\langle(-1,-1)\rangle$. We distinguish the following two cases:

1. $\mathrm{SO}(\tilde{\psi})$ is not simple over E. The fact that is most relevant to us is that there exists a quaternion algebra D/E such that $\mathrm{SO}(\tilde{\psi}) \cong (N \times N^{\mathrm{op}})/\langle(-1,-1)\rangle$ where N is the group over E of elements in $D^\star$ that have norm 1, and $N^{\mathrm{op}} \subset (D^{\mathrm{op}})^\star$ is the group of units with norm 1 in D^{op}. One can read more about the details of this claim in section 8.1 of [15]. This situation is also described in section 26.B of [10], where the quaternion algebra is replaced by $D \times D$ viewed as quaternion algebra over $E \times E$. This might be slightly more natural, but it requires bookkeeping of étale algebras which makes the proof in section 7 more difficult than necessary.
2. $\mathrm{SO}(\tilde{\psi})$ is simple over E. This means that the action of $\mathrm{Gal}(\overline{E}/E)$ on $\mathrm{SO}(\tilde{\psi})_{\overline{E}}$ interchanges the two factors $\mathrm{SL}_{2,\overline{E}}$. The stabilisers of these factors are subgroups of index 2 that coincide. This subgroup fixes a quadratic extension F/E. From our description of the geometric situation, together with the description of the stabilisers, we see that $\mathrm{Spin}(\tilde{\psi}) = \mathrm{Res}_{F/E}\,\mathscr{G}$ is a $(2:1)$-cover of $\mathrm{SO}(\tilde{\psi})$, where $\mathscr{G}$ is an absolutely simple, simply connected group of Lie type A_1 over F.

What we have gained is that in all cases we have a description (up to isogeny) of $\mathrm{G_B}(V)^{\mathrm{der}}$ as Weil restriction of a group that is an *almost direct product* of groups that are *absolutely simple*. This allows us to apply lemma 3.6, which will play an important rôle in section 7.

6.4 Notation and terminology. — Let V, E and $\tilde{\psi}$ be as in lemma 6.2. To harmonise the proof in section 7, we unify notation as follows:

$$F = \begin{cases} E_0 & \text{if } E \text{ is CM,} \\ E & \text{if } E \text{ is TR and } \dim_E(V) \neq 4, \\ E & \text{if } E \text{ is TR, } \dim_E(V) = 4, \text{ and we are in case 6.3.1,} \\ F & \text{if } E \text{ is TR, } \dim_E(V) = 4, \text{ and we are in case 6.3.2.} \end{cases}$$

Similarly

$$\mathscr{G} = \begin{cases} \mathrm{U}(\tilde{\psi}) & \text{if } E \text{ is CM,} \\ \mathrm{SO}(\tilde{\psi}) & \text{if } E \text{ is TR and } \dim_E(V) \neq 4, \\ \mathrm{SO}(\tilde{\psi}) & \text{if } E \text{ is TR, } \dim_E(V) = 4, \text{ and we are in case 6.3.1,} \\ \mathscr{G} & \text{if } E \text{ is TR, } \dim_E(V) = 4, \text{ and we are in case 6.3.2.} \end{cases}$$

We stress that $\mathscr{G}^{\mathrm{der}}$ is an almost direct product of absolutely simple groups over F. In section 7, most of the time it is enough to know that $\mathrm{G_B}(V)$ is isogenous to $\mathrm{Res}_{F/\mathbb{Q}}\,\mathscr{G}$. When we need more detailed information, it is precisely the case that E is TR and $\dim_E(V) = 4$. For this case we gave a description of $\mathscr{G}$ in the previous remark.

6.5 — Let V, E and $\tilde{\psi}$ be as in lemma 6.2. Write n for $\dim_E(V)$. If E is TR, then we say that the group $\mathrm{SO}(\tilde{\psi})$ over E is a group of type $\mathrm{SO}_{n,E}$. We also say that $\mathrm{G_B}(V)$ is of type $\mathrm{Res}_{E/\mathbb{Q}}\,\mathrm{SO}_{n,E}$. Similarly, if E is CM, with maximal totally real subfield E_0, then we say that the group $\mathrm{U}_{E/E_0}(\tilde{\psi})$ over E_0 is a group of type U_{n,E_0}, and that $\mathrm{G_B}(V)$ is of type $\mathrm{Res}_{E_0/\mathbb{Q}}\,\mathrm{U}_{n,E_0}$.

6.6 Remark. — Let A be an abelian surface over $\mathbb{C}$. Recall that $\mathrm{H}^2_\mathrm{B}(A)(1)$ has dimension 6. Let H be the transcendental part of $\mathrm{H}^2_\mathrm{B}(A)(1)$ and let ρ denote the Picard number of A, so that $\dim_\mathbb{Q}(H) + \rho = 6$. If A is simple, then the Albert classification of endomorphism algebras of abelian varieties states that $\mathrm{End}(A) \otimes \mathbb{Q}$ can be one of the following:

1. The field of rational numbers, $\mathbb{Q}$. In this case $\rho = 1$ and $\mathrm{G_B}(H)$ is of type $\mathrm{SO}_{5,\mathbb{Q}}$.
2. A real quadratic extension $F/\mathbb{Q}$. In this case $\rho = 2$ and $\mathrm{G_B}(H)$ is of type $\mathrm{SO}_{4,\mathbb{Q}}$. By exemple 3.2.2(a) of [7], we see that $\mathrm{Nm}_{F/\mathbb{Q}}(\mathrm{H}^1(A)) \hookrightarrow \bigwedge^2 \mathrm{H}^1(A) \cong \mathrm{H}^2(A)$, where $\mathrm{Nm}(\_)$ is the norm functor studied in [7]. This norm map identifies $\mathrm{Nm}_{F/\mathbb{Q}}(\mathrm{H}^1(A))(1)$ with the transcendental part H. Observe that consequently the Hodge group $\mathrm{Hdg_B}(\mathrm{H}^1(A)) = \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{SL}_{2,F}$ is a $(2:1)$-cover of $\mathrm{G_B}(H)$.
3. An indefinite quaternion algebra $D/\mathbb{Q}$. (This means that $D \otimes_\mathbb{Q} \mathbb{R} \cong M_2(\mathbb{R})$.) In this case $\rho = 3$ and $\mathrm{G_B}(H)$ is of type $\mathrm{SO}_{3,\mathbb{Q}}$.
4. A CM field $E/\mathbb{Q}$ of degree 4. In this case $\rho = 2$ and $\mathrm{G_B}(H)$ is of type $\mathrm{Res}_{E_0/\mathbb{Q}}\,\mathrm{U}_{1,E_0}$.

(Note that the endomorphism algebra of A cannot be an imaginary quadratic field, by theorem 5 of [22].) If A is isogenous to the product of two elliptic curves $Y_1 \times Y_2$, then there are the following options:

5. The elliptic curves are not isogenous, and neither of them is of CM type, in which case $\rho = 2$ and $\mathrm{G_B}(H)$ is of type $\mathrm{SO}_{4,\mathbb{Q}}$. Indeed, $\mathrm{Hdg_B}(Y_1)$ and $\mathrm{Hdg_B}(Y_2)$ are isomorphic to $\mathrm{SL}_{2,\mathbb{Q}}$. Note that $H = \mathrm{H}^2_\mathrm{B}(A)(1)^{\mathrm{tra}}$ is isomorphic to the exterior tensor product $\big(\mathrm{H}^1_\mathrm{B}(Y_1) \boxtimes \mathrm{H}^1_\mathrm{B}(Y_2)\big)(1)$, We find that $\mathrm{G_B}(H)$ is the image of the canonical map $\mathrm{SL}_{2,\mathbb{Q}} \times \mathrm{SL}_{2,\mathbb{Q}} \to \mathrm{GL}(H)$. The kernel of this map is $\langle(-1,-1)\rangle$.
6. The elliptic curves are not isogenous, one has endomorphism algebra $\mathbb{Q}$, and the other has CM by an imaginary quadratic extension $E/\mathbb{Q}$. In this case $\rho = 2$ and $\mathrm{G_B}(H)$ is of type $\mathrm{U}_{2,\mathbb{Q}}$.
7. The elliptic curves are not isogenous, and Y_i (for $i = 1, 2$) has CM by an imaginary quadratic extension $E_i/\mathbb{Q}$. Observe that $E_1 \ncong E_2$, since Y_1 and Y_2 are not isogenous. Let $E/\mathbb{Q}$ be the compositum of E_1 and E_2, which is a CM field of degree 4 over $\mathbb{Q}$. In this case $\rho = 2$ and $\mathrm{G_B}(H)$ is of type $\mathrm{Res}_{E_0/\mathbb{Q}}\,\mathrm{U}_{1,E_0}$.
8. The elliptic curves are isogenous and have trivial endomorphism algebra. In this case $\rho = 3$ and $\mathrm{G_B}(H)$ is of type $\mathrm{SO}_{3,\mathbb{Q}}$.
9. The elliptic curves are isogenous and have CM by an imaginary quadratic extension $E/\mathbb{Q}$. In this case $\rho = 4$ and $\mathrm{G_B}(H)$ is of type $\mathrm{U}_{1,\mathbb{Q}}$.

7 The main theorem: the Mumford–Tate conjecture for the product of an abelian surface and a K3 surface

7.1 — Let K be a finitely generated subfield of $\mathbb{C}$. Let A be an abelian surface over K, and let M_A denote the transcendental part of the motive $\mathrm{H}^2(A)(1)$. (The Hodge structure H in remark 6.6 is the Betti realisation $\mathrm{H_B}(M_A)$ of M_A.) Let X be a K3 surface over K, and let M_X denote the transcendental part of the motive $\mathrm{H}^2(X)(1)$.

Recall from §6.4 that we associated a field F and a group $\mathscr{G}$ with every Hodge structure V of K3 type. The important properties of F and G are that

» $\mathscr{G}^{\mathrm{der}}$ is an almost direct product of absolutely simple groups over F; and

» $\mathrm{Res}_{F/\mathbb{Q}}\mathscr{G}$ is isogenous to $\mathrm{G_B}(V)$.

Let F_A and $\mathscr{G}_A$ be the field and group associated with $\mathrm{H_B}(M_A)$ as in §6.4. Similarly, let F_X and $\mathscr{G}_X$ be the field and group associated with $\mathrm{H_B}(M_X)$. Concretely, for F_A this means that

$$F_A \cong \begin{cases} \mathrm{End}(A)\otimes\mathbb{Q} & \text{in case 6.6.2 (so } F_A \text{ is TR of degree 2)} \\ E_{A,0} & \text{in cases 6.6.4 and 6.6.7 (so } F_A \text{ is TR of degree 2)} \\ \mathbb{Q} & \text{otherwise.} \end{cases}$$

Let E_X be the endomorphism algebra of M_X. We summarise the notation for easy review during later parts of this section:

K	finitely generated subfield of $\mathbb{C}$
A	abelian surface over K
M_A	transcendental part of the motive $\mathrm{H}^2(A)(1)$
F_A	field associated with the Hodge structure $\mathrm{H_B}(M_A)$, as in §6.4
$\mathscr{G}_A$	group over F_A such that $\mathrm{Res}_{F_A/\mathbb{Q}}\mathscr{G}_A$ is isogenous to $\mathrm{G_B}(M_A)$, as in §6.4
X	K3 surface over K
M_X	transcendental part of the motive $\mathrm{H}^2(X)(1)$
F_X	field associated with the Hodge structure $\mathrm{H_B}(M_X)$, as in §6.4
$\mathscr{G}_X$	group over F_X such that $\mathrm{Res}_{F_X/\mathbb{Q}}\mathscr{G}_X$ is isogenous to $\mathrm{G_B}(M_X)$, as in §6.4
E_X	the endomorphism algebra of M_X

The proof of the main theorem (1.1) will take the remainder of this article. There are four main parts going into the proof, which are lemmas 7.4, 7.6, 7.7 and 7.9. The lemmas 7.2, 7.3 and 7.8 and corollary 7.10 are small reductions and intermediate results. Together lemmas 7.4, 7.6 and 7.7 deal with almost all combinations of abelian surfaces and K3 surfaces. Lemma 7.9 is rather technical, and is the only place in the proof where we use that M_X really is a motive coming from a K3 surface.

7.2 Lemma. — » *The Mumford–Tate conjecture for* $\mathrm{H}^2(A\times X)(1)$ *is equivalent to* $\mathrm{MTC}(M_A \oplus M_X)$.

» *The ℓ-adic realisations of $M_A \oplus M_X$ form a compatible system of ℓ-adic representations.*

Proof. The first claim follows from lemma 4.2. The $\mathrm{H}^2_\ell(A\times X)(1)$ form a compatible system of ℓ-adic representations and we only remove Tate classes to obtain $\mathrm{H}_\ell(M_A\oplus M_X)$; hence the ℓ-adic realisations of $M_A \oplus M_X$ also form a compatible system of ℓ-adic representation. □

7.3 Lemma. — *If for some prime ℓ, the natural morphism*

$$\iota_\ell \colon \mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}} \longrightarrow \mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}}$$

is an isomorphism, then the Mumford–Tate conjecture for $M_A \oplus M_X$ is true.

Proof. By lemma 4.1, we know that the Mumford–Tate conjecture for $M_A \oplus M_X$ is true on the centres of $\mathrm{G_B}(M_A \oplus M_X) \otimes \mathbb{Q}_\ell$ and $\mathrm{G}_\ell^\circ(M_A \oplus M_X)$. By Deligne's theorem on absolute Hodge cycles, we know that $\mathrm{G}_\ell^\circ(M_A \oplus M_X) \subset \mathrm{G_B}(M_A \oplus M_X) \otimes \mathbb{Q}_\ell$. Hence if $\iota_\ell \colon \mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}} \hookrightarrow \mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}}$ is an isomorphism, then $\mathrm{MTC}_\ell(M_A \oplus M_X)$ is true, and by lemma 4.4, so is $\mathrm{MTC}(M_A \oplus M_X)$. □

7.4 Lemma. — *The Mumford–Tate conjecture for $M_A \oplus M_X$ is true if $F_A \ncong F_X$.*

Proof. By lemma 7.3 we are done if $\iota_\ell \colon \mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}} \hookrightarrow \mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}}$ is an isomorphism for some prime ℓ. We proceed by assuming that for all ℓ, the morphism ι_ℓ is not an isomorphism. This will imply that $F_A \cong F_X$.

By lemma 3.7, we see that $F_{A,\ell} = F_A \otimes \mathbb{Q}_\ell$ and $F_{X,\ell} = F_X \otimes \mathbb{Q}_\ell$ have an isomorphic factor, since we assume that ι_ℓ is not an isomorphism. If F_A is isomorphic to $\mathbb{Q}$, then $F_{X,\ell}$ has a factor $\mathbb{Q}_\ell$ for each ℓ, and we win by lemma 2.3.

Next suppose that $F_A \ncong \mathbb{Q}$, in which case it is a real quadratic extension of $\mathbb{Q}$. In particular F_A is Galois over $\mathbb{Q}$ and $\mathscr{G}_A^{\mathrm{der}}$ is an absolutely simple group over F_A of Lie type A_1. Using remark 3.3 we find, for each prime ℓ, semisimple Lie algebras $\mathfrak{s}_{A,\ell}$, $\mathfrak{t}_\ell$ and $\mathfrak{s}_{X,\ell}$ such that

$$\begin{aligned}
\mathrm{Lie}(\mathscr{G}_A^{\mathrm{der}}) \cong \mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}}) &\cong \mathfrak{s}_{A,\ell} \oplus \mathfrak{t}_\ell \\
\mathrm{Lie}(\mathscr{G}_X^{\mathrm{der}}) \cong \mathrm{Lie}(\mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}}) &\cong \mathfrak{t}_\ell \oplus \mathfrak{s}_{X,\ell} \\
\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}}) &\cong \mathfrak{s}_{A,\ell} \oplus \mathfrak{t}_\ell \oplus \mathfrak{s}_{X,\ell}.
\end{aligned}$$

The absolute ranks of these Lie algebras do not depend on ℓ, by lemma 4.1 and remark 6.13 of [11] (or the letters of Serre to Ribet in [20]).

If ℓ is a prime that is inert in F_A, then $\mathscr{G}_A^{\mathrm{der}} \otimes_{F_A} F_{A,\ell}$ is an absolutely simple group. Since $\mathfrak{t}_\ell \neq 0$, we conclude that $\mathfrak{s}_{A,\ell} = 0$. By the independence of the absolute ranks, $\mathfrak{s}_{A,\ell} = 0$ for all primes ℓ. Consequently, if ℓ is a prime that splits in F_A, then $\mathfrak{t}_\ell$ has two simple factors that are absolutely simple Lie algebras over $\mathbb{Q}_\ell$ of Lie type A_1.

If $\mathscr{G}_X^{\mathrm{der}}$ is an absolutely simple group over F_X, then $F_{X,\ell}$ contains two copies of $\mathbb{Q}_\ell$, for each ℓ that splits in F_A. Recall that by lemma 3.7, for all primes ℓ, we know that $F_{A,\ell}$ and $F_{X,\ell}$ have an isomorphic factor. In particular, for inert primes ℓ, $F_{A,\ell}$ is a factor of $F_{X,\ell}$. Hence $F_{A,\ell}$ is a factor of $F_{X,\ell}$ for all primes ℓ, and we are done, by lemma 2.3.

If $\mathscr{G}_X^{\mathrm{der}}$ is not an absolutely simple group, then it is of type SO_{4,E_X}. In particular $\dim_{E_X}(M_X) = 4$ and $F_X \cong E_X$. It follows from lemma 6.2 and the fact that $\dim_{\mathbb{Q}}(M_X) \leq 22$ that $[F_X : \mathbb{Q}] \leq 5$. By lemma 2.4 we conclude that $F_A \cong F_X$. □

7.5 — From now on, we assume that $F_A \cong F_X$, which we will simply denote with F. We single out the following cases, and prove the Mumford–Tate conjecture for $M_A \oplus M_X$ for all other cases in the next lemma.

1. $\mathrm{G_B}(M_A)$ and $\mathrm{G_B}(M_X)$ are both of type $\mathrm{SO}_{5,\mathbb{Q}}$;
2. $\mathrm{G_B}(M_A)$ is of type $\mathrm{SO}_{3,\mathbb{Q}}$, or $\mathrm{SO}_{4,\mathbb{Q}}$, or $\mathrm{U}_{2,\mathbb{Q}}$, and the type of $\mathrm{G_B}(M_X)$ is also one of these types;
3. F is a real quadratic extension of $\mathbb{Q}$, A is an absolutely simple abelian surface with endomorphisms by F (so $\mathscr{G}_A \cong \mathrm{SL}_{2,F}$), and
   1. $\mathscr{G}_X$ is of type $\mathrm{SO}_{3,F}$ or $\mathrm{U}_{2,F}$; or
   2. $\mathscr{G}_X$ is non-simple of type $\mathrm{SO}_{4,F}$ as in case 6.3.1 of remark 6.3.

We point out that in the first two cases $\dim(M_X) \leq 5$, which can be deduced from lemma 6.2.

7.6 Lemma. — *If we are not in one of cases listed in §7.5, then the Mumford–Tate conjecture for $M_A \oplus M_X$ is true.*

Proof. By lemma 7.3 we are done if $\iota_\ell \colon \mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}} \hookrightarrow \mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}}$ is an isomorphism for some prime ℓ.

The crucial ingredient in this lemma is corollary 3.4. Recall that $\mathbb{C} \cong \overline{\mathbb{Q}_\ell}$, as fields. If the Dynkin diagram of $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}})_\mathbb{C}$ has no components in common with the Dynkin diagram of $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}})_\mathbb{C}$, by corollary 3.4, we see that ι_ℓ is an isomorphism, and we win. Recall that $\mathrm{MTC}(M_A)$ and $\mathrm{MTC}(M_X)$ are known. Thus ι_ℓ is an isomorphism when the Dynkin diagram of $\mathrm{Lie}(\mathrm{G_B}(M_A)^{\mathrm{der}})_\mathbb{C}$ has no components in common with the Dynkin diagram of $\mathrm{Lie}(\mathrm{G_B}(M_X)^{\mathrm{der}})_\mathbb{C}$. By inspection of lemma 6.2 and remark 6.6, we see that this holds, except for the cases listed in §7.5. □

7.7 Lemma. — *The Mumford–Tate conjecture for $M_A \oplus M_X$ is true if* $\dim(M_X) \leq 5$. *In particular, the Mumford–Tate conjecture is true for the first two cases listed in §7.5.*

Proof. Let B be the Kuga–Satake variety associated with $\mathrm{H_B}(M_X)$. This is a complex abelian variety of dimension $2^{\dim(M_X)-2}$. Up to a finitely generated extension of K, we may assume that B is defined over K. (In fact, B *is* defined over K, by work of Rizov, [19].) By lemma 4.3, we may and do allow ourselves a finite extension of K, to assure that B is isogenous to a product of absolutely simple abelian varieties over K. By proposition 6.3.3 of [8], we know that $\mathrm{H_B}(M_X)$ is a sub-$\mathbb{Q}$-Hodge structure of $\mathrm{End}(\mathrm{H}^1_\mathrm{B}(B))$. Since M_X is an abelian motive, we deduce that M_X is a submotive of $\mathrm{End}(\mathrm{H}^1(B))$, by André's "Hodge = motivated" theorem (see théorième 0.6.2 of [2]). Consequently, $\mathrm{MTC}(A \times B)$ implies $\mathrm{MTC}(M_A \oplus M_X)$.

Recall that the even Clifford algebra $C^+(M_X) = C^+(\mathrm{H_B}(M_X))$ acts on B. Theorem 7.7 of [8] gives a description of $C^+(M_X)$; thus describing a subalgebra of $\mathrm{End}^0(B)$.

» If $\dim(M_X) = 3$, then $\dim(B) = 2$ and $C^+(M_X)$ is a quaternion algebra over $\mathbb{Q}$.
» If $\dim(M_X) = 4$, then $\dim(B) = 4$ and $C^+(M_X)$ is either a product $D \times D$, where D is a quaternion algebra over $\mathbb{Q}$; or $C^+(M_X)$ is a quaternion algebra over a totally real quadratic extension of $\mathbb{Q}$.
» If $\dim(M_X) = 5$, then $\dim(B) = 8$ and $C^+(M_X)$ is a matrix algebra $M_2(D)$, where D is a quaternion algebra over $\mathbb{Q}$.

We claim that $A \times B$ satisfies the conditions of lemma 5.5. First of all, observe that A satisfies those conditions, which can easily be seen by reviewing remark 6.6. We are done if we check that B satisfies the conditions as well.

- If $\dim(M_X) = 3$, then B is either a simple abelian surface, or isogenous to the square of an elliptic curve. In both cases, B satisfies the conditions of lemma 5.5.
- If $\dim(M_X) = 4$, and $C^+(M_X)$ is $D \times D$ for some quaternion algebra D over $\mathbb{Q}$, then B splits (up to isogeny) as $B_1 \times B_2$. In particular $\dim(B_i) = 2$, since D cannot be the endomorphism algebra of an elliptic curve. Hence both B_i satisfy the conditions of lemma 5.5.

  On the other hand, if $\dim(M_X) = 4$ and $C^+(M_X)$ is a quaternion algebra over a totally real quadratic extension of $\mathbb{Q}$, then there are two options.
  - If B is not absolutely simple, then all simple factors have dimension ≤ 2; since $\mathrm{End}^0(B)$ is non-commutative. Indeed, the product of an elliptic curve and a simple abelian threefold has commutative endomorphism ring (see, *e.g.,* section 2 of [17]).
  - If B is absolutely simple, then it has relative dimension 1. This abelian fourfold must be of type II(2), since type III(2) does not occur (see proposition 15 of [22], or table 1 of [16] which also proves $\mathrm{MTC}(B)$).

  In both of these cases, B satisfies the conditions of lemma 5.5.
- If $\dim(M_X) = 5$, then B is the square of an abelian fourfold C with endomorphism algebra containing a quaternion algebra over $\mathbb{Q}$.
  - If C is not absolutely simple, then all simple factors have dimension ≤ 2; since $\mathrm{End}^0(C)$ is non-commutative.
  - If C is simple, then we claim that C must be of type II. Indeed, since $\mathrm{H_B}(M_X)$ is a sub-$\mathbb{Q}$-Hodge structure of $\mathrm{End}(\mathrm{H}^1_\mathrm{B}(B))$, the Mumford–Tate group of B must surject onto $\mathrm{G_B}(M_X)$. In this case, $\dim(M_X) = 5$, hence $\mathrm{G_B}(M_X)$ is of type $\mathrm{SO}_{5,\mathbb{Q}}$, with Lie type B_2. But §6.1 of [16] shows that if C is of type III, then $\mathrm{G_B}(C)$ has Lie type $D_2 \cong A_1 \oplus A_1$. This proves our claim. Since $\mathrm{End}^0(C)$ is a quaternion algebra and C is an abelian fourfold, table 1 of [16] shows that $\mathrm{MTC}(C)$ is true and D_4 does not occur in the Lie type of $\mathrm{G_B}(C)$.

We conclude that $\mathrm{MTC}(A \times B)$ is true, and therefore $\mathrm{MTC}(M_A \oplus M_X)$ is true as well. □

The only cases left are those listed in case 7.5.3 of §7.5. Therefore, we may and do assume that F is a real quadratic field extension of $\mathbb{Q}$; and that A is an absolutely simple abelian surface with endomorphisms by F (*i.e.,* case 6.6.2). In particular $\mathscr{G}_A = \mathrm{SL}_{2,F}$.

7.8 Lemma. — *If X falls in one of the subcases listed in case 7.5.3, then there exists a place λ of F such that $\mathscr{G}_X^{\mathrm{der}} \otimes_F F_\lambda$ does not contain a split factor.*

Proof. In case 7.5.3.1, $\mathscr{G}_X^{\mathrm{der}}$ is of Lie type A_1. In case 7.5.3.2, $\mathscr{G}_X \sim N \times N^{\mathrm{op}}$, where N is a form of $\mathrm{SL}_{2,F}$, as explained in remark 6.3. By theorem 26.9 of [10], there is an equivalence between forms of SL_2 over a field, and quaternion algebras over the same field. We find a quaternion algebra D over F corresponding to $\mathscr{G}_X^{\mathrm{der}}$, respectively N, in case 7.5.3.1, respectively case 7.5.3.2. In particular $\mathscr{G}_X^{\mathrm{der}}$ contains a split factor if and only if the quaternion algebra is split.

Let $\{\sigma, \tau\}$ be the set of embeddings $\mathrm{Hom}(F, \mathbb{R})$. Since F acts on $\mathrm{H_B}(M_X)$, we see that $F \otimes_\mathbb{Q} \mathbb{R} \cong \mathbb{R}^{(\sigma)} \oplus \mathbb{R}^{(\tau)}$ acts on

$$\mathrm{H_B}(M_X) \otimes_\mathbb{Q} \mathbb{R} \cong W^{(\sigma)} \oplus W^{(\tau)}.$$

Here $W^{(\sigma)}$ and $W^{(\tau)}$ are $\mathbb{R}$-Hodge structure of dimension $\dim_F(M_X)$. Observe that the polarisation form is definite on one of the terms, while it is non-definite on the other. Without loss of generality we may assume that the polarisation form is definite on $W^{(\sigma)}$, and non-definite on $W^{(\tau)}$.

Thus, the group $\mathrm{G_B}(M_X) \otimes_{\mathbb{Q}} \mathbb{R}$ is the product of a compact group and a non-compact group; and therefore, $\mathrm{Res}_{F/\mathbb{Q}} \mathscr{G}_X \otimes_{\mathbb{Q}} \mathbb{R}$ is the product of a compact group and a non-compact group. Indeed $\mathscr{G}_X \otimes_F \mathbb{R}^{(\sigma)}$ is compact, while $\mathscr{G}_X \otimes_F \mathbb{R}^{(\tau)}$ is non-compact. By the first paragraph of the proof, this means that $D \otimes_F \mathbb{R}^{(\sigma)}$ is non-split, while $D \otimes_F \mathbb{R}^{(\tau)}$ is split.

Since the Brauer invariants of D at the infinite places do not add up to 0, there must be a finite place λ of F such that D_λ is non-split. Therefore $\mathscr{G}_X^{\mathrm{der}} \otimes_F F_\lambda$ does not contain a split factor. □

7.9 Lemma. — *Assume that K is a number field. If X falls in one of the subcases listed in case 7.5.3, then there is a prime number ℓ for which the natural map*

$$\iota_\ell \colon \mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}} \hookrightarrow \mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}}$$

is an isomorphism.

Proof. The absolute rank of $\mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}}$ does not depend on ℓ, by lemmas 4.1 and 7.2 and remark 6.13 of [11] (or the letters of Serre to Ribet in [20]). Let ℓ be a prime that is inert in F. Observe that all simple factors of $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}})$ are $\mathbb{Q}_\ell$-Lie algebras with even absolute rank (since $[F : \mathbb{Q}] = 2$). By remark 3.3, the Lie algebra of $\mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}}$ is a summand of $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}})$, and therefore the absolute rank of $\mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}}$ must be even.

Let λ be one of the places of F found in lemma 7.8, and let ℓ be the place of $\mathbb{Q}$ lying below λ. Since $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}})$ must surject to $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A))$ (which is split, and has absolute rank 2), and $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}})$ must also surject onto $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}})$, which has no split factor, by lemma 7.8, we conclude that the absolute rank of $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}})$ must be at least 3. By the previous paragraph, we find that the absolute rank must be at least 4.

If $\dim_{E_X}(M_X) \neq 4$ (case 7.5.3.1) then $\mathscr{G}_X^{\mathrm{der}}$ is a group of Lie type A_1, and therefore the product $\mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}}$ has absolute rank 4. Hence $\mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}}$ must have absolute rank 4, which means that ι_ℓ is an isomorphism, by remark 3.3 and lemma 3.1.

If $\dim_{E_X}(M_X) = 4$ (case 7.5.3.2), then $\mathscr{G}_X$ is a group of Lie type $D_2 = A_1 \oplus A_1$. (Note that in this final case $\mathrm{G_B}(M_A)$ and $\mathrm{G_B}(M_X)$ are semisimple, and therefore we may drop all the superscripts $(\_)^{\mathrm{der}}$ from the notation.) Since in this case $\mathrm{G}_\ell^\circ(M_A) \times \mathrm{G}_\ell^\circ(M_X)$ has absolute rank 6, and the absolute rank of $\mathrm{G}_\ell^\circ(M_A \oplus M_X)$ is ≥ 4, it must be 4 or 6 (since it is even).

Suppose $\mathrm{G}_\ell^\circ(M_A \oplus M_X)$ has absolute rank 4. We apply remark 3.3 to the current situation, and find Lie algebras $\mathfrak{t}$ and $\mathfrak{s}_2$ over $\mathbb{Q}_\ell$ such that $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A)) \cong \mathfrak{t}$ and $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A \oplus M_X)) \cong \mathrm{Lie}(\mathrm{G}_\ell^\circ(M_X)) \cong \mathfrak{t} \oplus \mathfrak{s}_2$. In particular, $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_X))$ which is isomorphic to $\mathrm{Lie}(\mathscr{G}_X) \otimes \mathbb{Q}_\ell$ has a split simple factor. By lemma 7.8 this means that ℓ splits in F as $\lambda \cdot \lambda'$. Observe that $F_\lambda \cong \mathbb{Q}_\ell \cong F_{\lambda'}$.

Note that in this case $\mathrm{G_B}(M_X) \cong \mathrm{Res}_{F_X/\mathbb{Q}} \mathscr{G}_X$, and since $\mathrm{MTC}(M_X)$ is known we find $\mathrm{G}_\ell^\circ(M_X) \cong \mathscr{G}_{X,\lambda} \times \mathscr{G}_{X,\lambda'}$ and a decomposition $\mathrm{H}_\ell(M_X) \cong \mathrm{H}_\lambda(M_X) \oplus \mathrm{H}_{\lambda'}(M_X)$. The group $\mathrm{Gal}(\overline{K}/K)$ acts on $\mathrm{H}_\lambda(M_X)$ via $\mathscr{G}_{X,\lambda}$, and on $\mathrm{H}_{\lambda'}(M_X)$ via $\mathscr{G}_{X,\lambda'}$.

To summarise, our situation is now as follows. The prime number ℓ splits in F as $\lambda \cdot \lambda'$. The group $\mathscr{G}_A$ is isomorphic to $\mathrm{SL}_{2,F}$, and is split and simply connected, The group $\mathscr{G}_{X,\lambda'}$ is split, of type $\mathrm{SO}_{4,\mathbb{Q}_\ell}$, with Lie algebra $\mathfrak{t}$. The group $\mathscr{G}_{X,\lambda}$ is non-split, of type $\mathrm{SO}_{4,\mathbb{Q}_\ell}$, with Lie algebra $\mathfrak{s}_2$. Recall the natural diagram:

$$\begin{array}{ccc} & \mathrm{G}_\ell^\circ(M_A \oplus M_X) & \\ & \downarrow{\scriptstyle \iota_\ell} & \\ & \mathrm{G}_\ell^\circ(M_A) \times \mathrm{G}_\ell^\circ(M_X) & \\ \swarrow & & \searrow \\ (\mathrm{SL}_{2,\mathbb{Q}_\ell} \times \mathrm{SL}_{2,\mathbb{Q}_\ell})/\langle(-1,-1)\rangle \cong \mathrm{G}_\ell^\circ(M_A) & & \mathrm{G}_\ell^\circ(M_X) \cong \mathscr{G}_{X,\lambda'} \times \mathscr{G}_{X,\lambda} \end{array}$$

We are now set for the attack. We claim that the Galois representations $\mathrm{H}_\ell(M_A)$ and $\mathrm{H}_{\lambda'}(M_X)$ are isomorphic. Indeed, from the previous paragraph we conclude that $\mathrm{G}_\ell^\circ(M_A \oplus M_X) \cong \Gamma \times \mathscr{G}_{X,\lambda}$, where Γ is a subgroup of $\mathrm{G}_\ell^\circ(M_A) \times \mathscr{G}_{X,\lambda'}$ with surjective projections. Thus $\mathrm{H}_\ell(M_A)$ and $\mathrm{H}_{\lambda'}(M_X)$ are both orthogonal representations of $\mathrm{Gal}(\overline{K}\,/K)$, and the action of Galois factors via $\Gamma(\mathbb{Q}_\ell)$.

The Lie algebra of Γ is isomorphic to $\mathfrak{t}$, and $\mathrm{Lie}(\Gamma)$ is the graph of an isomorphism $\mathrm{Lie}(\mathrm{G}_\ell^\circ(M_A) \to \mathrm{Lie}(\mathscr{G}_{X,\lambda'})$. Since $\mathrm{G}_\ell^\circ(M_A)$ and $\mathscr{G}_{X,\lambda'}$ have $(2:1)$-covers by $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SL}_{2,F} \cong \mathrm{Hdg}_\ell(A)$ with kernels $\{\pm 1\}$, and Γ is a subgroup of $\mathrm{G}_\ell^\circ(M_A) \times \mathscr{G}_{X,\lambda'}$, we find that Γ also has a $(2:1)$-cover by $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SL}_{2,F}$. Hence Γ is the graph of an isomorphism $\mathrm{G}_\ell^\circ(M_A) \to \mathscr{G}_{X,\lambda'}$. Because $\mathrm{H}_\ell(M_A)$ and $\mathrm{H}_{\lambda'}(M_X)$ are 4-dimensional faithful orthogonal representations of Γ, they must be isomorphic; for up to isomorphism, there is a unique such representation.

As a consequence, for any place v of K, the characteristic polynomial of Frob_v acting on $\mathrm{H}_\ell(M_A)$ coincides with its characteristic polynomial when acting on $\mathrm{H}_{\lambda'}(M_X)$. We conclude that $\mathrm{charpol}_{F_{\lambda'}}(\mathrm{Frob}_v|\mathrm{H}_{\lambda'}(M_X))$ has coefficients in $\mathbb{Q}$. But then the same is true for $\mathrm{charpol}_{F_\lambda}(\mathrm{Frob}_v|\mathrm{H}_\lambda(M_X))$ since their product is $\mathrm{charpol}_{\mathbb{Q}_\ell}(\mathrm{Frob}_v|\mathrm{H}_\ell(M_X))$, which has coefficients in $\mathbb{Q}$.

Since we assumed that K is a number field, we may apply the following results:

» Theorem 1 (item 1) of [5], which tells us that (up to a finite extension of K, which does not matter, by lemma 4.3) there exists a set $\mathscr{V}$ of places of K with density 1 such that X has good reduction at places $v \in \mathscr{V}$, and the Picard number of the reduction X_v is the same as that of X (which, in our case is $22 - 8 = 14$).

» Proposition 3.2 of [27], which says that if X has good and ordinary reduction at v, then the characteristic polynomial $\mathrm{charpol}_{\mathbb{Q}_\ell}(\mathrm{Frob}_v|\mathrm{H}_\ell^2(X_v)^{\mathrm{tra}})$ is a power of an irreducible polynomial with coefficients in $\mathbb{Q}$.

We find that $\mathrm{charpol}_{F_{\lambda'}}(\mathrm{Frob}_v|\mathrm{H}_{\lambda'}(M_X)) = \mathrm{charpol}_{F_\lambda}(\mathrm{Frob}_v|\mathrm{H}_\lambda(M_X))$, for all places $v \in \mathscr{V}$. Since $\mathrm{Gal}(\overline{K}\,/K)$ is compact, we may apply the argument given on the first pages of [21], and find that $\mathrm{H}_{\lambda'}(M_X) \cong \mathrm{H}_\lambda(M_X)$ as Galois representations. This contradicts the fact that $\mathscr{G}_{X,\lambda'}$ is split, while $\mathscr{G}_{X,\lambda}$ is not. We conclude that the rank must be 6, which implies, by remark 3.3 and lemma 3.1, that ι_ℓ is an isomorphism. □

7.10 Corollary. — *If X falls in one of the subcases listed in case 7.5.3, then the Mumford–Tate conjecture is true for $M_A \oplus M_X$.*

Proof. By lemma 7.3 we are done if $\iota_\ell \colon \mathrm{G}_\ell^\circ(M_A \oplus M_X)^{\mathrm{der}} \hookrightarrow \mathrm{G}_\ell^\circ(M_A)^{\mathrm{der}} \times \mathrm{G}_\ell^\circ(M_X)^{\mathrm{der}}$ is an isomorphism for some prime ℓ. This result follows from lemmas 4.8 and 7.9. □

7.11 Proof of theorem 1.1. — We know have all tools in place to prove the main theorem. By lemma 7.2 we reduce to the Mumford–Tate conjecture for $M_A \oplus M_X$. The theorem follows from lemmas 7.4, 7.6 and 7.7 and corollary 7.10. □

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