PRICING BARRIER OPTIONS WITH DISCRETE DIVIDENDS

D. JASON GIBSON AND AARON WINGO

Abstract. The presence of discrete dividends complicates the derivation and form of pricing formulas even for vanilla options. Existing analytic, numerical, and theoretical approximations provide results of varying quality and performance. Here, we compare the analytic approach, developed and effective for European puts and calls, of Buryak and Guo with the formulas, designed in the context of barrier option pricing, of Dai and Chiu.

1. Introduction

Following Buryak and Guo [3], we focus on the analysis of a stock process S_t that jumps down by dividend amounts d_i at times t_i. At non-dividend times, S_t follows a geometric Brownian motion with flat volatility σ. In this context, we have

$$dS_t = \left(rS_t - \sum_{0<t_i\leq T} d_i\, \delta(t-t_i) \right) dt + \sigma S_t\, dW_t, \tag{1.1}$$

where r is the risk-free interest rate, δ is the Dirac delta function, and W_t is a Wiener process. (The book by Hull [8] serves as a standard reference on these matters.) The Black-Scholes partial differential equation

$$\frac{\partial V}{\partial t} - rV + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = 0 \tag{1.2}$$

models the dynamics of the option price. Here, S denotes the (spot) asset price, σ denotes volatility, and r denotes the flat interest rate.

The presence of discrete dividends complicates the derivation and form of pricing formulas even for vanilla options, let alone barrier options or more exotic instruments. Existing analytic, numerical, and

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theoretical approximations provide results of varying quality and performance. One possibility, discussed by Frishling [5] and used under the name "Model 1" for the sake of comparison by Dai and Chiu [4], holds that the difference between the stock price and the present value of future dividends over the life of the option follows a lognormal diffusion process. More involved approaches, like those of Buryak and Guo [3] and Dai and Chiu [4], respectively, allow more sophisticated and sensitive incorporation of factors influencing the option price (volatility, barriers, etc.). Numerical approximations, including Monte-Carlo methods, lattice methods, and Crank-Nicolson schemes, often provide benchmarks for other methods, whether sophisticated or naive.

In §2, we describe the analytic approximations from Buryak and Guo [3]: the spot volatility adjusted, strike volatility adjusted, hybrid, and hybrid volatility adjusted approximations. The latter approximation originates in the paper [3], where it performed well in pricing calls and puts. In §3, we set up the analytic pricing formula, valid in the absence of discrete dividends, for up and out barrier options. In §4, the performance of the Buryak and Guo hybrid volatility adjusted approximation, adapted to the setting of barrier options, can be seen in charts that incorporate data from Dai and Chiu [4]. In §5, we briefly sketch directions for further work on these and related problems.

2. The analytic approximations

The conventional Black-Scholes formulas

$$\begin{aligned} C &= S_0\,\Phi(b_1) - K\exp(-rT)\Phi(b_2), \\ P &= K\exp(-rT)\Phi(b_2) - S_0\,\Phi(-b_1) \end{aligned} \tag{2.1}$$

do not provide for the possibility of stocks with discrete dividends. Here, C and P denote Call and Put, respectively. We also have stock (spot) price S_0, strike price K, time T to maturity for the option, Φ the cumulative Gaussian distribution function, and b_i that satisfy

$$\begin{aligned} b_1 &= \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T\right) \\ b_2 &= b_1 - \sigma\sqrt{T}. \end{aligned} \tag{2.2}$$

2.1. **Spot volatility adjusted approximation.**

Beneder and Vorst [1] use an approximation that, roughly speaking, adjusts some of the Black-Scholes parameters and then adjusts the volatility to refine the correction. To incorporate the dividend information, one might subtract the present value of the dividends

$$D = \sum_{0<t_i \le T} d_i \exp(-rt_i) \tag{2.3}$$

from S_0, producing the adjusted value

$$\begin{aligned} \tilde{S}_0 &= S_0 - D \\ &= S_0 - \sum_{0<t_i \le T} d_i \exp(-rt_i). \end{aligned} \tag{2.4}$$

They observed that, provided the local volatility (of a stock process with discrete dividends) is constant, the process without dividend-induced jumps should then have non-constant local volatilities

$$\tilde{\sigma}_S(S, D, t) = \sigma(T) \frac{S}{S - D_j^{(S)}}, \tag{2.5}$$

where

$$D_j^{(S)} = \sum_{i=j(t)}^{N} d_i \exp(-rt_i), \tag{2.6}$$

with N being the number of dividend payments in $(0, T)$ and the sum restricted to include only those payments occurring after time t, with $j(t)$ the index of the first dividend payment at or after time t.

In light of this volatility adjustment, the corresponding variance can be averaged on $(0, T)$, yielding

$$\overline{\sigma}_S = \sigma \sqrt{\left(\frac{S}{S - D_1^{(S)}}\right)^2 \frac{t_1}{T} + \sum_{1<j\le N} \left(\frac{S}{S - D_j^{(S)}}\right)^2 \frac{t_j - t_{j-1}}{T} + \frac{T - T_N}{T}}, \tag{2.7}$$

with t_N being the time of the last dividend payment in $(0, T)$.

Replacing S_0 by $\tilde{S}_0$ and σ by $\overline{\sigma}_S$ in (2.1) and in (2.2) yields

$$\begin{aligned} C &= \tilde{S}_0 \, \Phi(b_1) - K \exp(-rT)\Phi(b_2), \\ P &= K \exp(-rT)\Phi(b_2) - \tilde{S}_0 \, \Phi(-b_1) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} b_1 &= \frac{1}{\overline{\sigma}_S \sqrt{T}} \left(\ln \frac{\tilde{S}_0}{K} + \left(r + \frac{\overline{\sigma}_S^2}{2} \right) T \right) \\ b_2 &= b_1 - \overline{\sigma}_S \sqrt{T}. \end{aligned} \tag{2.9}$$

Call the above scheme the spot volatility adjusted approximation, the spot VA approximation.

2.2. **Strike volatility adjusted approximation.** Buryak and Guo [3] introduce a different approximation based on the set-up of Beneder and Vorst [1]. First, following Frishling's description [5] of a strike approximation, they modify the strike price from K to $\tilde{K}$ by setting

$$\tilde{K} = K + \sum_{0<t_i\leq T} d_i \exp(r(T-t_i)), \tag{2.10}$$

a natural analog of (2.4).

Next, the volatilities get adjusted by considering the non-constant local volatilities

$$\tilde{\sigma}_K(S,D,t) = \sigma(T)\frac{S}{S+D_j^{(K)}}, \tag{2.11}$$

where

$$D_j^{(K)} = \sum_{i=1}^{j(t)} d_i \exp(-rt_i), \tag{2.12}$$

with N being the number of dividend payments in $(0,T)$ and the sum restricted to include only those payments occurring before time t, with $j(t)$ the index of the first dividend payment at or before time t. With this volatility adjustment, the corresponding variance can be averaged on $(0,T)$, yielding

$$\overline{\sigma}_K = \sigma\sqrt{\frac{t_1}{T} + \sum_{1\leq j<N}\left(\frac{S}{S+D_j^{(K)}}\right)^2\frac{t_{j+1}-t_j}{T} + \left(\frac{S}{S+D_N^{(K)}}\right)^2\frac{T-t_N}{T}}, \tag{2.13}$$

with t_N being the time of the last dividend payment in $(0,T)$.

Replacing K by $\tilde{K}$ and σ by $\overline{\sigma}_K$ in (2.1) and in (2.2) yields

$$\begin{aligned} C &= S_0\,\Phi(b_1) - \tilde{K}\exp(-rT)\Phi(b_2),\\ P &= \tilde{K}\exp(-rT)\Phi(b_2) - S_0\,\Phi(-b_1) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} b_1 &= \frac{1}{\overline{\sigma}_K\sqrt{T}}\left(\ln\frac{S_0}{\tilde{K}} + \left(r + \frac{\overline{\sigma}_K^2}{2}\right)T\right)\\ b_2 &= b_1 - \overline{\sigma}_K\sqrt{T}. \end{aligned} \tag{2.15}$$

Call the above scheme the strike volatility adjusted approximation, the strike VA approximation.

2.3. **Hybrid approximation.** Bos and Vandermark [2] offered a different approximation, one supported with some theoretical analysis. Specifically, take

$$\begin{aligned} C &= \overline{S}_0\,\Phi(b_1) - \overline{K}\exp(-rT)\Phi(b_2), \\ P &= \overline{K}\exp(-rT)\Phi(b_2) - \overline{S}_0\,\Phi(-b_1), \end{aligned} \tag{2.16}$$

with

$$\begin{aligned} \overline{S}_0 &= S_0 - D_S \\ \overline{K} &= K + D_K\exp(rT). \end{aligned} \tag{2.17}$$

Here,

$$\begin{aligned} D_S &= \sum_{0<t_i\le T} \frac{T-t_i}{T}\,d_i\exp(-rt_i) \\ D_K &= \sum_{0<t_i\le T} \frac{t_i}{T}\,d_i\exp(-rt_i). \end{aligned} \tag{2.18}$$

In contrast with the Spot VA and Strike VA approximations described above, this method does not adjust the volatility. Call the above scheme the hybrid approximation.

2.4. **Hybrid volatility adjusted approximation.** A new method described by Buryak and Guo takes the Hybrid approximation above as a starting point, but then also adjusts the volatilities in a manner related to the volatility adjustment schemes mentioned earlier. A key difference between this new method and those other methods lies in the individual treatment of the D_S and D_K terms, where the discounted dividend stream D satisfies $D = D_S + D_K$.

Specifically, the volatilities get adjusted by considering the non-constant local volatilities

$$\tilde{\sigma}_S(S,D,t) = \sigma(T)\frac{S}{S - D_j^{(S)}}, \tag{2.19}$$

where

$$D_j^{(S)} = \sum_{i=j(t)}^{N} \frac{T-t_i}{T} d_i\exp(-rt_i), \tag{2.20}$$

and

$$\tilde{\sigma}_K(S,D,t) = \sigma(T)\frac{S}{S + D_j^{(K)}}, \tag{2.21}$$

where

$$D_j^{(K)} = \sum_{i=1}^{j(t)} \frac{t_i}{T} d_i \exp(-rt_i). \tag{2.22}$$

Here, with N being the number of dividend payments in $(0, T)$, the spot sum gets restricted to include only those payments occurring after time t, with $j(t)$ the index of the first dividend payment at or after time t, and the strike sum restricted to include only those payments occurring before time t, with $j(t)$ the index of the first dividend payment at or before time t.

In both instances, the corresponding variance can be averaged on $(0, T)$,

$$\begin{aligned}\overline{\sigma}_S &= \sigma\sqrt{\left(\frac{S}{S - D_1^{(S)}}\right)^2 \frac{t_1}{T} + \sum_{1<j\le N}\left(\frac{S}{S - D_j^{(S)}}\right)^2 \frac{t_j - t_{j-1}}{T} + \frac{T - T_N}{T}} \\ &= \sigma(1 + \varepsilon_S^{(h)}),\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}\overline{\sigma}_K &= \sigma\sqrt{\frac{t_1}{T} + \sum_{1\le j<N}\left(\frac{S}{S + D_j^{(K)}}\right)^2 \frac{t_{j+1} - t_j}{T} + \left(\frac{S}{S + D_N^{(K)}}\right)^2 \frac{T - t_N}{T}} \\ &= \sigma(1 - \varepsilon_K^{(h)}),\end{aligned} \tag{2.24}$$

with t_N being the time of the last dividend payment in $(0, T)$.

Finally, set

$$\overline{\sigma}_H = \sigma(1 + \epsilon_S^{(h)})(1 - \epsilon_K^{(h)}), \tag{2.25}$$

and use the set-up described by (2.16). For calls, this approximation then requires no further adjustments. Puts require further adjustment. (See the Buryak-Guo [3] discussion of Liquidator and Survivor dividend policies, based on considerations described in Haug [6].) Call this scheme the hybrid VA approximation.

3. Barrier options

We follow the books of Haug [7] and Levy [9], and we use the notational structure of Haug.

An up and out call option ceases to exist when the asset price reaches or goes above the barrier level B. For the up and out call, we follow the formulas available on p.152–153 of Haug [6] (also see (9.77) in Levy [9]). These formulas, in turn, originate in the work of Merton [10] and Reiner and Rubinstein [11]. The up and out call option pays $\max(S-K,0)$ if $S<B$ holds for all times up to T, and, otherwise, it pays a rebate R. Then

$$\begin{aligned} C_{K>B} &= F \\ C_{K<B} &= A - B + C - D + F, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
A &= \phi S e^{(b-r)T}\Phi(\phi x_1) - \phi K e^{-rT}\Phi(\phi x_1 - \phi\sigma\sqrt{T}) \\
B &= \phi S e^{(b-r)T}\Phi(\phi x_2) - \phi K e^{-rT}\Phi(\phi x_2 - \phi\sigma\sqrt{T}) \\
C &= \phi S e^{(b-r)T}\left(\frac{B}{S}\right)^{2(\mu+1)}\Phi(\eta y_1) - \phi K e^{-rT}\left(\frac{B}{S}\right)^{2\mu}\Phi(\eta y_1 - \eta\sigma\sqrt{T}) \\
D &= \phi S e^{(b-r)T}\left(\frac{B}{S}\right)^{2(\mu+1)}\Phi(\eta y_2) - \phi K e^{-rT}\left(\frac{B}{S}\right)^{2\mu}\Phi(\eta y_2 - \eta\sigma\sqrt{T}) \\
F &= R\left[\left(\frac{B}{S}\right)^{\mu+\lambda}\Phi(\eta z) + \left(\frac{B}{S}\right)^{\mu-\lambda}\Phi(\eta z - 2\eta\lambda\sigma\sqrt{T})\right],
\end{aligned} \tag{3.2}$$

and

$$x_1 = \frac{\ln(S/K)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T} \qquad x_2 = \frac{\ln(S/B)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T} \tag{3.3}$$

$$y_1 = \frac{\ln(B^2/(SK))}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T} \quad y_2 = \frac{\ln(B/S)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T} \tag{3.4}$$

$$z = \frac{\ln(B/S)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \tag{3.5}$$

$$\mu = \frac{b - \frac{\sigma^2}{2}}{\sigma^2} \qquad \lambda = \sqrt{\mu^2 + \frac{2r}{\sigma^2}}. \tag{3.6}$$

4. Performance

We compare the Hybrid VA approach, applying the adjusted spot, strike, and volatility parameters described in Section 2.4 to the barrier option formulas in Section 3, with some work from Dai and Chiu [4]. The Hybrid VA and HVA Error columns are new, and the other columns first appear in the work of Dai and Chiu.

The details of this test example appear first in their Figure 2 on page 1377, and it provides the default values for our later comparisons. There, the risk-free rate is 3%, the volatility is 20%, the strike price is 50, the barrier is 65, and the time to maturity is 1 year. A discrete dividend 1 is paid at 0.5 year. We use Maximum Absolute Error and Root-Mean-Squared Error as performance indicators.

$S(0)$	MC	Dai-Chiu	Model1	Hybrid VA	DC Error	M1 Error	HVA Error
46	1.1265	1.1260	1.1336	1.0717	0.0005	0.0071	0.0548
48	1.3456	1.3427	1.3641	1.2736	0.0029	0.0184	0.0720
50	1.5054	1.5026	1.5417	1.4219	0.0028	0.0363	0.0835
52	1.5829	1.5796	1.6401	1.4938	0.0033	0.0572	0.0891
54	1.5661	1.5571	1.6422	1.4758	0.0089	0.0762	0.0903
56	1.4389	1.4310	1.5423	1.3649	0.0079	0.1034	0.0740
58	1.2112	1.2093	1.3463	1.1686	0.0019	0.1352	0.0426
60	0.9164	0.9106	1.0700	0.9030	0.0059	0.1536	0.0134
62	0.5667	0.5602	0.7358	0.5898	0.0065	0.1691	0.0231
64	0.1932	0.1868	0.3697	0.2529	0.0065	0.1765	0.0597
MAE					0.0089	0.1765	0.0903
RMSE					0.0054	0.1109	0.0654

TABLE 1. Varying initial stock price for barrier call, single discrete dividend

As shown in Table 1, as the initial stock price, $S(0)$, increased, the Model1 formula tended to increase in error as compared with the HVA model, which maintained some stability in its error magnitude. The RMSE and MAE of each model's performance indicate this as well.

d_1	MC	Dai-Chiu	Model1	Hybrid VA	DC Error	M1 Error	HVA Error
0.3	1.5759	1.5730	1.5857	1.5467	0.0029	0.0098	0.0292
0.6	1.5438	1.5435	1.5680	1.4928	0.0003	0.0242	0.0510
0.9	1.5202	1.5129	1.5486	1.4395	0.0073	0.0283	0.0807
1.2	1.4868	1.4815	1.5273	1.3870	0.0053	0.0405	0.0998
1.5	1.4478	1.4493	1.5044	1.3353	0.0015	0.0566	0.1125
1.8	1.4147	1.4163	1.4798	1.2844	0.0017	0.0652	0.1303
2.1	1.3843	1.3828	1.4538	1.2345	0.0015	0.0695	0.1498
2.4	1.3459	1.3488	1.4262	1.1855	0.0030	0.0804	0.1604
MAE					0.0073	0.0804	0.1604
RMSE					0.0036	0.0523	0.1105

TABLE 2. Varying payout amounts for barrier call, single discrete dividend

However, with Table 2, the HVA error in this case increased while the Model1 error remained relatively flat as the payout amounts increased. The MAE and RMSE of each show a drastic difference, with the HVA error reflecting around double the size in comparison with the Model1 error.

σ	MC	Dai-Chiu	Model1	Hybrid VA	DC Error	M1 Error	HVA Error
0.1	2.0707	2.0756	2.0612	2.0412	0.0049	0.0094	0.0295
0.2	1.5054	1.5026	1.5417	1.4219	0.0028	0.0363	0.0835
0.3	0.7215	0.7167	0.7534	0.6742	0.0047	0.0320	0.0473
0.4	0.3625	0.3611	0.3846	0.3395	0.0014	0.0221	0.0230
0.5	0.2035	0.1998	0.2144	0.1881	0.0037	0.0109	0.0154
0.6	0.1205	0.1197	0.1292	0.1129	0.0007	0.0087	0.0076
0.7	0.0767	0.0764	0.0827	0.0721	0.0003	0.0060	0.0046
0.8	0.0526	0.0511	0.0556	0.0483	0.0014	0.0030	0.0042
0.9	0.0366	0.0356	0.0388	0.0337	0.0010	0.0021	0.0001
1.0	0.0255	0.0255	0.0279	0.0242	0.0000	0.0024	0.0013
MAE					0.0049	0.0363	0.0835
RMSE					0.0027	0.0178	0.0331

Table 3. Varying volatility for barrier call, single discrete dividend

In Table 3, both models performed well, with the Model1 approach doing slightly better.

t_1	MC	Dai-Chiu	Model1	Hybrid VA	DC Error	M1 Error	HVA Error
0.1	1.5425	1.5378	1.5408	1.5165	0.0047	0.0016	0.0260
0.2	1.5347	1.5335	1.5410	1.4926	0.0012	0.0063	0.0401
0.3	1.5291	1.5262	1.5412	1.4689	0.0029	0.0121	0.0602
0.4	1.5236	1.5160	1.5415	1.4453	0.0076	0.0179	0.0783
0.5	1.5054	1.5026	1.5417	1.4219	0.0028	0.0363	0.0835
0.6	1.4903	1.4861	1.5419	1.3986	0.0042	0.0516	0.0917
0.7	1.4737	1.4658	1.5421	1.3756	0.0079	0.0684	0.0981
0.8	1.4391	1.4399	1.5423	1.3527	0.0007	0.1032	0.0864
0.9	1.4036	1.4029	1.5425	1.3300	0.0007	0.1389	0.0736
MAE					0.0079	0.1389	0.0981
RMSE					0.0042	0.0625	0.0745

Table 4. Varying payout times for a barrier call, single discrete dividend

Table 4 shows that increasing the date of payouts does not yield much better model performance when evaluating either HVA or Model1. The RMSE of each are almost identical, and the MAE values are not much different.

$S(0)$	MC	Dai-Chiu	Model1	Hybrid VA	DC Error	M1 Error	HVA Error
46	0.9156	0.9122	0.9493	0.8126	0.0034	0.0338	0.1030
48	1.0033	1.0028	1.0619	0.8907	0.0005	0.0586	0.1126
50	1.0538	1.0493	1.1322	0.9307	0.0045	0.0783	0.1231
52	1.0484	1.0438	1.1519	0.9276	0.0046	0.1035	0.1208
54	0.9880	0.9843	1.1178	0.8806	0.0037	0.1298	0.1074
56	0.8771	0.8737	1.0316	0.7924	0.0035	0.1545	0.0847
58	0.7241	0.7192	0.8990	0.6692	0.0049	0.1749	0.0549
60	0.5364	0.5315	0.7287	0.5189	0.0049	0.1924	0.0175
62	0.3249	0.3233	0.5317	0.3508	0.0016	0.2068	0.0259
64	0.1104	0.1077	0.3192	0.1744	0.0032	0.2088	0.0640
MAE					0.0049	0.2088	0.1231
RMSE					0.0038	0.1470	0.0893

TABLE 5. Varying initial stock price for barrier call, two discrete dividends

When studying the difference between the two models, Model1 and HVA, in the situation of changing $S(0)$ for a barrier call with two dividend payouts, the differences in the errors become apparent. Both performed poorly, but the HVA error was smaller in magnitude.

$d_1 = d_2$	MC	Dai-Chiu	Model1	Hybrid VA	DC Error	M1 Error	HVA Error
0.3	1.1305	1.1238	1.1514	1.0841	0.0067	0.0210	0.0464
0.6	1.0948	1.0933	1.1462	1.0173	0.0015	0.0514	0.0775
0.9	1.0585	1.0606	1.1364	0.9521	0.0021	0.0780	0.1064
1.2	1.0279	1.0269	1.1222	0.8885	0.0010	0.0943	0.1394
1.5	0.9864	0.9897	1.1038	0.8267	0.0033	0.1174	0.1597
1.8	0.9552	0.9535	1.0812	0.7670	0.0018	0.1260	0.1882
2.1	0.9147	0.9156	1.0548	0.7095	0.0009	0.1401	0.2052
2.4	0.8769	0.8772	1.0247	0.6542	0.0003	0.1478	0.2227
MAE					0.0068	0.1478	0.2227
RMSE					0.0029	0.1056	0.1547

TABLE 6. Varying payout amounts for barrier call, two discrete dividends

When changing the payout amounts for two discrete dividend payouts in a barrier call, again both did poorly, but, in this instance, in contrast with Table 6 just before, Model1 yielded tighter results.

5. Further Work

The tables in Section 4 show that the method of Dai and Chiu handily outperforms the Hybrid VA method. It can also be seen that the Model 1 approach and the Hybrid VA approach each have strengths in certain parameter regions, but neither really compete with the method of Dai and Chiu.

The paper of Buryak and Guo [3] introduces the Hybrid VA to price European calls and puts, and, for that purpose, the method performs reasonably well. Because their approach doesn't see or make use of barrier information, it should not be surprising that the modified spot, strike, and volatility information on their own do not perform as well for barrier options. Further, the sharp results of the method of Dai and Chiu, while derived with solid theoretical justification, also require nontrivial calculations to deduce. In particular, their paper [4] treats only the single- and double-dividend cases in detail.

It would be interesting to modify or to refine the Hybrid VA method in a manner that shows greater sensitivity to the context of barrier options, say by incorporating the barrier value. Ideally, an improvement would retain the analytic flavor of the existing Hybrid VA method of Buryak and Guo and would achieve results closer to the method of Dai and Chiu (and to numerical benchmarks like Monte Carlo methods or Crank-Nicolson schemes).

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Department of Mathematics and Statistics, Eastern Kentucky University, KY 40475, USA
E-mail address: jason.gibson@eku.edu

Department of Mathematics and Statistics, Eastern Kentucky University, KY 40475, USA
E-mail address: aaron_wingo@mymail.eku.edu