REPRESENTATION OF LARGE MATCHINGS IN BIPARTITE GRAPHS

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Abstract. Let $f(n)$ be the smallest number such that every collection of n matchings, each of size at least $f(n)$, in a bipartite graph, has a full rainbow matching. Generalizing famous conjectures of Ryser, Brualdi and Stein, Aharoni and Berger [2] conjectured that $f(n) = n+1$ for every $n > 1$. Clemens and Ehrenmüller [7] proved that $f(n) \leq \frac{3}{2}n + o(n)$. We show that the $o(n)$ term can be reduced to a constant, namely $f(n) \leq \lceil \frac{3}{2}n \rceil + 1$.

1. Introduction

Given sets $F_1, F_2, \ldots, F_n$ of edges in a graph, a *(partial) rainbow matching* is a choice of disjoint edges from some of the F_is. In other words, it is a partial choice function whose range is a matching. If the rainbow matching represents all F_is then we say that it is *full*. For a comprehensive survey on rainbow matchings and the related subject of transversals in Latin squares see [13].

As in the abstract, we assume the graph is bipartite and define $f(n)$ to be the least number such that if $|F_i| \geq f(n)$ for all i, then there exists a full rainbow matching. A greedy choice of representatives shows that if $|F_i| \geq 2n-1$ for all $i \leq n$ then there is a rainbow matching, namely $f(n) \leq 2n-1$. On the other hand, for every $n > 1$ there exits a family $F_1, \ldots, F_n$ of matchings of size n with no full rainbow matching: for an arbitrary $1 \leq k \leq n$ let $F_1, \ldots, F_k$ be all equal to the perfect matching in the cycle C_{2n} consisting of the odd edges, and let $F_{k+1}, \ldots, F_n$ be all equal to the perfect matching in C_{2n} consisting of the even edges. This shows that $f(n) \geq n+1$ for all $n > 1$ (in fact, this example can be modified to produce $2n-2$ matchings of size n with no rainbow matching of size n). In [2] it was conjectured that this bound is sharp:

Conjecture 1.1. [2] *$f(n) = n+1$ for all $n > 1$.*

If true, this would easily imply:

Conjecture 1.2. *A family of n matchings in a bipartite graph, each of size n, has a rainbow matching of size $n-1$.*

This strengthens a famous conjecture of Ryser-Brualdi-Stein.

Conjecture 1.3. [6, 15, 16] *A partition of the edges of the complete bipartite graph $K_{n,n}$ into n matchings, each of size n, has a rainbow matching of size $n-1$.*

Another strengthening of the last conjecture is due to Stein:

Conjecture 1.4. [16] *A partition of the edges of the complete bipartite graph $K_{n,n}$ into n subsets, each of size n, has a rainbow matching of size $n-1$.*

In our terminology, the weaker condition that Stein demands on sets F_i is not that they are matchings, but that each has degree at most 1 in one side of the graph, and that jointly their degree at each vertex in the other side is at most n. Possibly the 'right' requirement is even more general: that the degree at each vertex is at most n, and that each F_i is a set, and not a multiset, namely it does not contain repeating edges. An even more general conjecture will be presented in the last section.

Successive improvements on the trivial bound $f(n) \leq 2n-1$ were $f(n) \leq \lfloor \frac{7}{4}n \rfloor$ [4], $f(n) \leq \lfloor \frac{5}{3}n \rfloor$ [12] and $f(n) \leq \lfloor \frac{3}{2}n \rfloor + o(n)$ [7]. The latter was extended in [8] to general graphs, and to the more general case in

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which the sets F_i are not assumed to be matchings, but disjoint unions of cliques, each containing $3n + o(n)$ vertices. Pokrovskiy [14] showed that if we add the requirement that the n matchings are edge disjoint, then $|F_i| \geq n + o(n)$ suffices. In this note we prove:

Theorem 1.5. $f(n) \leq \lceil \frac{3}{2}n \rceil + 1$.

2. Proof of Theorem 1.5

The following was shown in [12]:

Proposition 2.1. *A family $\mathcal{F} = \{F_1, \ldots, F_n\}$ of n matchings in a bipartite graph, each of size at least $\lfloor \frac{3}{2}n \rfloor$, has a rainbow matching of size $n-1$.*

Proof of Theorem 1.5. Let R be a rainbow matching of maximal size. By Proposition 2.1 $|R| \geq n-1$. Assume, for contradiction, that $|R| = n-1$. Without loss of generality we may assume that $R \cap F_n = \emptyset$. For each $i = 1, \ldots, n-1$ let $F_i \cap R = \{r_i\}$ and $r_i = \{u_i, w_i\}$ where $u_i \in U$ and $w_i \in W$. Let $X \subset U$ and $Y \subset W$ be the sets of vertices of G not covered by R. Let F_n^Y be the subset of F_n consisting of edges matching vertices in Y. Since R has maximal size, F_n^Y matches vertices in Y to $U \setminus X$. Let U' be the set of vertices in $U \setminus X$ that are matched by F_n^Y. Let R' be the subset of R that matches the elements in U' and let W' be the set of vertices in W that are matched by R'. Our strategy is to replace some edges in R' by edges having one endpoint in X and the other endpoint in $W \setminus Y$, thus freeing vertices in U'. This will allow us to add an edge from F_n^Y to the rainbow matching.

Let $\ell = |F_n^Y|$. Since $|W \setminus Y| = n-1$ and $|F_n| = \lceil 3n/2 \rceil + 1$ we have $\ell \geq \lceil n/2 \rceil + 2$. So,

$$|R'| \geq \lceil n/2 \rceil + 2. \tag{1}$$

Define,

$$\mathcal{F}' = \{F_i \in \mathcal{F} | F_i \cap R' \neq \emptyset\}.$$

Notation 2.2. For each $F_i \in \mathcal{F}'$ let e_i be the edge of F_n^Y such that $e_i \cap r_i \neq \emptyset$. Let y_i be the endpoint of e_i in Y (Figure 1).

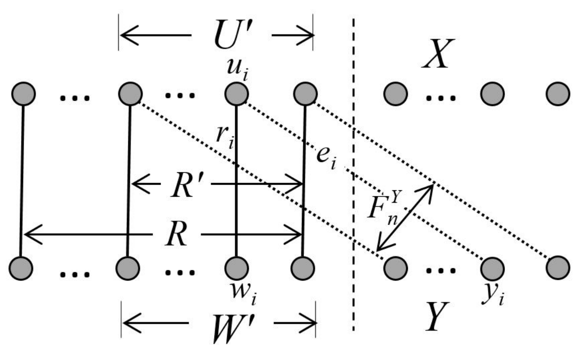


Figure 1

Claim 1. For every $F_i \in \mathcal{F}'$ the matching F_i has at most one edge between X and Y.

Proof. Suppose F_i has two edges e and f between X and Y. The edge e_i is disjoint from one of them, say e. Thus, $(R \setminus \{r_i\}) \cup \{e_i, e\}$ is a rainbow matching of size n, contradicting the maximality of R (Figure 2(a)).

□

Corollary 2.3. *Each $F_i \in \mathcal{F}'$ has at least $\ell - 1 \geq \lceil n/2 \rceil + 1$ edges with one endpoint in X and the other endpoint in $W \setminus Y$ and at least $\lceil n/2 \rceil + 1$ edges with one endpoint in Y and the other endpoint in $U \setminus X$.*

Remark 2.4. In all the figures below, dashed lines represent edges that are candidates to be removed from the rainbow matching, and solid and dotted lines represent edges that are candidates for being added in.

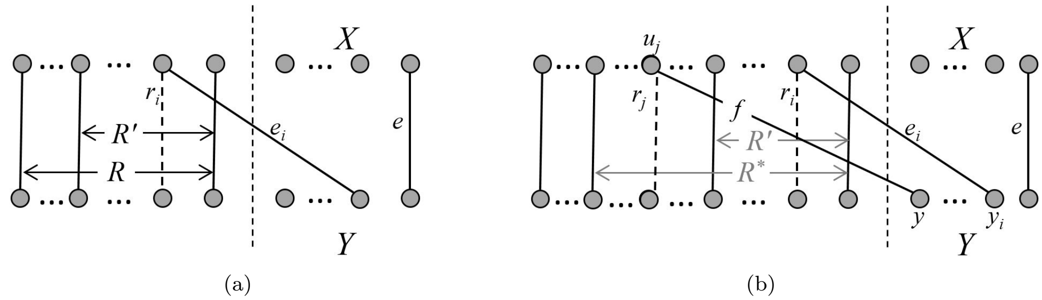


FIGURE 2

Notation 2.5. For each $F_i \in \mathcal{F}'$ let F_i^Y be the subset of F_i consisting of edges with one endpoint in $Y \setminus \{y_i\}$ and the other in $U \setminus X$. Let U^* be the union of U' and the set of vertices in $U \setminus X$ that are endpoints of edges in $\bigcup\{F_i^Y \mid F_i \in \mathcal{F}\}$. Let R^* be the subset of R that matches the elements in U^*. Let W^* be the set of vertices in W that are matched by R^*. We define

$$\mathcal{F}^* = \{F_j \in \mathcal{F} | F_j \cap R^* \neq \emptyset\}.$$

(Note that $U' \subset U^*$, $W' \subset W^*$, $R' \subset R^*$ and $\mathcal{F}' \subset \mathcal{F}^*$.) Let $\mathcal{F}'' = \mathcal{F}^* \setminus \mathcal{F}'$ and let $d = |\mathcal{F}''|$ (it is possible that $d = 0$).

Claim 2. Each $F \in \mathcal{F}''$ has at least $\ell - 2 \geq \lceil n/2 \rceil$ edges with one endpoint in X and the other endpoint in $W \setminus Y$ and at least $\lceil n/2 \rceil$ edges with one endpoint in Y and the other endpoint in $U \setminus X$.

Proof. Let $F_j \in \mathcal{F}''$. We show that F_j has at most two edges between X and Y. By the definition of $\mathcal{F}^*$, there exists $F_i \in \mathcal{F}'$ and an edge $f \in F_i$ such that $f \cap r_j = \{u_j\} \subset U \setminus X$ and the other endpoint y of f is in $Y \setminus \{y_i\}$. Now suppose F_j has three edges between X and Y. Then one of them, say e, has an endpoint in $Y \setminus \{y_i, y\}$. We can now augment R by taking $R \setminus \{r_i, r_j\} \cup \{f, e_i, e\}$ (Figure 2(b)). □

Claim 3. Each $F_i \in \mathcal{F}^*$ has at least $d + 3$ edges with one endpoint in X and the other endpoint in W^*.

Proof. We know that $|R^*| = |R'| + d$. Hence, by (1), we have $|R \setminus R^*| \leq n - 1 - (\lceil n/2 \rceil + 2 + d) = \lfloor n/2 \rfloor - d - 3$. By Claim 2, the edges of F_i with one endpoint in X and the other endpoint in $W \setminus Y$ meet at least $\lceil n/2 \rceil - (\lfloor n/2 \rfloor - d - 3) \geq d + 3$ edges of R^*. □

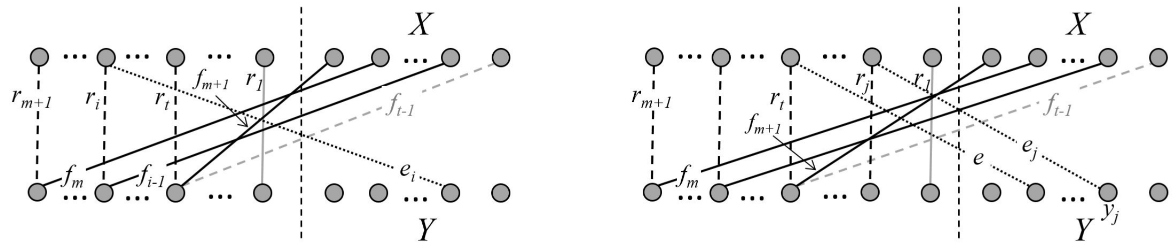
We shall inductively choose edges $f_1, f_2, \ldots, f_i$ and $r_1, r_2, \ldots, r_i, r_{i+1}$, as follows. Without loss of generality we assume that $F_1 \in \mathcal{F}'$. By Claim 3, there exists $f_1 \in F_1$ connecting a vertex $x_1 \in X$ and a vertex in W^*. Denote this vertex by w_2, and without loss of generality we may assume that $w_2 \in r_2$, where $r_2 \in R^* \cap F_2$. Again, by Claim 3, F_2 has at least $d+2$ edges with one endpoint in $X \setminus \{x_1\}$ and the other endpoint in W^*. Let $f_2 \in F_2$ be such an edge. We continue this way, choosing at each step an edge f_i, disjoint from all f_j, $j < i$, and belonging to the same matching as r_i, with endpoints in X and in W^*, and an edge $r_{i+1} \in R^*$, such that $f_i \cap r_{i+1} \cap W^* \neq \emptyset$. The process ends when we have obtain a set of disjoint edges $F = \{f_1, f_2, \ldots, f_m\}$, each with endpoints in X and in W^*, and a set of distinct edges $P = \{r_1, r_2, \ldots, r_m, r_{m+1}\} \subseteq R^*$ such that $f_i \cap r_{i+1} \cap W^* \neq \emptyset$ for $i = 1, \ldots, m$, and for each i, f_i and r_i belong to the same matching (without loss of generality we assume that $f_i, r_i \in F_i$ for $i = 1, \ldots, m$, and $r_{m+1} \in F_{m+1}$), so that one of two options holds:

(1) $m < d + 3$ and the matching F_{m+1} has an edge f_{m+1} with one endpoint in $X \setminus (f_1 \cup f_2 \cup \ldots \cup f_m)$ such $f_{m+1} \cap r_t \cap W^* \neq \emptyset$ for some $t \in \{1, \ldots, m\}$, or
(2) $m = d + 3$.

(Note that by Claim 3 one of these two options must hold.)

In Case (1) the partial rainbow matching R can be augmented as follows: If $F_i \in \mathcal{F}'$ for some $i \in \{t, \ldots, m+1\}$, then $(R \setminus \{r_t, \ldots, r_{m+1}\}) \cup \{f_t, \ldots, f_{m+1}, e_i\}$ is a full rainbow matching (Figure 3(a)). If $F_i \in \mathcal{F}''$ for all $i \in \{t, \ldots, i+1\}$, then, by the definition of $\mathcal{F}^*$, there exists $F_j \in \mathcal{F}'$ and an edge $e \in F_j^Y$

so that $e \cap r_t \in U^*$. In this case $(R \setminus \{r_t, \ldots, r_{m+1}, r_j\}) \cup \{f_t, \ldots, f_{m+1}, e, e_j\}$ is a full rainbow matching (Figure 3(b)). (Note that e and e_j are disjoint by the definition of F_j^Y.)



(a) (b)

FIGURE 3

In Case (2) let $Q = (R \setminus P) \cup F$. Then, Q is a partial rainbow matching of size $n-2$, since it excludes the matchings F_{m+1} and F_n. We shall augment Q with edges from F_{m+1} and F_n respectively, having endpoints in Y and U^*.

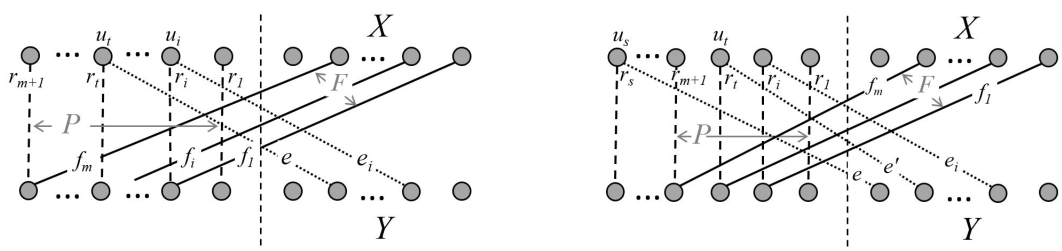
Claim 4. If $F_i \in \mathcal{F}'$, then the size of the set $\{e \in F_i^Y : e \cap (\cup_{j=1}^m r_j) \neq \emptyset\}$ is at least 2.

Proof. Let U^i be the set of endpoints in $U \setminus X$ of the edges in F_i^Y. Note that $|U^i| \geq \ell - 1$ (Corollary 2.3), $U^i \subset U^*$ (since $F_i \in \mathcal{F}'$), and $|U^*| = \ell + d$. Recall that for each edge $r_j \in R \cap F_j$ its endpoint in $U \setminus X$ was denoted u_j. Since $|U^* \setminus \{u_1, \ldots, u_m\}| = \ell + d - m = \ell + d - (d+3) = \ell - 3$, the claim follows. □

There are two sub-cases to consider: (2a) $F_{m+1} \in \mathcal{F}'$, and (2b) $F_{m+1} \in \mathcal{F}''$.

(2a) Assume $F_{m+1} \in \mathcal{F}'$. By Claim 4, there exists and edge $e \in F_{m+1}$ connecting a vertex in $Y \setminus \{y_{m+1}\}$ with some u_t, which is the endpoint in U of some $r_t \in P \setminus \{r_{m+1}\}$. Since $m = d+3$ and $|P| = m+1 = d+4$, at least four of the edges in P are in R' (actually, three are enough in this case). For at least one of these four edges, say r_i, its corresponding e_i (the edge of F_n^Y meeting r_i in U) avoids both endpoints of e. Then, $Q \cup \{e, e_i\}$ is a rainbow matching of size n (Figure 4(a)).

(2b) Assume $F_{m+1} \in \mathcal{F}''$ and let F_{m+1}^Y be the subset of F_{m+1} consisting of edges having one endpoint in Y and one endpoint in $U \setminus X$. By Claim 2, $|F_{m+1}^Y| \geq \lceil n/2 \rceil$. Since $|R \setminus R'| \leq n - 1 - (\lceil n/2 \rceil + 2) = \lfloor n/2 \rfloor - 3$, there is an edge $e \in F_{m+1}^Y$ sharing an endpoint with an edge $r_s \in R'$. Assume first that $s \in \{1, \ldots, m\}$. As in the previous paragraph, there exists e_i disjoint from r_s and e, so that $Q \cup \{e, e_i\}$ is a rainbow matching of size n. Now assume that $s \notin \{1, \ldots, m\}$ and let again e be the edge of F_{m+1} sharing an endpoint with r_s. Since $r_s \in R'$, there exists, by Claim 4, an edge $e' \in F_s^Y$, disjoint from e, sharing an endpoint with some r_t with $t \in \{1, \ldots, m\}$. Since $|P \cap R'| \geq 4$, there exists an edge $e_i \in F_n^Y$, avoiding both endpoints of e' and the endpoint of e in Y, such that $u_i \in \{u_1, \ldots, u_{m+1}\}$. Then, $Q \setminus \{r_t\} \cup \{e, e', e_i\}$ is a rainbow matching of size n (Figure 4(b)). This completes the proof.



(a) (b)

FIGURE 4

□

3. Related conjectures on matchings in 3-partite hypergraphs

A rainbow matching for sets F_i of edges in a graph is in fact a matching in a 3-uniform hypergraph, with a special set of vertices, each representing a set F_i. A hypergraph is called *simple* if it does not contain repeating edges. If we want to indicate explicitly that a hypergraph is not necessarily simple, we say that it is a *multihypergraph*. Given a hypergraph H and a set S of vertices, let $\Delta_H(S) = \max_{s \in S} deg_H(s)$ and $\delta_H(S) = \min_{s \in S} deg_H(s)$. Let $\nu(H)$ be the maximal size of a matching in H.

Re-phrased in hypergraph terminology, Conjecture 1.2 may be substantially generalized.

Conjecture 3.1. *Let H be a simple tripartite hypergraph with sides A, B, C. If $\delta_H(A) \geq \Delta_H(B \cup C)$ then*

$$\nu(H) \geq \frac{\delta_H(A) - 1}{\delta_H(A)}|A|.$$

This strengthens Conjecture 1.4, and would also imply generalizations of the following theorem of Drisko:

Theorem 3.2. [9] *$2n-1$ matchings of size n in a bipartite graph have a partial rainbow matching of size n.*

To obtain this theorem from the conjecture, duplicate the set $V \setminus A$, calling x' the duplicate of every vertex x, and adding, for every edge axy, the edge $ax'y'$. In fact, it is possible that Conjecture 3.1 may be true for all 3-uniform hypergraphs in which every edge meets the special set A at precisely one vertex. This case of the conjecture would imply the following conjecture of Barat, Gyarfas and Sarkozy:

Conjecture 3.3. [5] *$2n$ matchings of size n in a general graph have a partial rainbow matching of size n.*

In [3] the following was proved:

Theorem 3.4. *Let H be an r-uniform hypergraph in which there is a set A of vertices such that every edge meets A at precisely one vertex. For every subset A' of A let $K[A'] = \{f \subseteq V \setminus A \mid f \cup \{a\} \in H \text{ for some } a \in A'\}$. If $\nu^*(K[A']) > (r-1)(|A'|-1)$ for every subset A' of A then $\nu(H) = |A|$.*

Here $\nu^*(K)$ denotes the fractional matching number of the hypergraph K, which is defined as

$$\nu^*(K) = \max\{\sum_{e \in E(K)} f(e) \mid f: \ E(K) \to \mathbb{R}^+, \sum_{v \in e} f(e) \leq 1 \ \text{ for every } v \in V(K)\}.$$

Using this theorem the following can be proved:

Theorem 3.5. *Let H be a 3-uniform hypergraph in which there is a set A of vertices such that every edge meets A at precisely one vertex. If $\delta_H(A) \geq \Delta_H(V \setminus A)$ then $\nu(H) \geq \frac{|A|}{2}$.*

Here topology can possibly go a bit further. In the case that H is 3-partite with A being one of the sides, Theorem 3.5 follows directly from the result of [1] that in a 3-partite hypergraph $\tau \leq 2\nu$. In [10, 11] a characterization of the equality case in this theorem was proved, from which it follows that if $\delta_H(A) > 2$ then in case of equality in Theorem 3.5 the hypergraph is not simple: in fact every edge repeats at least $\frac{\delta_H(A)}{2}$ times. This poses the following challenge:

Use the methods of [10, 11] *to prove that if H is simple and 3-partite, with A one of its sides and $\delta_H(A) > 2$ and $\delta_H(A) \geq \Delta_H(V \setminus A)$, then $\nu(H) \geq \frac{|A|(1+\epsilon)}{2}$ for some positive number ϵ.*

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