GROUPS QUASI-ISOMETRIC TO RAAG'S

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Abstract. We characterize groups quasi-isometric to a right-angled Artin group G with finite outer automorphism group. In particular all such groups admit a geometric action on a CAT(0) cube complex that has an equivariant "fibering" over the Davis building of G.

1. Introduction

Overview. In this paper we will study right angled Artin groups (RAAG's). Like other authors, our motivation for considering these groups stems from the fact that they are an easily defined yet remarkably rich class of objects, exhibiting interesting features from many different vantage points: algebraic structure (subgroup structure, automorphism groups) [Dro87, Ser89, Lau95, CCV07], finiteness properties [BB97, BM01], representation varieties [KM98], $CAT(0)$ geometry [CK00], cube complex geometry [Wis11, HW08], and coarse geometry [Wis96, BM00, BKS08a, BN08, BJN10, Hua14b, Hua14a]. Further impetus for studying RAAG's comes from their role in the theory of special cube complexes, which was a key ingredient in Agol's spectacular solution of Thurston's virtual Haken and virtual fibered conjectures [AGM13, Wis11, HW08, Sag95, KM12].

Our focus here is on quasi-isometric rigidity, which is part of Gromov's program for quasi-isometric classification of groups and metric spaces. In this paper we build on [BKS08b, BKS08a, Hua14b, Hua14a], which analyzed the structure of individual quasi-isometries $G \to G$, where G is a RAAG with finite outer automorphism group. Our main results are a structure theorem for groups of quasi-isometries (more precisely quasi-actions), and a characterization of finitely generated groups quasi-isometric to such RAAG's. Both are formulated using a geometric description in terms of Caprace-Sageev restriction quotients [CS11] and the Davis building [Dav98].

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Background. Prior results on quasi-isometric classification of RAAG's may be loosely divided into two types: internal quasi-isometry classification among (families of) RAAG's, and quasi-isometry rigidity results characterizing arbitrary finitely generated groups quasi-isometric to a given RAAG. In the former category, it is known that to classify RAAG's up to quasi-isometry, it suffices to consider the case when the groups are 1-ended and do not admit any nontrivial direct product decomposition, or equivalently, when their defining graphs are connected, contain more than one vertex, and do not admit a nontrivial join decomposition ([Hua14a, Theorem 2.9], [PW02, KKL98]). This covers, for instance, the classification up to quasi-isometry of RAAG's that may be formed inductively by taking products or free products, starting from copies of $\mathbb{Z}$. Beyond this, internal classification is known for RAAG's whose defining graph is a tree [BN08] or a higher dimension analog [BJN10], or when the outer automorphism group is finite [Hua14a, BKS08a]. General quasi-isometric classification results in the literature are much more limited; if H is a finitely generated group quasi-isometric to a RAAG G then:

(i) If G is free or free abelian, H is virtually free or free abelian, respectively [Sta68, Dun85, Bas72, Gro81a].
(ii) If $G = F_k \times \mathbb{Z}^\ell$, then H is virtually $F_k \times \mathbb{Z}^\ell$ [Why10].
(iii) If the defining graph of G is a tree, then H is virtually a RAAG having a tree as defining graph [BN08, KL97a].
(iv) If G is a product of free groups, then H acts geometrically on a product of trees [Ahl02, KKL98, MSW03].

Unlike (i)-(iii), which give characterizations up to commensurability, the characterization in (iv) only asserts the existence of an action on a good geometric model; the stronger commensurability assertion is false, in view of examples of Wise and Burger-Mozes [Wis96, BM00].

The setup. We now recall some terminology and notation; see Section 3 for more detail.

If Γ is a finite simplicial graph with vertex set $V(\Gamma)$, we denote the associated right-angled Artin group by $G(\Gamma)$. This is the fundamental group of the Salvetti complex $S(\Gamma)$, a nonpositively curved cube complex that may be constructed by choosing a pointed unit length circle $(S^1_v, \star_v)$ for every vertex $v \in V(\Gamma)$, forming the pointed product torus $\prod_v (S^1_v, \star_v)$, and passing to the union of the product subtori corresponding to the cliques (complete subgraphs) in Γ. The clique subtori are the *standard tori* in $S(\Gamma)$.

We denote the universal covering by $X(\Gamma) \to S(\Gamma)$; here $X(\Gamma)$ is a CAT(0) cube complex on which $G(\Gamma)$ acts geometrically by deck transformations. The inverse image of a standard torus in $S(\Gamma)$ under the universal covering $X(\Gamma) \to S(\Gamma)$ breaks up into connected components; these are the *standard flats* in $X(\Gamma)$ which we partial order by inclusion. Note that we include standard tori and standard flats of dimension 0.

The poset of standard flats in $X(\Gamma)$ turns out to be crucial to our story. Using it one may define a locally infinite CAT(0) cube complex $|\mathcal{B}|(\Gamma)$ whose cubes of dimension $k \geq 0$ are indexed by inclusions $F_1 \subset F_2$, and F_1, F_2 are standard flats where $\dim F_2 = \dim F_1 + k$. Elements of the 0-skeleton $|\mathcal{B}|^{(0)}(\Gamma)$ correspond to the trivial inclusions $F \subset F$ where F is a standard flat, so we will identify $|\mathcal{B}|^{(0)}(\Gamma)$ with the collection of standard flats, and define the rank of a vertex of $|\mathcal{B}|(\Gamma)$ to be the dimension of the corresponding standard flat; in particular we may identify the 0-skeleton $X^{(0)}(\Gamma)$ with the set of rank 0 vertices of $|\mathcal{B}|^{(0)}$. Since $G(\Gamma) \curvearrowright X(\Gamma)$ preserves the collection of standard flats, there is an induced action $G(\Gamma) \curvearrowright |\mathcal{B}|(\Gamma)$ by cubical isomorphisms. The above description is a slight variation on the original construction of the same object given by Davis, in which one views $|\mathcal{B}|(\Gamma)$ as the Davis realization of a certain right-angled building $\mathcal{B}(\Gamma)$ associated with $G(\Gamma)$, where the apartments of $\mathcal{B}(\Gamma)$ are modelled on the right-angled Coxeter group $W(\Gamma)$ with defining graph Γ; see [Dav98] and Section 3. By abuse of terminology we will refer to this cube complex as the *Davis building associated with* $G(\Gamma)$; it has been called the modified Deligne complex in [CD95b] and flat space in [BKS08b].

The following lemma is not difficult to prove.

Lemma 1.1.

- *Every isomorphism $|\mathcal{B}|^{(0)}(\Gamma) \to |\mathcal{B}|^{(0)}(\Gamma)$ of the poset of standard flats extends to a unique cubical isomorphism $|\mathcal{B}|(\Gamma) \to |\mathcal{B}|(\Gamma)$ (Section 3.4).*
- *Every cubical isomorphism of $|\mathcal{B}| \to |\mathcal{B}|$ induces a poset isomorphism $|\mathcal{B}|^{(0)} \to |\mathcal{B}|^{(0)}$ (Lemma 3.15).*
- *A bijection $\phi^{(0)} : |\mathcal{B}|^{(0)}(\Gamma) \supset X^{(0)}(\Gamma) \to X^{(0)}(\Gamma) \subset |\mathcal{B}|^{(0)}(\Gamma)$ induces/extends to a poset isomorphism $|\mathcal{B}|^{(0)}(\Gamma) \to |\mathcal{B}|^{(0)}(\Gamma)$ iff it is flat-preserving in the sense that for every standard flat $F_1 \subset X(\Gamma)$, the 0-skeleton $F_1^{(0)}$ is mapped bijectively by $\phi^{(0)}$ onto the 0-skeleton of some standard flat F_2 (Section 5.1).*

Remark 1.2. We caution the reader that a cubical isomorphism $|\mathcal{B}|(\Gamma) \to |\mathcal{B}|(\Gamma)$ need not arise from an isomorphism $\mathcal{B}(\Gamma) \to \mathcal{B}(\Gamma)$ of the right-angled building.

Rigidity and flexibility. We now fix a finite graph Γ such that the outer automorphism group $\mathrm{Out}(G(\Gamma))$ is finite; by work of [CF12, Day12], one may view this as the generic case. The reader may find it helpful to keep in mind the case when Γ is a pentagon.

Since there is no ambiguity in Γ we will often suppress it in the notation below.

It is known that in this case $X = X(\Gamma)$ is not quasi-isometrically rigid: there are quasi-isometries that are not at finite sup distance from isometries, and there are finitely generated groups H that are quasi-isometric to X, but do not admit geometric actions on X (Corollary 6.12). On the other hand, quasi-isometries exhibit a form of partial rigidity that is captured by the building $|\mathcal{B}|$:

Theorem 1.3 ([Hua14a, BKS08a])**.** *Suppose* $\mathrm{Out}(G(\Gamma))$ *is finite and* $G(\Gamma) \not\simeq \mathbb{Z}$*. If* $\phi : X^{(0)} \to X^{(0)}$ *is an* (L, A)*-quasi-isometry, then there is a unique cubical isomorphism* $|\mathcal{B}| \to |\mathcal{B}|$ *such that associated flat-preserving bijection* $\bar{\phi} : X^{(0)} \to X^{(0)}$ *is at finite sup distance from* ϕ*, and moreover*

$$d(\bar{\phi}, \phi) = \sup\{v \in X^{(0)} \mid d(\bar{\phi}(v), \phi(v))\} < D = D(L, A)\,.$$

By the uniqueness assertion, we obtain a cubical action $\mathrm{QI}(X) \curvearrowright |\mathcal{B}|$ *of the quasi-isometry group of* X *on* $|\mathcal{B}|$*.*

We point out that the partial rigidity statement of the theorem does not hold for general RAAG's: it only holds for the RAAG's covered by the theorem in [Hua14a].

The main results. We will produce good geometric models quasi-isometric to $X(\Gamma)$ that are simultaneously compatible with group actions, the underlying building $|\mathcal{B}|$, and cubical structure. The key idea for expressing this is:

Definition 1.4. A cubical map $q : Y \to Z$ between CAT(0) cube complexes (see Definition 3.4) is a *restriction quotient* if it is surjective, and the point inverse $q^{-1}(z)$ is a convex subset of Y for every $z \in Z$.

It turns out that restriction quotients as defined above are essentially equivalent to the class of mappings introduced by Caprace-Sageev [CS11] with a different definition (see Section 4 for the proof that the definitions are equivalent). Restriction quotients $Y \to |\mathcal{B}|$ provide a means to "resolve" or "blow-up" the locally infinite building $|\mathcal{B}|$ to a locally finite CAT(0) cube complex.

Theorem 1.5. *(See Section 3 for definitions.) Let $H \curvearrowright X$ be a quasi-action of an arbitrary group on $X = X(\Gamma)$, where* $\mathrm{Out}(G(\Gamma))$ *is finite and $G(\Gamma) \not\simeq \mathbb{Z}$. Then there is an H-equivariant restriction quotient $H \curvearrowright Y \xrightarrow{q} H \curvearrowright |\mathcal{B}|$ where:*

- *(a) $H \curvearrowright |\mathcal{B}|$ is the cubical action arising from the quasi-action $H \curvearrowright X$ using Theorem 1.5, and $H \curvearrowright Y$ is a cubical action.*
- *(b) The point inverse $q^{-1}(v)$ of every rank k vertex $v \in |\mathcal{B}|^{(0)}$ is a copy of $\mathbb{R}^k$ with the usual cubulation.*
- *(c) $H \curvearrowright X$ is quasiconjugate to the cubical action $H \curvearrowright Y$.*

Theorem 1.6. *If $|\,\mathrm{Out}(G(\Gamma))| < \infty$ and $G(\Gamma) \not\simeq \mathbb{Z}$, then a finitely generated group H is quasi-isometric to $G(\Gamma)$ iff there is an H-equivariant restriction quotient $H \curvearrowright Y \xrightarrow{q} H \curvearrowright |\mathcal{B}|$ where*

- *(a) $H \curvearrowright Y$ is a geometric cubical action.*
- *(b) $H \curvearrowright |\mathcal{B}|$ is cubical.*
- *(c) The point inverse $q^{-1}(v)$ of every rank k vertex $v \in |\mathcal{B}|^{(0)}$ is a copy of $\mathbb{R}^k$ with the usual cubulation.*

Remark 1.7. In fact the restriction quotient $Y \to |\mathcal{B}|$ in Theorems 1.5 and 1.6 has slightly more structure, see Theorem 6.4.

In particular, we have:

Corollary 1.8. *Any group quasi-isometric to G is cocompactly cubulated, i.e. it has a geometric cubical action on a* CAT(0) *cube complex.*

One may compare Theorem 1.6 with rigidity theorems for symmetric spaces or products of trees, which characterize a quasi-isometry class of groups by the existence of a geometric action on a model space of a specified type [Sul81, Gro81b, Tuk86, Pan89, KL97b, Sta68, Dun85, KKL98, MSW03, Ahl02]. As in the case of products of trees — and unlike the case of symmetric spaces — there are finitely generated groups H as in Theorem 1.6 which do not admit a geometric action on the original model space X, so one is forced to pass to a different space Y [BKS08a, Hua14a]. Also, Theorems 1.5 and 1.6 fail for general

RAAG's, for instance for free abelian groups of rank ≥ 2, and for products of nonabelian free groups $\prod_{1\leq j\leq k} G_j$, for $k \geq 1$.

The quasi-isometry invariance of the existence of a cocompact cubulation as asserted in Corollary 1.8 is false in general. Some groups quasi-isometric to $\mathbb{H}^2 \times \mathbb{R}$ admit a cocompact cubulation, while others are not virtually $CAT(0)$ [BH99]. Combining [Lee95], [BN08] and [HP], it follows that there is a pair of quasi-isometric $CAT(0)$ graph manifold groups, one of which is the fundamental group of a compact special cube complex, while the other is not virtually cocompactly cubulated. The quasi-isometry invariance of cocompact cubulations fails to hold even among RAAG's: for $n > 1$ there are groups quasi-isometric $\mathbb{R}^n$ that are not cocompactly cubulated [Hag14].

Earlier cocompact cubulation theorems in the spirit of Corollary 1.8 include the cases of groups quasi-isometric to trees, products of trees, and hyperbolic k-space $\mathbb{H}^k$ for $k \in \{2, 3\}$ [Sta68, Dun85, KKL98, MSW03, Ahl02, GMRS98, KM12, BW12]. It is worth noting that each case requires different ingredients that are specific to the spaces in question.

Further results. We briefly discuss some further results here, referring the reader to the body of the paper for details.

One portion of the proof of Theorem 1.5 has to do with the geometry of restriction quotients, and more specifically, restriction quotients with a right-angled building as target. We view this as a contribution to cube complex geometry, and to the geometric theory of graph products; beyond the references mentioned already, our treatment has been influenced by the papers of Januszkiewicz-Swiatkowski and Haglund [JŚ01, Hag08]. The main results on this are:

(a) We show in Section 4 that restriction quotients may be characterized in several different ways.
(b) We show that having a restriction quotient $q : Y \to Z$ is equivalent to knowing certain "fiber data" living on the target complex Z.
(c) When $|\mathcal{B}|$ is the Davis realization of a right-angled building $\mathcal{B}$ and $Y \to |\mathcal{B}|$ is a restriction quotient whose fibers are copies of $\mathbb{R}^k$ with dimension specified as in Theorems 1.5 and 1.6, the fiber data in (b) may be distilled even more, leading to what we call "blow-up data".

As by-products of (a)-(c), we obtain:

- A characterization of the quasi-actions $H \curvearrowright X(\Gamma)$ that are quasiconjugate to isometric actions $H \curvearrowright X(\Gamma)$ (Section 6.2).
- A characterization of the restriction quotients $Y \to |\mathcal{B}|$ satisfying (b) of Theorem 1.5 for which Y is quasi-isometric to X (Corollary 6.5 and Theorem 6.6).
- A proof of uniqueness of the right-angled building modelled on the right-angled Coxeter group $W(\Gamma)$ with defining graph Γ, with countably infinite rank 1 residues (Corollary 5.23).
- Applications to more general graph products (Theorem 8.8).

It follows from [KL01] that a finitely generated group H quasi-isometric to a symmetric space of noncompact type X admits an epimorphism $H \to \Lambda$ with finite kernel, where Λ is a cocompact lattice in the isometry group $\mathrm{Isom}(X)$. In contrast to this, we have the following result, which is inspired by [MSW03, Theorem 9, Corollary 10]:

Theorem 1.9. *(See Section 6.2) Suppose G is a RAAG with $|\mathrm{Out}(G)| < \infty$. Then there are finitely generated groups H and H' quasi-isometric to G that do not admit discrete, virtually faithful cocompact representations into the same locally compact topological group.*

Open questions. As mentioned above, Corollary 1.8 may be considered part of the quasi-isometry classification program for finitely generated groups. The leads to:

Question 1.10. If $\mathrm{Out}(G(\Gamma))$ is finite, what is the commensurability classification of groups quasi-isometric to $G(\Gamma)$? Are they all commensurable to $G(\Gamma)$? What about cocompact lattices in the automorphism group of $X(\Gamma)$?

For comparison, we recall that any group quasi-isometric to a tree is commensurable to a free group, but there are groups quasi-isometric to a product of trees that contain no nontrivial finite index subgroups, and are therefore not commensurable to a product of free groups [Wis96, BM00].

We mention that Theorem 1.6 will be used in forthcoming work to answer Question 1.10 in certain cases.

Discussion of the proofs. Before sketching the arguments for Theorems 1.5 and 1.6, we first illustrate them in the tautological case when $H = G$ and the quasi-action is the deck group action $G \curvearrowright X$. In this case we cannot take $Y = X$, as there is no H-equivariant restriction quotient $H \curvearrowright X \to H \curvearrowright |\mathcal{B}|$ satisfying (c) of Theorem 1.6. Instead, we use a different geometric model.

Definition 1.11 (Graph products of spaces [Hag08])**.** For every vertex $v \in V(\Gamma)$, choose a pointed geodesic metric space $(Z_v, \star_v)$. The Γ-*graph product of* $\{(Z_v, \star_v)\}_{v\in V(\Gamma)}$ is obtained by forming the product $\prod_v (Z_v, \star_v)$, and passing to the union of the subproducts corresponding to the cliques in Γ. We denote this by $\prod_\Gamma (Z_v, \star_v)$. When the Z_v's are nonpositively curved, then so is the graph product [Hag08, Corollary 4.6].

There are three graph products that are useful here:

(1) The Salvetti complex $S(\Gamma)$ is the graph product $\prod_\Gamma (S^1_v, \star_v)$, where $(S^1_v, \star_v)$ is a pointed unit circle.
(2) For every $v \in V(\Gamma)$, let $(L_v, \star_v)$ be a pointed lollipop, i.e. L_v is the wedge of the unit circle S^1_v and a unit interval I_v, and the basepoint $\star_v \in L_v$ is the vertex of valence 1. Then the graph product $\prod_\Gamma (L_v, \star_v)$ is the *exploded Salvetti complex* $S_e = S_e(\Gamma)$. We denote its universal covering by $X_e \to S_e$.
(3) If $(Z_v, \star_v)$ is a unit interval and $\star_v \in Z_v$ is an endpoint for every $v \in V(\Gamma)$, then the graph product $\prod_\Gamma (Z_v, \star_v)$ is the *Davis chamber*, i.e. it is a copy of the Davis realization $|c|$ of a chamber c in $|\mathcal{B}|(\Gamma)$; for this reason we will denote it by $|c|_\Gamma$.

By collapsing the interval I_v in each lollipop L_v to a point, we obtain a cubical map $S_e \to S$; this has contractible point inverses, and is therefore a homotopy equivalence. If we collapse the circles $S^1_v \subset L_v$ to points instead, we get a map $S_e \to |c|_\Gamma$ to the Davis chamber whose point inverses are closed, locally convex tori. The point inverses of the composition $X_e \to S_e \to |c|_\Gamma$ cover the torus point inverses of $S_e \to |c|_\Gamma$, and their connected components form a "foliation" of X_e by flat convex subspaces. It turns out that by collapsing X_e along these flat subspaces, we obtain a copy of $|\mathcal{B}|$, and the quotient map $X_e \to |\mathcal{B}|$ is a restriction quotient $X_e \to |\mathcal{B}|$. Alternately, one may take the collection $\mathcal{K}$ of hyperplanes of X_e dual to edges $\sigma \subset X_e$ whose projection under $X_e \to |c|_\Gamma$ is an edge, and form the restriction quotient using the Caprace-Sageev construction.

Remark 1.12. The exploded Salvetti complex and the restriction quotient $X_e \to |\mathcal{B}|$ were discussed in [BKS08a] in the 2-dimensional case, using an ad hoc construction that was initially invented for "ease of visualization". However, the authors were unaware of the general description above, and the notion of restriction quotient had not yet appeared.

We now discuss the proofs of Theorem 1.5 and the forward direction of 1.6.

The forward direction of Theorem 1.6 reduces to Theorem 1.5, by the standard observation that a quasi-isometry $H \to G \overset{qi}{\simeq} X$ allows us to quasiconjugate the left translation action $H \curvearrowright H$ to a quasi-action $H \curvearrowright X$. Therefore we focus on Theorem 1.5.

Let $H \curvearrowright X$ be as in Theorem 1.5. By a bounded perturbation, we may assume that this quasi-action preserves the 0-skeleton $X^{(0)} \subset X$. Applying Theorem 1.3, we may further assume that we have an action $H \curvearrowright X^{(0)}$ by flat-preserving quasi-isometries. The fact the we have an action, rather than just a quasi-action, comes from the uniqueness in Theorem 1.5; this turns out to be a crucial point in the sequel.

Before proceeding further, we remark that if one is only interested in Corollary 1.8 as opposed to the more refined statement in Theorem 1.6, then there is an alternate approach using wallspaces. This is treated in Sections 9 and 10.

Given a standard geodesic $\ell \subset X$, the parallel set $P_\ell \subset X$ decomposes as a product $\mathbb{R}_\ell \times Q_\ell$, where $\mathbb{R}_\ell$ is a copy of $\mathbb{R}$; likewise there is a product decomposition of 0-skeleta $P_\ell^{(0)} \simeq \mathbb{Z}_\ell \times Q_\ell^{(0)}$. One argues that the action $H \curvearrowright X^{(0)}$ permutes the collection of 0-skeleta $\{P_\ell^{(0)}\}_\ell$, and that for any ℓ, the stabilizer $\mathrm{Stab}(P_\ell^{(0)}, H)$ of $P_\ell^{(0)}$ in H acts on $P_\ell^{(0)} \simeq \mathbb{Z}_\ell \times Q_\ell^{(0)}$ preserving the product structure. We call the action $\mathrm{Stab}(P_\ell^{(0)}, H) \curvearrowright \mathbb{Z}_\ell$ a *factor action*. The factor actions are by bijections with quasi-isometry constants bounded uniformly independent of ℓ.

It turns out that factor actions play a central role in the story. For instance, when the action $H \curvearrowright X^{(0)}$ is the restriction of an action $H \curvearrowright X$ by cubical isometries, then the factor actions $H_{[\ell]} \curvearrowright \mathbb{Z}_{[\ell]}$ are also actions by isometries. In general the factor actions can be arbitrary: up to isometric conjugacy, any action $A \curvearrowright \mathbb{Z}$ by quasi-isometries with uniform constants can arise as a factor action for some action as in Theorem 1.5. A key step in the proof is to show that such actions have a relatively simple structure:

Proposition 1.13 (Semiconjugacy)**.** *Let $U \overset{\rho_0}{\curvearrowright} \mathbb{Z}$ be an action of an arbitrary group by (L, A)-quasi-isometries. Then there is an isometric action $U \overset{\rho_1}{\curvearrowright} \mathbb{Z}$ and surjective equivariant (L', A')-quasi-isometry*

$$U \overset{\rho_0}{\curvearrowright} \mathbb{Z} \longrightarrow U \overset{\rho_1}{\curvearrowright} \mathbb{Z}\,,$$

where L' and A' depend only on L and A.

The assumption that ρ_0 is an action, as opposed to a quasi-action, is crucial: if a group U has a nontrivial quasihomomorphism $\alpha : U \to \mathbb{R}$, then the translation quasi-action $U \overset{\hat{\alpha}}{\curvearrowright} \mathbb{R}$ defined by $\hat{\alpha}(u)(x) = x+\alpha(u)$ is quasiconjugate to a quasi-action on $\mathbb{Z}$, but not to an isometric action on $\mathbb{Z}$.

It follows immediately from the Proposition 1.13 that $U \overset{\rho_0}{\curvearrowright} \mathbb{Z}$ is quasiconjugate to an isometric action on the tree $\mathbb{R}$. In that respect Proposition 1.13 is similar to the theorem of Mosher-Sageev-Whyte about promoting quasi-actions on bushy trees to isometric actions on trees [MSW03, Theorem 1]. Since $\mathbb{R}$ is not bushy [MSW03, Theorem 1] does not apply, and indeed the example above shows that the assumption of bushiness is essential in that theorem.

Continuing with the proof of Theorem 1.5, Proposition 1.13 gives a good geometric model for the factor action $\mathrm{Stab}(P^{(0)}_\ell, H) \curvearrowright \mathbb{Z}_\ell$: we simply extend each isometry $\mathbb{Z}_\ell \to \mathbb{Z}_\ell$ to an isometry $\mathbb{R}_\ell \to \mathbb{R}_\ell$, thereby obtaining a cubical action $\mathrm{Stab}(P^{(0)}_\ell, H) \curvearrowright \mathbb{R}_\ell$. In vague terms, the remainder of the proof is concerned with combining these cubical models into models for the fibers of a restriction quotient $Z \to |\mathcal{B}|$, in an H-equivariant way. This portion of the proof is covered by more general results about restriction quotients, see (b)-(c) in the subsection on further results above.

Organization of the paper. The paper is divided into three relatively independent parts. We first outline the content of each part, then give suggestions for readers who want to focus on one particular part.

A summary of notation can found in Section 2. Section 3 contains some background material on quasi-actions, $CAT(0)$ cube complexes, RAAG's and buildings. One can proceed directly to later sections with Section 2 and Section 3 as references.

The main part of the paper is Section 4 to Section 7, where we prove Theorem 1.6. In Section 4 we discuss restriction quotients, showing how to construct a restriction quotient $Y \to Z$ starting from the target

Z and an admissible assignment of fibres to the cubes of Z. Then we discuss equivariance properties and the coarse geometry of restriction quotients.

In Section 5, we introduce blow-ups of buildings based on Section 4. These are restriction quotients $Y \to |\mathcal{B}|$ where the target is a right-angled building and the fibres are Euclidean spaces of varying dimension. We motivate our construction in Section 5.1 and Section 5.2. Blow-ups of buildings are constructed in Section 5.3. Several properties of them are discussed in Section 5.4 and Section 5.5. We incorporate a group action into our construction in Section 5.6.

In Section 6.1, we apply the construction in Section 5.6 to RAAG's and prove Theorem 1.6 modulo Theorem 1.13, which is postponed until Section 7. In Section 6.2 we answer several natural questions motivated by Theorem 1.6, and prove Theorem 1.9.

The second part of the paper is Section 8. We discuss an alternative construction of blow-ups of buildings. In certain cases, this is more general than the construction in Section 5. We also discuss several applications of this construction to graph products.

The third part of the paper is Section 9 and Section 10. Using wallspaces we give an alternative way to cubulate groups quasi-isometric to RAAG's, and prove a weaker version of Theorem 1.6.

The reader can proceed directly to Section 8 with reference to Section 3.4 and Definition 5.35. The reader can also start with Section 9 with reference to Section 3.3, and come back to Section 7 when needed.

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Contents

2. Index of notation

- $\mathcal{B}$: A combinatorial building (Section 3.4).
- $|\mathcal{B}|$: The Davis realization of a building (Section 3.4).
- Chambers in the combinatorial building $\mathcal{B}$ are c, c', d.
- $|c|_\Gamma$: the Davis chamber (the discussion after Definition 1.11, Section 3.4).
- S^r: the collection of all spherical residues in the building $\mathcal{B}$.
- $\text{proj}_\mathcal{R} : \mathcal{B} \to \mathcal{R}$: the nearest point projection from $\mathcal{B}$ to a residue $\mathcal{R}$ (Section 3.4).
- $\Lambda_\mathcal{B}$: the collection of parallel classes of rank 1 residues in the combinatorial building $\mathcal{B}$. We also write Λ when the building $\mathcal{B}$ is clear (Section 5.3).
- T: a type map which assigns each residue of $\mathcal{B}$ a subset of $\Lambda_\mathcal{B}$ (Section 5.3).
- β: a branched line (Section 9).
- CCC: the category of nonempty CAT(0) cube complexes with morphisms given by convex cubical embeddings.
- P_C: the parallel set of a closed convex subset of a CAT(0) space (Section 3.2).
- $W(\Gamma)$: The right-angled Coxeter group with defining graph Γ (Section 3.4).
- $G(\Gamma)$ the right-angled Artin group with defining graph Γ.
- $X(\Gamma) \to S(\Gamma)$ the universal covering of the Salvetti complex (Section 3.3)
- $X_e(\Gamma) \to S_e(\Gamma)$ the universal covering of the exploded Salvetti complex (after Definition 1.11 and Section 5.1). We also write $X_e \to S_e$ when the graph Γ is clear.

- $\mathcal{P}(\Gamma)$: the extension complex (Definition 3.5).
- $X \to X(\mathcal{K})$: the restriction quotient arising from a set $\mathcal{K}$ of hyperplanes in a CAT(0) cube complex (Definition 4.1).
- $Lk(x, X)$ or $Lk(c, X)$: the link of a vertex x or a cell c in a polyhedral complex X.
- $\Gamma_1 \circ \Gamma_2$: the join of two graphs.
- $K_1 * K_2$: the join of two simplicial complexes.

3. Preliminaries

3.1. **Quasi-actions.** We recall several definitions from coarse geometry.

Definition 3.1. An (L, A)-quasi-action of a group G on a metric space Z is a map $\rho : G \times Z \to Z$ so that $\rho(\gamma, \cdot) : Z \to Z$ is an (L, A) quasi-isometry for every $\gamma \in G$, $d(\rho(\gamma_1, \rho(\gamma_2, z)), \rho(\gamma_1\gamma_2, z)) < A$ for every $\gamma_1, \gamma_2 \in G$, $z \in Z$, and $d(\rho(e, z), z) < A$ for every $z \in Z$.

The action ρ is *discrete* if for any point $z \in Z$ and any $R > 0$, the set of all $\gamma \in G$ such that $\rho(\gamma, z)$ is contained in the ball $B_R(z)$ is finite; ρ is *cobounded* if Z coincides with a finite tubular neighbourhood of the "orbit" $\rho(G, z)$. If ρ is a discrete and cobounded quasi-action of G on Z, then the orbit map $\gamma \in G \to \rho(\gamma, z)$ is a quasi-isometry. Conversely, given a quasi-isometry between G and Z, it induces a discrete and cobounded action of G on Z.

Two quasi-actions ρ and ρ' are *equivalent* if there exists a constant D so that $d(\rho(\gamma), \rho'(\gamma)) < D$ for all $\gamma \in G$.

Definition 3.2. Let ρ and ρ' be quasi-actions of G on Z and Z' respectively, and let $\phi : Z \to Z'$ be a quasi-isometry. Then ρ is *quasiconjugate* to ρ' via ϕ if there is a D so that $d(\phi \circ \rho(\gamma), \rho'(r) \circ \phi) < D$ for all $\gamma \in G$.

3.2. CAT(0) **cube complexes.** We refer to [BH99] for background about CAT(0) spaces (Chapter II.1) and cube complexes (Chapter II.5), and [Sag95, Sag12] for CAT(0) cube complexes and hyperplanes.

A unit Euclidean n-cube is $[0, 1]^n$ with the standard metric. A *mid-cube* is the set of fixed points of a reflection with respect to some $[0, 1]$ factor of $[0, 1]^n$. A cube complex Y is obtained by taking a collection of unit Euclidean cubes and gluing them along isometric faces. The gluing metric on Y is CAT(0) if and only if Y is simply connected and the link of each vertex in Y is a flag simplicial complex ([Gro87]), in this case, Y is called a CAT(0) *cube complex*.

Let X be a CAT(0) space and let $C \subset X$ be a closed convex subset. Then there is a well-defined nearest point projection from X to C, which we denote by $\pi_C : X \to C$. Two convex subsets C_1 and C_2 are *parallel* if $d(\cdot, C_2)|_{C_1}$ and $d(\cdot, C_1)|_{C_2}$ are constant functions. In this case, the convex hull of C_1 and C_2 is isometric to $C_1 \times [0, d(C_1, C_2)]$.

For closed convex subset $C \subset X$, we define P_C, the *parallel set* of C, to be the union of all convex subsets of X which are parallel to C. If C has geodesic extension property, then P_C is also a closed convex subset and admits a canonical splitting $P_C \cong C \times C^{\perp}$ ([BH99, Chapter II.2.12]).

Suppose Y is a CAT(0) cube complex. Then two edges e and e' are parallel if and only if there exists sequences of edges $\{e_i\}_{i=1}^{n}$ such that $e_1 = e$, $e_n = e'$, and e_i, e_{i+1} are the opposite sides of a 2-cube in Y. For each edge $e \subset Y$. Let N_e be the union of cubes in Y which contain an edge parallel to e. Then N_e is a convex subcomplex of Y, moreover, N_e has a natural splitting $N_e \cong h_e \times [0, 1]$, where $[0, 1]$ corresponds to the e direction. The subset $h_e \times \{1/2\}$ is called the *hyperplane* dual to e, and N_e is called the *carrier* of this hyperplane. Each hyperplane is a union of mid-cubes, hence has a natural cube complex structure, which makes it a CAT(0) cube complex. The following are true for hyperplanes:

(1) Each hyperplane h is a convex subset of Y. Moreover, $Y \setminus h$ has exactly two connected components. The closure of each connected component is called a *halfspace*. Each halfspace is also a convex subset.
(2) Pick an edge $e \subset Y$. We identify e with $[0, 1]$ and consider the CAT(0) projection $\pi_e : Y \to e \cong [0, 1]$. Then $h = \pi_e^{-1}(1/2)$ is the hyperplane dual to e, and $\pi_e^{-1}([0, 1/2]), \pi_e^{-1}([1/2, 1])$ are two halfspaces associated with h. The closure of $\pi_e^{-1}((0, 1))$ is the carrier of h.

Let Y be a CAT(0) cube complex and let $l \in Y$ be a geodesic line (with respect to the CAT(0) metric) in the 1-skeleton of Y. Let $e \subset l$ be an edge and pick $x \in e$. We claim that if x is in the interior of e, then $\pi_l^{-1}(x) = \pi_e^{-1}(x)$. It is clear that $\pi_l^{-1}(x) \subset \pi_e^{-1}(x)$. Suppose $y \in \pi_e^{-1}(x)$. It follows from the splitting $N_e \cong h_e \times [0, 1]$ as above that the geodesic segment $\overline{xy}$ is orthogonal to l, i.e. $\angle_x(y, y') = \pi/2$ for any $y' \in l \setminus \{x\}$, thus $y \in \pi_l^{-1}(x)$.

The above claim implies $\pi_l^{-1}(x)$ is a convex subset for any $x \in l$. Moreover, the following lemma is true.

Lemma 3.3. *Let Y and l be as before. Pick an edge $e \subset Y$. If e is parallel to some edge $e' \subset l$, then $\pi_l(e) = e'$, otherwise $\pi_l(e)$ is a vertex of l.*

Now we define an alternative metric on the CAT(0) cube complex Y, which is called the l^1-metric. One can view the 1-skeleton of Y as a metric graph with edge length $= 1$, and this metric extends naturally to a metric on Y. The distant between two vertices in Y with respect to this metric is equal to the number of hyperplanes separating these two vertices.

A *combinatorial geodesic* in Y is an edge path in $Y^{(1)}$ which is a geodesic with respect to the l^1 metric. However, we always refer to the CAT(0) metric when we talk about a geodesic.

If Y is finite dimensional, the l^1 metric and the CAT(0) metric on Y are quasi-isometric ([CS11, Lemma 2.2]). In this paper, we will use the CAT(0) metric unless otherwise specified.

Definition 3.4. ([CS11, Section 2.1]) A cellular map between cube complexes is *cubical* if its restriction $\sigma \to \tau$ between cubes factors as $\sigma \to \eta \to \tau$, where the first map $\sigma \to \eta$ is a natural projection onto a face of σ and the second map $\eta \to \tau$ is an isometry.

3.3. **Right-angled Artin groups.** Pick a finite simplicial graph Γ, recall that $G(\Gamma)$ is the right-angled Artin group with defining graph Γ. Let S be a standard generating set for $G(\Gamma)$ and we label the vertices of Γ by elements in S. $G(\Gamma)$ has a nice Eilenberg-MacLane space $S(\Gamma)$, called the Salvetti complex (see [CD95a, Cha07]). Recall that $S(\Gamma)$ is the graph product $\prod_\Gamma(S^1_v, \star_v)$, where $(S^1_v, \star_v)$ is a pointed unit circle (see Definition 1.11).

The 2-skeleton of $S(\Gamma)$ is the usual presentation complex of $G(\Gamma)$, so $\pi_1(S(\Gamma)) \cong G(\Gamma)$. The 0-skeleton of $S(\Gamma)$ consists of one point whose link is a flag complex, so $S(\Gamma)$ is non-positively curved and $S(\Gamma)$ is an Eilenberg-MacLane space for $G(\Gamma)$ by the Cartan-Hadamard theorem ([BH99, Theorem II.4.1]).

The closure of each k-cell in $S(\Gamma)$ is a k-torus. Tori of this kind are called *standard tori.* There is a 1-1 correspondence between the k-cells (or standard torus of dimension k) in $S(\Gamma)$ and k-cliques in Γ. We define the *dimension* of $G(\Gamma)$ to be the dimension of $S(\Gamma)$.

Denote the universal cover of $S(\Gamma)$ by $X(\Gamma)$, which is a CAT(0) cube complex. Our previous labelling of vertices of Γ induces a labelling of the standard circles of $S(\Gamma)$, which lifts to a labelling of edges of $X(\Gamma)$.

A *standard k-flat* in $X(\Gamma)$ is a connected component of the inverse image of a standard k-torus under the covering map $X(\Gamma) \to S(\Gamma)$. When $k = 1$, we also call it a *standard geodesic*.

For each simplicial graph Γ, there is a simplicial complex $\mathcal{P}(\Gamma)$ called the *extension complex*, which captures the combinatorial pattern of how standard flats intersect each other in $X(\Gamma)$. This object was first introduced in [KK13]. We will define it in a slightly different way (see [Hua14a, Section 4.1] for more discussion).

Definition 3.5 (Extension complex)**.** The vertices of $\mathcal{P}(\Gamma)$ are in 1-1 correspondence with the parallel classes of standard geodesics in $X(\Gamma)$. Two distinct vertices $v_1, v_2 \in \mathcal{P}(\Gamma)$ are connected by an edge if and only if there is standard geodesic l_i in the parallel class associated with v_i $(i = 1, 2)$ such that l_1 and l_2 span a standard 2-flat. Then $\mathcal{P}(\Gamma)$ is defined to be the flag complex of its 1-skeleton, namely we build $\mathcal{P}(\Gamma)$ inductively from its 1-skeleton by filling a k-simplex whenever we see the $(k-1)$-skeleton of a k-simplex.

Since each complete subgraph in the 1-skeleton of $\mathcal{P}(\Gamma)$ gives rise to a collection of mutually orthogonal standard geodesics lines, there is a 1-1 correspondence between k-simplexes in $\mathcal{P}(\Gamma)$ and parallel classes of standard $(k+1)$-flats in $X(\Gamma)$. In particular, there is a 1-1 correspondence between maximal simplexes in $\mathcal{P}(\Gamma)$ and maximal standard flats in $X(\Gamma)$. Given standard flat $F \subset X(\Gamma)$, we denote the simplex in $\mathcal{P}(\Gamma)$ associated with the parallel class containing F by $\Delta(F)$.

Lemma 3.6. *Pick non-adjacent vertices $v, u \in \mathcal{P}(\Gamma)$ and let l_u be a standard geodesic such that $\Delta(l_u) = u$. Then for any standard geodesic line l with $\Delta(l) = v$, $\pi_{l_u}(l)$ is a vertex of l_u which does not depend on the choice of l in the parallel class. Thus we define this vertex to be the projection of v on l_u and denote it by $\pi_{l_u}(v)$.*

Proof. It follows from Lemma 3.3 that if two standard geodesic lines l_1 and l_2 are not parallel, then $\pi_{l_1}(l_2)$ is a vertex of l_2. Thus $\pi_{l_u}(l)$ is a vertex of l_u. Suppose l' is parallel to l, then there exists a chain of parallel standard geodesic lines $\{l_i\}_{i=1}^n$ such that $l_1 = l$, $l_n = l'$ and l_i, l_{i+1} are in the same standard 2-flat F_i. Let $l^\perp \subset F_i$ be a standard geodesic orthogonal to l_i and l_{i+1}. Since v and u are not adjacent in $\mathcal{P}(\Gamma)$, $l^\perp$ and l_u are not parallel, thus $\pi_{l_u}(l^\perp)$ is a point. It follows that $\pi_{l_u}(l_i) = \pi_{l_u}(l_{i+1})$, hence $\pi_{l_u}(l) = \pi_{l_u}(l')$. □

3.4. **Right-angled buildings.** We will follow the treatment in [Dav98, AB08, Ron09]. In particular, we refer to Section 1.1 to Section 1.3 of

[Dav98] for the definitions of chamber systems, galleries, residues, Coxeter groups and buildings. We will focus on right-angled buildings, i.e. the associated Coxeter group is right-angled, though most of the discussion below is valid for general buildings.

Let $W = W(\Gamma)$ be a right-angled Coxeter group with (finite) defining graph Γ. Let $\mathcal{B} = \mathcal{B}(\Gamma)$ be a right-angled building with the associated W-distance function denoted by $\delta : \mathcal{B} \times \mathcal{B} \to W$. We will also call $\mathcal{B}(\Gamma)$ a *right-angled* Γ*-building* for simplicity.

Let I be the vertex set of Γ. Recall that a subset $J \subset I$ is *spherical* if the subgroup of W generated by J is finite. Let S be the poset of spherical subsets of I (including the empty set) and let $|\mathsf{S}|_\Delta$ be the geometric realization of S, i.e. $|\mathsf{S}|_\Delta$ is a simplicial complex such that its vertices are in 1-1 correspondence to elements in S and its n-simplices are in 1-1 correspondence to $(n+1)$-chains in S. Note that $|\mathsf{S}|_\Delta$ is isomorphic the simplicial cone over the barycentric subdivision of the flag complex of Γ.

Recall that for elements $x \leq y$ in S, the *interval* I_{xy} between x and y is a poset consist of elements $z \in \mathsf{S}$ such that $x \leq z \leq y$ with the induced order from S. There is a natural simplicial embedding $|I_{xy}|_\Delta \hookrightarrow |\mathsf{S}|_\Delta$. Each $|I_{xy}|_\Delta$ is a simplicial cone over the barycentric subdivision of a simplex, thus can be viewed a subdivision of a cube into simplices. It is not hard to check the collection of all intervals in S gives rise to a structure of cube complex on $|\mathsf{S}|_\Delta$. Let $|\mathsf{S}|$ be the resulting cube complex, then $|\mathsf{S}|$ is $CAT(0)$.

A residue is *spherical* if it is a J-residue with $J \in \mathsf{S}$. Let S^r be the poset of all spherical residues in $\mathcal{B}$. For $x \in \mathsf{S}^r$ which comes from a J-residue, we define the *rank* of x to be the cardinality of J, and define a *type map* $t : \mathsf{S}^r \to \mathsf{S}$ which maps x to $J \in \mathsf{S}$. Let $|\mathsf{S}^r|_\Delta$ be the geometric realization of S^r, then the type map induces a simplicial map $t : |\mathsf{S}^r|_\Delta \to |\mathsf{S}|_\Delta$. For $x \in \mathsf{S}^r$, let S^r_x be the sub-poset made of elements in S^r which is $\geq x$. If x is of rank 0, then S^r_x is isomorphic to S, moreover, there is a natural simplicial embedding $|\mathsf{S}^r_x|_\Delta \to |\mathsf{S}^r|_\Delta$ and t maps the image of $|\mathsf{S}^r_x|_\Delta$ isomorphically onto $|\mathsf{S}|_\Delta$.

As before, the geometric realization of each interval in S^r is a subdivision of a cube into simplices. Moreover, the intersection of two intervals in S^r is also an interval. Thus one gets a cube complex $|\mathcal{B}|$ whose cubes are in 1-1 correspondence with intervals in S^r. $|\mathcal{B}|$ is called the *Davis realization* of the building $\mathcal{B}$ and $|\mathcal{B}|$ is a $CAT(0)$ cube complex by [Dav98]. Moreover, the above type map induces a cubical map $t : |\mathcal{B}| \to |\mathsf{S}|$. Let $\mathcal{R} \subset \mathcal{B}$ be a residue. Since $\mathcal{R}$ also has the structure of a building, there is an isometric embedding $|\mathcal{R}| \to |\mathcal{B}|$ between their Davis realizations. $|\mathcal{R}|$ is called a *residue* in $|\mathcal{B}|$.

In the special case when $\mathcal{B}$ is equal to the associated Coxeter group W, there is a natural embedding from the Cayley graph of W to $|\mathcal{B}|$ such that vertices of Cayley graph are mapped to vertices of rank 0 in $|\mathcal{B}|$. And $|\mathcal{B}|$ can be viewed as the first cubical subdivision of the cubical completion of the Cayley graph of W (the cubical completion means we attach an n-cube to the graph whenever there is a copy of the 1-skeleton of an n-cube inside the graph).

Each vertex of $|\mathcal{B}|$ corresponds to a J-residue in $\mathcal{B}$, thus has a well-defined rank. For a vertex x of rank 0, the space $|\mathsf{S}^r_x|_\Delta$ discussed in the previous paragraph induces a subcomplex $|\mathcal{B}_x| \subset |\mathcal{B}|$. Note that t maps $|\mathcal{B}_x|$ isomorphically onto $|\mathsf{S}|$. $|\mathcal{B}_x|$ is called a *chamber* in $|\mathcal{B}|$, and there is a 1-1 correspondence between chambers in $|\mathcal{B}|$ and chambers in $\mathcal{B}$. Let $|\mathcal{B}_x|$ and $|\mathcal{B}_y|$ be two chambers in $|\mathcal{B}|$. Since there is an apartment $\mathcal{A} \subset \mathcal{B}$ which contains both x and y, this induces an isometric embedding $|\mathcal{A}| \to |\mathcal{B}|$ whose image contains $|\mathcal{B}_x|$ and $|\mathcal{B}_y|$, here $|\mathcal{A}|$ is isomorphic to the Davis realization of the Coxeter group W. $|\mathcal{A}|$ is called an *apartment* in $|\mathcal{B}|$.

Definition 3.7. For $c_1, c_2 \in \mathcal{B}$, define $d(c_1, c_2)$ to be the minimal length of word in W (with respect to the generating set I) that represents $\delta(c_1, c_2)$. For any two residues $\mathcal{R}_1, \mathcal{R}_2 \in \mathcal{B}$, we define $d(\mathcal{R}_1, \mathcal{R}_2) = \min\{d(c, d) \mid c \in \mathcal{R}_1, d \in \mathcal{R}_2\}$. It turns out that for any $c \in \mathcal{R}_1$ and $d \in \mathcal{R}_2$ with $d(c, d) = d(\mathcal{R}_1, \mathcal{R}_2)$, $\delta(c, d)$ gives rise to the same element in W ([AB08, Chapter 5.3.2]), this element is defined to be $\delta(\mathcal{R}_1, \mathcal{R}_2)$.

Lemma 3.8. *$d(c_1, c_2) = 2d_{l^1}(c_1, c_2)$, here d_{l^1} means the l^1-distance in $|\mathcal{B}|$. Since c_1 and c_2 can be also viewed as vertex of rank 0 in $|\mathcal{B}|$, $d_{l^1}(c_1, c_2)$ makes sense.*

Proof. If $\mathcal{B} = W$, then this lemma follows from the above description of the Davis realization of a Coxeter group. The general case can be reduced to this case by considering an apartment $|\mathcal{A}| \subset |\mathcal{B}|$ which contains c_1 and c_2. Note that $|\mathcal{A}|$ is convex in $|\mathcal{B}|$. □

Given a residue $\mathcal{R} \subset \mathcal{B}$, there is a well-defined nearest point projection map as follows.

Theorem 3.9 (Proposition 5.34, [AB08])**.** *Let $\mathcal{R}$ be a residue and c a chamber. Then there exists a unique $c' \in \mathcal{R}$ such that $d(c, c') = d(\mathcal{R}, c)$.*

This projection is compatible with several other projections in the following sense. Let $|\mathcal{R}| \subset |\mathcal{B}|$ be the convex subcomplex corresponding to $\mathcal{R}$. Let c and c' be as above. We also view them as vertex of rank 0 in $|\mathcal{B}|$. Let c_1 be the combinatorial projection of c onto $|\mathcal{R}|$ (see [HW08, Lemma 13.8]) and let c_2 be the $CAT(0)$ projection of c onto $|\mathcal{R}|$.

Lemma 3.10. $c' = c_1 = c_2$.

Proof. $c_1 = c_2$ is actually true for any $CAT(0)$ cube complexes. By [Hua14a, Lemma 2.3], c_2 is a vertex. If $c_2 \neq c_1$, by [HW08, Lemma 13.8], the concatenation of the combinatorial geodesic ω_1 which connects c_2 and c_1 and the combinatorial geodesic ω_2 which connects c_1 and c is a combinatorial geodesic connecting c and c_2. Note that $\omega_1 \subset |\mathcal{R}|$. Let $e \subset \omega_1$ be the edge that contains c_2 and let x be the other endpoint of e. Then x and c are in the same side of the hyperplane dual to e. It is easy to see $d(x, c) < d(c_2, c)$ (here d denotes the $CAT(0)$ distance), which yields a contradiction.

To see $c' = c_1$, by Lemma 3.8, it suffices to prove c_1 is of rank 0. When $\mathcal{B} = W$, this follows from $c_1 = c_2$, since we can work with the cubical completion of the Cayley graph of W instead of $|W|$ (the latter is the cubical subdivision of the former) and apply [Hua14a, Lemma 2.3]. The general case follows by considering an apartment $|\mathcal{A}| \subset |\mathcal{B}|$ which contains c_1 and c, note that in this case $|\mathcal{A}| \cap |\mathcal{R}|$ can be viewed as a residue in $|\mathcal{A}|$. □

Definition 3.11. Let $\mathrm{proj}_{\mathcal{R}}$ be the map defined in Theorem 3.9. Two residues $\mathcal{R}_1$ and $\mathcal{R}_2$ are *parallel* if $\mathrm{proj}_{\mathcal{R}_1}(\mathcal{R}_2) = \mathcal{R}_1$ and $\mathrm{proj}_{\mathcal{R}_2}(\mathcal{R}_1) = \mathcal{R}_2$. In this case $\mathrm{proj}_{\mathcal{R}_1}$ and $\mathrm{proj}_{\mathcal{R}_2}$ induce mutually inverse bijections between $\mathcal{R}_1$ and $\mathcal{R}_2$. These bijections are called *parallelism maps* between $\mathcal{R}_1$ and $\mathcal{R}_2$. They are also isomorphisms of chamber system i.e. they map residues to residues ([AB08, Proposition 5.37]).

It follows from the uniqueness of the projection map that if $f : \mathcal{R} \to \mathcal{R}'$ is the parallelism map between two parallel residues and $\mathcal{R}_1 \subset \mathcal{R}$ is a residue, then $\mathcal{R}_1$ and $f(\mathcal{R}_1)$ are parallel, and the parallelism map between $\mathcal{R}_1$ and $f(\mathcal{R}_1)$ is induced by f.

Lemma 3.12. *If $\mathcal{R}_1$ and $\mathcal{R}_2$ are parallel, then $|\mathcal{R}_1|$ and $|\mathcal{R}_2|$ are parallel with respect to the $CAT(0)$ metric on $|\mathcal{B}|$. Moreover, the parallelism maps between $\mathcal{R}_1$ and $\mathcal{R}_2$ induces by $proj_{\mathcal{R}_1}$ and $proj_{\mathcal{R}_2}$ is compatible with the $CAT(0)$ parallelism between $|\mathcal{R}_1|$ and $|\mathcal{R}_2|$ induced by $CAT(0)$ projections.*

Proof. By Lemma 3.10, it suffices to show for any residue $\mathcal{R} \in \mathcal{B}$, $|\mathcal{R}|$ is the convex hull of the vertices of rank 0 inside $|\mathcal{R}|$. This is clear when

$\mathcal{B} = W$ if one consider the cubical completion of the Cayley graph of W. The general case also follows since $|\mathcal{R}|$ is a union of apartments in $|\mathcal{R}|$, and $|\mathcal{R}|$ is convex in $|\mathcal{B}|$. □

It follows that if $\mathcal{R}_1$ and $\mathcal{R}_2$ are parallel residues, and $\mathcal{R}_2$ and $\mathcal{R}_3$ are parallel residues, then $\mathcal{R}_1$ is parallel to $\mathcal{R}_3$. Moreover, let f_{ij} be the parallelism map from $\mathcal{R}_i$ to $\mathcal{R}_j$ induced by the projection map, then $f_{13} = f_{23} \circ f_{12}$.

Given chamber systems $C_1, \cdots, C_k$ over $I_1, \cdots, I_k$, their direct product $C_1 \times \cdots \times C_k$ is a chamber system over the disjoint union $I_1 \sqcup \cdots \sqcup I_k$. Its chambers are k-tuples $(c_1, \cdots, c_k)$ with $c_t \in C_t$. For $i \in I_t$, $(c_1, \cdots, c_k)$ is i-adjacent to $(d_1, \cdots, d_k)$ if $c_j = d_j$ for $j \neq t$ and c_t and d_t are i-adjacent.

Suppose the defining graph Γ of the right-angled Coxeter group W admits a join decomposition $\Gamma = \Gamma_1 \circ \Gamma_2 \circ \cdots \circ \Gamma_k$. Let $I = \cup_{i=1}^{k} I_i$ be the corresponding decomposition of the vertex set of Γ and $W = \prod_{i=1}^{k} W_i$ be the induced product decomposition of W. Pick chamber $c \in \mathcal{B}$, and let $\mathcal{B}_i$ be the I_i-residue that contains c. Define a map $\phi : \mathcal{B} \to \mathcal{B}_1 \times \mathcal{B}_2 \times \cdots \times \mathcal{B}_k$ by $\phi(d) = (\mathrm{proj}_{\mathcal{B}_1}(d), \mathrm{proj}_{\mathcal{B}_2}(d), \cdots, \mathrm{proj}_{\mathcal{B}_k}(d))$ for any chamber $d \in \mathcal{B}$.

Theorem 3.13 (Theorem 3.10, [Ron09]). *The definition of ϕ does not depend on the choice of c, and ϕ is an isomorphism of buildings.*

It follows from the definition of the Davis realization that there is a natural isomorphism $|\mathcal{B}_1 \times \mathcal{B}_2 \times \cdots \times \mathcal{B}_k| \cong |\mathcal{B}_1| \times |\mathcal{B}_2| \times \cdots \times |\mathcal{B}_k|$, thus we have a product decomposition $|\mathcal{B}| \cong |\mathcal{B}_1| \times |\mathcal{B}_2| \times \cdots \times |\mathcal{B}_k|$, where the isomorphism is induced by $CAT(0)$ projections from $|\mathcal{B}|$ to $|\mathcal{B}_i|$'s (this is a consequence of Lemma 3.12).

We define the *parallel set* of a residue $\mathcal{R} \subset \mathcal{B}$ to be the union of all residues in $\mathcal{B}$ that are parallel to $\mathcal{R}$.

Lemma 3.14. *Suppose $\mathcal{R}$ is a J-residue. Let $J^{\perp} \subset I$ be the collection of vertices in Γ which are adjacent to every vertex in J. Then:*

(1) If $\mathcal{R}'$ is parallel to $\mathcal{R}$, then $\mathcal{R}'$ is a J-residue.
(2) The parallel set of $\mathcal{R}$ is the $J \cup J^{\perp}$-residue that contains $\mathcal{R}$.

Note that this lemma is not true if the building under consideration is not right-angled.

Proof. Suppose $\mathcal{R}'$ is a J_1-residue. Let $w = \delta(\mathcal{R}, \mathcal{R}')$ (see Definition 3.7). It follows from (2) of [AB08, Lemma 5.36] that $\mathcal{R}'$ is a $(J \cap$

wJ_1w^{-1})-residue. Since $\mathcal{R}$ and $\mathcal{R}'$ are parallel, they have the same rank, thus $J = wJ_1w^{-1}$. By considering the abelianization of the right-angled Coxeter group W, we deduce that $J = J_1$ (this proves the first assertion of the lemma) and w commutes with each element in J. Thus w belongs to the subgroup generated by $J^{\perp}$ and $\mathcal{R}'$ is in the $J \cup J^{\perp}$-residue $\mathcal{S}$ that contains $\mathcal{R}$. Then the parallel set of $\mathcal{R}$ is contained in $\mathcal{S}$. It remains to prove every J-residue in $\mathcal{S}$ is parallel to $\mathcal{R}$, but this follows from Theorem 3.13. □

Pick a vertex $v \in |\mathcal{B}|$ of rank k and let $\mathcal{R} = \prod_{i=1}^{k} \mathcal{R}_i$ be the associated residue with its product decomposition. Let $\{v_\lambda\}_{\lambda\in\Lambda}$ be the collection of vertices that are adjacent to v. Then there is a decomposition $\{v_\lambda\}_{\lambda\in\Lambda} = \{v_\lambda \leq v\} \sqcup \{v_\lambda > v\}$, where $\{v_\lambda > v\}$ denotes the collection of vertices whose associated residues contain $\mathcal{R}$. This induces a decomposition $Lk(v, |\mathcal{B}|) = K_1 * K_2$ of the link of v in $|\mathcal{B}|$ ([BH99, Definition I.7.15]) into a spherical join of two $CAT(1)$ all-right spherical complexes. Note that K_2 is finite, since $\{v_{\lambda>v}\}$ is finite. Moreover, $K_1 \cong Lk(v, |\mathcal{R}|)$. However, $|\mathcal{R}| \cong \prod_{i=1}^{k} |\mathcal{R}_i|$, thus K_1 is the spherical join of k discrete sets such that elements in each of these discrete sets are in 1-1 correspondence to elements in some $\mathcal{R}_i$. Now we can deduce from this the following result.

Lemma 3.15. *Suppose $\mathcal{B}$ is a right-angled building such that each of its residues of rank 1 contains infinitely many elements. If $\alpha : |\mathcal{B}| \to |\mathcal{B}|$ is a cubical isomorphism, then α preserves the rank of vertices in $|\mathcal{B}|$.*

Part I: A fibration approach to cubulating RAAG's

4. Restriction quotients

In this section we study restriction quotients, a certain type of mapping between CAT(0) cube complexes introduced by Caprace and Sageev in [CS11]. These play a central role in our story.

We first show in Subsection 4.1 that restriction quotients can be characterized in several different ways, see Theorem 4.4. We then show in Subsection 4.2 that a restriction quotient $f : Y \to Z$ determines fiber data that satisfies certain conditions; conversely, given such fiber data, one may construct a restriction quotient inducing the given data, which is unique up to equivalence. This correspondence will later be applied to construct restriction quotients over right-angled buildings. Subsections 4.3 and 4.4 deals with the behavior of restriction quotients under group actions and quasi-isometries.

4.1. **Quotient maps between** CAT(0) **cube complexes.** We recall the notion of restriction quotient from [CS11, Section 2.3]; see Section 9.1 for the background on wallspaces.

Definition 4.1. Let Y be a CAT(0) cube complex and let $\mathcal{H}$ be the collection of walls in the 0-skeleton $Y^{(0)}$ corresponding to the hyperplanes in Y. Pick a subset $\mathcal{K} \subset \mathcal{H}$ and let $Y(\mathcal{K})$ be the CAT(0) cube complex associated with the wallspace $(Y^{(0)}, \mathcal{K})$. Then every 0-cube of the wallspace $(Y^{(0)}, \mathcal{H})$ gives rise to a 0-cube of $(Y^{(0)}, \mathcal{K})$ by restriction. This can be extended to a surjective cubical map $q : Y \to Y(\mathcal{K})$, which is called the *restriction quotient* arising from the subset $\mathcal{K} \subset \mathcal{H}$.

The following example motivates many of the constructions in this paper:

Example 4.2 (The canonical restriction quotient of a RAAG)**.** For a fixed graph Γ, let $S_e \to |c|_\Gamma$ and $X_e \to S_e$ be the mappings associated with the exploded Salvetti complex, as defined in the introduction after Definition 1.11. Let $\mathcal{K}$ be the collection of hyperplanes in $X_e(\Gamma)$ dual to edges $e \subset X_e$ that project to edges under the composition $X_e \to S_e \to |c|_\Gamma$. Then the *canonical restriction quotient of* $G = G(\Gamma)$ is the restriction quotient arising from $\mathcal{K}$.

Let $q : Y \to Y(\mathcal{K})$ be a restriction quotient. Pick an edge $e \subset Y$. If e is dual to some element in $\mathcal{K}$, then $q(e)$ is an edge, otherwise $q(e)$ is a point. The edge e is called *horizontal* in the former case, and *vertical* in the latter case. We record the following simple observation.

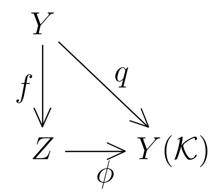
Lemma 4.3. *Let $\alpha : Y \to Y$ be a cubical* CAT(0) *automorphism of Y that maps vertical edges to vertical edges and horizontal edges to horizontal edges. Then α descends to an automorphism $Y(\mathcal{K}) \to Y(\mathcal{K})$.*

The following result shows that restriction quotients may be characterized in several different ways.

Theorem 4.4. *If $f : Y \to Z$ is a surjective cubical map between two* CAT(0) *cube complexes, then the following conditions are equivalent:*

1. *The inverse image of each vertex of Z is convex.*
2. *The inverse image of every point in Z is convex.*
3. *The inverse image of every convex subcomplex of Z is convex.*
4. *The inverse image of every hyperplane in Z is a hyperplane.*
5. *f is equivalent to a restriction quotient, i.e. for some set of walls $\mathcal{K}$ in Y, there is a cubical isomorphism $\phi : Z \to Y(\mathcal{K})$*

making the following diagram commute:



The proof of Theorem 4.4 will take several lemmas. For the remainder of this subsection we fix CAT(0) cube complexes Y and Z and a (not necessarily surjective) cubical map $f : Y \to Z$.

Lemma 4.5. *Let $\sigma \subset Z$ be a cube and let Y_σ be the be the union of cubes in Y whose image under f is exactly σ. Then:*

(1) If $y \in \sigma$ is an interior point, then $f^{-1}(y) \subset Y_\sigma$.
(2) $f^{-1}(y)$ has a natural induced structure as a cube complex; moreover, there is a natural isomorphism of cube complexes $Y_\sigma \cong f^{-1}(y) \times \sigma$.
(3) If $\sigma_1 \subset \sigma_2$ are cubes of Z and $y_i \in \sigma_i$ are interior points, then there is a canonical embedding $f^{-1}(y_2) \hookrightarrow f^{-1}(y_1)$. Moreover, these embeddings are compatible with composition of inclusions.

Lemma 4.6.

(1) For every $y \in Z$, every connected component of $f^{-1}(y)$ is a convex subset of Y.
(2) For every convex subcomplex $A \subset Z$, every connected component of $f^{-1}(A)$ is a convex subcomplex of Y.

Proof. First we prove (1). Let σ be the support of y and let $Y_\sigma \cong f^{-1}(y) \times \sigma$ be the subcomplex defined as above. It suffices to show Y_σ is locally convex. Pick vertex $x \in Y_\sigma$, and let $\{e_i\}_{i=1}^n$ be a collection of edges in Y_σ that contains x. It suffices to show if these edges span an n-cube $\eta \subset Y$, then $\eta \subset Y_\sigma$. It suffices to consider the case when all e_i's are orthogonal to σ, in which case it follows from Definition 3.4 that $\eta \times \sigma \subset Y_\sigma$.

To see (2), pick an n-cube $\eta \subset Y$ and let $\{e_i\}_{i=1}^n$ be the edges of η at one corner $c \subset \eta$. It suffices to show if $f(e_i) \subset A$, then $f(\eta) \subset A$. Note that $f(\eta)$ is a cube, and every edge of this cube which emanates from the corner $f(c)$ is contained in A. Thus $f(\eta) \subset A$ by the convexity of A. $\square$

Lemma 4.7. *Let $f : Y \to Z$ be a cubical map as above. Then:*

(1) The inverse image of each hyperplane of Z is a disjoint union of hyperplanes in Y.
(2) If the inverse image of each hyperplane of Z is a single hyperplane, then for each point $y \in Z$, the point inverse $f^{-1}(y)$ is connected, and hence convex.

Proof. It follows from Definition 3.4 that the inverse image of each hyperplane of Z is an union of hyperplanes. If two of them were to intersect, then there would be a 2-cube in Y with two consecutive edges mapped to the same edge in Z, which is impossible.

Now we prove (2). It suffices to consider the case that y is the center of some cube in Z. In this case, y is a vertex in the first cubical subdivision of Z, and f can viewed as a cubical map from the first cubical subdivision of Y to the first cubical subdivision of Z such that the inverse image of each hyperplane is a single hyperplane, thus it suffices to consider the case that y is a vertex of Z.

Suppose $f^{-1}(y)$ contains two connected components A and B. Pick a combinatorial geodesic ω of shortest distant that connects vertices in A and vertices in B. Note that $f(\omega)$ is a non-trivial edge-loop in Z, otherwise we will have $\omega \subset f^{-1}(y)$. It follows that there exists two different edges e_1 and e_2 of ω mapping to parallel edges in Y. The hyperplanes dual to e_1 and e_2 are different, yet they are mapped to the same hyperplane in Y, which is a contradiction. □

Lemma 4.8. *If f is surjective, and for any vertex $v \in Z$, $f^{-1}(v)$ is connected, then the inverse image of each hyperplane of Z is a single hyperplane.*

Proof. Let $h \subset Z$ be a hyperplane, by Lemma 4.7, $f^{-1}(h) = \sqcup_{\lambda\in\Lambda} h_\lambda$ where each h_λ is a hyperplane in Y. Since f is surjective, $\{f(h_\lambda)\}_{\lambda\in\Lambda}$ is a collection of subcomplexes of h that cover h. Thus there exists $h_1, h_2 \in \{h_\lambda\}_{\lambda\in\Lambda}$ and vertex $u \in h$ such that $u \subset f(h_1) \cup f(h_2)$. Let $e \subset Z$ be the edge such that $u = e \cap h$, then there exist edges $e_1, e_2 \subset Y$ such that $e_i \cap h_i \neq \emptyset$ and $f(e_i) = e$ for $i = 1, 2$. Since $h_1 \cap h_2 = \emptyset$, a case study implies there exist x_1 and x_2 which are endpoints of e_1 and e_2 respectively such that

(1) these two points are separated by at least one of h_1 and h_2;
(2) they are mapped to the same end point $y \in e$.

It follows that $f^{-1}(y)$ is disconnected, which is a contradiction. □

Remark 4.9. If f is not surjective, then the above conclusion is not necessarily true. Consider the map from $A = [0,3] \times [0,1]$ to the unit square which collapses the $[0,1]$ factor in A and maps $[0,3]$ to 3 consecutive edges in the boundary of the unit square.

Lemma 4.10. *If $q : Y \to Y(\mathcal{K})$ is the restriction quotient as Definition 4.1, then the inverse image of each hyperplane in $Y(\mathcal{K})$ is a single hyperplane in Y. Conversely, suppose $f : Y \to Z$ is a surjective cubical map between* CAT(0) *cube complexes such that the inverse image of each hyperplane is a hyperplane. Let $\mathcal{K}$ be the collection of walls arising from inverse images of hyperplanes in Z. Then there is a natural isomorphism $i : Z \cong Y(\mathcal{K})$ which fits into the following commutative diagram:*

$$\begin{array}{ccc} Y & \xrightarrow{\ f\ } & Z \\ & \searrow^{q} & \downarrow{\scriptstyle i} \\ & & Y(\mathcal{K}) \end{array}$$

Proof. Define two vertices of Y to be $\mathcal{K}$-equivalent if and only if they are not separated by any wall in $\mathcal{K}$. This defines an equivalence relation on vertices of Y, and the corresponding equivalent classes are called $\mathcal{K}$-classes. For each $\mathcal{K}$-class C and every wall in $\mathcal{K}$, we may choose the halfspace that contains C; it follows that the points in C are exactly the set of vertices contained in the intersection of such halfspaces, and thus C is the vertex set of a convex subcomplex of Y. Note that each $\mathcal{K}$-class determines a 0-cube of $(Y^0, \mathcal{K})$, hence is mapped to this 0-cube under q. It follows that the inverse image of every vertex in $Y(\mathcal{K})$ is convex, thus by Lemma 4.8, the inverse image of a hyperplane is a hyperplane.

It remains to prove the converse. Note that the inverse image of each halfspace in Z under f is a halfspace of Y. Moreover, the surjectivity of f implies that f maps hyperplane to hyperplane and halfspace to halfspace. Pick vertex $y \in Z$, let $\{H_\lambda\}_{\lambda \in \lambda}$ be the collection of hyperplanes in Z that contains y. Then $f^{-1}(y) \subset \cap_{\lambda\in\Lambda} f^{-1}(H_\lambda)$, and every vertex of $\cap_{\lambda\in\Lambda} f^{-1}(H_\lambda)$ is mapped to y by f, and thus the vertex set of $f^{-1}(y)$ is a $\mathcal{K}$-class. This induces a bijective map from $Z^{(0)}$ to the vertex set of $Y(\mathcal{K})$, which extends to an isomorphism. The above diagram commutes since it commutes when restricted to the 0-skeleton. $\square$

Proof of Theorem 4.4. The equivalence of (4) and (5) follows from Lemma 4.10. (1) $\Rightarrow$ (4) follows from Lemma 4.8, (4) $\Rightarrow$ (2) follows from Lemma

4.7, (3) $\Rightarrow$ (1) is obvious. It suffices to show (2) $\Rightarrow$ (3). Pick a convex subcomplex $K \subset Z$ and let $\{R_\lambda\}_{\lambda\in\Lambda}$ be the collection of cubes in K. For each R_λ, let Y_{R_λ} be the subcomplex defined after Definition 3.4. $Y_{R_\lambda} \neq \emptyset$ since f is surjective and Y_{R_λ} is connected by (2). If $R_\lambda \subset R_{\lambda'}$, then $Y_{R_\lambda} \cap Y_{R_{\lambda'}} \neq \emptyset$. Thus $f^{-1}(K) = \cup_{\lambda\in\Lambda} Y_{R_\lambda}$ is connected, hence convex. □

4.2. **Restriction maps versus fiber functors.** If $q : Y \to Z$ is a restriction quotient between CAT(0) cube complexes, then we may express the fiber structure in categorical language as follows. Let Face(Z) denote the face poset of Z, viewed as a category, and let CCC denote the category whose objects are nonempty CAT(0) cube complexes and whose morphisms are convex cubical embeddings. By Lemma 4.5, we obtain a contravariant functor $\Psi_q : \mathrm{Face}(Z) \to \mathrm{CCC}$.

Definition 4.11. The contravariant functor Ψ_q is the *fiber functor* of the restriction quotient $q : Y \to Z$.

For notational brevity, for any inclusion $i : \sigma_1 \to \sigma_2$, we will often denote the map $\Psi(i) : \Psi(\sigma_2) \to \Psi(\sigma_1)$ simply by $\Psi(\sigma_2) \to \Psi(\sigma_1)$, suppressing the name of the map.

Note that if $\sigma_1 \subset \sigma_2 \subset \sigma_3$, then the functor property implies that the image of $\Psi(\sigma_3) \to \Psi(\sigma_1)$ is a convex subcomplex of the image of $\Psi(\sigma_2) \to \Psi(\sigma_1)$. In particular, if v is a vertex of a cube σ, then the image of $\Phi(\sigma) \to \Psi(v)$ is a convex subcomplex of the intersection

$$\bigcap_{v \subsetneq e \subset \sigma^{(1)}} \mathrm{Im}(\Psi(e) \to \Psi(v))$$

Definition 4.12. Let Z be a cube complex. A contravariant functor $\Psi : \mathrm{Face}(Z) \to \mathrm{CCC}$ is **1-determined** if for every cube $\sigma \in \mathrm{Face}(Z)$, and every vertex $v \in \sigma^{(0)}$,

$$\mathrm{Im}(\Psi(\sigma) \longrightarrow \Psi(v)) = \bigcap_{v \subsetneq e \subset \sigma^{(1)}} \mathrm{Im}(\Psi(e) \to \Psi(v)) \,. \tag{4.13}$$

Lemma 4.14. *If $q : Y \to Z$ is a restriction quotient, then the fiber functor $\Psi : \mathrm{Face}(Z) \to \mathrm{CCC}$ is 1-determined.*

Proof. Pick $\sigma \in \mathrm{Face}(Z)$, $v \in \sigma^{(0)}$. We know that $\mathrm{Im}(\Psi(\sigma) \to \Psi(v))$ is a nonempty convex subcomplex of $\cap_{v \subsetneq e \subset \sigma^{(1)}} \mathrm{Im}(\Psi(e) \to \Psi(v))$, so to establish (4.13) we need only show that the two convex subcomplexes have the same 0-skeleton.

Pick a vertex $w \in \mathrm{Im}(\Psi(\sigma) \to \Psi(v))$, and let $w' \in \cap_{v \subsetneq e \subset \sigma^{(1)}} \mathrm{Im}(\Psi(e) \to \Psi(v))$ be a vertex adjacent to w. We let $\tau \in \mathrm{Face}(Y)$ denote the edge spanned by w, w'. For every edge e of Z with $v \subsetneq e \subset \sigma^{(1)}$, let $\hat{e} \subset Y^{(1)}$ denote the edge with $q(\hat{e}) = e$ that contains w. By assumption, the collection of edges $\{\tau\} \cup \{\hat{e}\}_{v \subsetneq e \subset \sigma^{(1)}}$ determines a complete graph in the link of w, and therefore is contained in a cube $\hat{\sigma}$ of dimension $1+\dim \sigma$. Then $q(\hat{\sigma}) = \sigma$ and $\tau \subset \hat{\sigma}$; this implies that $\tau \subset \mathrm{Im}(\Psi(\sigma) \to \Psi(v))$.

Since the 1-skeleton of $\cap_{v \subsetneq e \subset \sigma^{(1)}} \mathrm{Im}(\Psi(e) \to \Psi(v))$ is connected, we conclude that it coincides with the 1-skeleton of $\mathrm{Im}(\Psi(\sigma) \to \Psi(v))$. By convexity, we get (4.13). □

Theorem 4.15. *Let Z be a* CAT(0) *cube complex, and $\Psi : \mathrm{Face}(Z) \to$ CCC be a 1-determined contravariant functor. Then there is a restriction quotient $q : Y \to Z$ such that the associated fiber functor $\Psi_q : \mathrm{Face}(Z) \to$ CCC is equivalent by a natural transformation to Ψ.*

Proof. We first construct the cube complex Y, and then verify that it has the desired properties.

We begin with the disjoint union $\bigsqcup_{\sigma \in \mathrm{Face}(Z)} (\sigma \times \Psi(\sigma))$, and for every inclusion $\sigma \subset \tau$, we glue the subset $\sigma \times \Psi(\tau) \subset \tau \times \Psi(\tau)$ to $\sigma \times \Psi(\sigma)$ by using the map

$$\sigma \times \Psi(\tau) \xrightarrow{\mathrm{id}_\sigma \times \Psi(\sigma \subset \tau)} \sigma \times \Psi(\sigma)\,.$$

One checks that the cubical structure on $\bigsqcup_{\sigma \in \mathrm{Face}(Z)} (\sigma \times \Psi(\sigma))$ descends to the quotient Y, the projection maps $\sigma \times \Psi(\sigma) \to \sigma$ descend to a cubical map $q : Y \to Z$, and for every $\sigma \in \mathrm{Face}(Z)$, the union of the cubes $\hat{\sigma} \subset Y$ such that $f(\hat{\sigma}) = \sigma$ is a copy of $\sigma \times \Psi(\sigma)$.

We now verify that links in Y are flag complexes.

Let v be a 0-cube in Y, and suppose $\sigma_1, \ldots, \sigma_k$ are 1-cubes containing v, such that for all $1 \leq i \neq j \leq k$ the 1-cubes σ_i, σ_j span a 2-cube σ_{ij} in the link of v. We may assume after reindexing that for some $h \geq 0$ the image $q(\sigma_i)$ is a 1-cube in Z if $i \leq h$ and a 0-cube if $i > h$.

Since $\Psi(v)$ is a CAT(0) cube complex, the edges $\{\sigma_i\}_{i>h}$ span a cube $\sigma_{vert} \subset q^{-1}(v)$.

For $1 \leq i \neq j \leq h$, the 2-cube σ_{ij} projects to a 2-cube $q(\sigma_{ij})$ spanned by the two edges $q(\sigma_i), q(\sigma_j)$. Since Z is a CAT(0) cube complex, the edges $\{q(\sigma_i)\}_{i \leq h}$ span an h-cube $\bar{\sigma}_{hor} \subset Z$. By the 1-determined property, we get that $\mathrm{Im}(\Psi(\bar{\sigma}_{hor}) \to \Psi(v))$ contains v, and so there is an h-cube $\sigma_{hor} \subset Y$ containing v such that $q(\sigma_{hor}) = \bar{\sigma}_{hor}$.

Fix $1 \leq i \leq h$. Then for $j > h$, the 2-cube σ_{ij} projects to $q(\sigma_i)$, and hence σ_j belongs to $\mathrm{Im}(\Psi(\sigma_i) \to \Psi(v))$. If $j, k > h$, then σ_j, σ_k both belong to $\mathrm{Im}(\Psi(\sigma_i) \to \Psi(v))$, and by the convexity of $\mathrm{Im}(\Psi(\sigma_i) \to \Psi(v))$ in $\Psi(v)$, we get that σ_{jk} also belongs to $\mathrm{Im}(\Psi(\sigma_i) \to \Psi(v))$. Applying convexity again, we get that $\sigma_{vert} \subset \mathrm{Im}(\Psi(\sigma_i) \to \Psi(v))$. By the 1-determined property, it follows that $\sigma_{vert} \subset \mathrm{Im}(\Psi(\bar{\sigma}_{hor}) \to \Psi(v))$. This yields a k-cube $\sigma \subset Y$ containing $\sigma_{hor} \cup \sigma_{vert}$, which is spanned by $\sigma_1, \ldots, \sigma_k$.

Thus we have shown that links in Y are flag complexes. The fact that the fibers of $f : Y \to Z$ are contractible implies that Y is contractible (in particular simply connected), so Y is CAT(0).

□

We now observe that the construction of restriction quotients is compatible with product structure:

Lemma 4.16 (Behavior under products). *For $i \in \{1, 2\}$ let $q_i : Y_i \to Z_i$ be a restriction quotient with fiber functor $\Psi_i : \mathrm{Face}(Z_i) \to \mathrm{CCC}$. Then the product $q_1 \times q_2 : Y_1 \times Y_2 \to Z_1 \times Z_2$ is a restriction quotient with fiber functor given by the product:*

$$\mathrm{Face}(Z_1 \times Z_2) \simeq \mathrm{Face}(Z_1) \times \mathrm{Face}(Z_2) \xrightarrow{\Psi_1 \times \Psi_2} \mathrm{CCC} \times \mathrm{CCC} \xrightarrow{\times} \mathrm{CCC} .$$

In particular, if one starts with CAT(0) *cube complexes Z_i and fiber functors $\Psi_i : Z_i \to \mathrm{CCC}$ for $i \in \{1, 2\}$, then the product fiber functor defined as above is the fiber functor of the product of the restriction quotients associated to the Ψ_i's.*

4.3. **Equivariance properties.** We now discuss isomorphisms between restriction quotients, and the naturality properties of the restriction quotient associated with a fiber functor.

Suppose we have a commutative diagram

$$\begin{array}{ccc} Y_1 & \xrightarrow{\hat{\alpha}} & Y_2 \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ Z_1 & \xrightarrow{\alpha} & Z_2 \end{array}$$

where the q_i's are restriction quotients and $\alpha, \hat{\alpha}$ are cubical isomorphisms. Let $\Psi_i : \mathrm{Face}(Z_i) \to \mathrm{CCC}$ be the fiber functor associated with q_i. Notice that the pair $\alpha, \hat{\alpha}$ allows us to compare the two fiber functors, since for every $\sigma \in \mathrm{Face}(Z_1)$, the map $\hat{\alpha}$ induces a cubical isomorphism between $\Psi_1(\sigma)$ and $\Psi_2(\alpha(\sigma))$, and this is compatible with maps induced with inclusions of faces. This may be stated more compactly by saying that $\hat{\alpha}$ induces a natural isomorphism between the fiber functors Ψ_1 and $\Psi_2 \circ \mathrm{Face}(\alpha)$, where $\mathrm{Face}(\alpha) : \mathrm{Face}(Z_1) \to \mathrm{Face}(Z_2)$ is the poset isomorphism induced by α. Here the term *natural isomorphism* is being used in the sense of category theory, i.e. a natural transformation that has an inverse that is also a natural transformation.

Now suppose that for $i \in \{1, 2\}$ we have a CAT(0) cube complex Z_i and a 1-determined fiber functor $\Psi_i : \mathrm{Face}(Z_i) \to \mathrm{CCC}$. Let $f_i : Y_i \to Z_i$ be the associated restriction quotients. If we have a pair α, β, where $\alpha : Z_1 \to Z_2$ is a cubical isomorphism, and β is a natural isomorphism between the fiber functors Ψ_1 and $\Psi_2 \circ \mathrm{Face}(\alpha)$, then we get an induced map $\hat{\alpha} : Y_1 \to Y_2$, which may be defined by using the description of Y_i as the quotient of the disjoint collection $\{\sigma \times \Psi_i(\sigma)\}_{\sigma \in \mathrm{Face}(Z_i)}$.

As a consequence of the above, having an action of a group G on a restriction quotient $f : Y \to Z$ is equivalent to having an action $G \curvearrowright Z$ together with a compatible "action" on the fiber functor Ψ_f, i.e. a family $\{(\alpha(g), \beta(g))\}_{g \in G}$ as above that also satisfies an appropriate composition rule.

4.4. **Quasi-isometric properties.** We now consider the coarse geometry of restriction quotients; this amounts to a "coarsification" of the discussion in the preceding subsection.

The relevant definition is a coarsification of the natural isomorphisms between fiber functors.

Definition 4.17. Let Z be a CAT(0) cube complex and $\Psi_i : \mathrm{Face}(Z) \to \mathrm{CCC}$ be fiber functors for $i \in \{1, 2\}$. An (L, A)-*quasi-natural isomorphism from* Ψ_1 *to* Ψ_2 is a collection $\{\phi(\sigma) : \Psi_1(\sigma) \to \Psi_2(\sigma)\}_{\sigma \in \mathrm{Face}(Z)}$ such that $\phi(\sigma)$ is an (L, A)-quasi-isometry for all $\sigma \in \mathrm{Face}(Z)$, and for every inclusion $\sigma \subset \tau$, the diagram

$$\begin{array}{ccc} \Psi_1(\tau) & \xrightarrow{\phi(\tau)} & \Psi_2(\tau) \\ \downarrow & & \downarrow \\ \Psi_1(\sigma) & \xrightarrow{\phi(\sigma)} & \Psi_2(\sigma) \end{array}$$

commutes up to error L.

Now for $i \in \{1, 2\}$ let $f_i : Y_i \to Z$ be a finite dimensional restriction quotient, with respective fiber functor $\Psi_i : \mathrm{Face}(Z) \to \mathrm{CCC}$. For any $\sigma \in \mathrm{Face}(Z)$, we identify $\Psi_i(\sigma)$ with $f_i(b_\sigma)$, where $b_\sigma \in \sigma$ is the barycenter.

Lemma 4.18. *Suppose we have a commutative diagram*

$$\begin{array}{ccc} Y_1 & \underset{\phi'}{\overset{\phi}{\rightleftarrows}} & Y_2 \\ & {\scriptstyle f_1}\searrow \quad \swarrow{\scriptstyle f_2} & \\ & Z & \end{array}$$

where ϕ, ϕ' are (L, A)-quasi-isometries that are A-quasi-inverses, i.e. the compositions $\phi \circ \phi'$, $\phi' \circ \phi$ are at distance $< A$ from the identity maps. Then the collection

$$\{\Psi_1(\sigma) = f_1^{-1}(b_\sigma) \overset{\phi\big|_{f_1^{-1}(b_\sigma)}}{\longrightarrow} f_2^{-1}(b_\sigma) = \Psi_2(\sigma)\}_{\sigma \in \mathrm{Face}(Z)}$$

is an (L', A')-quasi-natural isomorphism where $L' = L'(L, A, \dim Y_i)$, $A' = A'(L, A, \dim Y_i)$.

Proof. By Theorem 4.4, the fiber $f_i^{-1}(b_\sigma)$ is a convex subset of Y_i, and hence is isometrically embedded. Therefore ϕ and ϕ' induce (L, A)-quasi-isometric embeddings $f_1^{-1}(b_\sigma) \to f_2^{-1}(b_\sigma)$, $f_2^{-1}(\sigma_b) \to f_1^{-1}(b_\sigma)$. If $\sigma \subset \tau$, then any point $x \in f_i^{-1}(b_\tau)$ lies at distance $< C = C(\dim Y_i)$ from a point in $f_i^{-1}(b_\sigma)$, and this implies that the collection of maps $\{\Psi_1(\sigma) \to \Psi_2(\sigma)\}_{\sigma \in \mathrm{Face}(Z)}$ is an (L', A')-quasi-natural isomorphism as claimed. □

Lemma 4.19. *If $\{\phi(\sigma) : \Psi_1(\sigma) \to \Psi_2(\sigma)\}_{\sigma \in \mathrm{Face}(Z)}$ is an (L, A)-quasi-natural isomorphism from Ψ_1 to Ψ_2, then it arises from a commutative diagram as in the previous lemma, where ϕ, ϕ' are (L', A')-quasi-isometries that are A'-quasi-inverses, and L', A' depend only on L, A, and $\dim Y_i$.*

Proof. For every $\sigma \in \mathrm{Face}(Z)$, we may choose a quasi-inverse $\phi'(\sigma) : \Psi_2(\sigma) \to \Psi_1(\sigma)$ with uniform constants; this is also a quasi-natural isomorphism. Identifying $f_i^{-1}(\mathrm{Int}(\sigma))$ with the product $\mathrm{Int}(\sigma) \times \Psi_i(\sigma)$, we define $\phi\big|_{f_1^{-1}(\mathrm{Int}(\sigma))}$ by

$$f_1^{-1}(\mathrm{Int}(\sigma)) = \mathrm{Int}(\sigma) \times \Psi_1(\sigma) \overset{\mathrm{id}_{\mathrm{Int}(\sigma)} \times \phi(\sigma)}{\longrightarrow} \mathrm{Int}(\sigma) \times \Psi_2(\sigma) = f_2^{-1}(\sigma)\,,$$

and ϕ' similarly using $\{\phi'(\sigma)\}_{\sigma \in \mathrm{Face}(Z)}$. One readily checks that ϕ, ϕ' are quasi-isometric embeddings that are also quasi-inverses, where the constants depend on L, A, and $\dim Y_i$. □

5. The $\mathbb{Z}$-blow-up of right-angled Building

In this section Γ will be an arbitrary finite simplicial graph, and all buildings will be right-angled buildings modelled on the right-angled Coxeter group $W(\Gamma)$ with defining graph Γ. The reader may wish to review Section 3.4 for terminology and notation regarding buildings, before proceeding.

The goal of this section is examine restriction quotients $q : Y \to |\mathcal{B}|$, where the fibers are Euclidean spaces satisfying a dimension condition as in Theorem 1.5 or 1.6. For such restriction quotients, the fiber functor may be distilled down to 1-data, see Definition 5.3; this is discussed in Subsection 5.2. Conversely, given a building $\mathcal{B}$ and certain blow-up data (Definition 5.8), one can construct a corresponding 1-determined fiber functor as in Section 4.2; see Subsection 5.3.

5.1. **The canonical restriction quotient for a RAAG.** Let $G(\Gamma)$ be the RAAG with defining graph Γ and let $\mathcal{B}(\Gamma)$ be the building associated with $G(\Gamma)$ (see [Dav98, Section 5]). Then $G(\Gamma)$ can identified with the set of chambers of $\mathcal{B}(\Gamma)$. Under this identification, the J-residues of $\mathcal{B}$, for J a collection of vertices in Γ, are the left cosets of the standard subgroups of $G(\Gamma)$ generated by J. Thus the poset of spherical residues is exactly the poset of left cosets of standard Abelian subgroups of $G(\Gamma)$, which is also isomorphic to the poset of standard flats in $X(\Gamma)$.

We now revisit the discussion after Definition 1.11 and Example 4.2 in more detail, and relate them to buildings. To simplify notation, we will write $G = G(\Gamma)$, $\mathcal{B} = \mathcal{B}(\Gamma)$ and $X = X(\Gamma)$.

Let $|\mathcal{B}|$ be the Davis realization of the building $\mathcal{B}$. Then we have an induced isometric action $G \curvearrowright |\mathcal{B}|$, which is cocompact, but not proper. It turns out there is natural way to blow-up $|\mathcal{B}|$ to obtain a space $X_e = X_e(\Gamma)$ such that there is a geometric action $G \curvearrowright X_e$ and a G-equivariant restriction quotient map $X_e \to |\mathcal{B}|$.

X_e can be constructed as follows. First we constructed the *exploded Salvetti complex* $S_e = S_e(\Gamma)$, which was introduced in [BKS08a], see also the discussion after Definition 1.11. Suppose L is the "lollipop", which is the union of a unit circle S and a unit interval I along one point. For each vertex v in the vertex set $V(\Gamma)$ of Γ, we associate a copy of $L_v = S_v \cup I_v$, and let $\star_v \in L_v$ be the free end of I_v. Let $T = \prod_{v \in V(\Gamma)} L_v$. Each clique $\Delta \subset \Gamma$ gives rise to a subcomplex $T_\Delta = \prod_{v\in\Delta} L_v \times \prod_{v \notin \Delta} \{\star_v\}$. Then S_e is the subcomplex of T which is the union of all such T_Δ's, here Δ is allowed to be empty. It is easy to check S_e is a non-positively curved cube complex. A *standard torus* in S_e is a subcomplex of form $\prod_{v\in\Delta} S_v \times \prod_{v\notin\Delta}\{\star_v\}$, where $\Delta \subset \Gamma$ is a clique. Note that there is a unique standard torus of dimension 0, which corresponds to the empty clique. There is a natural map $S_e = S_e(\Gamma) \to S(\Gamma)$ by collapsing the I_v-edge in each L_v-factor. This maps induces a 1-1 correspondence between standard tori in S_e and standard tori in $S(\Gamma)$. Notice that there is also a 1-1 correspondence between vertices in S_e and standard tori in S_e.

Let X_e be the universal cover of S_e. Then X_e is a $CAT(0)$ cube complex and the action $G \curvearrowright X_e$ is geometric. The inverse images of standard tori in S_e are called *standard flats*. Note that each vertex in X_e is contained in a unique standard flat. We define a map between the 0-skeletons $p : X_e^{(0)}(\Gamma) \to |\mathcal{B}|^{(0)}$ as follow. Pick a G-equivariant identification between 0-dimensional standard flats in X and elements in G, and pick a G-equivariant map $\phi : X_e \to X$ induced by $S_e = S_e(\Gamma) \to S(\Gamma)$ described as above. Note that c induces a 1-1 correspondence between standard flats in X_e and standard flats in X. This gives rise to a 1-1 correspondence between standard flats in X_e and left cosets of standard Abelian subgroups of G. For each $x \in X_e^{(0)}(\Gamma)$, we define $p(x)$ to be the vertex in $|\mathcal{B}|^{(0)}$ that represents the left coset of the standard Abelian subgroup of G which corresponds to the unique standard flat that contains x.

A *vertical edge* of X_e is an edge which covers some S_v-circle in S_e. A *horizontal edge* of X_e is an edge which covers some I_v-interval in S_e. Two endpoints of every vertical edge are in the same standard flat, thus they are mapped by p to the same point in $|\mathcal{B}|^{(0)}$. More generally, for any given *vertical cube*, i.e. every edge in this cube is a vertical edge, its vertex set is mapped by p to one point in $|\mathcal{B}|^{(0)}$. Pick a horizontal edge and let $F_1, F_2 \subset X_e$ be standard flats which contain the two endpoints of this edge respectively. Then $\phi(F_1)$ and $\phi(F_2)$ are two standard flats in X such that one is contained as a codimension 1 flat inside another. More generally, if σ is a horizontal cube, i.e. each edge of σ is a horizontal edge, then by looking the image of σ under the covering map $X_e \to S_e$, we know the vertex set of σ corresponds to an interval in the poset of standard flats of X. Every cube in X_e splits as a product of a vertical cube and a horizontal cube (again this is clear by looking at cells in S_e). Thus we can extend p to a cubical map $p : X_e \to |\mathcal{B}|$.

By construction, for a vertex $v \in |\mathcal{B}|$ of rank k, $p^{-1}(v)$ is isometric to $\mathbb{E}^n$. It follows from Theorem 4.4 that p arises from a restriction quotient, and this is called the *canonical restriction quotient* for the RAAG G. This restriction quotient is exactly the one described in Example 4.2, since hyperplanes in $\mathcal{K}$ of Example 4.2 are those which are dual to horizontal edges. We record following immediate consequence of the this construction.

Lemma 5.1. *Let $\sigma \subset |\mathcal{B}|$ be a cube and let $v \in \sigma$ be the vertex of minimal rank in σ. Then for any interior point $x \in \sigma$, $p^{-1}(x)$ is isometric to $\mathbb{E}^{rank(\sigma)}$.*

Remark 5.2. In the literature, there is a related cubical map $X \to |\mathcal{B}|$ defined as follows. First we recall an alternative description of X. Actually similar spaces can be defined for all Artin groups (not necessarily right-angled) and was introduced by Salvetti. We will follow the description in [Cha]. Let $G \to W(\Gamma)$ be the natural projection map. This map has a set theoretic section defined by representing an element $w \in W$ by a minimal length positive word with respect to the standard generating set and setting $\sigma(w)$ to be the image of this word G. It follows from fundamental facts about Coxeter groups that σ is well-defined. Let I be the vertex set of Γ, and for any $J \subset I$, let $W(J)$ be the subgroup of $W(\Gamma)$ generated by J. Let K be the geometric realization of the following poset:

$$\{g\sigma(W(J)) \mid g \in G, J \subset I, W(J) \text{ is finite}\}.$$

It turns out that K is isomorphic to the first barycentric subdivision of X. Let $G(J) \leq G$ be the subgroup generated by J. We associate each $g\sigma(W(J))$ with the left coset $gG(J)$, and this induces a cubical map from the first cubical subdivision of X to $|\mathcal{B}|$. However, this map is not a restriction quotient, since it has a lot of foldings (think of the special case when $G \cong \mathbb{Z}$).

5.2. **Restriction quotients with Euclidean fibers.** We reminder the reader that in this section, $W = W(\Gamma)$ will be the right-angled Coxeter group with defining graph Γ and standard generating set I. Let $\mathcal{B}$ be an arbitrary right-angled building modelled on W. Let S be the poset of spherical subsets of I and let $|\mathcal{B}|$ be the Davis realization of $\mathcal{B}$.

Our next goal is to generalize the canonical restriction quotient mentioned in the previous subsection. However, to motivate our construction, we will first consider a restriction quotient $q : Y' \to |\mathcal{B}|$ which satisfies the conclusion of Lemma 5.1, and identify several key features of q.

Let Φ be the fiber functor associated with q (see Section 4.2). For any vertices $v, w \in |\mathcal{B}|$, we will write $v \leq w$ if and only if the residue associated with v is contained in the residue associated with w.

Let S^r be the poset of spherical residues in $\mathcal{B}$. Then Φ induces a functor Φ' from S^r to CCC (Section 4.2) as follows. Each element in S^r is associated with the fiber of the corresponding vertex in $|\mathcal{B}|$. If $s, t \in \mathsf{S}^r$ are two elements such that $rank(t) = rank(s) + 1$ and $s < t$, then the associated vertices in $v_s, v_t \in |\mathcal{B}|$ are joined by an edge e_{st}. In this case $\Phi(e_{st}) \to \Phi(v_s)$ is an isomorphism, so we define the morphism $\Phi'(s) \to \Phi'(t)$ to be the map induced by $\Phi(e_{st}) \to \Phi(v_t)$. If $s, t \in \mathsf{S}^r$ are arbitrary two elements with $s \leq t$, then we find an ascending chain from s to t such that the difference between the ranks of adjacent elements in the chain is 1, and define $\Phi'(s) \to \Phi'(t)$ be the composition of those maps induced by the chain. It follows from the functor property of Φ that $\Phi'(s) \to \Phi'(t)$ does not depend on the choice of the chain, and Φ' is a functor. Recall that there is a 1-1 correspondence between elements in S^r and vertices of $|\mathcal{B}|$, so we will also view Φ' as a functor from the vertex set of $|\mathcal{B}|$ to CCC. Let $\sigma_1 \subset \sigma_2$ be faces in $|\mathcal{B}|$ and let v_i be the vertex of minimal rank in σ_i for $i = 1, 2$. Then by our construction, then morphism $\Phi(\sigma_2) \to \Phi(\sigma_1)$ is the same as $\Phi'(v_2) \to \Phi'(v_1)$.

Definition 5.3 (1-data)**.** Pick a vertex $v \in |\mathcal{B}|$ of rank 1, and let $\mathcal{R}_v$ be the associated residue. Let $\{v_\lambda\}_{\lambda \in \Lambda}$ be the collection of vertices in $|\mathcal{B}|$ which is $< v$ and let e_λ be the edge joining v and v_λ. Then there is a 1-1 correspondence between elements in $\mathcal{R}_v$ and v_λ's. Each v_λ determines a point in $\Phi(v)$ by consider the image of $\Phi(e_\lambda) \to \Phi(v)$. This induced a map $f_{\mathcal{R}_v} : \mathcal{R}_v \to \Phi(v)$. The collection of all such $f_{\mathcal{R}_v}$'s with v ranging over all rank 1 vertices of $|\mathcal{B}|$ is called the *1-data* associated with the restriction quotient $q : Y' \to |\mathcal{B}|$.

Lemma 5.4. *Pick two vertices $v, u \in |\mathcal{B}|$ of rank 1, and let $\mathcal{R}_v, \mathcal{R}_u$ be the corresponding residues. Suppose these two residues are parallel with the parallelism map given by $p : \mathcal{R}_v \to \mathcal{R}_u$. Then:*

1. *(1) $\Phi(v)$ and $\Phi(u)$, considered as convex subcomplexes of Y', are parallel.*
2. *(2) If $p' : \Phi(v) \to \Phi(u)$ is the parallelism map, then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{R}_v & \xrightarrow{p} & \mathcal{R}_u \\ {\scriptstyle f_{\mathcal{R}_v}}\big\downarrow & & {\scriptstyle f_{\mathcal{R}_u}}\big\downarrow \\ \Phi(v) & \xrightarrow{p'} & \Phi(u) \end{array}$$

Proof. It follows from Lemma 3.14 that there is a finite chain of residues, starting at $\mathcal{R}_v$ and ending at $\mathcal{R}_u$, such that adjacent elements in the chain are parallel residues in a spherical residue of rank 2. Thus we can assume without loss of generality that $\mathcal{R}_v$ and $\mathcal{R}_u$ are contained in the a spherical residue $\mathcal{S}$ of type $J = \{j, j'\}$, and we assume both $\mathcal{R}_v$ and $\mathcal{R}_u$ are j-residues.

Pick $x \in \mathcal{R}_v$. By Theorem 3.13, there is a j'-residue $\mathcal{W}$ which contains both x and $p(x)$. Let $s, w \in |\mathcal{B}|$ be the vertex corresponding to $\mathcal{S}$ and $\mathcal{W}$. Note that there is a 2-cube in $|\mathcal{B}|$ such that v, w, s are its vertices. Since Φ is 1-determined, $\mathrm{Im}(\Phi'(v) \to \Phi'(s))$ and $\mathrm{Im}(\Phi'(w) \to \Phi'(s))$ are orthogonal lines in the 2-flat $\Phi'(s)$. Moreover, the intersection these two lines is the image of $f_{\mathcal{R}_v}(x)$ under the morphism $\Phi'(v) \to \Phi'(s)$. Similarly, the images of $\Phi'(u) \to \Phi'(s)$ and $\Phi'(w) \to \Phi'(s)$ are orthogonal lines $\Phi'(s)$, and their intersection is the image of $f_{\mathcal{R}_u}(p(x))$ under $\Phi'(v) \to \Phi'(s)$. It follows that $\mathrm{Im}(\Phi'(v) \to \Phi'(s))$ and $\mathrm{Im}(\Phi'(u) \to \Phi'(s))$ are parallel, hence $\Phi(v)$ and $\Phi(u)$, considered as convex subcomplexes of Y', are parallel. Moreover, since image of $f_{\mathcal{R}_v}(x)$ under $\Phi'(v) \to \Phi'(s)$ and the image of $f_{\mathcal{R}_u}(p(x))$ under $\Phi'(v) \to \Phi'(s)$ are in the line $\mathrm{Im}(\Phi'(w) \to \Phi'(s))$, the diagram in (2) commutes. ☐

Pick a vertex $u \in |\mathcal{B}|$ of rank $= k$ and let $\mathcal{R}_u$ be the corresponding J-residue with $J = \cup_{i=1}^{k} j_i$. Then there is a map $f_{\mathcal{R}_u} : \mathcal{R}_u \to \Phi'(\mathcal{R}_u) = \Phi(u)$ defined by considering $\Phi'(x) \to \Phi'(\mathcal{R}_u)$ for each element $x \in \mathcal{R}_u$. This map coincides with the $f_{\mathcal{R}_u}$ defined before when u is rank 1. For $1 \leq i \leq k$, let $\mathcal{R}_i$ be a j_i-residue in $\mathcal{R}_u$. Since Φ is 1-determined, $\{\mathrm{Im}(\Phi'(\mathcal{R}_i) \to \Phi'(\mathcal{R}_u))\}_{i=1}^{k}$ are mutually orthogonal lines in $\Phi'(\mathcal{R}_u)$. This induces an isomorphism

$$i : \prod_{i=1}^{k} \Phi'(\mathcal{R}_i) \to \Phi'(\mathcal{R}_u). \tag{5.5}$$

The following is a consequence of (2) of Lemma 5.4.

Corollary 5.6. *The map $f_{\mathcal{R}_u}$ satisfies $f_{\mathcal{R}_u} = i \circ (\prod_{i=1}^{k} f_{\mathcal{R}_i}) \circ g$, where $g : \mathcal{R}_u \to \prod_{i=1}^{k} \mathcal{R}_i$ is the map in Theorem 3.13. In this case, we will write $f_{\mathcal{R}_u} = \prod_{i=1}^{k} f_{\mathcal{R}_i}$ for simplicity.*

Pick $J' \subset J$ and let $\mathcal{R}'_u$ be a J'-residue in $\mathcal{R}_u$. By Theorem 3.13, $\mathcal{R}'_u = \prod_{i\in J'} \mathcal{R}_i \times \prod_{i\notin J'}\{a_i\}$ for $a_i \in \mathcal{R}_i$. Then the following is a consequence of Corollary 5.6 and the functorality of Φ'.

Corollary 5.7. *Let h be the morphism between $\Phi'(\mathcal{R}_{u'})$ and $\Phi'(\mathcal{R}_u) = \Phi'(\mathcal{R}_{u'}) \times \prod_{i\notin J'} \Phi'(\mathcal{R}_i)$. Then for $x \in \Phi'(\mathcal{R}_{u'})$, we have $h(x) = \{x\} \times \prod_{i\notin J'}\{f_{\mathcal{R}_i}(a_i)\}$.*

5.3. **Construction of the $\mathbb{Z}$-blow-up.** In the previous section, we started from a restriction quotient $q : Y' \to |\mathcal{B}|$, and produced associated 1-data (Definition 5.3), which is compatible with parallelism in the sense of Lemma 5.4. In this section, we will consider the inverse, namely we want construct a restriction quotient from this data.

Let $\Lambda_{\mathcal{B}}$ be the collection of parallel sets of i-residues in $\mathcal{B}$ (i could be any element in I). There is another type map T which maps a spherical J-residue $\mathcal{R}$ to $\{\lambda \in \Lambda_{\mathcal{B}} \mid \lambda \text{ contains a representative in } \mathcal{R}\}$. In other words, let $\mathcal{R} \cong \prod_{i\in I} \mathcal{R}_i$ be the product decomposition as in Theorem 3.13, where each $\mathcal{R}_i$ is an i-residue in $\mathcal{R}$ ($i \in J$). Then $T(\mathcal{R})$ is the collection of parallel sets represented by those $\mathcal{R}_i$'s. Let $\mathbb{Z}^{T(\mathcal{R})}$ be the collection of maps from $T(\mathcal{R})$ to $\mathbb{Z}$, and let $\mathbb{Z}^{\emptyset}$ be a single point.

Our goal in this section is to construct a restriction quotient from the following data.

Definition 5.8 (Blow-up data)**.** For each i-residue $\mathcal{R} \subset \mathcal{B}$, we associate a map $h_{\mathcal{R}} : \mathcal{R} \to \mathbb{Z}^{T(\mathcal{R})}$ such that if two i-residues $\mathcal{R}_1$ and $\mathcal{R}_2$ are parallel, let $h_{12} : \mathcal{R}_1 \to \mathcal{R}_2$ be the parallelism map, then $h_{\mathcal{R}_1} = h_{\mathcal{R}_2} \circ h_{12}$.

If $\mathcal{R}$ is a spherical residue with product decomposition given by $\mathcal{R} \cong \prod_{i\in I} \mathcal{R}_i$, then the maps $h_{\mathcal{R}_i} : \mathcal{R}_i \to \mathbb{Z}$ induces a map $h_{\mathcal{R}} : \mathcal{R} \to \mathbb{Z}^{T(\mathcal{R})}$. It follows from the definition of $h_{\mathcal{R}}$, and the discussion after Definition 3.11 that if $\mathcal{R}, \mathcal{R}' \in C$ are parallel and let $h : \mathcal{R} \to \mathcal{R}'$ be the parallelism map, then $h_{\mathcal{R}} = h_{\mathcal{R}'} \circ h$.

The following result is a consequence of Theorem 3.13:

Lemma 5.9. *Let $\mathcal{T} \in C$ be an H-residue. Let $g : \mathcal{T} \cong \prod_{i=1}^n \mathcal{T}_i$ be the product decomposition induced by $H = \sqcup_{i=1}^n H_i$ (see Theorem 3.13). Then $h_{\mathcal{T}} = (\prod_{i=1}^n h_{\mathcal{T}_i}) \circ g$.*

To simplify notation, we will write $h_{\mathcal{T}} = \prod_{i=1}^n h_{\mathcal{T}_i}$ instead of $h_{\mathcal{T}} = (\prod_{i=1}^n h_{\mathcal{T}_i}) \circ g$.

Let J and $\mathcal{R} = \prod_{i\in J} \mathcal{R}_i$ be as before. A J'-residue $\mathcal{R}' \subset \mathcal{R}$ can be expressed as $(\prod_{i\in J'} \mathcal{R}_i) \times (\prod_{i\in J\setminus J'}\{c_i\})$, here c_i is a chamber in

$\mathcal{R}_i$. We define an inclusion $h_{\mathcal{R}'\mathcal{R}} : \mathbb{Z}^{T(\mathcal{R}')} \to \mathbb{Z}^{T(\mathcal{R})}$ by $h_{\mathcal{R}'\mathcal{R}}(a) = \{a\} \times \prod_{i\in J\setminus J'}\{h_{\mathcal{R}_i}(c_i)\}$. Since $h_{\mathcal{R}} = h_{\mathcal{R}'} \times (\prod_{i\in J\setminus J'} h_{\mathcal{R}_i})$, $h_{\mathcal{R}'\mathcal{R}}$ fits into the following commutative diagram:

$$\begin{array}{ccc} \mathcal{R}' & \longrightarrow & \mathcal{R} \\ {\scriptstyle h_{\mathcal{R}'}}\downarrow & & {\scriptstyle h_{\mathcal{R}}}\downarrow \\ \mathbb{Z}^{T(\mathcal{R}')} & \xrightarrow{h_{\mathcal{R}'\mathcal{R}}} & \mathbb{Z}^{T(\mathcal{R})} \end{array}$$

Suppose $\mathcal{R}''$ is a J''-residue such that $\mathcal{R}'' \subset \mathcal{R}' \subset \mathcal{R}$. Since $h_{\mathcal{R}} = h_{\mathcal{R}'} \times (\prod_{i\in J\setminus J'} h_{\mathcal{R}_i}) = h_{\mathcal{R}''} \times (\prod_{i\in J'\setminus J''} h_{\mathcal{R}_i}) \times (\prod_{i\in J\setminus J'} h_{\mathcal{R}_i})$, we have

$$h_{\mathcal{R}''\mathcal{R}} = h_{\mathcal{R}'\mathcal{R}} \circ h_{\mathcal{R}''\mathcal{R}'}. \tag{5.10}$$

Now we define a contravariant functor Ψ : Face($|\mathcal{B}|$) $\to$ CCC as follows. Let f be a face of $|\mathcal{B}|$ and let $v_f \in f$ be unique vertex which has minimal rank among the vertices of f. Let $\mathcal{R}_f \subset \mathcal{B}$ be the residue associated with v_f. We define $\Psi(f) = \mathbb{R}^{T(\mathcal{R}_f)}$ ($\mathbb{R}^{\emptyset}$ is a single point), here $\mathbb{R}^{T(\mathcal{R}_f)}$ is endowed with the standard cubical structure and we identify $\mathbb{Z}^{T(\mathcal{R}_f)}$ with the 0-skeleton of $\mathbb{R}^{T(\mathcal{R}_f)}$.

An inclusion of faces $f \to f'$ induces an inclusion $\mathcal{R}_{f'} \to \mathcal{R}_f$. We define the morphism $\Psi(f') \to \Psi(f)$ to be the embedding induce by $h_{\mathcal{R}_{f'}\mathcal{R}_f} : \mathbb{Z}^{T(\mathcal{R}_{f'})} \to \mathbb{Z}^{T(\mathcal{R}_f)}$.

Lemma 5.11. Ψ *is contravariant functor.*

Proof. It is easy to check that passing from an inclusion of faces $f \to f'$ to $\mathcal{R}_{f'} \to \mathcal{R}_f$ is a functor. And it follows from (5.10) that passing from $\mathcal{R}_{f'} \to \mathcal{R}_f$ to $h_{\mathcal{R}_{f'}\mathcal{R}_f} : \mathbb{Z}^{T(\mathcal{R}_{f'})} \to \mathbb{Z}^{T(\mathcal{R}_f)}$ is a functor. □

Lemma 5.12. Ψ *is 1-determined.*

Proof. Let $\sigma \subset |\mathcal{B}|$ be a face and pick a vertex $v \in \sigma$. Let $\{v_i\}_{i=1}^k$ be the vertices in σ that are adjacent to v along an edge e_i. Let $\sigma_{\leq v}$ be the sub-cube of σ that is spanned by e_i's such that $v_i \geq v$. We define $\sigma_{>v}$ similarly ($\sigma_{>v}$ could be empty). Then $\sigma = \sigma_{\leq v} \times \sigma_{>v}$. Moreover, v is the maximal vertex in $\sigma_{\leq v}$ and the minimum vertex in $\sigma_{>v}$. Note that $\Psi(e_i) \to \Psi(v)$ is an isometry if $v_i > v$. Thus it suffices to consider the case where v is the maximal vertex of σ.

Let v_m be the minimal vertex of σ. Note that $\mathrm{Im}(\Psi(\sigma) \to \Psi(v)) \subset \cap_{i=1}^k \mathrm{Im}(\Psi(e) \to \Psi(v))$ is a cubical convex embedding of Euclidean subspaces, it suffices to show they have the same dimension. Let $\mathcal{R}(v) \subset C$ be the residue corresponding to the vertex v. Note that $T(\mathcal{R}(v_m)) = \cap_{i=1}^k T(\mathcal{R}(v_i))$ (T is the type map defined on the beginning of Section

5.3). Thus the dimension of $\cap_{i=1}^{k} \operatorname{Im}(\Psi(e) \to \Psi(v))$ equals to the cardinality of $T(\mathcal{R}(v_m))$, which is the dimension of $\operatorname{Im}(\Psi(\sigma) \to \Psi(v))$. □

Ψ is called the *fiber functor associated with the blow-up data* $\{h_{\mathcal{R}}\}$, and the restriction quotient $q : Y \to |\mathcal{B}|$ which arises from the fiber functor Ψ (see Theorem 4.15) is called the *restriction quotient associated with the blow-up data* $\{h_{\mathcal{R}}\}$. It is clear from the construction that the 1-data of q (Definition 5.3) is the blow-up data $\{h_{\mathcal{R}}\}$ (we naturally identify $\mathbb{Z}^{T(\mathcal{R})}$'s in the blow-up data with the 0-skeleton of the q-fibers of rank 1 vertices in $|\mathcal{B}|$). We summarize the above discussion in the following theorem.

Theorem 5.13. *Given the blow-up data* $\{h_{\mathcal{R}}\}$ *as in Definition 5.8, there exists a restriction quotient* $q : Y \to |\mathcal{B}|$ *whose 1-data is the blow-up data we start with.*

Remark 5.14. Here we blow up the building $\mathcal{B}$ with respect to a collection of $\mathbb{Z}$'s since we want to apply the construction for RAAG's. However, in other cases, it may be natural to blow up with respect to other objects. Here is a variation. For each parallel class of rank 1 residues $\lambda \in \Lambda_{\mathcal{B}}$, we associate a $CAT(0)$ cube complex Z_λ. For each rank 1 residue $\mathcal{R}$ in the class λ, we define a map $h_{\mathcal{R}}$ which assigns each element of $\mathcal{R}$ a convex subcomplex of Z_λ. We require these $\{h_{\mathcal{R}}\}$ to be compatible with parallelism between rank 1 residues. Given this set of blow-up data, we can repeat the previous construction to obtain a restriction quotient over $|\mathcal{B}|$. It is also possible to blow-up buildings with respect to spaces more general than $CAT(0)$ cube complexes. We give a slightly different approach in Section 8.

Now we show that the construction in this section is indeed a converse to Section 5.2 in the following sense. Let $q : Y' \to |\mathcal{B}|$ be a restriction quotient as in Section 5.2 and let Φ and Φ' be the functors introduced there. For each vertex $v \in |\mathcal{B}|$ of rank 1 and its associated residue $\mathcal{R}_v$, we pick an isometric embedding $\eta_v : \mathbb{Z}^{T(\mathcal{R}_v)} \to \Phi(v)$ such that its image is vertex set of $\Phi(v)$. We also require these η_v's respect parallelism. More precisely, let $u \in |\mathcal{B}|$ be a vertex of rank 1 such that $\Phi(v)$ and $\Phi(u)$ (understood as subcomplexes of Y') are parallel with the parallelism map given by $p : \Phi(v) \to \Phi(u)$. Then $p \circ \eta_v = \eta_u$ (note that $T(\mathcal{R}_v) = T(\mathcal{R}_u)$ by Lemma 5.4).

Let Ψ be the functor constructed in this section from the blow-up data $\{h_{\mathcal{R}_v} = \eta_v^{-1} \circ f_{\mathcal{R}_v} : \mathcal{R}_v \to \mathbb{Z}^{T(\mathcal{R}_v)}\}_{v \in |\mathcal{B}|}$, here v ranges over all vertices of rank 1 in $|\mathcal{B}|$, $\mathcal{R}_v$ is the residue associated with v and $f_{\mathcal{R}_v}$ is the map in Definition 5.3. Pick a face $\sigma \in |\mathcal{B}|$ and let $u \in \sigma$ be the vertex of minimal rank. Let $\mathcal{R}_u$ be the associated J-residue with its product decomposition given by $\mathcal{R}_u = \prod_{j\in J}\mathcal{R}_{v_j}$ (v_j's are rank 1 vertices $\leq u$). Let $\xi_\sigma : \Psi(\sigma) \to \Phi(\sigma)$ be the isometry induced by

$$\prod_{j\in J}\eta_{v_j} : \mathbb{Z}^{T(\mathcal{R}_u)} \to \prod_{j\in J}\Phi(v_j)$$

and the product decomposition $\prod_{j\in J}\Phi(v_j) \cong \Phi(u) \cong \Phi(\sigma)$ which comes from (5.5). The following is a consequence of Corollary 5.6, Corollary 5.7 and the discussion in this section.

Corollary 5.15. *The maps $\{\xi_\sigma\}_{\sigma\in \mathrm{Face}(|\mathcal{B}|)}$ induce a natural isomorphism between Φ and Ψ. Thus for any restriction quotient $q : Y' \to |\mathcal{B}|$ which satisfies the conclusion of Lemma 5.1, if q' is the restriction quotient whose blow-up data is the 1-data of q, then q' is equivalent to q up to a natural isomorphism between their fiber functors.*

Corollary 5.16. *Let $q : Y \to |\mathcal{B}|$ be a restriction quotient which satisfies the conclusion of Lemma 5.1. Let $\mathcal{B} \cong \mathcal{B}_1 \times \mathcal{B}_2$ be a product decomposition of the building $\mathcal{B}$ induced by the join decomposition $\Gamma = \Gamma_1 \circ \Gamma_2$ of the defining graph of the associated right-angled Coxeter group. Then there are two restriction quotients $q_1 : Y_1 \to |\mathcal{B}_1|$ and $q_2 : Y_2 \to |\mathcal{B}_2|$ such that $Y = Y_1 \times Y_2$ and $q = q_1 \times q_2$. Moreover, q_1 and q_2 also satisfy the conclusion of Lemma 5.1.*

Proof. By Corollary 5.15, we can assume q is the restriction quotient associated with a set of blow-up data $\{h_{\mathcal{R}}\}$. For every $\mathcal{B}_1$-slice in $\mathcal{B}$, we can restrict $\{h_{\mathcal{R}}\}$ to $\mathcal{B}_1$ to obtain a blow-up data for $\mathcal{B}_1$. This does not depend on our choice of the $\mathcal{B}_1$-slice, since the blow-up data respects parallelism. We obtain a blow-up data for $\mathcal{B}_2$ in a similar way. It follows from the above construction that the fiber functor associated with $\{h_{\mathcal{R}}\}$ is the product of the fiber functors associated the blow-up data on $\mathcal{B}_1$ and $\mathcal{B}_2$. Thus this corollary is a consequence of Lemma 4.16. □

5.4. **More properties of the blow-up buildings.** In this section, we look at the restriction quotient $q : Y \to |\mathcal{B}|$ associated with the blow-up data $\{h_{\mathcal{R}}\}$ as in Definition 5.8 (or equivalently, a restriction quotient $q : Y \to |\mathcal{B}|$ which satisfies the conclusion of Lemma 5.1) in more detail, and record several basic properties of Y. A hurried reader can go through Definition 5.17, then proceed directly to Section 5.5 and come back to this part later.

Definition 5.17. A vertex $y \in Y$ is of *rank k* if $p(y)$ is a vertex of rank k. Thus q induces a bijection between rank 0 vertices in Y and rank 0 vertices in $|\mathcal{B}|$. Since rank 0 vertices in $|\mathcal{B}|$ can be identified with chambers in $\mathcal{B}$, q^{-1} induces a well-defined map $q^{-1}: \mathcal{B} \to Y$ from the set of chambers of $\mathcal{B}$ (or rank 0 vertices of $|\mathcal{B}|$) to rank 0 vertices in Y.

Lemma 5.18. *For any residue $\mathcal{R} \subset \mathcal{B}$, we view $\mathcal{R}$ as a building and restrict the blow-up data over $\mathcal{B}$ to a blow-up data over $\mathcal{R}$. Let $q_{\mathcal{R}}: Y_{\mathcal{R}} \to |\mathcal{R}|$ be the associated restriction quotient. Then there exists an isometric embedding $i: Y_{\mathcal{R}} \to Y$ which fits into the following commutative diagram:*

$$\begin{array}{ccc} Y_{\mathcal{R}} & \xrightarrow{i} & Y \\ {\scriptstyle q_{\mathcal{R}}}\big\downarrow & & {\scriptstyle q}\big\downarrow \\ |\mathcal{R}| & \xrightarrow{i'} & |\mathcal{B}| \end{array}$$

Moreover, $i(Y_{\mathcal{R}}) = q^{-1}(i'(|\mathcal{R}|))$.

The lemma is a direct consequence of the construction in Section 5.3.

Pick a vertex $v \in |\mathcal{B}|$. The *downward complex of* v is the smallest convex subcomplex of $|\mathcal{B}|$ which contains all vertices which are $\leq v$. If $\mathcal{R}_v$ is the residue associated with v, then the downward complex is the image of the embedding $|\mathcal{R}_v| \hookrightarrow |\mathcal{B}|$. The next result follows from Lemma 5.18 and Corollary 5.16.

Lemma 5.19. *Let D_v be the downward complex of a vertex $v \in \mathcal{B}$ and let $\mathcal{R}_v = \prod_{i=1}^{k} \mathcal{R}_i$ be the product decomposition of residue associated with v. Then $q^{-1}(D_v)$ is isomorphic to the product of the mapping cylinders of $\mathcal{R}_i \xrightarrow{h_{\mathcal{R}_i}} \mathbb{Z}^{T(\mathcal{R}_i)} \to \mathbb{R}^{T(\mathcal{R}_i)}$ $(1 \leq i \leq k)$.*

Lemma 5.20.

(1) *If $h_{\mathcal{R}}^{-1}(x)$ is finite for any rank 1 residue $\mathcal{R}$ and $x \in \mathbb{Z}^{T(\mathcal{R})}$, then Y is locally finite. If there is a uniform upper bound for the cardinality of $h_{\mathcal{R}}^{-1}(x)$, then Y is uniformly locally finite.*
(2) *If there exists $D > 0$ such that the image of each $h_{\mathcal{R}}$ is D-dense in $\mathbb{Z}^{T(\mathcal{R})}$, then there exists D' which depends on D and the dimension of $|\mathcal{B}|$ such that the collection of inverse images of rank 0 vertices in $|\mathcal{B}|$ is D'-dense in Y.*

Proof. We prove (1) first. Pick a vertex $y \in Y$. Let $v = q(y)$. It suffices to show the set of edges in $|\mathcal{B}|$ which contain v, and can be lifted to an edge in Y that contains y, is finite. Since there are only finitely many vertices in $|\mathcal{B}|$ which are $\geq v$, it suffices to consider the edges of the form $\overline{v_\lambda v}$ with $v_\lambda < v$. It follows from our assumption and Lemma 5.19

that there are only finitely many such edges which have the required lift. The proof of uniform local finiteness is similar.

To see (2), notice that $\cup_{v\in|\mathcal{B}|}\Psi(v)$ is 1-dense in Y, here v ranges over all vertices of $|\mathcal{B}|$. It follows from Lemma 5.19 that every point in $\Psi(v)$ be can approximated by the inverse image of some rank 0 vertex up to distance D'. □

Next we discuss the relation between Y and the exploded Salvetti complex $S_e = S_e(\Gamma)$ introduced in Section 5.1. Let Ψ be the fiber functor associated with $q : Y \to |\mathcal{B}|$.

First we label each vertex $v \in Y$ by a clique in Γ as follows. Recall that $q(v)$ is associated with a J-residue $\mathcal{R} \subset \mathcal{B}$, where J is the vertex set of a clique in Γ. Thus we label v by this clique. We also label each vertex of S_e by a clique. Any vertex $v \in S_e$ is contained in a unique standard torus. Recall that a standard torus arises from a clique in Γ, thus we label v by this clique. Note some vertices of Y and S_e are labelled by the empty set. There is a unique label-preserving map $p : Y^{(0)} \to S_e^{(0)}(\Gamma)$.

An edge in Y or S_e is *horizontal* if the labels on its two endpoints are different, otherwise, this edge is *vertical*. When $Y = X_e$, this definition coincides with the one in Section 5.1. Moreover, horizontal (or vertical) edges in X_e are lifts of horizontal (or vertical) edges in S_e.

Horizontal edges in Y are exactly those ones whose dual hyperplanes are mapped by q to hyperplanes in $|\mathcal{B}|$, and the q-image of any vertical edge is a point. Now we label each edge vertical edge of Y by vertices in Γ as follows. Pick vertical edge $e \subset Y$ and let $v = q(e)$. Let $\mathcal{R} = \prod_{i=1}^{k}\mathcal{R}_i$ be the product decomposition of the residue associated with v. There is a corresponding product decomposition $\Psi(v) = \prod_{i=1}^{k}\ell_i$, where ℓ_i is a line which is parallel to $\Psi(v_i)$, here $v_i \in |\mathcal{B}|$ is the vertex associated with $\mathcal{R}_i$, and we view $\Psi(v_i)$ and $\Psi(v)$ as subcomplexes of Y. If e is in the ℓ_i-direction, then we label e by the type of $\mathcal{R}_i$, which is a vertex in Γ. A case study implies if two vertical edges are the opposite sides of a 2-cube, then they have the same label. Hence all parallel vertical edges have the same label. Now we label vertical edges in S_e. Recall that the map $S_e \to S(\Gamma)$ induces a 1-1 correspondence between vertical edges in S_e and edges in $S(\Gamma)$, and edges in $S(\Gamma)$ are labelled by vertices of Γ. This induces a labelling of vertical edges in S_e.

We pick an orientation for each vertical edge in S_e, and orient every vertical edge in Y in the following way. A *vertical line* is a geodesic line made of vertical edges. It is easy to see every vertical edge is contained in a vertical line. For two vertical ℓ_1 and ℓ_2, if there exist edges $e_i \in \ell_i$ for $i = 1, 2$ such that they are parallel, then ℓ_1 and ℓ_2 are parallel. To see this, it suffices to consider the case where e_1 and e_2 are the opposite sides of a 2-cube, and this follows from a similar case study as before. Now we pick an orientation for each parallel class of vertical lines, and this induces well-define orientation on each vertical edge of Y, moreover, this orientation respects parallelism of edges.

There is a unique way to extend $p : Y^{(0)} \to S_e^{(0)}(\Gamma)$ to $p : Y^{(1)} \to S_e^{(1)}(\Gamma)$ such that p preserves the orientation and labelling of vertical edges. One can further extend p to higher-dimensional cells as follows. A cube $\sigma \subset Y$ is of *type* (m, n) if σ is the product of m vertical edges and n horizontal edges. We extend p according to the type:

(1) If σ is of type $(m, 0)$, then we can define p on σ since the orientation of vertical edges in Y respects parallelism, and p preserves labelling and orientation of vertical edges. In this case, $p(\sigma)$ is an m-dimensional standard torus.
(2) If σ is of type $(0, n)$, then we can define p on σ since p preserves labelling of vertices. In this case, $p(\sigma) \cong [0, 1]^n$.
(3) If σ is of type (m, n), then we can define p on σ for similar reasons as before. In this case, $p(\sigma) \cong \mathbb{T}^m \times [0, 1]^n$.

Pick vertex $y \in Y$, then p induces a simplicial map between the vertex links $p_y : Lk(y, Y) \cong Lk(p(y), S_e)$. The above case study implies p_y is a *combinatorial* map, i.e. p_y maps each simplex isomorphically onto its image.

Theorem 5.21. *If each map $h_{\mathcal{R}}$ in the blow-up data is a bijection, then Y is isomorphic to $X_e = X_e(\Gamma)$, which is the universal cover of the exploded Salvetti complex $S_e = S_e(\Gamma)$.*

Proof. We prove the theorem by showing $p : Y \to S_e$ is a covering map. It suffices to show for each vertex $y \in Y$, the above map p_y is an isomorphism. Suppose y is labelled by a clique $\Delta \subset \Gamma$. We look at edges which contain y, which fall into three classes:

(1) vertical edges;
(2) horizontal edges whose other endpoints are labelled by cliques in Δ;
(3) horizontal edges whose other endpoints are labelled by cliques that contain Δ.

Note that there is a 1-1 correspondence between edges in (3) and cliques which contain Δ and have exactly one vertex not in Δ. For any clique

$\Delta' \subset \Delta$ which contains all but one vertex of Δ, there exists a unique edge in (2) such that its other endpoint is labelled by Δ', since if such edge does not exist, then some $h_{\mathcal{R}}$ will not be surjective; if there exists more than one such edges, then some $h_{\mathcal{R}}$ will not be injective. Thus there is a 1-1 correspondence between horizontal edges which contains y and horizontal edges which contains $p(y)$. Hence p_y induces bijection between the 0-skeletons. Moreover, edges in (3) are orthogonal to edges in (1) and (2), so a case study implies if two edges at $p(y)$ form the corner of a 2-cube, then their lifts at y (if any exist) also form the corner of a 2-cube. It follows that p_y induces isomorphism between the 1-skeletons. Since both $Lk(y, Y)$ and $Lk(p(y), S_e)$ are flag complexes, p_y is an isomorphism. □

Remark 5.22. If each map $h_{\mathcal{R}}$ is injective (or surjective), then p is locally injective (or locally surjective).

Corollary 5.23. *Let $\mathcal{B}_1 = \mathcal{B}_1(\Gamma)$ and $\mathcal{B}_2 = \mathcal{B}_2(\Gamma)$ be two right-angled Γ-buildings with countably infinite rank 1 residues. Then they are isomorphic as buildings.*

Proof. We pick a blow-up for $\mathcal{B}_1$ such that each map in the blow-up data is a bijection. Let $Y \to |\mathcal{B}_1|$ be the associated restriction quotient and let $p : Y \to S_e$ be the covering map as in Theorem 5.21. Note that p sends vertical edges to vertical edges and horizontal edges to horizontal edges, and p preserves the labelling of vertices and edges. So does the lift $\tilde{p} : Y \to X_e$ of p. Lemma 4.3 implies $\tilde{p}$ descends to a cubical isomorphism $|\mathcal{B}_1| \to |\mathcal{B}|$, where $|\mathcal{B}|$ is the building associated with $G(\Gamma)$. Since $\tilde{p}$ is label-preserving, this cubical isomorphism induces a building isomorphism $\mathcal{B}_1 \to \mathcal{B}$. Similarly, we can obtain a building isomorphism $\mathcal{B}_2 \to \mathcal{B}$. Hence the corollary follows. □

Theorem 5.24. *Suppose Γ does not admit a join decomposition $\Gamma = \Gamma_1 \circ \Gamma_2$ where that Γ_1 is a discrete graph with more than one vertex. If $\mathcal{B}$ is a Γ-building and $q : Y \to |\mathcal{B}|$ is a restriction quotient with blow-up data $\{h_{\mathcal{R}}\}$, then any automorphism $\alpha : Y \to Y$ descends to an automorphism $\alpha' : |\mathcal{B}| \to |\mathcal{B}|$.*

Proof. By Lemma 4.3, it suffices to show α preserves the rank (Definition 5.17) of vertices of Y. Let $F(\Gamma)$ be the flag complex of Γ. Here we change the label of each vertex in Y from some clique in Γ to the associated simplex in $F(\Gamma)$. Suppose $y \in Y$ is vertex of rand k labelled by Δ. Then Lemma 5.19 and the proof of Theorem 5.21 imply $Lk(y, Y) \cong K_1 * K_2 * \cdots * K_k * Lk(\Delta, F(\Gamma))$, where each K_i is discrete with cardinality ≥ 2, and $Lk(\Delta, F(\Gamma))$ is understood to be $F(\Gamma)$ when

$\Delta = \emptyset$. Note that $\{K_i\}_{i=1}^k$ comes from vertices adjacent to y of rank $\leq k$, and $Lk(\Delta, F(\Gamma))$ comes from vertices adjacent to y of rank $> k$. Thus α preserves the collection of rank 0 vertices.

Now we assume α preserves the collection of rank i vertices for $i \leq k-1$. A rank k vertex in Y is of *type I* if it is adjacent to a vertex of rank $k-1$, otherwise it is a vertex of *type II*. It is clear that α preserves the collection of rank k vertices of type I. Before we deal with type II vertices, we need the following claim. Suppose $w \in Y$ is a vertex of rank k such that $\alpha(w)$ is also of rank k. If there exist k vertices $\{z_i\}_{i=1}^k$ adjacent to w such that

(1) $rank(z_i) \leq k$ and $rank(\alpha(z_i)) \leq k$;
(2) the edges $\{\overline{z_i w}\}_{i=1}^k$ are mutually orthogonal,

then $rank(\alpha(z)) \leq k$ for any z adjacent to w with $rank(z) \leq k$.

Let $w' = \alpha(w)$. Suppose w and w' are labelled by Δ and Δ'. Then α induces an isomorphism between the links of w and w' in Y:

$$\alpha_* : K_1 * \cdots * K_k * Lk(\Delta, F(\Gamma)) \to K'_1 * \cdots * K'_k * Lk(\Delta', F(\Gamma)).$$

Each edge $\overline{z_i w}$ gives rise to a vertex in K_i, and each edge $\overline{\alpha(z_i) w'}$ gives rise to a vertex in K'_i. Thus $\alpha_*(K_1 * \cdots * K_k) = K'_1 * \cdots * K'_k$. Since the edge $\overline{zw}$ gives rise to a vertex in $K_1 * \cdots * K_k$, the edge $\overline{\alpha(z) w'}$ gives rise to a vertex in $K'_1 * \cdots * K'_k$. Then $\alpha(z)$ is of rank $\leq k$.

Let $y \in Y$ be a rank k vertex of type II. Then there exists an edge path ω from y to a type I vertex y_1 such that every vertex in ω is of rank k. Let $\{y_i\}_{i=1}^m$ be consecutive vertices in ω such that $y_m = y$. Note that there are k vertices of rank $k-1$ adjacent to y_1. By the induction assumption, they are send to vertices of rank $k-1$ by α. Moreover, $rank(\alpha(y_1)) = k$ since y_1 is of type I. Thus the assumption of the claim is satisfied for y_1. Then $rank(\alpha(y_2)) \leq k$, hence $rank(\alpha(y_2)) = k$ by the induction assumption. Next we show y_2 satisfies the assumption of the claim. Let $\{z_i\}_{i=1}^k$ be vertices of rank k such that they are adjacent to y_1 and $\{\overline{z_i y_1}\}_{i=1}^k$ are mutually orthogonal. We also assume $y_2 = z_1$. Then $rank(\alpha(z_i)) = k$ for all i. Hence all $\alpha(\overline{z_i y_1})$'s are vertical edges. For $i \geq 2$, let z'_i be the vertex adjacent to y_2 such that $\overline{z'_i y_2}$ and $\overline{z_i y_1}$ are parallel. Then $\alpha(\overline{z'_i y_2})$ is a vertical edge for $i \geq 2$. Thus $rank(\alpha(z'_i)) = k$ and the assumption of the claim is satisfies for y_2. We can repeat this argument finite many times to deduce that $rank(\alpha(y)) = k$. $\square$

Remark 5.25. If the assumption on Γ in Theorem 5.24 is not satisfied, then there exists a blow-up $Y \to |\mathcal{B}|$ and an automorphism of Y such that it does not descend to an automorphism of $|\mathcal{B}|$. By Corollary 5.16, it suffices to construct an example in the case when Γ be a discrete graph with n vertices with $n \geq 2$. If $n \geq 3$, then we define each $h_{\mathcal{R}}$ to be a surjective map such that the inverse image of each point has $n-2$ points. Then Y is a tree with valence $= n$. If $n = 2$, then we define $h_{\mathcal{R}}$ to be an injective map whose image is the set of even integers. Then Y is isomorphic to the first subdivision of a tree of valence 3. In both case, it is not hard to find an automorphism of Y which maps some vertex of rank 0 to a vertex of rank 1.

5.5. **Morphisms between blow-up data.** Let $\mathcal{B}$ and $\mathcal{B}'$ be two buildings modelled on the same right-angled Coxeter group $W(\Gamma)$. An isomorphism $\eta : |\mathcal{B}| \to |\mathcal{B}'|$ is *rank-preserving* if for each vertex $v \in |\mathcal{B}|$, v and $\eta(v)$ have the same rank. Note that such η induces a bijection $\eta' : \mathcal{B} \to \mathcal{B}'$ which preserves the spherical residues. Conversely, every bijection $\mathcal{B} \to \mathcal{B}'$ which preserves the spherical residues induces a rank-preserving isomorphism $|\mathcal{B}| \to |\mathcal{B}'|$. Note that η' maps parallel residues of rank 1 to parallel residues of rank 1, thus η' induces a bijection $\bar{\eta} : \Lambda_{\mathcal{B}} \to \Lambda_{\mathcal{B}'}$, where $\Lambda_{\mathcal{B}}$ and $\Lambda_{\mathcal{B}'}$ denote the collection of parallel classes of residues of rank 1 in $\mathcal{B}$ and $\mathcal{B}'$ respectively (see Section 5.3).

Definition 5.26 (η-isomorphism)**.** Suppose the blow-up data (Definition 5.8) of $|\mathcal{B}|$ and $|\mathcal{B}'|$ are given by $\{h_{\mathcal{R}}\}$ and $\{h'_{\mathcal{R}}\}$ respectively. An *η-isomorphism* between the blow-up data is defined to be a collection of isometries $\{f_\lambda : \mathbb{Z}^\lambda \to \mathbb{Z}^{\bar{\eta}(\lambda)}\}_{\lambda \in \Lambda_{\mathcal{B}}}$ such that the following diagram commutes for every rank 1 residue $\mathcal{R} \subset \mathcal{B}$:

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{h_{\mathcal{R}}} & \mathbb{Z}^{T(\mathcal{R})} \\ {\scriptstyle \eta'}\downarrow & & \downarrow {\scriptstyle f_{T(\mathcal{R})}} \\ \eta'(\mathcal{R}) & \xrightarrow{h'_{\eta'(\mathcal{R})}} & \mathbb{Z}^{\bar{\eta}(T(\mathcal{R}))} \end{array}$$

Here T is the type map defined in the beginning of Section 5.3. The map $h_{\mathcal{R}}$ is *nondegenerate* if its image contains more than one point. In this case, if $f_{T(\mathcal{R})}$ exists, then it is unique. If $h_{\mathcal{R}}$ is degenerate, then we have two choices for $f_{T(\mathcal{R})}$.

Let $\eta_1 : |\mathcal{B}_1| \to |\mathcal{B}_2|$, $\eta_2 : |\mathcal{B}_2| \to |\mathcal{B}_3|$ and $\eta : |\mathcal{B}_1| \to |\mathcal{B}_3|$ be rank-preserving isomorphisms such that $\eta = \eta_2 \circ \eta_1$. We fix a blow-up data for each $\mathcal{B}_i$. Let $\{f_\lambda : \mathbb{Z}^\lambda \to \mathbb{Z}^{\bar{\eta}_1(\lambda)}\}_{\lambda \in \Lambda_{\mathcal{B}_1}}$ and $\{g_\lambda : \mathbb{Z}^\lambda \to \mathbb{Z}^{\bar{\eta}_2(\lambda)}\}_{\lambda \in \Lambda_{\mathcal{B}_2}}$ be the η_1-isomorphism and η_2-isomorphism between the corresponding blow-up data. We define the *composition* of them to be $\{g_{\bar{\eta}_1(\lambda)} \circ f_\lambda\}_{\lambda\in\Lambda}$, which turns out to be an η-isomorphism.

Let Ψ and Ψ' be the fiber functor associated with the blow-up data $\{h_{\mathcal{R}}\}$ and $\{h'_{\mathcal{R}}\}$ respectively, and let $Y \to |\mathcal{B}|$ and $Y' \to |\mathcal{B}'|$ be the associated restriction quotient.

Lemma 5.27. *Every η-isomorphism induces a natural isomorphism from Ψ to Ψ', hence by Section 4.3, it induces an isomorphism $Y \to Y'$ which is a lift of $\eta : |\mathcal{B}| \to |\mathcal{B}'|$. Moreover, composition of η-isomorphisms gives rise to composition of natural transformations of the associated fiber functors.*

Proof. For every spherical residue $\mathcal{R} \subset \mathcal{B}$, η' respects the product decomposition of $\mathcal{R}$. Thus the following diagram commutes:

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{h_{\mathcal{R}}} & \mathbb{Z}^{T(\mathcal{R})} \\ {\scriptstyle\eta'}\downarrow & & \downarrow{\scriptstyle\prod_{\lambda\in T(\mathcal{R})} f_\lambda} \\ \eta'(\mathcal{R}) & \xrightarrow{h'_{\eta'(\mathcal{R})}} & \mathbb{Z}^{\bar{\eta}(T(\mathcal{R}))} \end{array}$$

Here $\prod_{\lambda\in T(\mathcal{R})} f_\lambda$ induces an isometry $\mathbb{R}^{T(\mathcal{R})} \to \mathbb{R}^{\bar{\eta}(T(\mathcal{R}))}$. This gives rise to a collection of isometries between objects of Ψ and Ψ'. It follows from the construction in Section 5.3 that these isometries give the required natural isomorphism between Ψ and Ψ'. The second assertion in the lemma is straightforward. $\square$

Remark 5.28. If we weaken the assumption of Definition 5.26 by assuming each f_λ is a bijection, then we can obtain a bijection between the vertex sets of Y and Y'. This bijection preserves the fibers, however, we may not be able to extend it to a cubical map.

Theorem 5.29. *If each map $h_{\mathcal{R}}$ in the blow-up data is a bijection, then Y is isomorphic to $X_e = X_e(\Gamma)$, which is the universal cover of the exploded Salvetti complex.*

Definition 5.30 (η-quasi-morphism)**.** We follow the notation in Definition 5.26. An (η, L, A)*-quasi-morphism* between the blow-up data $\{h_{\mathcal{R}}\}$ and $\{h'_{\mathcal{R}}\}$ is a collection of (L, A)-quasi-isometries $\{f_\lambda : \mathbb{Z}^\lambda \to \mathbb{Z}^{\bar{\eta}(\lambda)}\}_{\lambda\in\Lambda_{\mathcal{B}}}$ such that the diagram in Definition 5.26 commutes up to error A.

Lemma 5.31. *Each (η, L, A)-quasi-morphism between $\{h_{\mathcal{R}}\}$ and $\{h'_{\mathcal{R}}\}$ induces an (L', A')-quasi-isometry $Y \to Y'$ with L', A' depending on L, A, and the dimension of $|\mathcal{B}|$.*

Proof. By Lemma 4.19, it suffices to produce an (L', A')-quasi-natural isomorphism from Ψ to Ψ'. This can be done by considering maps of form $\prod_{\lambda \in T(\mathcal{R})} f_\lambda$ as in Lemma 5.27. $\square$

Remark 5.32 (A nice representative). Let Y_0 be the collection of rank 0 vertices in Y (Definition 5.17). We define Y_0' similarly. If the assumption in Lemma 5.20 (2) is satisfied, then Y_0 and Y_0' are D-dense in Y and Y' respectively. In this case, the quasi-isometry $Y \to Y'$ in Lemma 5.31 can be represented by $\phi : Y_0 \to Y_0'$, where ϕ is the bijection induced by $\eta : |\mathcal{B}| \to |\mathcal{B}'|$ (recall that we can identify Y_0 and Y_0' with rank 0 vertices in $|\mathcal{B}|$ and $|\mathcal{B}'|$ respectively, see Definition 5.17). The fact that ϕ is a quasi-isometry follows from the construction in the proof Lemma 4.19.

Corollary 5.33. *If there exists constant $D > 0$ such that each map $h_\mathcal{R}$ in the blow-up data satisfies:*

(1) For any $x \in \mathbb{Z}^{T(\mathcal{R})}$, $|h_\mathcal{R}^{-1}(x)| \leq D$.
(2) The image of $h_\mathcal{R}$ is D-dense in $\mathbb{Z}^{T(\mathcal{R})}$.

Then Y is quasi-isometric to $G(\Gamma)$.

Proof. By the assumptions, there exists another set of blow-up data $\{h'_\mathcal{R}\}$ such that each $h'_\mathcal{R}$ is a bijection, and an (η, L, A)-quasi-isomorphism $\{f_\lambda\}_{\lambda \in \Lambda_\mathcal{B}}$ from $\{h'_\mathcal{R}\}$ to $\{h_\mathcal{R}\}$ where η is the identity map. It follows from Lemma 5.31 and Theorem 5.21 that Y is quasi-isometric to X_e, the universal cover of the exploded Salvetti complex; hence Y is quasi-isometric to $G(\Gamma)$. $\square$

In the rest of this section, we look at the special case when $\mathcal{B} = \mathcal{B}(\Gamma)$ is the Davis building of $G(\Gamma)$ (see the beginning of Section 5.1), and record an observation for later use. In this case, we identify points in $G(\Gamma)$ with chambers in $\mathcal{B}$.

We denote the word metric on $G(\Gamma)$ by d_w. If we identify $G(\Gamma)$ with chambers of the building $\mathcal{B} = \mathcal{B}(\Gamma)$, then there is another metric on $G(\Gamma)$ defined in Definition 3.7. We caution the reader that these two metrics are not the same. We pick a set of blow-up data $\{h_\mathcal{R}\}$ on $\mathcal{B}$ and let $q : Y \to |\mathcal{B}|$ be the associated restriction quotient. Recall that vertices of rank 0 on $|\mathcal{B}|$ can be identified with chambers in $\mathcal{B}$, hence can be identified with $G(\Gamma)$. Thus the map $q^{-1} : G(\Gamma) \to Y$ is well-defined.

Lemma 5.34. *If there exist $L, A > 0$ such that all $\{h_\mathcal{R} : \mathcal{R} \to \mathbb{Z}^{T(\mathcal{R})}\}$ are (L, A)-quasi-isometries (here we identify chambers in $\mathcal{R}$ with a subset of $G(\Gamma)$, hence $\mathcal{R}$ is endowed with an induced metric from d_w), then*

$q^{-1}: (G(\Gamma), d_w) \to Y$ is an (L', A')-quasi-isometry with its constants depending on L, A and Γ.

Proof. Let $q': X_e \to |\mathcal{B}|$ be the $G(\Gamma)$-equivariant canonical restriction quotient constructed in Section 5.1. In this case, $(q')^{-1}: G(\Gamma) \to X_e$ is a quasi-isometry whose constants depend on Γ. Let $h'_{\mathcal{R}}$ be the blow-up data which arises from the 1-data (Definition 5.3) of q'. Then each $h'_{\mathcal{R}}$ is an isometry. It follows from the assumption that there exists an (η, L, A)-quasi-isomorphism from the blow-up data $\{h'_{\mathcal{R}}\}$ to $\{h_{\mathcal{R}}\}$ with η being the identity map. Thus there exists a quasi-isometry $X_e \to Y$ which can be represented by a map ϕ of the form in Remark 5.32. Since $q^{-1} = \phi \circ (q')^{-1}$, the lemma follows. □

5.6. **An equivariant construction.** Let $\mathcal{B} = \mathcal{B}(\Gamma)$ be a right-angled building. Let K be group which acts on $|\mathcal{B}|$ by automorphisms which preserve the rank of its vertices and let $K \curvearrowright \mathcal{B}$ and $K \curvearrowright \Lambda_{\mathcal{B}}$ be the induced actions ($\Lambda_{\mathcal{B}}$ is defined in the beginning of Section 5.3).

Definition 5.35 (Factor actions)**.** Pick $\lambda \in \Lambda$ and let $\mathcal{R}_\lambda \subset \mathcal{B}$ be a residue of rank 1 such that $T(\mathcal{R}_\lambda) = \lambda$ (T is the type map defined in Section 5.3). Let K_λ be the stabilizer of λ with respect to the action $K \curvearrowright \Lambda_{\mathcal{B}}$ and let $P(\mathcal{R}_\lambda) = \mathcal{R}_\lambda \times \mathcal{R}^\perp_\lambda$ be the parallel set of $\mathcal{R}_\lambda$ with its product decomposition (see Lemma 3.14 and Theorem 3.13). Then $P(\mathcal{R}_\lambda)$ is K_λ-invariant, and K_λ respects the product decomposition of $P(\mathcal{R}_\lambda)$. Let $\rho_\lambda : K_\lambda \curvearrowright \mathcal{R}_\lambda$ be the action of K_λ on the $\mathcal{R}_\lambda$-factor. This action ρ_λ is called a *factor action.*

We construct equivariant blow-up data as follows. Pick one representative from each K-orbit of $K \curvearrowright \Lambda_{\mathcal{B}}$ and form the set $\{\lambda_u\}_{u\in U}$. Let K_u be the stabilizer of λ_u. Pick residue $\mathcal{R}_u \subset \mathcal{B}$ of rank 1 such that $T(\mathcal{R}_u) = \lambda_u$ and let $\rho_u : K_u \curvearrowright \mathcal{R}_u$ be the factor action defined as above.

To obtain a K-equivariant blow-up data, we pick an isometric action $K_u \curvearrowright \mathbb{Z}^{\lambda_u}$ and a K_u-equivariant map $h_{\mathcal{R}_u}: \mathcal{R}_u \to \mathbb{Z}^{\lambda_u}$. If $\mathcal{R}$ is parallel to $\mathcal{R}_u$ with the parallelism map given by $p: \mathcal{R} \to \mathcal{R}_u$, we define $h_{\mathcal{R}} = h_{\mathcal{R}_u} \circ p$. By the previous discussion, there is a factor action $K_u \curvearrowright \mathcal{R}$, and $h_{\mathcal{R}}$ is K_u-equivariant. We run this process for each element in $\{\lambda_u\}_{u\in U}$. If $\lambda \notin \{\lambda_u\}_{u\in U}$, then we fix an element $g_\lambda \in K$ such that $g_\lambda(\lambda) \in \{\lambda_u\}_{u\in U}$. For rank 1 element $\mathcal{R}$ with $T(\mathcal{R}) = \lambda$, we define $h_{\mathcal{R}} = \mathrm{Id} \circ h_{g_\lambda(\mathcal{R})} \circ g_\lambda$, here $\mathrm{Id}: \mathbb{Z}^{g_\lambda(\lambda)} \to \mathbb{Z}^\lambda$ is the identity map. Let $K_\lambda = g^{-1}_\lambda K_{g_\lambda(\lambda)} g_\lambda$ be the stabilizer of λ. We define the action

$K_\lambda \curvearrowright \mathbb{Z}^\lambda$ by letting $g_\lambda^{-1} g g_\lambda$ acts on $\mathbb{Z}^\lambda$ by $\mathrm{Id} \circ g \circ \mathrm{Id}^{-1}$ ($g \in K_{g_\lambda(\lambda)}$). Then $h_{\mathcal{R}}$ becomes K_λ-equivariant.

It follows from the above construction that we can produce an isometry $f_{g,\mathcal{R}} : \mathbb{Z}^{T(\mathcal{R})} \to \mathbb{Z}^{T(g(\mathcal{R}))}$ for each $g \in K$ and rank 1 residue $\mathcal{R} \in \mathcal{B}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{h_{\mathcal{R}}} & \mathbb{Z}^{T(\mathcal{R})} \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f_{g,\mathcal{R}}} \\ g(\mathcal{R}) & \xrightarrow{h_{g(\mathcal{R})}} & \mathbb{Z}^{T(g(\mathcal{R}))} \end{array}$$

and $f_{g_1 g_2,\mathcal{R}} = f_{g_1, g_2(\mathcal{R})} \circ f_{g_2,\mathcal{R}}$ for any $g_1, g_2 \in K$. Let Ψ be the fiber functor associated with the above blow-up data and let $q : Y \to |\mathcal{B}|$ be the corresponding restriction quotient. Lemma 5.27 implies K acts on Ψ by natural transformations, hence there is an induced action $K \curvearrowright Y$ and q is K-equivariant.

Remark 5.36. The previous construction depends on several choices:

(1) The choice of the set $\{\lambda_u\}_{u \in U}$.
(2) The choice of the isometric action $K_u \curvearrowright \mathbb{Z}^{\lambda_u}$ and the K_u-equivariant map $h_{\mathcal{R}_u} : \mathcal{R}_u \to \mathbb{Z}^{\lambda_u}$.
(3) The choice of the elements g_λ's.

6. Quasi-actions on RAAG's

In this section we will apply the construction in Section 5.6 to study quasi-actions on RAAG's.

We assume $G(\Gamma) \neq \mathbb{Z}$ throughout Section 6.

6.1. **The cubulation.**

Throughout Subsection 6.1 we assume $G(\Gamma) \not\simeq \mathbb{Z}$ is a RAAG with $|\operatorname{Out}(G(\Gamma))| < \infty$, and $\rho : H \curvearrowright G(\Gamma)$ is an (L, A)-quasi-action.

Recall that $G(\Gamma)$ acts on $X(\Gamma)$ by deck transformations, and this action is simply transitive on the vertex set of $X(\Gamma)$. By picking a base point in $X(\Gamma)$, we identify $G(\Gamma)$ with the 0-skeleton of $X(\Gamma)$.

Definition 6.1. A quasi-isometry $\phi : G(\Gamma) \to G(\Gamma)$ is *flat-preserving* if it is a bijection and for every standard flat $F \subset X(\Gamma)$ there is a standard flat $F' \subset X(\Gamma)$ such that ϕ maps the 0-skeleton of F bijectively onto the 0-skeleton of F'. The standard flat F' is uniquely determined, and we denote it by $\phi_*(F)$. Note that if ϕ is flat-preserving, then ϕ^{-1} is also flat-preserving.

By Theorem 1.3, without loss of generality we can assume $\rho : H \curvearrowright G(\Gamma)$ is an action by flat-preserving bijections which are also (L, A)-quasi-isometries.

On the one hand, we want to think $G(\Gamma)$ as a metric space with the word metric with respect to its standard generating set, or equivalently, with the induced l^1-metric from $X(\Gamma)$; on the other hand, we want to treat $G(\Gamma)$ as a right-angled building (see Section 5.1), more precisely, we want to identify points in $G(\Gamma)$ with chambers in the associated right-angled building of $G(\Gamma)$. Then the ρ preserves the spherical residues in $G(\Gamma)$, thus there is an induced $\rho_{|\mathcal{B}|} : H \curvearrowright |\mathcal{B}|$ on the Davis realization $|\mathcal{B}|$ of $G(\Gamma)$.

Let Λ be the collection of parallel classes of standard geodesic lines in $X(\Gamma)$, in other words, Λ is the collection of parallel classes of rank 1 residues in $G(\Gamma)$, and let T be the type map defined in the beginning of Section 5.3. There is another induced action $\rho_\Lambda : H \curvearrowright \Lambda$. For each $\lambda \in \Lambda$, let H_λ be the stabilizer of λ. Pick a residue $\mathcal{R}$ in the parallel class λ, and let $\rho_\lambda : H_\lambda \curvearrowright \mathcal{R}$ be the factor action in Definition 5.35. Note that $\mathcal{R}$ is an isometrically embedded copy of $\mathbb{Z}$ with respect to the metric on $G(\Gamma)$; moreover, ρ_λ is an action by (L', A')-quasi-isometries. Here we can choose L' and A' such that they depend only on L and A, so in particular they do not depend on λ and $\mathcal{R}$.

For the action $\rho_\Lambda : H \curvearrowright \Lambda$, we pick a representative from each H-orbit and form the set $\{\lambda_u\}_{u \in U}$. By the construction in Section 5.6, it remains to choose an isometric action $G_u \curvearrowright \mathbb{Z}^{\lambda_u}$ and a G_u-equivariant map $h_{\mathcal{R}_u} : \mathcal{R}_u \to \mathbb{Z}^{\lambda_u}$ for each $u \in U$ ($\mathcal{R}_u$ is a residue in the parallel class λ_u). The choice is provided by the following result, whose proof is postponed to Section 7.

Proposition 6.2. *If a group K has an action on $\mathbb{Z}$ by (L, A)-quasi-isometries, then there exists another action $K \curvearrowright \mathbb{Z}$ by isometries which is related to the original action by a surjective equivariant (L', A')-quasi-isometry $f : \mathbb{Z} \to \mathbb{Z}$ where L', A' depend on L and A.*

From the above data, we produce H-equivariant blow-up data $h_{\mathcal{R}} : \mathcal{R} \to \mathbb{Z}^{T(\mathcal{R})}$ for each rank 1 residue $\mathcal{R} \subset G(\Gamma)$ as in Section 5.6. Note that each $h_{\mathcal{R}}$ is an (L'', A'')-quasi-isometry with constants depending only on L and A.

Let $q : Y \to |\mathcal{B}|$ be the restriction quotient associated with the above blow-up data. Then there is an induced action $H \curvearrowright Y$ by isomorphisms, and q is H-equivariant. It follows from Lemma 5.20 that Y is uniformly locally finite.

Claim. There exists an (L_1, A_1)-quasi-isometry $G(\Gamma) \to Y$ with L_1, A_1 depending only on L and A.

Proof of claim. Let $h'_{\mathcal{R}} : \mathcal{R} \to \mathbb{Z}^{T(\mathcal{R})}$ be another blow-up data such that each $h'_{\mathcal{R}}$ is an isometry (such blow-up data always exists), and let $q' : Y' \to |\mathcal{B}|$ be the associated restriction quotient. By Theorem 5.21, Y' is isomorphic to X_e, which is the universal cover of the exploded Salvetti complex introduced in Section 5.1. For any $\lambda \in \Lambda$, we define $f_\lambda = h_{\mathcal{R}} \circ (h'_{\mathcal{R}})^{-1}$, here $\mathcal{R}$ is a residue such that $T(\mathcal{R}) = \lambda$ and the definition of f_λ does not depend on $\mathcal{R}$. Each f_λ is an (L'', A'')-quasi-isometry and the collection of all f_λ's induces a quasi-isomorphism between the blow-up data $\{h'_{\mathcal{R}}\}$ and $\{h_{\mathcal{R}}\}$. It follows from Lemma 5.31 that there exists a quasi-isometry between $\varphi : Y' \cong X_e \to Y$, and the claim follows. □

Let B_0 be the set of vertices of rank 0 in $|\mathcal{B}|$. There is a natural identification of B_0 with $G(\Gamma)$. Letting $Y_0 = q^{-1}(B_0)$, we get that q induces a bijection between Y_0 and B_0. We define Y'_0 similarly. It follows from (2) of Lemma 5.20 that Y'_0 and Y_0 are D-dense in Y' and Y respectively for D depending on L and A. Note that $q^{-1} : G(\Gamma) \to Y_0$ is H-equivariant, and if the action $\rho : H \curvearrowright G(\Gamma)$ is cobounded, then $H \curvearrowright Y$ is cocompact.

The above quasi-isometry φ can be represented by $q^{-1} \circ q' : Y'_0 \to Y_0$ (Remark 5.32). By Lemma 5.34, $(q')^{-1} : B_0 = G(\Gamma) \to Y'_0$ is also a quasi-isometry, and thus $q^{-1} : G(\Gamma) \to Y_0$ is a quasi-isometry. This map is H-equivariant, so if $\rho : H \curvearrowright G(\Gamma)$ is proper, then $H \curvearrowright Y$ is also proper.

Remark 6.3. Here we discuss a refinement of the above construction. Instead of requiring each $h'_{\mathcal{R}}$ to be an isometry, it is possible to choose each $h'_{\mathcal{R}}$ such that:

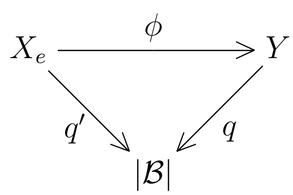
(1) $h'_{\mathcal{R}}$ is a bijection.
(2) $h'_{\mathcal{R}}$ is an (L_2, A_2)-quasi-isometry.
(3) $f_\lambda : \mathbb{Z}^\lambda \to \mathbb{Z}^\lambda$ is a surjective map which respects the order of the $\mathbb{Z}$, hence can be extended to a surjective cubical map $\mathbb{R}^\lambda \to \mathbb{R}^\lambda$.

The surjectivity in (3) follows from our choice in Proposition 6.2. In this case, the space Y' is still isomorphic to X_e (Theorem 5.21). Let Ψ and Ψ' be the fiber functors associated with the blow-up data $\{h_{\mathcal{R}}\}$ and $\{h'_{\mathcal{R}}\}$. As in the proof of Lemma 5.27, the f_λ's induce a natural transformation from Ψ' to Ψ which is made of a collection of surjective cubical maps from objects in Ψ' to objects in Ψ; moreover, these maps are quasi-isometries with uniform quasi-isometry constants. Recall that we can describe Y as the quotient of the disjoint collection $\{\sigma \times \Psi(\sigma)\}_{\sigma\in \mathrm{Face}(|\mathcal{B}|)}$ (see the proof of Theorem 4.15), and a similar description holds for Y'. Thus there is a surjective cubical map $\phi : Y' \to Y$ induced by the natural transformation. Actually ϕ is a restriction quotient, since the inverse image of each hyperplane is a hyperplane. We also know ϕ is a quasi-isometry by Lemma 4.19.

The following theorem summarizes the above discussion.

Theorem 6.4. *If the outer automorphism group* $\mathrm{Out}(G(\Gamma))$ *is finite and* $G(\Gamma) \not\simeq \mathbb{Z}$*, then any quasi-action* $\rho : H \curvearrowright X(\Gamma)$ *is quasiconjugate to an action* $\hat{\rho}$ *of* H *by cubical isometries on a uniformly locally finite* $CAT(0)$ *cube complex* Y*. Moreover:*

(1) If ρ *is cobounded, then* $\hat{\rho}$ *is cocompact.*
(2) If ρ *is proper, then* $\hat{\rho}$ *is proper.*
(3) Let $|\mathcal{B}|$ *be the Davis realization of the right-angled building associated with* $G(\Gamma)$*, let* $H \curvearrowright |\mathcal{B}|$ *be the induced action, and let* $X_e = X_e(\Gamma)$ *be the universal cover of the exploded Salvetti complex for* $G(\Gamma)$*. Then* Y *fits into the following commutative diagram:*



Here q'*,* q *and* ϕ *are restriction quotients. The map* ϕ *is a quasi-isometry whose constants depend on the constants of the quasi-action* ρ*, and* q *is* H*-equivariant.*

Corollary 6.5. *Suppose the outer automorphism group* $\mathrm{Out}(G(\Gamma))$ *is finite. Then* H *is quasi-isometric to* $G(\Gamma)$ *if and only if there exists an* H*-equivariant restriction quotient map* $q : Y \to |\mathcal{B}|$ *such that:*

(1) $|\mathcal{B}|$ *is the Davis realization of some right-angled* Γ*-building.*
(2) The action $H \curvearrowright Y$ *is geometric.*
(3) If $v \in |\mathcal{B}|$ *is a vertex of rank* k*, then* $q^{-1}(v) = \mathbb{E}^k$*.*

Proof. The only if direction follows from Theorem 6.4. For the if direction, it suffices to show Y is quasi-isometric to $G(\Gamma)$. Let Φ be the fiber functor associated with q.

Pick a vertex $v \in |\mathcal{B}|$ of rank k and let $F_v = q^{-1}(v)$. We claim $\mathrm{Stab}(v)$ acts cocompactly on F_v. By a standard argument, to prove this it suffices to show that $\{h(F_v)\}_{h\in H}$ is a locally finite family in Y. Suppose there exists an R-ball $N \subset Y$ such that there are infinitely many distinct elements in $\{h(F_v)\}_{h\in H}$ which have nontrivial intersection of N. Since Y admits a geometric action, it is locally finite, and thus there exists a vertex $x \in |\mathcal{B}|$ which is contained in infinitely many distinct elements in $\{h(F_v)\}_{h\in H}$. This is impossible, since if $h(F_v) \neq h'(F_v)$, then $q(h(F_v))$ and $q(h'(F_v))$ are distinct vertices in $|\mathcal{B}|$ by the H-equivariance of q.

Consider a cube $\sigma \subset |\mathcal{B}|$ and let v be its vertex of minimal rank. We claim $\Phi(\sigma) \to \Phi(v)$ is surjective, hence is an isometry. By (3), the action $H \curvearrowright |\mathcal{B}|$ preserves the rank of the vertices, thus $\mathrm{Stab}(\sigma) \subset \mathrm{Stab}(v)$. We know that $\mathrm{Stab}(v)$ acts cocompactly on $q^{-1}(v)$; since the poset $\{w \geq v\}$ is finite, $\mathrm{Stab}(\sigma)$ has finite index in $\mathrm{Stab}(v)$, and so $\mathrm{Stab}(\sigma)$ also acts cocompactly on $q^{-1}(v)$. Now the image of $\Phi(\sigma) \to \Phi(v)$ is a convex subcomplex of $q^{-1}(v)$ that is $\mathrm{Stab}(\sigma)$-invariant, so it coincides with $q^{-1}(v)$.

By Corollary 5.15, we can assume q is the restriction quotient of a set of blow-up data $\{h_{\mathcal{R}}\}$. Pick a vertex $v \in |\mathcal{B}|$ of rank 1 and let $D(v)$ be the downward complex of v (see Section 5.3). Let $\mathcal{R}_v \subset \mathcal{B}$ be the associated residue and let $\mathcal{R}_v \to \mathbb{R}^{T(\mathcal{R})}$ be the map induced by $h_{\mathcal{R}_v}$. Then $q^{-1}(D_v)$ is isomorphic to the mapping cylinder of this map. Since the $\mathrm{Stab}(v)$ acts cocompactly on $q^{-1}(D_v)$, there are only finite many orbits of vertices of rank 1, and the assumptions of Corollary 5.33 are satisfied. It follows that Y is quasi-isometric to $G(\Gamma)$. □

It is possible to drop the H-equivariant assumption on q under the following conditions. Here we do not put any assumption on Γ.

Theorem 6.6. *Let $\mathcal{B}$ be a right-angled Γ-building. Suppose $q : Y \to |\mathcal{B}|$ is a restriction quotient such that for every cube $\sigma \subset |\mathcal{B}|$, and every interior point $x \in \sigma$, the point inverse $q^{-1}(x)$ is a copy of $\mathbb{E}^{rank(v)}$, where $v \in \sigma$ is the vertex of minimal rank.*

If H acts geometrically on Y by automorphisms, then H is quasi-isometric to $G(\Gamma)$.

Proof. First we assume Γ satisfies the assumption of Theorem 5.24. Then the above result is a consequence of Corollary 5.15, Theorem 5.24 and the argument in Corollary 6.5.

For arbitrary Γ, we make a join decomposition $\Gamma = \Gamma_1 \circ \Gamma_2 \circ \cdots \Gamma_k \circ \Gamma'$ where Γ' satisfies the assumption of Theorem 5.24, and all Γ_i's are discrete graphs with more than one vertex. By Corollary 5.16, there are induced cubical product decomposition $Y = Y_1 \times Y_2 \times \cdots Y_k \times Y'$ and restriction quotients $q_i : Y_i \to |\mathcal{B}_i|$ and $q' : Y' \to |\mathcal{B}'|$ which satisfy the assumption of the theorem. By [CS11, Proposition 2.6], we assume H respects this product decomposition by passing to a finite index subgroup. Since Y' is locally finite and cocompact, the same argument in Corollary 6.5 implies Y' is quasi-isometric to $G(\Gamma')$. Each Y_i is a locally finite and cocompact tree which is not quasi-isometric to a line. So Y_i is quasi-isometric to $G(\Gamma')$. Thus Y is quasi-isometric to $G(\Gamma)$. □

Corollary 6.7. *Suppose* $\mathrm{Out}(G(\Gamma))$ *is finite and* $G(\Gamma) \not\simeq \mathbb{Z}$. *Let* $\mathcal{B}$ *be the right-angled building of* $G(\Gamma)$. *Then* H *is quasi-isometric to* $G(\Gamma)$ *if and only if* H *acts geometrically on a blow-up of* $\mathcal{B}$ *in the sense of Section 5.3 by automorphisms.*

6.2. **Reduction to nicer actions.** Though every action $\rho : H \curvearrowright G(\Gamma)$ by flat-preserving bijections which are also (L, A)-quasi-isometries is quasiconjugate to an isometric action $H \curvearrowright Y$ as in Theorem 6.4, it is in general impossible to take $Y = X(\Gamma)$, even if the action ρ is proper and cobounded.

Definition 6.8. Let $H = \mathbb{Z}/2 \oplus \mathbb{Z}$ with the generator of $\mathbb{Z}/2$ and $\mathbb{Z}$ denoted by a and b respectively. Let $H \overset{\rho_0}{\curvearrowright} \mathbb{Z}$ be the action where $\rho_0(b)(n) = n + 2$, and $\rho_0(a)$ acts on $\mathbb{Z}$ by flipping $2n$ and $2n+1$ for all $n \in \mathbb{Z}$. An action $K \curvearrowright \mathbb{Z}$ is *2-flipping* if it factors through the action $H \overset{\rho_0}{\curvearrowright} \mathbb{Z}$ via an epimorphism $K \to H$.

Lemma 6.9. *Let* $\rho_K : K \curvearrowright \mathbb{Z}$ *be a 2-flipping action. Then* ρ_K *is not conjugate to an action by isometries on* $\mathbb{Z}$ *(with respect to the word metric on* $\mathbb{Z}$*).*

Proof. Suppose there exists a permutation $p : \mathbb{Z} \to \mathbb{Z}$ which conjugates ρ_K to an isometric action. Let $h : K \to G$ be the epimorphism in Definition 6.8. Pick $k_1, k_2 \in K$ such that $h(k_1)$ is of order 2 and $h(k_2)$ is of order infinity. Then pk_1p^{-1} is a reflection of $\mathbb{Z}$ and pk_2p^{-1} is a translation. However, this is impossible since $h(k_1)$ and $h(k_2)$ commute. □

Lemma 6.10. *There does not exists an action* $\rho_K : K \curvearrowright \mathbb{Z}$ *by* (L, A)*-quasi-isometries with the following property.* K *has two subgroup* K_1

and K_2 such that $\rho_K|_{K_1}$ is conjugate to a 2-flipping action and $\rho_K|_{K_2}$ is conjugate to a transitive action on $\mathbb{Z}$ by translations.

Proof. By Proposition 6.2, there exists an isometric action $\rho'_K : K \curvearrowright \mathbb{Z}$ and a K-equivariant surjective map $f : K \overset{\rho_K}{\curvearrowright} \mathbb{Z} \longrightarrow K \overset{\rho'_K}{\curvearrowright} \mathbb{Z}$. We claim f is also injective. Given this claim, we can deduce a contradiction to Lemma 6.9 by restricting the action to K_1. To see the claim, we restrict the action to K_2. Thus we can assume without loss of generality that ρ_K is a transitive action by translations. Suppose $f(a_1) = f(a_1 + k)$ for $a_1, k \in \mathbb{Z}$ and $k \neq 0$. Then the equivariance of f implies $f(a_1) = f(a_1 + nk)$ for any integer $n \in \mathbb{Z}$, which contradicts that f is a quasi-isometry. □

Theorem 6.11. *Suppose $G(\Gamma)$ is a RAAG with $|\operatorname{Out}(G(\Gamma))| < \infty$ and $G(\Gamma) \not\simeq \mathbb{Z}$. Then there is a pair H, H' of finitely generated groups quasi-isometric to $G(\Gamma)$ that does not admit discrete, virtually faithful cocompact representations into the same locally compact topological group.*

Recall that a discrete, virtually faithful cocompact representation from H to a locally compact group $\hat{G}$ is a homomorphism $h : H \to \hat{G}$ such that its kernel is finite, and its image is a cocompact lattice.

Proof. Pick a vertex $u \in \Gamma$ and let Γ' be a graph obtained by taking two copies of Γ and gluing them along the closed star of u. There is a graph automorphism $\alpha : \Gamma' \to \Gamma'$ which fixes the closed star of u pointwise and flips the two copies of Γ. Then α induces an involution $\alpha : G(\Gamma') \to G(\Gamma')$, which gives rise to a semi-product $H = G(\Gamma') \rtimes \mathbb{Z}/2$.

Note that $G(\Gamma')$ is a subgroup of index 2 in both $G(\Gamma)$ and H. Therefore this induces a quasi-isometry $q : H \to G(\Gamma)$, an also a quasi-action $\rho_H : H \curvearrowright G(\Gamma)$. By the previous discussion, we can assume ρ_H is an action by flat-preserving quasi-isometries. We look at the associated collection of factor actions $\{H_\lambda \curvearrowright \mathbb{Z}\}_{\lambda \in \Lambda}$ (see Definition 5.35), recall that Λ is the collection of parallel classes of rank 1 residues in $G(\Gamma)$, and a rank 1 residue in some class λ can be identified with a copy of $\mathbb{Z}$. Up to conjugacy by bijective quasi-isometries, these factor actions are either transitive actions on $\mathbb{Z}$ or 2-flipping actions.

We claim that $G(\Gamma)$ and H do not admit discrete, virtually faithful cocompact representations into the same locally compact topological group. Suppose such a topological group $\hat{G}$ exists. Then by [MSW03, Chapter 6], $\hat{G}$ has a quasi-action on $G(\Gamma)$. We assume $\hat{G}$ acts on $G(\Gamma)$

by flat-preserving quasi-isometries as before. Then there are restriction actions $\rho'_{G(\Gamma)} : G(\Gamma) \curvearrowright G(\Gamma)$ and $\rho'_H : H \curvearrowright G(\Gamma)$ which are discrete and cobounded. Since any two discrete and cobounded quasi-actions $H \curvearrowright G(\Gamma)$ are quasi-conjugate, it follows from Theorem 1.3 that ρ_H and ρ'_H are conjugate by a flat-preserving quasi-isometry. Thus factor actions of ρ'_H is conjugate to factor actions of ρ_H by bijective quasi-isometries. Similarly, we deduce that the factor actions of $\rho'_{G(\Gamma)}$ are conjugate to transitive actions by left translations on $\mathbb{Z}$ via bijective quasi-isometries. Note that the factor actions of $\rho'_{G(\Gamma)}$ and the factor actions of ρ'_H are both restrictions of factor actions of $\hat{G} \curvearrowright G(\Gamma)$, however, this is impossible by Lemma 6.10. □

Corollary 6.12. *The group $H = G(\Gamma') \rtimes \mathbb{Z}/2$ cannot act geometrically on $X(\Gamma)$.*

We now give a criterion for when one can quasi-conjugate a quasi-action on $X(\Gamma)$ to an isometric action $H \curvearrowright X(\Gamma)$.

Theorem 6.13. *Let $\rho : H \curvearrowright G(\Gamma)$ be an action by flat-preserving bijections. If for each $\lambda \in \Lambda$, the factor action $\rho_\lambda : H_\lambda \curvearrowright \mathbb{Z}$ can be conjugate to an action by isometries with respect to the word metric of $\mathbb{Z}$, then there is an flat-preserving bijection $g : G(\Gamma) \to G(\Gamma)$ which conjugates $\rho : H \curvearrowright G(\Gamma)$ to an action $\rho' : H \curvearrowright X(\Gamma)$ by flat-preserving isometries. If ρ is also an action by (L, A)-quasi-isometries, then g can be taken to be a quasi-isometry.*

Proof. We repeat the construction in Section 6.1 and assume each $h_{\mathcal{R}} : \mathcal{R} \to \mathbb{Z}^{T(\mathcal{R})}$ is a bijection. Let $q : Y \to |\mathcal{B}|$, Y_0, $q^{-1} : G(\Gamma) \to Y_0$ and the action $\hat{\rho} : H \curvearrowright Y$ by automorphisms be as in Section 6.1. Recall that q^{-1} is H-equivariant. There is an isomorphism $i : Y \to X_e$ by Theorem 5.21, moreover, $i(Y_0)$ is exactly the collection of 0-dimensional standard flats X_0 in X_e. We deduce from the construction of i that the isometric action $H \curvearrowright X_e$ induced by $\hat{\rho}$ preserves standard flats in X_e. By the construction of X_e, there exists a natural identification $f : X_0 \to G(\Gamma)$ such that any automorphism of X_e which preserves its standard flats induces a flat-preserving isometry of $G(\Gamma)$ (with respect to the word metric) via f. It suffices to take $g = f \circ i \circ q^{-1}$. □

Suppose we have already conjugated the flat-preserving action $\rho : H \curvearrowright G(\Gamma)$ to an action $\rho' : H \curvearrowright X(\Gamma)$ (or $H \curvearrowright G(\Gamma)$) by flat-preserving isometries. We ask whether it is possible to further conjugate ρ' to an action by left translations.

We can oriented each 1-cell in $S(\Gamma)$ and label it by the associated generator. This lifts to orientations and labels of edges of $X(\Gamma)$. If H preserves this orientation and labelling, then ρ' is already an action by left translations. In general, it suffices to require H preserves a possibly different orientation and labelling which satisfy several compatibility conditions. Now we recall the following definitions from [Hua14a].

Definition 6.14 (Coherent ordering)**.** A *coherent ordering* for $G(\Gamma)$ is a blow-up data for $G(\Gamma)$ such that each map $h_{\mathcal{R}}$ is a bijection. Two coherent orderings are *equivalent* if the their maps agree up to translations.

Let $\mathcal{P}(\Gamma)$ be the extension complex defined in Section 3.3. Note that we can identify $\Lambda_{G(\Gamma)}$ with the 0-skeleton of $\mathcal{P}(\Gamma)$. Any flat-preserving action $H \curvearrowright G(\Gamma)$ induces an action $H \curvearrowright \mathcal{P}(\Gamma)$ by simplicial isomorphisms. Let $F(\Gamma)$ be the flag complex of Γ.

Definition 6.15 (Coherent labelling)**.** Recall that for each vertex $x \in X(\Gamma)$, there is a natural simplicial embedding $i_x : F(\Gamma) \to \mathcal{P}(\Gamma)$ by considering the standard flats passing through x. A *coherent labelling* of $G(\Gamma)$ is a simplicial map $L : \mathcal{P}(\Gamma) \to F(\Gamma)$ such that $L \circ i_x : F(\Gamma) \to F(\Gamma)$ is a simplicial isomorphism for every vertex $x \in X(\Gamma)$.

The next result follows from [Hua14a, Lemma 5.7].

Lemma 6.16. *Let $\rho' : H \curvearrowright G(\Gamma)$ be an action by flat-preserving bijections and let $H \curvearrowright \mathcal{P}(\Gamma)$ be the induced action. If there exists an H-invariant coherent ordering and an H-invariant coherent labelling, then ρ' is conjugate to an action by left translations.*

Since each vertex of $\mathcal{P}(\Gamma)$ corresponds to a parallel class of v-residues for vertex $v \in \Gamma$, this gives a labelling of vertices of $\mathcal{P}(\Gamma)$ by vertices of Γ. We can extend this labelling map to a simplicial map $L : \mathcal{P}(\Gamma) \to F(\Gamma)$, which gives rise to a coherent labelling.

Corollary 6.17. *Let $\rho : H \curvearrowright G(\Gamma)$ be an action by flat-preserving bijections. Suppose:*

(1) The induced action $H \curvearrowright \mathcal{P}(\Gamma)$ preserves the vertex labelling of $\mathcal{P}(\Gamma)$ as above.
(2) For each vertex $v \in \mathcal{P}(\Gamma)$, the action $\rho_v : H_v \curvearrowright \mathbb{Z}$ is conjugate to an action by translations.

Then ρ is conjugate to an action $H \curvearrowright G(\Gamma)$ by left translations.

Note that condition (2) is equivalent to the existence of an H-invariant coherent ordering.

7. Actions by quasi-isometries on $\mathbb{Z}$

In this section we prove Proposition 6.2.

7.1. **Tracks.**

Tracks were introduced in [Dun85]. They are hypersurface-like objects in 2-dimensional simplicial complexes.

Definition 7.1 (Tracks)**.** Let K be 2-dimensional simplicial complex. A *track* $\tau \subset K$ is a connected embedded finite simplicial graph such that:

(1) For each 2-simplex $\Delta \subset K$, $\tau \cap \Delta$ is a finite disjoint union of curves such that the end points of each curve are in the interior of edges of Δ.
(2) For each edge $e \in K$, $\tau \cap e$ is a discrete set in the interior of e. Let $\{\Delta_\lambda\}_{\lambda \in \Lambda}$ be the collection of 2-simplices that contains e. If $v \in \tau \cap e$, then for each λ, $\tau \cap \Delta_\lambda$ contains a curve that contains v.

Given a track $\tau \subset K$, we defined the *support* of τ, denoted $\mathrm{Spt}(\tau)$, to be the minimal subcomplex of K which contains τ.

We can view hyperplanes defined in Section 3.2 as analogue of tracks in the cubical setting. Each track $\tau \subset K$ has a regular neighbourhood which fibres over τ. When K is simply-connected, $K \setminus \tau$ has two connected components, moreover, the regular neighbourhood of τ is homeomorphic to $\tau \times (-\epsilon, \epsilon)$.

Two tracks τ_1 and τ_2 are *parallel* if $\mathrm{Spt}(\tau_1) = \mathrm{Spt}(\tau_2)$ and there is a region homeomorphic to $\tau_1 \times (0, \epsilon)$ bounded by τ_1 and τ_2. A track $\tau \subset K$ is *essential* if the components of $K \setminus \tau$ are unbounded. The following result follows from [Dun85, Proposition 3.1]:

Lemma 7.2. *If K is simply-connected and has more than one end, then there exists an essential track $\tau \subset K$.*

Next we look at essential tracks which are "minimal"; these turn out to behave like minimal surfaces. First we metrize K as in [SS96].

Let $\Delta = \Delta(\xi_1\xi_2\xi_3)$ be an ideal triangle in the hyperbolic plane. We mark a point in each side of the triangle as follows. Let ϕ be the unique isometry which fixes ξ_3 and flips ξ_1 and ξ_2, we mark the unique point in $\overline{\xi_1\xi_2}$ which is fixed by ϕ. Other sides of Δ are marked similarly. This is called a *marked ideal triangle.*

We identify each 2-simplex of K with a marked ideal triangle in the hyperbolic plane and glue these triangles by isometries which identify the marked points. This gives a collection of complete metrics on each connected component of $K - K^{(0)}$ which is not an interval. We denote this collection of metrics by $d_{\mathbb{H}}$. If a group G acts on K by simplicial isomorphisms, then G also acts by isometries on $(K, d_{\mathbb{H}})$. The original definition in [SS96] does not required these marked points, see Remark 7.4 for why we add them.

Each track τ has a well-defined length under $d_{\mathbb{H}}$, which we denote by $l(\tau)$. We also define the *weight* of τ, denoted by $w(\tau)$, to be cardinality of $\tau \cap K^{(1)}$. The *complexity* $c(\tau)$ is defined to be the ordered pair $(w(\tau), l(\tau))$. We order the complexity lexicographically, namely $c(\tau_1) < c(\tau_2)$ if and only if $w(\tau_1) < w(\tau_2)$ or $w(\tau_1) = w(\tau_2)$ and $l(\tau_1) < l(\tau_2)$.

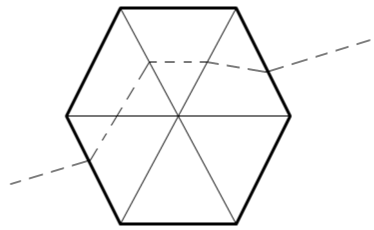
The following result follows from [SS96, Lemma 2.11 and Lemma 2.14]:

Lemma 7.3. *Suppose K is a uniformly locally finite and simply-connected simplicial 2-complex with at least two ends. Suppose K does not contain separating vertices. Then there exists an essential track $\tau \subset K$ which has the least complexity with respect to $d_{\mathbb{H}}$ among all essential tracks in K.*

Remark 7.4. Let $\{\tau_i\}_{i=1}^{\infty}$ be a minimizing sequence. Since K is uniformly locally finite, there are only finitely many combinatorial possibilities for $\{\tau_i\}_{i=1}^{\infty}$. Thus we can assume all the τ_i's are inside a finite subcomplex L. Moreover, we can construct a hyperbolic metric $d_{\mathbb{H}}$ on L as above and it suffices to work in the space $(L, d_{\mathbb{H}})$. However, if we do not use marked points in the construction of the hyperbolic metric on K, then each τ_i may sit inside a copy of L with different shears along the edges of L.

In [SS96], K is assumed to be cocompact, so one does not need to worry about the above issue.

Remark 7.5. If we metrize each simplex in K with the Euclidean metric, then Lemma 7.3 and Lemma 7.6 may not be true. For example, one can take the following picture, where the dotted line is part of some track τ. Once we shorten τ, it may hit the central vertex of the hexagon. However, this cannot happen if we have hyperbolic metrics on each simplex. Once τ gets too close to some vertex, then it takes a large amount of length for τ to escape that vertex since $d_{\mathbb{H}}$ is complete (actually it does not matter if $d_{\mathbb{H}}$ is not complete, since we also have a upper bound on the weight of τ).



The next result can be proved in a similar fashion:

Lemma 7.6. *Let K be a simply-connected simplicial 2-complex. Let $A \subset K$ be a uniformly locally finite subcomplex such that*

(1) A contains an essential track of K.
(2) A does not contain any separating vertex of K.

Then there exists an essential track τ of K which has the least complexity among all essential tracks of K with support in A.

Lemma 7.7. [SS96, Lemma 2.7] *Let τ_1 and τ_2 be two essential tracks of K which are minimal in the sense of Lemma 7.3 or Lemma 7.6, then either $\tau_1 = \tau_2$, or $\tau_1 \cap \tau_2 = \emptyset$.*

7.2. **The proof of Proposition 6.2.** First we briefly recall the notion of Rips complex. See [BH99, Chapter III.Γ.3] for more detail. Let (X, d) be a metric space and pick $R > 0$. The *Rips complex* $P_R(X, d)$ is the geometric realization of the simplicial complex with vertex set X whose n-simplices are the $(n+1)$-element subsets $\{x_0, \cdots, x_n\} \subset X$ of diameter at most R.

Let d be the usual metric on $\mathbb{Z}$. Define a new metric $\bar{d}$ on $\mathbb{Z}$ by

$$\bar{d}(x, y) = \sup_{g \in G} d(g(x), g(y))$$

Note that $(\mathbb{Z}, \bar{d})$ is quasi-isometric to $(\mathbb{Z}, d)$, and G acts on $(\mathbb{Z}, \bar{d})$ by isometries. Since $(\mathbb{Z}, \bar{d})$ is Gromov-hyperbolic, the Rips complex $P_R(\mathbb{Z}, \bar{d})$ is contractible for some $R = R(L, A)$ (see [BH99, Proposition III.Γ.3.23]). Let K be the 2-skeleton of $P_R(\mathbb{Z}, \bar{d})$. Then K is simply-connected, uniformly locally finite and 2-ended.

We make K a piecewise Euclidean complex by identifying each 2-face with an equilateral triangle and identifying each edge with $[0, 1]$. Let $d_{\mathbb{E}}$ be the resulting length metric. There is an inclusion map $i : (\mathbb{Z}, d) \to (K, d_{\mathbb{E}})$ which is a quasi-isometry with quasi-isometry constants depending only on L and A.

Claim 7.8. There exist $D_1 = D_1(L, A)$ and a collection of disjoint essential tracks $\{\tau_i\}_{i\in I}$ of K such that

(1) $\{\tau_i\}_{i\in I}$ is G-invariant.
(2) The diameter of each τ_i with respect to $d_{\mathbb{E}}$ is $\leq D_1$.
(3) Each connected component of $K \setminus (\cup_{i\in I}\tau_i)$ has diameter $\leq D_1$.

In the following proof, we denote the ball of radius D centered at x in K with respect to $d_{\mathbb{E}}$ by $B_{\mathbb{E}}(x, D)$. Let $\text{diam}_{\mathbb{E}}$ be the diameter with respect to $d_{\mathbb{E}}$.

Proof of Claim 7.8. First we assume K does not have separating vertices. Since K is quasi-isometric to $\mathbb{Z}$, there exists $D = D(L, A)$ such that $K \setminus B_{\mathbb{E}}(x, D)$ has at least two unbounded components for each $x \in K$. Thus every $(D+1)$-ball contains an essential track with weight bounded above by $D' = D'(L, A)$. We put a G-invariant hyperbolic metric $d_{\mathbb{H}}$ on K as in Section 7.1. By Lemma 7.3, there exists an essential track $\tau \subset K$ of least complexity. Note that $\text{diam}_{\mathbb{E}}(\tau) \leq D'$ since the weight $w(\tau) \leq D'$. Lemma 7.7 implies the G-orbits of τ give rise to collection of disjoint essential tracks in K.

A collection of tracks $\{\tau_i\}_{i\in I}$ of K is *admissible* if

(1) Each track in $\{\tau_i\}_{i\in I}$ is essential and different tracks have empty intersection.
(2) No two tracks in $\{\tau_i\}_{i\in I}$ are parallel.
(3) The collection $\{\tau_i\}_{i\in I}$ is G-invariant.
(4) $\text{diam}_{\mathbb{E}}(\tau_i) \leq D'$ for each $i \in I$.

There exists a non-empty admissible collection of tracks by previous discussion.

Let $\{\tau_i\}_{i\in I}$ be a maximal admissible collection of tracks. Then this collection satisfies the above claim with $D_1 = 2D' + 5D$. To see this, let C be one connected component of $K \setminus (\cup_{i\in I}\tau_i)$. Since each track is essential and K is 2-ended, either $\text{diam}_{\mathbb{E}}(C) < \infty$ and $\bar{C} \setminus C$ ($\bar{C}$ is the closure of C) is made of two tracks τ_1 and τ_2, or $\text{diam}_{\mathbb{E}}(C) = \infty$ and $\bar{C} \setminus C$ is made of one track. Let us assume the former case is true. The latter case can be dealt in a similar way. Let A be the maximal subcomplex of K which is contained in C. Then A is uniformly locally finite and $C \setminus A$ is contained in the 1-neighbourhood of $\tau_1 \cup \tau_2$.

Suppose $\text{diam}_{\mathbb{E}}(C) \geq 2D' + 5D$. Since $\text{diam}_{\mathbb{E}}(\tau_i) \leq D'$ for $i = 1, 2$, there exists $x \in A$ such that $B_{\mathbb{E}}(x, 2D) \subset A$. Thus A contains an essential track of X with its weight bounded above by D'. Let $\eta \subset A$

be an essential track of K which has the least complexity in the sense of Lemma 7.6, then $w(\eta) \leq D'$, hence $\mathrm{diam}_{\mathbb{E}}(\eta) \leq D'$. Moreover, by Lemma 7.7, for each $g \in \mathrm{Stab}(A) = \mathrm{Stab}(C)$, either $g \cdot \eta = \eta$ or $g \cdot \eta \cap \eta = \emptyset$. Thus we can enlarge the original admissible collection of tracks by adding the G-orbits of η, which yields a contradiction.

The case when K has separating vertices is actually easier, since one can find essential tracks on the ϵ-sphere of each separating vertices. The rest of the proof is identical. □

We now continue with the proof of Proposition 6.2.

Pick a regular neighbourhood $N(\tau_i)$ for each τ_i such that collection $\{N(\tau_i)\}_{i\in I}$ is disjoint and G-invariant. Then we define a map ϕ from K to a tree T by collapsing each component of $Y \setminus \cup_{i\in I} N(\tau_i)$ to a vertex and collapsing each $N(\tau_i)$, which is homeomorphic to $\tau_i \times (0,1)$, to the $(0,1)$ factor. It is easy to make ϕ equivariant under G, and by the above claim, ϕ is a quasi-isometry with quasi-isometry constants depending only on L and A. Note that T is actually a line since τ is essential and K is 2-ended. Then Proposition 6.2 follows by considering the G-equivariant map $\phi \circ i : (\mathbb{Z}, d) \to T$.

Remark 7.9. If the action $G \curvearrowright \mathbb{Z}$ by (L,A)-quasi-isometries in Proposition 6.2 is not cobounded, then the resulting isometric action $\Lambda : G \curvearrowright \mathbb{Z}$ is also not cobounded, hence there are two possibilities:

(1) if G coarsely preserve the orientation of $\mathbb{Z}$, then Λ is trivial;
(2) otherwise Λ factors through a $\mathbb{Z}/2$-action by reflection.

Part II: An alternate blow-up construction for buildings

8. Blowing-up buildings by metric spaces

In Section 5 we discussed a construction for blowing-up a right-angled building. Here we give a somewhat different approach, which allows us to deal with more general situations. It unifies several other objects discussed in this paper and has interesting applications to graph products.

8.1. Construction.

We are motivated by the fact that the universal cover of the Salvetti complex can obtained by gluing a collection of standard flats in a way determined by the associated building. Similarly, we can construct the universal cover of exploded Salvetti complex by gluing a collection of branched flats. Here we specify the gluing pattern, and replace standard flats or branched flats by other product structures, which allows us to generalize the construction of these objects.

Let Γ be a finite simplicial graph, and let $\mathcal{B}$ be a building modelled on the right-angled Coxeter group $W(\Gamma)$ with finite defining graph Γ.

Step 1: For every vertex v of Γ, and every v-residue $r_v \subset \mathcal{B}$, we associate a metric space Z_{r_v}, and a map f_{r_v} which maps chambers in r_v to Z_{r_v}. If another v-residue r'_v is parallel to r_v, then we associate r'_v with the same space Z_{r_v}, and a map $f_{r'_v}$ which is induced by f_{r_v} and the parallelism between r_v and r'_v.

Let M_{r_v} be the mapping cylinder of f_{r_v}. M_{r_v} is obtained by attaching edges to Z_{r_v}, and the endpoints of these edges which correspond to the domain of f_{r_v} are called *tips* of M_{r_v}. There is a 1-1 correspondence between the tips of M_{r_v} and chambers in r_v. Each edge in M_{r_v} has length $= 1$, and M_{r_v} is endowed with the quotient metric (see [BH99, Chapter I.5.19]).

In summary, for each parallel class of v-residues, we have associated a space M_{r_v}, whose tips are in 1-1 correspondence with chambers in any chosen v-residue in this parallel class. The spaces Z_{r_v}'s and the maps f_{r_v}'s is called the *blow-up data* on building $\mathcal{B}$.

Step 2:

Let S^r be the poset of spherical residues in $\mathcal{B}$. We define a functor Φ from S^r to the category of nonempty metric spaces and isometric embeddings as follows.

Let $r_J = \prod_{j\in J} r_j$ be the product decomposition of a spherical J-residue into its rank 1 residues. Define $\Phi(r_J) = \prod_{j\in J} M_{r_j}$, which is the Cartesian product of the metric spaces M_{r_j}'s.

Suppose $r_{J'}$ is a spherical J'-residue such that $r_{J'} \subset r_J$. In this case $J' \subset J$. Moreover, $r_{J'}$ can be expressed as a product $\prod_{j\notin J'} u_j \times \prod_{j\in J'} r_j$, here $u_j \in r_j$ is a chamber. By step 1, each u_j corresponds to a tip $t_j \in M_{r_j}$. Then we define the morphism $\Phi(r_{J'}) \to \Phi(r_J)$ to be the isometric embedding $\prod_{j\notin J'} t_j \times \prod_{j\in J'} M_{r_j} \to \prod_{j\in J} M_{r_j}$.

Step 3:

We begin with the disjoint union $\bigsqcup_{r_J\in \mathsf{S}^r} \Phi(r_J)$, and for every inclusion $r_{J'} \subset r_J$, we glue $\Phi(r_{J'})$ with a subset of $\Phi(r_J)$ by using the map $\Phi(r_{J'}) \to \Phi(r_J)$ defined in step 2.

We denote the resulting space with the quotient metric ([BH99, Definition I.5.19]) by Π. Π is called the *blow-up* of $\mathcal{B}$ with respect to the blow-up data Z_{r_v}'s and f_{r_v}'s.

The following lemma gives an alternative description of our gluing process.

Lemma 8.1. *Two points $x_1 \in \Phi(r_{J_1})$ and $x_2 \in \Phi(r_{J_2})$ are glued together if and only if $r_J = r_{J_1} \cap r_{J_2} \neq \emptyset$ and there exists a point $x \in \Phi(r_J)$ such that for $i = 1, 2$, its image under $\Phi(r_J) \to \Phi(r_{J_i})$ is x_i.*

Proof. The if direction is clear. For the only if direction, define $x_1 \sim x_2$ if the condition in the lemma is satisfied, it suffices to show $\sim$ is an equivalence relation. However, this follows from our construction. □

We can collapse each mapping cylinder in the construction to its range, namely $M_{r_v} \to Z_{r_v}$. This induces a collapsing $\Pi \to \bar{\Pi}$. $\bar{\Pi}$ is called the *reduced blow-up* of $\mathcal{B}$ with respect to Z_{r_v}'s and f_{r_v}'s.

Remark 8.2. There is some flexibility in this construction; for example, we can also collapse certain collections of mapping cylinders and keep other mapping cylinders to obtain a "semi-reduced" blow-up. Also instead of requiring $\Phi(r_J)$ to be the Cartesian product of the metric spaces M_{r_j}'s, we can use l^p-product for $1 \leq p \leq \infty$. We will see an example later where it is more natural to use l^1-product.

Let $\mathcal{B}_1 \subset \mathcal{B}$ be a residue. We restrict the blow-up data on $\mathcal{B}$ to $\mathcal{B}_1$ and obtain a blow-up Π_1 of $\mathcal{B}_1$. There is a natural map $\Pi_1 \to \Pi$, which is injective by Lemma 8.1. If $\mathcal{B}_1$ is a spherical residue, then the image of $\Pi_1 \to \Pi$ is a product of mapping cylinders. It is called a *standard product*, and there is a 1-1 correspondence between spherical residues in $\mathcal{B}$ and standard products in Π.

8.2. **Properties.**

Proposition 8.3. *If each Z_{r_v} in the above construction is a point, then the resulting blow-up Π is isometric to the Davis realization of $\mathcal{B}$.*

Proof. Let $|\mathcal{B}|$ be the Davis realization of $\mathcal{B}$. Let $r_J \subset \mathcal{B}$ be a spherical residue and let $D(r_J)$ be the subcomplex of $|\mathcal{B}|$ spanned by those vertices corresponding to residues inside r_J. Note that $D(r_J)$ is a convex subcomplex, and there is a naturally defined isomorphism $h_{r_J} : D(r_J) \to \Phi(r_J)$. One readily verifies the following:

(1) The collection of all such $D(r_J)$'s covers $|\mathcal{B}|$.

(2) There is a 1-1 correspondence between $D(r_J)$'s and $\Phi(r_J)$'s in step 2 of Section 8.1.
(3) The gluing pattern of these $D(r_J)$'s is compatible with step 3 of Section 8.1 via the maps h_{r_J}'s.

It follows that h_{r_J}'s induce a cubical isomorphism $|\mathcal{B}| \to \Pi$. $\square$

Remark 8.4. If each Z_{r_v} is a point, then the corresponding reduced blow-up is a point. However, later we will see cases where the reduced blow-up is more interesting than the blow-up.

It follows that for any blow-up Π of $\mathcal{B}$, there is a surjective projection map from Π to the Davis realization of $\mathcal{B}$ which is induced by collapsing each Z_{r_v} to a point. This is called the *canonical projection*. This map is a special case of the following situation.

Let Π_1 be the blow-up of $\mathcal{B}_1$ with respect to Z_{r_v}'s and f_{r_v}'s, and let Π_2 be the blow-up of $\mathcal{B}_2$ with respect to Y_{r_v}'s and g_{r_v}'s. Let $\bar{\Pi}_1$ and $\bar{\Pi}_2$ be the corresponding reduced blow-ups. Suppose:

(1) There exists a bijection $\eta : \mathcal{B}_1 \to \mathcal{B}_2$ such that both η and η^{-1} preserve spherical residues.
(2) For each r_v, there exists a map $h_{r_v} : Z_{r_v} \to Y_{\eta(r_v)}$ such that the following diagram commutes:

$$\begin{array}{ccc} r_v & \xrightarrow[\eta]{} & \eta(r_v) \\ \Big\downarrow{\scriptstyle f_{r_v}} & & \Big\downarrow{\scriptstyle g_{\eta(r_v)}} \\ Z_{r_v} & \xrightarrow[h_{r_v}]{} & Y_{\eta(r_v)} \end{array}$$

Then there are induced maps $h : \Pi_1 \to \Pi_2$ and $\bar{h} : \bar{\Pi}_1 \to \bar{\Pi}_2$. Let $\eta_* : |\mathcal{B}_1| \to |\mathcal{B}_2|$ be the map between Davis realizations induced by η. Then h fits into the following commuting diagram:

$$\begin{array}{ccc} \Pi_1 & \xrightarrow[h]{} & \Pi_2 \\ \Big\downarrow & & \Big\downarrow \\ |\mathcal{B}_1| & \xrightarrow[\eta_*]{} & |\mathcal{B}_2| \end{array}$$

Let $\mathcal{B}' \subset \mathcal{B}$ be a residue. Let $\mathrm{proj}_{\mathcal{B}'} : \mathcal{B} \to \mathcal{B}'$ be the projection map defined in Section 3.4. Pick a spherical residue $r \in \mathcal{B}$. Suppose $r_0 = \mathrm{proj}_r(\mathcal{B}')$ and $\mathcal{B}'_0 = \mathrm{proj}_{\mathcal{B}'}(r)$. Let $r = r_0 \times r_0^{\perp}$ be the product decomposition of r. It follows from [AB08, Lemma 5.36] that the map $\mathrm{proj}_{\mathcal{B}'} : r \to \mathcal{B}'_0$ is a composition of the factor projection $r \to r_0$ and

$\text{proj}_{\mathcal{B}'} : r_0 \to \mathcal{B}'_0$. Recall that r_0 and $\mathcal{B}'_0$ are parallel and the second map induces the parallelism map between r_0 and $\mathcal{B}'_0$.

Let Π be a blow-up of $\mathcal{B}$. We restrict the blow-up data on $\mathcal{B}$ to $\mathcal{B}'$, and let Π' be the resulting blow-up of $\mathcal{B}'$. Since there is a 1-1 correspondence between spherical residues in $\mathcal{B}$ or $\mathcal{B}'$ with standard products in Π or Π', then above discussion implies that we can assign to each standard product of Π a standard product in Π', together with a map between them. This assignment is compatible with the gluing pattern of these standard products, thus induces a map $\rho : \Pi \to \Pi'$. This map ρ is a 1-Lipschitz retraction map, and is called a *residue retraction* map. The following is an consequence of the existence of such map.

Corollary 8.5. *The inclusion $\Pi' \to \Pi$ is an isometric embedding.*

Theorem 8.6. *If each Z_{r_v} is $CAT(0)$, then the blow-up Π of a right-angled building $\mathcal{B}$ with the quotient metric is also $CAT(0)$.*

Proof. We start with the following observation. If $\mathcal{R} \subset \mathcal{B}$ is a residue, then $\mathcal{R}$ is also a building. We can restrict any blow-up data of $\mathcal{B}$ to a blow-up data of $\mathcal{R}$. Let Π_B and $\Pi_{\mathcal{R}}$ be the resulting blow-ups. And let $|\mathcal{B}|$ and $|\mathcal{R}|$ be the Davis realization of $\mathcal{B}$ and $\mathcal{R}$. Then the following diagram commutes:

$$\begin{array}{ccc} \Pi_{\mathcal{R}} & \xrightarrow[i_1]{} & \Pi_B \\ \downarrow & & \downarrow \\ |\mathcal{R}| & \xrightarrow[i_2]{} & |\mathcal{B}| \end{array}$$

The two vertical maps are canonical projections, and i_1 and i_2 are isometric embeddings (Corollary 8.5). Moreover, the image of i_1 is exactly the inverse image of $|\mathcal{R}|$ under the canonical projection (here $|\mathcal{R}|$ is viewed as a subspace of $|\mathcal{B}|$).

We first show Π_B is locally $CAT(0)$. Define the *rank* of $\mathcal{B}$ to be the maximum possible number of vertices in a clique of Γ. We induct on the rank of $\mathcal{B}$. The rank 1 case is clear. Now assume $\mathcal{B}$ is of rank n, and all blow-ups of buildings of rank $\leq n-1$ are locally $CAT(0)$.

Pick vertex $x \in |\mathcal{B}|$ and suppose x corresponds to a J-residue $r \subset \mathcal{B}$. Let $\pi : \Pi_B \to |\mathcal{B}|$ be the canonical projection and let $\text{st}(x)$ be the open star of x in $|\mathcal{B}|$. It suffices to show $\pi^{-1}(\text{st}(x))$ is locally $CAT(0)$.

Case 1: $J \neq \emptyset$. Let $J^{\perp}$ be the vertices in Γ which are adjacent to every vertex in J. Let $\mathcal{R}$ be the $J \cup J^{\perp}$-residue that contains r. We restrict the blow-up data on $\mathcal{B}$ to $\mathcal{R}$, and let $\Pi_{\mathcal{R}}$ be resulting metric space. Let $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2$ be the product decomposition (see Section 3.4) induced by $J \cup J^{\perp}$. Then we also have $|\mathcal{R}| = |\mathcal{R}_1| \times |\mathcal{R}_2|$ and $\Pi_{\mathcal{R}} = \Pi_{\mathcal{R}_1} \times \Pi_{\mathcal{R}_i}$ where $\Pi_{\mathcal{R}_i}$ is some blow-up of $\mathcal{R}_i$ for $i = 1, 2$. By the induction assumption, $\Pi_{\mathcal{R}_2}$ is locally $CAT(0)$. Moreover, $\mathcal{R}_1$ can be further decomposed into a product of buildings of rank 1, thus $\Pi_{\mathcal{R}_1}$ is locally $CAT(0)$. It follows that Π_R is locally $CAT(0)$. Since $\mathrm{St}(x) \subset |\mathcal{R}|$, $\pi^{-1}(\mathrm{st}(x)) \subset \Pi_R$. Since $\Pi_R \to \Pi_B$ is a local isometry, it follows that $\pi^{-1}(\mathrm{st}(x))$ is locally $CAT(0)$.

Case 2: $J = \emptyset$. In this case, π induces a simplicial isomorphism between $\pi^{-1}(\mathrm{St}(x))$ and $\mathrm{St}(x)$. It follows from the $CAT(0)$ property of $|\mathcal{B}|$ and [BH99, Lemma I.5.27] that $\pi^{-1}(\mathrm{st}(x))$ is locally $CAT(0)$.

It remains to prove Π_B is simply connected. Actually we show the canonical projection $\pi : \Pi_B \to |\mathcal{B}|$ is a homotopy equivalence. We construct a homotopy inverse $\phi : |\mathcal{B}| \to \Pi_B$ as follows.

Recall that $|\mathcal{B}|$ be also be viewed as a blow-up of $\mathcal{B}$ where the associated metric spaces Z'_{r_v}'s are single points. Let M'_{r_v} be the mapping cylinder of the constant map $r_v \to Z'_{r_v}$. There is a natural map $h_{r_v} : M_{r_v} \to M'_{r_v}$ induced by collapsing each Z_{r_v} to Z'_{r_v}.

For each r_v, we choose a map $h'_{r_v} : Z'_{r_v} \to Z_{r_v}$ such that if r_v and r_u are parallel residues of rank 1, then h'_{r_v} and h'_{r_u} have the same image. We extend h'_{r_v} to a map $M'_{r_v} \to M_{r_v}$ as follows. Pick a chamber $c \in r_v$, and let e_c and e'_c be the edges in M_{r_v} and M'_{r_v} corresponding to the chamber c respectively. We identify these two edges with $[0, 1]$ such that the 0-endpoints are the tips. We map the $[1/2, 1]$ half of e'_c to the constant speed geodesic joining the midpoint of e_c and the image of h'_{r_v} in Z_{r_v} (such geodesic is unique since M_{r_v} is $CAT(0)$), and map the $[0, 1/2]$ half of e'_c identically to the $[0, 1/2]$ half of e_c.

One readily verify that one can take suitable products of the maps $h'_{r_v} : M'_{r_v} \to M_{r_v}$ and glue them together to obtain a map $\phi : |\mathcal{B}| \to \Pi_B$. Moreover, for each r_v, there is a geodesic homotopy between $h'_{r_v} \circ h_{r_v}$ to identity, and these homotopies induces a homotopy between $\phi \circ \pi$ and identity. Similarly we can produce a homotopy between $\pi \circ \phi$ and identity. □

8.3. **An equivariant construction.** The construction in this section is similar to Section 5.6.

Definition 8.7. Let $\mathcal{B}$ be a right-angled building. A bijection $f : \mathcal{B} \to \mathcal{B}$ is *flat-preserving* if both f and f^{-1} map spherical residues to spherical residues.

This definition is motivated by Definition 6.1.

Suppose H acts on $\mathcal{B}$ by flat-preserving bijections. Let Λ be the collection of all parallel classes of rank 1 residues, then there is an induced action $H \curvearrowright \Lambda$. We pick a representative from each orbit of this action and denote the resulting set by $\{\lambda_u\}_{u\in U}$.

Let H_u be the stabilizer of λ_u. Pick rank 1 residue r_u in the parallel class Λ_u and let $\rho_u : H_u \curvearrowright r_u$ be the induced factor action (see Section 5.6). We pick a metric space Z_{r_u}, an isometric action $H_u \curvearrowright Z_{r_u}$ and a H_u-equivariant map $f_{r_u} : r_u \to Z_{r_u}$. And we deal with residues which are parallel to r_u as in step 1 of Section 8.1.

We repeat this process for each element in $\{\lambda_u\}_{u\in U}$. If $\lambda \notin \{\lambda_u\}_{u\in U}$, then we fix an element $g_\lambda \in H$ such that $g_\lambda(\lambda) \in \{\lambda_u\}_{u\in U}$. For each rank 1 residue r_v in the parallel class λ, we associate the metric space $Z_{g_\lambda(r_v)}$ and the map $f_{r_v} = f_{g_\lambda(r_v)} \circ g_\lambda$, and we deal with residues which are parallel to r_v as before.

Let Π be the blow-up of $\mathcal{B}$ with respect to the above choice of spaces and maps and let $\bar{\mathcal{B}}$ be the corresponding reduced blow-up. Then there are induced actions $H \curvearrowright \Pi$ and $H \curvearrowright \bar{\Pi}$ by isometries. Moreover, the canonical projection $\Pi \to |\mathcal{B}|$ is H-equivariant.

Next we apply this construction to graph products of groups. Pick a finite simplicial graph Γ with its vertex set denoted by I, and pick a collection of groups $\{H_i\}_{i\in I}$, one for each vertex of Γ. Let H be the graph product of the H_i's with respect to Γ and let $\mathcal{B} = \mathcal{B}(\Gamma)$ be the right-angled building associated with this graph product (see [Dav98, Section 5]). We can identify H with the set of chambers in $\mathcal{B}$. Under this identification, the rank 1 residues in $\mathcal{B}$ correspond to left cosets of H_i's and spherical residues in $\mathcal{B}$ correspond to left cosets of $\prod_{j\in J} H_j$'s where J is the vertex set of some clique in Γ.

In the following discussion, we will slightly abuse notation. When we say a residue in H, we actually mean the corresponding object in $\mathcal{B}$ under the above identification.

Let Λ_H be the collection of all parallel classes of rank 1 residues in H, and let $\lambda_i \in \Lambda_H$ be the parallel class represented by H_i. Then there is a 1-1 correspondence between λ_i's and H-orbits of the induced action $H \curvearrowright \Lambda_H$. Thus we take $\{\lambda_u\}_{u\in U}$ in the above construction to be $\{\lambda_i\}_{i\in I}$. For each H_i, pick a metric space Z_i, an isometric action $H_i \curvearrowright Z_i$, and a H_i-equivariant map $f_i : H_i \to Z_i$. Given such data, we can produce a blow-up Π_H of the building $\mathcal{B}$.

Theorem 8.8. *Let H be the graph product of $\{H_i\}_{i\in I}$ with respect to a finite simplicial graph Γ. Suppose one of the following properties is satisfied by all of H_i's:*

(1) It acts geometrically on a $CAT(0)$ space;
(2) It acts properly on a $CAT(0)$ space;
(3) It acts geometrically on a $CAT(0)$ cube complex;
(4) It acts properly on a $CAT(0)$ cube complex.

Then H also has the same property.

Proof. We prove (1). The other assertions are similar. Suppose each H_i acts geometrically on a $CAT(0)$ space Z_i. By Theorem 8.6, it suffices to show the action $H \curvearrowright \Pi_H$ is geometric.

We first show that $H \curvearrowright \Pi_H$ is cocompact. Note that Π_H is locally compact, since each mapping cylinder in the construction of Π_H is locally compact. We deduce that Π_H is proper since Π_H is $CAT(0)$. Thus it suffices to show $H \curvearrowright \Pi_H$ is cobounded. Let $|H|$ be the Davis realization of $\mathcal{B}$ and let $\pi : \Pi_H \to |H|$ be the H-equivariant canonical projection. Let $V_0 \subset |H|$ be the collection of rank 0 vertices in $|H|$. Note that $\pi^{-1}(v)$ is exactly one point for any $v \in V_0$. The H-action on V_0, hence on $\pi^{-1}(V_0)$ is transitive. Since each action $H_i \curvearrowright Z_i$ is cobounded, the set of tips in the mapping cylinder of $f_i : H_i \to Z_i$ is cobounded. Thus $\pi^{-1}(V_0)$ is cobounded in Π_H since it arises from tips of mapping cylinders in the construction of Π_H. It follows that $H \curvearrowright \Pi_H$ is cobounded, hence cocompact.

Now we prove $H \curvearrowright \Pi_H$ is proper. It suffices to show the intersection of $\pi^{-1}(V_0)$ with any bounded metric ball in Π_H is finite. For each $\lambda \in \Lambda_H$, pick a residue r_v in the parallel class λ. Let $\xi_\lambda : \Pi_H \to M_{r_v}$ be the residue projection. Note that ξ_λ does not depend on the choice of r_v in the parallel class. We claim there exists constant $L > 0$ such that

$$L^{-1}d(x,y) \leq \Sigma_{\lambda\in\Lambda_H} d(\xi_\lambda(x), \xi_\lambda(y)) \leq Ld(x,y)$$

for any $x, y \in \pi^{-1}(V_0)$.

To see the second inequality, pick a geodesic line $\ell \in \Pi_H$ connecting x and y. Since each standard product in Π_H is convex (Corollary 8.5), the intersection of ℓ with each standard product is a (possibly empty) segment. Thus we can assume ℓ is an concatenation of geodesic segments $\{\ell_i\}_{i=1}^k$ such that each ℓ_i is contained in a standard product P_i. Recall that each P_i is a product of mapping cylinders. For each i, we construct another piecewise geodesic $\bar{\ell}_i$ such that:

(1) $\bar{\ell}_i$ and ℓ_i have the same endpoints.
(2) $\bar{\ell}_i$ is a concatenation of geodesic segments $\{\bar{\ell}_{ij}\}_{j=1}^{k_i}$ such that the projection of $\bar{\ell}_{ij}$ to all but one of the product factors of P_i are trivial.
(3) There exists D which is independent of x, y and ℓ_i such that $\text{length}(\bar{\ell}_i) \leq D \text{ length}(\ell_i)$.

It follows from the construction of residue projection that for each λ, $\xi_\lambda(\bar{\ell}_{ij})$ is either a point, or a segment which has the same length as $\bar{\ell}_{ij}$, and there is exactly one λ such that $\xi_\lambda(\bar{\ell}_{ij})$ is non-trivial. Thus the second inequality follows.

To see the first inequality, let c_x and c_y be chambers in $\mathcal{B}$ that correspond to x and y respectively. Let $\mathcal{G} \subset \mathcal{B}$ be a shortest gallery connecting c_x and c_y. Every two adjacent elements in the gallery are contained in a rank 1 residue, and different pairs of adjacent elements give rise to rank 1 residues in different parallel class (otherwise we can shorten the gallery). Note that $\mathcal{G}$ corresponds to a chain of points in $\pi^{-1}(V_0)$. Connecting adjacent points in this chain by geodesics induces a piecewise geodesic $\ell \subset \Pi_H$ connecting x and y, moreover, the length of ℓ is equal to $\Sigma_{\lambda\in\Lambda_H} d(\xi_\lambda(x), \xi_\lambda(y))$, thus the first inequality holds.

Given $x \in \pi^{-1}(V_0)$, any other point $y \in \pi^{-1}(V_0)$ in the R-ball can be connected to x via a chain of points in $\pi^{-1}(V_0)$ which is induced by a shortest gallery as above. The above claim implies that the size of the chain is bounded above in terms of R. Two adjacent points have distance in Π_H bounded above in terms of R, and they correspond to chambers in the same rank 1 residue. There are only finite many such chains since our assumption implies there are only finitely many choices for each successive point of such a chain. Thus the intersection of $\pi^{-1}(V_0)$ with any bounded metric ball in Π_H is finite. □

Remark 8.9. Suppose we use l^1-product instead of Cartesian product in step 2 of Section 8.1. Let d_{l^1} be the resulting metric on Π_H. Then $d_{l^1}(x, y) = \Sigma_{\lambda\in\Lambda_H} d(\xi_\lambda(x), \xi_\lambda(y))$ for any $x, y \in \pi^{-1}(V_0)$.

The construction in this section unifies several other constructions as follows:

(1) The Davis realization of right-angled building (Proposition 8.3).

(2) If $\mathcal{B}$ is the right-angled building associated with a graph product of $\mathbb{Z}$'s, each space is $\mathbb{R}$ and each map is a bijection between chambers and integer points in $\mathbb{R}$, then the blow-up of $\mathcal{B}$ is the universal covering of the exploded Salvetti complex with the correct group action.
(3) The corresponding reduced blow-up in (2) is the universal covering of Salvetti complex.
(4) Let $\mathcal{B}$ be as in (2). If each space is $\mathbb{R}$, and each map arises from Proposition 6.2, then the blow-up of $\mathcal{B}$ is the geometric model we construct for group quasi-isometric to RAAG's in Section 6.
(5) Suppose H is a graph product of a collection of groups $\{H_i\}_{i\in I}$ and $\mathcal{B}$ is the associated building. Suppose each H_i acts on a $CAT(0)$ cube complex Z_i with a free orbit. And we pick a H_i-equivariant map from H_i to the free orbit. Then the action of the H on the associated reduced blow-up of $\mathcal{B}$ is equivariantly isomorphic to the graph product of group actions defined on [Hag08, Section 4.2]

Part III: A wallspace approach to cubulating RAAG's

In the remainder of the paper, we use wallspaces to give a different approach to cubulating quasi-actions on RAAG's. This gives a shorter path to the cubulation, but gives less precise information about its structure.

We assume $G(\Gamma) \neq \mathbb{Z}$ in Section 9 and Section 10.

9. Construction of the wallspace

9.1. **Background on wallspaces.**
We will be following [HW14] in this section.

Definition 9.1. (Wallspaces) Let Z be a nonempty set. A *wall* of Z is a partition of Z into two subsets $\{U, V\}$, each of which is called a *halfspace*. Two points $x, y \in Z$ are *separated* by a wall if x and y are in different halfspaces. $(Z, \mathcal{W})$ is a *wallspace* if Z is endowed with a collection of walls $\mathcal{W}$ such that every two distinct points $x, y \in Z$ are separated by finitely many walls. Here we allow the situation that two points are not separated by any wall.

Two distinct walls $W = \{U_1, U_2\}$ and $W' = \{V_1, V_2\}$ are *transverse* if $U_i \cap V_j \neq \emptyset$ for all $1 \leq i, j \leq 2$.

An *orientation* of a wall is a choice of one of its halfspaces. An orientation σ of the wallspace $(Z, \mathcal{W})$ is a choice of orientation $\sigma(W)$ for each $W \in \mathcal{W}$.

Definition 9.2. A *0-cube* of $(Z, \mathcal{W})$ is an orientation σ such that

(1) $\sigma(W) \cap \sigma(W') \neq \emptyset$ for all $W, W' \in \mathcal{W}$.
(2) For each $x \in Z$, we have $x \in \sigma(W)$ for all but finitely many $W \in \mathcal{W}$.

Each CAT(0) cube complex C gives rise to a wallspace $(Z, \mathcal{W})$ where Z is the vertex set of C and the walls are induced by hyperplanes of C. Conversely, each wallspace $(Z, \mathcal{W})$ naturally give rises to a dual CAT(0) cube complex $C(Z, \mathcal{W})$. The vertices of $C(Z, \mathcal{W})$ are in 1-1 correspondence with 0-cubes of $(Z, \mathcal{W})$. Two 0-cubes are joined by an edge if and only if their orientations are different on exactly one wall.

Lemma 9.3. ([HW14, Corollary 3.13]) *$C(Z, \mathcal{W})$ is finite dimensional if and only if there is a finite upper bound on the size of collections of pairwise transverse walls. In this case,* $\dim(C(Z, \mathcal{W}))$ *is equal to the largest possible number of pairwise transverse walls.*

There is a 1-1 correspondence between walls in $(Z, \mathcal{W})$ and hyperplanes in $C(Z, \mathcal{W})$. We can define a map which associates each finite dimensional cube in $C(Z, \mathcal{W})$ a collection of pairwise transverse walls in $(Z, \mathcal{W})$ by considering the hyperplanes which have non-trivial intersection with the cube.

Lemma 9.4. ([HW14, Proposistion 3.14]) *The above map induces a 1-1 correspondence between finite dimensional maximal cubes in $C(Z, \mathcal{W})$ and finite maximal collections of pairwise transverse walls in $(Z, \mathcal{W})$.*

Two 0-cubes σ_x and σ_y of $(Z, \mathcal{W})$ are *separated* by a wall $W \in \mathcal{W}$ if and only if $\sigma_x(W)$ and $\sigma_y(W)$ are different halfspaces of W. In this case, the hyperplane in $C(Z, \mathcal{W})$ corresponding to W separates the vertices associated with σ_x and σ_y.

9.2. **Preservation of levels.** In the rest of this section, we assume $\mathrm{Out}(G(\Gamma))$ is finite and $\rho : H \curvearrowright G(\Gamma)$ is an (L, A)-quasi-action. We refer to Definition 6.1 for flat preserving bijections.

By Theorem 1.3, we can assume ρ is an action by flat-preserving bijections which are also (L, A)-quasi-isometries.

Let $\mathcal{P}(\Gamma)$ be the extension complex, and for every standard flat $F \subset X(\Gamma)$, let $\Delta(F) \in \mathcal{P}(\Gamma)$ be the parallel class; see Section 3.3. Note that every flat preserving bijection $h : G(\Gamma) \to G(\Gamma)$ induces a simplicial isomorphism $\partial h : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$. Thus ρ induces an action $H \curvearrowright \mathcal{P}(\Gamma)$ by simplicial isomorphisms.

Definition 9.5 (Levels)**.** Pick a vertex $v \in \mathcal{P}(\Gamma)$ and let $l \subset X(\Gamma)$ be a standard geodesics such that $\Delta(l) = v$. For every vertex $x \in l$, a *v-level* (*of height* x) is defined to be $\{z \in G(\Gamma) \mid \pi_l(z) = x\}$, here π_l is the CAT(0) projection onto l_v. Note that $\pi_l^{-1}(x)$ is a convex subcomplex of $X(\Gamma)$ and the v-level of height x is exactly the vertex set of this convex subcomplex. The definition of v-level does not depend on the choice of the standard geodesic l in the parallel class.

Lemma 9.6. *Let $h : G(\Gamma) \to G(\Gamma)$ be a flat-preserving projection. Then for any $v \in \mathcal{P}(\Gamma)$, h sends v-levels to $\partial h(v)$-levels.*

Proof. Pick standard geodesic l such that $\Delta(l) = v$ and pick vertex $x \in l$. Suppose l' is the standard geodesic such that $\alpha(v(l)) = v(l')$. Let $y \in G(\Gamma)$ be such that $\pi_l(y) = x$. It suffices to show $\pi_{l'}(h(y)) = h(x)$.

Let ω be a combinatorial geodesic joining y and x and let $\{x_i\}_{i=0}^n$ be vertices in ω such that (1) $x_0 = y$ and $x_n = x$; (2) for $1 \leq i \leq n$, there exists a standard geodesic l_i such that $x_i \in l_i$ and $x_{i-1} \in l_i$. Note that $\omega \subset \pi_l^{-1}(x)$ since $\pi_l^{-1}(x)$ is a convex subcomplex. Thus $l_i \subset \pi_l^{-1}(x)$ for all i, hence $\Delta(l_i) \neq v$ for all i. Suppose $x'_i = h(x_i)$ and let l'_i be the standard geodesic which contains x'_i and x'_{i-1}. Then $\partial h(\Delta(l_i)) = \Delta(l'_i)$, thus $\Delta(l'_i) \neq \Delta(l')$. It follows that $\pi_{l'}(l'_i)$ is a point, and hence $\pi_{l'}(x'_i) = \pi_{l'}(x'_{i-1})$ for all i. □

Lemma 9.7. *Let $K \subset X(\Gamma)$ be a standard subcomplex and define $\Delta(K)$ to be the union of all $\Delta(F)$ with F ranging over all standard flats in K. Pick vertex $v \in \mathcal{P}(\Gamma)$. If $v \notin \Delta(K)$, then $K^{(0)}$ is contained in a v-level.*

Proof. Let $l \in X(\Gamma)$ be a standard geodesic such that $\Delta(l) = v$. It suffices to prove for each edge $e \in K$, $\pi_l(e)$ is a point. Suppose the contrary is true. Then $\pi_l(e)$ is an edge of l by Lemma 3.3, and hence the standard geodesic l_e that contains e is parallel to l. However, $l_e \subset K$ since K is standard, which implies $v \in \Delta(K)$. □

9.3. **A key observation.** The flat-preserving action $H \curvearrowright G(\Gamma)$ induces an action $H \curvearrowright \mathcal{P}(\Gamma)$. For each vertex $v \in \mathcal{P}(\Gamma)$, let $H_v \subset H$ be the subgroup which stabilizes v. In other words, if l is a standard geodesic such that $\Delta(l) = v$, then H_v is the stabilizer of the parallel set of l.

By Lemma 9.6, H_v permutes v-levels. Recall that the CAT(0) projection π_l induces a 1-1 correspondence between v-levels and vertices of l; moreover, for any two v-levels L_1 and L_2, $d(L_1, L_2) = d(\pi_l(L_1), \pi_l(L_2))$ (d is the word metric). Thus the collections of v-levels can be identified with $\mathbb{Z}$ (endowed with the standard metric), and we have an induced action $\rho_v : H_v \curvearrowright \mathbb{Z}$ by (L', A')-quasi-isometries; here L', A' can be chosen to be independent of v.

We recall the following result which is proved in Section 7.

Proposition 9.8. *If a group G has an action on $\mathbb{Z}$ by (L, A)-quasi-isometries, then there exists another action $G \curvearrowright \mathbb{Z}$ by isometries which is related to the original action by a surjective equivariant (L', A')-quasi-isometry $f : \mathbb{Z} \to \mathbb{Z}$ with L', A' depending on L and A.*

Definition 9.9 (Branched lines and flats)**.** A *branched line* is constructed from a copy of $\mathbb{R}$ by attaching finitely many edges of length 1 to each integer point. We also require the valence of each vertex in a branched line is bounded above by a uniform constant. This space has a natural simplicial structure and is endowed with the path metric. A *branched flat* is a product of finitely many branched lines.

Definition 9.10 (Branching number)**.** Let β be a branched line. We define the *branching number* of β, denoted by $b(\beta)$, to be the maximum valence of vertices in β.

The following picture is a branched line with branching number = 5:

Remark 9.11. A branched line can be roughly thought as a "mapping cylinder" of the map f in Proposition 9.8.

Definition 9.12 (Tips and wall structure on tips)**.** Let β be a branched line. We call the vertices of valence ≤ 2 in β the *tips* of β, and the collection of all tips is denoted by $t(\beta)$. The set of hyperplanes in β (namely the midpoints of edges) induces a natural wall structure on $t(\beta)$.

For a branched flat F, we define $t(F)$ to be the product of the tips of its branched line factors.

Corollary 9.13. *If a group G has an action on $\mathbb{Z}$ by (L, A)-quasi-isometries, then there exist a branched line β and an isometric action*

$G \curvearrowright \beta$ which is related to the original action by a bijective equivariant map $q : \mathbb{Z} \to t(\beta)$. Moreover, there exists a constant M depending only on L and A such that q is an M-bi-Lipschitz map and each vertex in β has valance $\leq M$.

Proof. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the map in Proposition 9.8. We identify the range of f as integer points of $\mathbb{R}$. Pick $x \in \mathbb{Z}$, if the cardinality $|f^{-1}(x)| \geq 2$, we attach $|f^{-1}(x)|$ many edges of length 1 to $\mathbb{R}$ along x. When $|f^{-1}(x)| = 1$, no edge will be attached. Let β be the resulting branched line. Then there is a natural bijective equivariant map $q : \mathbb{Z} \to t(\beta)$ and the corollary follows from Proposition 9.8. □

In the proof of Corollary 9.13, the CAT(0) cube complex dual to $t(\beta)$ with the wall structure described in Definition 9.12 is isomorphic to β. Note that this is not true for general branched lines.

9.4. **An invariant wallspace.** $G(\Gamma)$ has a natural wallspace structure which is induced from $X(\Gamma)$; however, H may not act on this wallspace. We want to find an alternative wallspace structure on $G(\Gamma)$ which is consistent with the action of H.

Definition 9.14 (v-walls and v-halfspaces)**.** Pick a vertex $v \in \mathcal{P}(\Gamma)$. It follows from Corollary 9.13 that there exist a branched line β_v, an isometric action $H \curvearrowright \beta_v$, and an H_v-equivariant surjective map $\eta_v : G(\Gamma) \to t(\beta_v)$ such that the inverse image of each point in $t(\beta_v)$ is a v-level. A *v-wall* (or a *v-halfspace*) of $G(\Gamma)$ is the pullback of some wall (or halfspace) of $t(\beta_v)$ (see Definition 9.12) under the map η_v. The collection of all v-walls is denoted by $\mathcal{W}_v$.

Now for each vertex $v \in \mathcal{P}(\Gamma)$, we want to choose a collection of v-walls, β_v and η_v in an H-equivariant way. Recall that $H \curvearrowright G(\Gamma)$ induces an action $H \curvearrowright \mathcal{P}(\Gamma)$. We pick one representative from each vertex orbit of $H \curvearrowright \mathcal{P}(\Gamma)$, which gives rise to a collection $\{v_i\}_{i \in I}$. We choose $\mathcal{W}_{v_i}$ as above and let $\mathcal{W}$ be the union of all walls in $\mathcal{W}_{v_i}$ for $i \in I$, together with the H-orbits of these walls.

Now we record several consequences of the above construction.

Lemma 9.15.

(1) *Each v-halfspace is a union of v-levels, thus a v-wall and a v'-wall do not induce the same partition of $G(\Gamma)$ if $v \neq v'$.*
(2) *Distinct v-walls in $\mathcal{W}$ are never transverse.*
(3) *$C(G(\Gamma), \mathcal{W}_v)$ is isomorphic to β_v.*

(4) *Pick a standard geodesic line l such that $\Delta(l) = v$, and identify the vertex set $v(l)$ with $\mathbb{Z}$. Then there exists $N > 0$ independent of $v \in \mathcal{P}(\Gamma)$ such that the map $\xi_v : \mathbb{Z} \cong v(l) \to C(G(\Gamma), \mathcal{W}_v) \cong \beta_v$ is an N-bi-Lipschitz bijection.*
(5) *There exists $M > 0$ independent of $v \in \mathcal{P}(\Gamma)$ such that the branching number $b(\beta_v)$ is at most M.*

(4) and (5) follow from Corollary 9.13.

Lemma 9.16. *$(G(\Gamma), \mathcal{W})$ is a wallspace.*

Proof. For any pair of points $x, y \in G(\Gamma)$, we need to show there are finitely many walls separating x and y. If a v-wall separates x from y, then x and y are in different v-level. There are only finitely many such vertices in $\mathcal{P}(\Gamma)$. Given such v, there are finitely many v-walls separating x and y. □

By construction, H acts on the wallspace $(G(\Gamma), \mathcal{W})$. Let C be the CAT(0) cube complex dual to $(G(\Gamma), \mathcal{W})$, then there is an induced action $H \curvearrowright C$. Now we look at several properties of C.

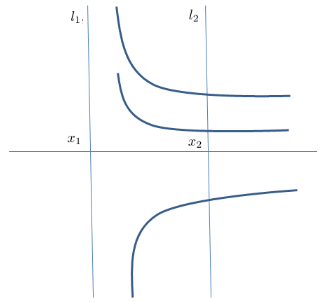
10. Properties of the cubulation

10.1. **Dimension.** We need the following lemmas before we compute the dimension of the dual complex C.

Lemma 10.1. *Let $v_1, v_2 \in \mathcal{P}(\Gamma)$ be distinct vertices. Then a v_1-wall and a v_2-wall are transverse if and only if v_1 and v_2 are adjacent.*

Proof. The if direction is clear (one can consider a standard 2-flat corresponding to the vertices v_1 and v_2). The other direction follows from Lemma 10.2 below. □

Lemma 10.2. *Let $v_1, v_2 \in \mathcal{P}(\Gamma)$ be non-adjacent vertices. For $i = 1, 2$, let l_i be standard geodesic such that $\Delta(l_i) = v_i$. Let $x_2 \in l_2$ be the image $\pi_{l_2}(l_1)$ of the* CAT(0) *projection π_{l_2} and let $x_1 = \pi_{l_1}(l_2)$. Then any v_2-level which is not of height x_2 is contained in the v_1-level of height x_1 (see the picture below).*



Proof. Suppose the contrary is true. Then there exists a vertex $x_2' \in l_2$ with $x_2' \neq x_2$, a v_2-level K of height x_2' and a vertex $x \in K$ such that x is not inside the v_1-level of height x_1. Let $\omega \subset X(\Gamma)$ be a combinatorial geodesic connecting x and x_2'. Since $\pi_{l_1}(x_2') = x_1 \neq \pi_{l_1}(x)$, there exists an edge $e \subset \omega$ and edge $e_1 \subset l_1$ such that e and e_1 are parallel. Thus $\pi_{l_2}(e) = \pi_{l_2}(e_1) = x_2$ by Lemma 3.6. On the other hand, since $\pi_{l_2}^{-1}(x_2')$ is convex, $\pi_{l_2}(e) \subset \pi_{l_2}(\omega) = x_2'$, which is a contradiction. □

Lemma 10.3. $\dim(C) = \dim(G(\Gamma))$.

Proof. Let $\{W_i\}_{i=1}^{l} \subset \mathcal{W}$ be a collection of pairwise intersecting walls and suppose W_i is a u_i-wall. Then $u_i \neq u_j$ for $i \neq j$. Thus $\{u_i\}_{i=1}^{l}$ are vertices of a simplex in $\mathcal{P}(\Gamma)$ by Lemma 10.1. It follows from Lemma 9.3 that $\dim(C) = \dim(\mathcal{P}(\Gamma)) + 1 = \dim(G(\Gamma))$. □

10.2. **Standard branched flats.** Pick a maximal simplex $\Delta \in \mathcal{P}(\Gamma)$. Let $\mathcal{W}_\Delta$ be the collection of u-walls in $\mathcal{W}$ with u ranging over all vertices of Δ. Let $C(G(\Gamma), \mathcal{W}_\Delta)$ be the CAT(0) cube complex dual to the wallspace $(G(\Gamma), \mathcal{W}_\Delta)$.

Lemma 10.4. *There exists a natural isometric embedding*

$$C(G(\Gamma), \mathcal{W}_\Delta) \hookrightarrow C(G(\Gamma), \mathcal{W}) = C.$$

Proof. First we want to assign an orientation to each wall $W \in \mathcal{W} \setminus \mathcal{W}_\Delta$. If W is a u-wall, then by the maximality of Δ, there exists vertex $v \in \Delta$ which is not adjacent to u. Pick standard geodesics l_u such that $\Delta(l_u) = u$. Let F be the maximal standard flat such that $\Delta(F) = \Delta$. Since $u \notin \Delta$, $F^{(0)}$ is contained in a u-level by Lemma 9.7. We orient W such that

$$v(F) \subset \sigma(W). \tag{10.5}$$

Pick a 0-cube σ of $(G(\Gamma), \mathcal{W}_\Delta)$. We claim one can obtain a 0-cube of $(G(\Gamma), \mathcal{W})$ by adding the orientations of walls in $\mathcal{W} \setminus \mathcal{W}_\Delta$ defined as above. Condition (1) of Definition 9.2 follows from (10.5). To see (2), suppose $W \in \mathcal{W} \setminus \mathcal{W}_\Delta$ such that $z \notin \sigma(W)$; then W separates z from $v(F)$ by (10.5) and there are only finitely many such W by Lemma 9.16. Now we have a map from 0-cubes of $(G(\Gamma), \mathcal{W}_\Delta)$ to 0-cubes of $(G(\Gamma), \mathcal{W})$, which extends to higher dimensional cells.

This embedding is isometric since two walls are transverse in the wallspace $(G(\Gamma), \mathcal{W}_\Delta)$ if and only if they are transverse in $(G(\Gamma), \mathcal{W})$. □

The image of $C(G(\Gamma), \mathcal{W}_\Delta)$ under the above embedding is called a *maximal standard branched flat.* The following properties are immediate:

(1) Each maximal standard branched flat is a convex subcomplex of C, since embedding in Lemma 10.4 is isometric.
(2) Each maximal standard branched flat splits as a product of branched lines $C(G(\Gamma), \mathcal{W}_\Delta) \cong \Pi_{v\in\Delta} C(G(\Gamma), \mathcal{W}_v)$ by Lemma 10.1.

Lemma 10.6. *Every maximal cube of C is contained in a maximal standard branched flat, and hence every point of C is contained in a maximal standard branched flat.*

Proof. By Lemma 9.4, every maximal cube is determined by a maximal collection of pairwise transverse walls. By Lemma 10.1, these walls correspond to a maximal simplex in $\Delta \subset \mathcal{P}(\Gamma)$. Thus the lemma follows from Lemma 10.4 since the image of the embedding $C(G(\Gamma), \mathcal{W}_\Delta) \to C$ contains a maximal cube whose dual hyperplanes are the required collection. □

There is a bijective map Φ: {maximal standard flats in $X(\Gamma)$} $\to$ {maximal standard branched flats in C}, since both sides of Φ are in 1-1 correspondence with maximal simplexes of $\mathcal{P}(\Gamma)$.

10.3. **An equivariant quasi-isometry.** Every point $x \in G(\Gamma)$ gives rise to a 0-cube of $(X, \mathcal{W})$ by considering the halfspaces containing x. This induces an H-equivariant map $\phi : G(\Gamma) \to C$. Let $F \subset X(\Gamma)$ be a maximal standard flat, so $\phi(v(F)) \subset \Phi(F)$ by (10.5). Actually $\phi(v(F))$ is exactly the collection of tips of $\Phi(F)$. To see this, note that $\phi|_{v(F)}$ can be written as a composition: $v(F) \to C(G(\Gamma), \mathcal{W}_{\Delta F}) \to C(G(\Gamma, \mathcal{W}))$. Thus there is a natural splitting $\phi|_{v(F)} = \Pi_{i=1}^n \xi_{v_i}$, here $\{v_i\}_{i=1}^n$ are the vertices of $\Delta(F)$ and $\xi_{v_i} : \mathbb{Z} \to C(G(\Gamma), \mathcal{W}_{v_i})$ is the map in Lemma 9.15.

Lemma 10.7. *The map ϕ is coarsely surjective.*

Proof. By Lemma 10.6, it suffices show there exists a constant D which does not depend on the maximal flat F such that $\phi(v(F))$ is D-dense in $\Phi(F)$. This follows from Lemma 9.15. □

Proposition 10.8. *The map $\phi : G(\Gamma) \to C$ is an H-equivariant injective quasi-isometry.*

Proof. We prove ϕ is a bi-Lipschitz embedding, then the proposition follows from Lemma 10.7. Pick $x, y \in G(\Gamma)$ and pick vertex $v \in \mathcal{P}(\Gamma)$. Suppose l is a standard geodesic with $\Delta(l) = v$. Let $d_v(x, y)$ be the number of *v-hyperplanes* in $X(\Gamma)$ which separate x from y (h is a *v-hyperplane* if and only if $l \cap h$ is one point), and let $d_v(\phi(x), \phi(y))$ be the number of walls in $\mathcal{W}_v$ that separate x from y. It suffices to show there exists $L > 0$ which does not depend on x, y and v such that

$$L^{-1} d_v(x, y) \leq d_v(\phi(x), \phi(y)) \leq L d_v(x, y). \tag{10.9}$$

Let $x_0 = \pi_l(x)$ and $y_0 = \pi_l(y)$. Then $d_v(x, y) = d(x_0, y_0)$. Recall that in Lemma 9.15 we define a map $\xi_v : v(l) \to C(G(\Gamma), \mathcal{W}_v) \cong \beta_v$, which is L-bi-Lipschitz with L independent of v. Moreover, $d_v(\phi(x), \phi(y)) = d(\xi(x_0), \xi(y_0))$. Hence (10.9) follows from Lemma 9.15. □

Lemma 10.10. *For any two maximal standard flats $F_1, F_2 \subset X(\Gamma)$, $F_1 \cap F_2 \neq \emptyset$ if and only if $\Phi(F_1) \cap \Phi(F_2) \neq \emptyset$.*

Proof. The only if direction follows from the previous discussion. If $F_1 \cap F_2 = \emptyset$, then there is a hyperplane h separating F_1 from F_2. Let l be a standard geodesic dual to h and let $v = \Delta(l)$. By the maximality of $\Delta(F_1)$, there exists a vertex $v_1 \in \Delta(F_1)$ which is not adjacent to v, and thus $v(F_1)$ is contained is some v-level V_1 by Lemma 3.6. Similarly, $v(F_2) \subset V_2$ for some v-level $V_2 \neq V_1$. If $W \in \mathcal{W}_v$ separates V_1 from V_2, then W also separates $\Phi(F_1)$ from $\Phi(F_2)$ by (10.5). □

If $x \in C$ is a vertex, since each edge that contains x is inside a maximal standard branched flat (Lemma 10.7), and these maximal standard flats intersect each other, Lemma 10.10 and (5) of Lemma 9.15 imply:

Corollary 10.11. *C is uniformly locally finite.*

We have actually proved the following result since the condition on $\text{Out}(G(\Gamma))$ has not been used after Section 9.2.

Theorem 10.12. *Let $H \curvearrowright G(\Gamma)$ be an H-action by flat-preserving bijections which are also (L, A)-quasi-isometries. Then there exists a uniformly locally finite* CAT(0) *cube complex C with* $\dim(G(\Gamma)) = \dim(C)$ *such that the above H-action is conjugate to an isometric action $H \curvearrowright C$. If the H-action is proper or cobounded, then the resulting isometric action is proper or cocompact respectively.*

Corollary 10.13. *Suppose* $\mathrm{Out}(G(\Gamma))$ *is finite and let $H \curvearrowright G(\Gamma)$ be a quasi-action. Then there exists a uniformly locally finite* CAT(0) *cube complex C with* $\dim(G(\Gamma)) = \dim(C)$ *such that the above quasi-action is quasiconjugate to an isometric action $H \curvearrowright C$. If the quasi-action is proper or cobounded, then the resulting isometric action is proper or cocompact respectively.*

Roughly speaking, the cube complex C is obtained by replacing each standard geodesic in $X(\Gamma)$ by a suitable branched line.

Remark 10.14. The cubulation in Corollary 10.13 is slightly different from the one in Theorem 6.4. However, if we modify Definition 9.12 such that tips are the vertices of valence 1 and repeat the whole construction, then cubulation in Corollary 10.13 coincides with the one in Theorem 6.4.

Since we will not be using it, we will not give the argument. However, it is instructive to think about the following example. Given an isometric action $H \curvearrowright X(\Gamma)$, the output cube complex is $X(\Gamma)$ in Corollary 10.13 and is $X_e(\Gamma)$ in Theorem 6.4. Here $X_e(\Gamma)$ is the universal cover of exploded Salvetti complex defined in Section 5.1.

Corollary 10.15. *Suppose* $\mathrm{Out}(G(\Gamma))$ *is finite. If G is a finitely generated group quasi-isometric to $G(\Gamma)$, then G acts geometrically on a* CAT(0) *cube complex.*

10.4. **Preservation of standard subcomplex.** We look at how standard subcomplexes behave under the map $\phi : G(\Gamma) \to C$ in Proposition 10.8.

Let $K \subset X(\Gamma)$ be a standard subcomplex and let $\mathcal{W}_K$ be the collection of walls in $\mathcal{W}$ that separate two vertices of K. Let $\Delta(K)$ be the same as in Lemma 9.7.

Lemma 10.16. *Let $v \in \mathcal{P}(\Gamma)$ be a vertex.*

(1) If $v \in \Delta(K)$, then $\mathcal{W}_v \subset \mathcal{W}_K$.
(2) If $v \notin \Delta(K)$, then $\mathcal{W}_v \cap \mathcal{W}_K = \emptyset$.
(3) There is a natural isometric embedding $C(G(\Gamma), \mathcal{W}_K) \hookrightarrow C$.

Proof. If $v \in \Delta(K)$, then there exists a standard geodesic $l \subset K$ with $\Delta(l) = v$. Thus $\mathcal{W}_v \subset \mathcal{W}_K$. If $v \notin \Delta(K)$, it follows from Lemma 9.7 that $K^{(0)}$ is contained in a v-level, thus (2) is true. It follows from (1) and (2) that for each wall $W \in \mathcal{W} \setminus \mathcal{W}_K$, we can orient W such that $K^{(0)} \subset \sigma(W)$. The rest of the proof for (3) is identical to Lemma 10.4. □

Theorem 10.17. *Let $\phi : G(\Gamma) \to C$ be as in Proposition 10.8. Then there exists $D > 0$ which only depends on the constants of the quasi-action such that for any standard subcomplex $K \subset X(\Gamma)$, there exists a convex subcomplex $K' \subset C$ such that $\phi(K^{(0)})$ is a D-dense subset of K'.*

Proof. By construction, the isometric embedding $C(G(\Gamma), \mathcal{W}_K) \hookrightarrow C$ fits into the following commuting diagram:

$$\begin{array}{ccc} K^{(0)} & \longrightarrow & G(\Gamma) \\ \Big\downarrow \phi' & & \Big\downarrow \phi \\ C(G(\Gamma), \mathcal{W}_K) & \longrightarrow & C \end{array}$$

Here ϕ' sends a vertex of K to the 0-cube of $(G(\Gamma), \mathcal{W}_K)$ which is consist of halfspaces containing this vertex. It suffices to show the image of ϕ' is D-dense.

This can be proved by the same arguments in Section 10.1. Namely, for each simplex Δ which is maximal in $\Delta(K)$, there is a natural isometric embedding $C(G(\Gamma), \mathcal{W}_\Delta) \to C(G(\Gamma), \mathcal{W}_K)$, which gives rise to standard branched flats which are maximal in $C(G(\Gamma), \mathcal{W}_K)$. These branched flats cover $C(G(\Gamma), \mathcal{W}_K)$ and they are in 1-1 correspondence with standard flats which are maximal in K. Moreover, given a standard flat F which is maximal in K, $\phi'(F^{(0)})$ is exactly the set of tips of the corresponding branched flat in $C(G(\Gamma), \mathcal{W}_K)$. Now we can conclude in the same way as Lemma 10.7. □

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