ODD ZETA MOTIVE AND LINEAR FORMS IN ODD ZETA VALUES

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Abstract. We study a family of mixed Tate motives over $\mathbb{Z}$ whose periods are linear forms in the zeta values $\zeta(n)$. They naturally include the Beukers−Rhin−Viola integrals for $\zeta(2)$ and the Ball−Rivoal linear forms in odd zeta values. We give a general integral formula for the coefficients of the linear forms which allows us to predict the vanishing of the coefficients of a given parity. The main underlying result is a geometric construction of a minimal ind-object in the category of mixed Tate motives over $\mathbb{Z}$ which contains all the non-trivial extensions between simple objects.

1. Introduction

1.1. **Constructing linear forms in zeta values.** The study of the values at integers $n \geqslant 2$ of the Riemann zeta function

$$\zeta(n) = \sum_{k\geqslant 1} \frac{1}{k^n}$$

goes back to Euler, who showed that the even zeta value $\zeta(2n)$ is a rational multiple of π^{2n}. Lindemann's theorem thus implies that the even zeta values are transcendental numbers. It is conjectured that the odd zeta values $\zeta(3)$, $\zeta(5)$, $\zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}[\pi]$.

Many of the results in the direction of this conjecture use as a key ingredient certain families of period integrals which evaluate to linear combinations of 1 and zeta values:

$$\int_\sigma \omega \;=\; a_0 + a_2\zeta(2) + \cdots + a_n\zeta(n) \;, \tag{1}$$

with $a_k \in \mathbb{Q}$ for every k. We can cite in particular the following results (see Fischler's Bourbaki talk [Fis04] for a more complete survey).

- Apéry's proof [Apé79] of the irrationality of $\zeta(2)$ and $\zeta(3)$ was simplified by Beukers [Beu79] by using a family of integrals evaluating to linear combinations $a_0 + a_2\zeta(2)$ and $a_0 + a_3\zeta(3)$;
- Ball and Rivoal's proof [Riv00, BR01] that infinitely many odd zeta values are irrational relies on a family of integrals evaluating to linear forms (1) for which all the even coefficients $a_2, a_4, a_6, \ldots$ vanish;
- Rhin and Viola's irrationality measures [RV96, RV01] for $\zeta(2)$ and $\zeta(3)$ are built on generalizations of the Beukers integrals and precise estimates for the coefficients a_2 and a_3.

In view of diophantine applications, it is crucial to have some control over the coefficients a_k appearing in linear forms (1), in particular to be able to predict the vanishing of certain coefficients.

In the present article, we study the family of integrals

$$\int_{[0,1]^n} \omega \qquad \text{with} \qquad \omega = \frac{P(x_1,\ldots,x_n)}{(1-x_1\cdots x_n)^N}\, dx_1\cdots dx_n \;, \tag{2}$$

where $n \geqslant 1$ and $N \geqslant 0$ are integers and $P(x_1,\ldots,x_n)$ is a polynomial with rational coefficients. This family contains the Beukers−Rhin−Viola integrals for $\zeta(2)$ and the Ball−Rivoal integrals. We say that an algebraic differential form ω as in (2) is *integrable* if the integral in (2) is absolutely convergent. Our first result is that such integrals evaluate to linear forms in 1 and zeta values, with an integral formula for the coefficients.

Theorem 1.1. *There exists a family $(\sigma_2,\ldots,\sigma_n)$ of relative n-cycles with rational coefficients in $(\mathbb{C}^*)^n - \{x_1\cdots x_n = 1\}$ such that for every integrable ω, we have*

$$\int_{[0,1]^n} \omega = a_0(\omega) + a_2(\omega)\zeta(2) + \cdots + a_n(\omega)\zeta(n)$$

with $a_k(\omega)$ a rational number for every k, given for $k = 2, \ldots, n$ by the formula

$$a_k(\omega) = \frac{1}{(2i\pi)^k} \int_{\sigma_k} \omega \ . \tag{3}$$

The case $n = k = 2$ of this theorem is Rhin and Viola's contour formula for $\zeta(2)$ [RV96, Lemma 2.6]. We note that in Theorem 1.1, the relative homology classes of the n-cycles σ_k are uniquely determined, see Theorem 4.6 for a precise statement. Furthermore, they are invariant, up to a sign, by the involution

$$\tau : (x_1, \ldots, x_n) \mapsto (x_1^{-1}, \ldots, x_n^{-1}) \ , \tag{4}$$

which implies a general vanishing theorem for the coefficients $a_k(\omega)$, as follows.

Theorem 1.2. *For $k = 2, \ldots, n$ the cycle $\tau.\sigma_k$ is homologous to $(-1)^{k-1}\sigma_k$. Thus, for every integrable ω:*

(1) if $\tau.\omega = \omega$ then $a_k(\omega) = 0$ for $k \neq 0$ even;
(2) if $\tau.\omega = -\omega$ then $a_k(\omega) = 0$ for k odd.

This allows us to construct families of integrals (2) which evaluate to linear forms in 1 and odd zeta values, or 1 and even zeta values. This is the case for the integrals (see Corollary 5.4)

$$\int_{[0,1]^n} \frac{x_1^{u_1-1} \cdots x_n^{u_n-1}(1-x_1)^{v_1-1} \cdots (1-x_n)^{v_n-1}}{(1-x_1 \cdots x_n)^N} \, dx_1 \cdots dx_n$$

where the integers $u_i, v_i \geqslant 1$ satisfy $2u_i + v_i = N + 1$ for every i. Depending on the parity of the product $(n+1)(N+1)$, the form is invariant or anti-invariant and we get the vanishing of even or odd coefficients. This gives a geometric interpretation of the vanishing of the coefficients in the Ball−Rivoal integrals [Riv00, BR01], which correspond to special values of the parameters u_i, v_i.

The fact that the vanishing of certain coefficients in the Ball−Rivoal integrals could be explained by the existence of (anti-)invariant relative cycles was suggested to me by Rivoal during a visit at Institut Fourier, Grenoble, in October 2015. The special role played by the involution τ was first remarked by Deligne in a letter to Rivoal [Del01].

We note that the evaluation of the integrals (2) as linear forms in 1 and zeta values can be proved by elementary methods; the same goes for the statement on the vanishing of even/odd coefficients (see Remark 5.3). What is more surprising, and follows from our geometric methods, is that the coefficients in the linear forms have the integral representations (3). We hope to convince the reader that this property could be studied for more general families of integrals, leading to a finer understanding of linear forms in zeta values.

1.2. **Constructing extensions in mixed Tate motives.** Recall that the category $\mathsf{MT}(\mathbb{Z})$ of mixed Tate motives over $\mathbb{Z}$ is a (neutral) tannakian category of motives (with rational coefficients) defined in [DG05] and whose abstract structure is well understood. The only simple objects in $\mathsf{MT}(\mathbb{Z})$ are the pure Tate objects $\mathbb{Q}(-k)$, for k an integer, and every object in $\mathsf{MT}(\mathbb{Z})$ has a canonical weight filtration whose graded quotients are sums of pure Tate objects. The only non-zero extension groups between the pure Tate objects are given by

$$\mathrm{Ext}^1_{\mathsf{MT}(\mathbb{Z})}(\mathbb{Q}(-(2n+1)), \mathbb{Q}(0)) \cong \mathbb{Q} \qquad (n \geqslant 1) \ . \tag{5}$$

Furthermore, a period matrix of the (essentially unique) non-trivial extension of $\mathbb{Q}(-(2n+1))$ by $\mathbb{Q}(0)$ has the form

$$\begin{pmatrix} 1 & \zeta(2n+1) \\ 0 & (2i\pi)^{2n+1} \end{pmatrix}.$$

The difficulty of constructing linear forms (1) with many vanishing coefficients reflects the difficulty of constructing objects of $\mathsf{MT}(\mathbb{Z})$ with many vanishing weight-graded quotients [Bro14, §1.4]. In particular, the difficulty of constructing small linear forms in 1 and $\zeta(2n+1)$ reflects the difficulty of giving a geometric construction of the extensions (5).

In this article, we construct a minimal ind-object $\mathcal{Z}^{\mathrm{odd}}$ in the category $\mathsf{MT}(\mathbb{Z})$ which contains all the non-trivial extensions (5). The construction goes as follows. We first define, for every integer n, an object $\mathcal{Z}_n \in$

$\mathsf{MT}(\mathbb{Z})$ whose periods include naturally all the integrals (2). More precisely, any integrable form ω defines a class in the de Rham realization $\mathcal{Z}_{n,\mathrm{dR}}$, and the unit n-cube $[0,1]^n$ defines a class in the dual of the Betti realization $\mathcal{Z}_{n,\mathrm{B}}^\vee$, the pairing between these classes being the integral (2). The technical heart of this article is the computation of the full period matrix of $\mathcal{Z}_n$.

Theorem 1.3. *We have a short exact sequence*

$$0 \to \mathbb{Q}(0) \to \mathcal{Z}_n \to \mathbb{Q}(-2) \oplus \cdots \oplus \mathbb{Q}(-n) \to 0$$

and a period matrix for $\mathcal{Z}_n$ is

$$\begin{pmatrix} 1 & \zeta(2) & \zeta(3) & \cdots & \cdots & \zeta(n-1) & \zeta(n) \\ & (2i\pi)^2 & & & & & \\ & & (2i\pi)^3 & & & 0 & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & 0 & & & & (2i\pi)^{n-1} & \\ & & & & & & (2i\pi)^n \end{pmatrix}.$$

Concretely, this theorem says that we can find a basis $(v_0, v_2, \ldots, v_n)$ of the de Rham realization $\mathcal{Z}_{n,\mathrm{dR}}$ (which we will compute explicitly in terms of a special family of integrable forms) and a basis $(\varphi_0, \varphi_2, \ldots, \varphi_n)$ of the dual of the Betti realization $\mathcal{Z}_{n,\mathrm{B}}^\vee$, such that the matrix of the integrals $\langle \varphi_i, v_j \rangle$ is the one given. The basis element φ_0 is the class of the unit n-cube $[0,1]^n$. Expressing the class $[\omega] \in \mathcal{Z}_{n,\mathrm{dR}}$ of an integrable form ω in the basis $(v_0, v_2, \ldots, v_n)$ as

$$[\omega] = a_0(\omega)v_0 + a_2(\omega)v_2 + \cdots + a_n(\omega)v_n$$

and pairing with the dual basis of the Betti realization gives the proof of Theorem 1.1, with $(\sigma_2, \ldots, \sigma_n)$ representatives of the classes $(\varphi_2, \ldots, \varphi_n)$.

The involution (4) plays an important role in the proof of Theorem 1.3. It induces a natural involution, still denoted by τ, on the quotient $\mathcal{Z}_n/\mathbb{Q}(0) \cong \mathbb{Q}(-2) \oplus \cdots \oplus \mathbb{Q}(-n)$.

Theorem 1.4. *For $k = 2, \ldots, n$, the involution τ acts on $\mathbb{Q}(-k)$ by multiplication by $(-1)^{k-1}$.*

This readily implies Theorem 1.2. Now if we write

$$\mathcal{Z}_n/\mathbb{Q}(0) = (\mathcal{Z}_n/\mathbb{Q}(0))^+ \oplus (\mathcal{Z}_n/\mathbb{Q}(0))^-$$

for the decomposition into invariant and anti-invariants with respect to τ and write $p : \mathcal{Z}_n \to \mathcal{Z}_n/\mathbb{Q}(0)$ for the natural projection, we may set

$$\mathcal{Z}_n^{\mathrm{odd}} := p^{-1}((\mathcal{Z}_n/\mathbb{Q}(0))^+)$$

whose period matrix only contains odd zeta values in the first row. The objects $\mathcal{Z}_n^{\mathrm{odd}} \in \mathsf{MT}(\mathbb{Z})$ form an inductive system, and the limit

$$\mathcal{Z}^{\mathrm{odd}} := \varinjlim_n \mathcal{Z}_n^{\mathrm{odd}}$$

has an infinite period matrix

$$\begin{pmatrix} 1 & \zeta(3) & \zeta(5) & \zeta(7) & \cdots & \cdots & \cdots \\ & (2i\pi)^3 & & & & & \\ & & (2i\pi)^5 & & & 0 & \\ & & & (2i\pi)^7 & & & \\ & & & & \ddots & & \\ & 0 & & & & \ddots & \\ & & & & & & \ddots \end{pmatrix}. \tag{6}$$

We call $\mathcal{Z}^{\mathrm{odd}}$ the *odd zeta motive*. It is uniquely determined by its period matrix since the Hodge realization functor is fully faithful on the category $\mathsf{MT}(\mathbb{Z})$, see Theorem 2.5 below.

1.3. **Related work and open questions.** This article follows the program initiated by Brown [Bro14], which aims at explaining and possibly producing irrationality proofs for zeta values by means of algebraic geometry. However, the motives that we are considering are different from the general motives considered by Brown, and in particular, easier to compute. It would be interesting to determine the precise relationship between our motives and those defined in [Bro14] in terms of the moduli spaces $\mathcal{M}_{0,n+3}$.

In another direction, an explicit description of the relative cycles σ_k defined in Theorem 1.1 could prove helpful in proving quantitative results on the irrationality measures of zeta values, in the spirit of [RV96, RV01].

It is also tempting to apply our methods to other families of integrals appearing in the literature, such as the Beukers integrals for $\zeta(3)$ and their generalizations. One should be able, for instance, to recover Rhin and Viola's contour integrals for $\zeta(3)$ [RV01, Theorem 3.1]. The symmetry properties studied by Cresson, Fischler and Rivoal [CFR08] can probably be explained geometrically via finite group actions as in the present article. The ad-hoc long exact sequences appearing here should be replaced by more systematic tools such as the Orlik−Solomon bi-complexes from [Dup14].

Finally, it should be possible to extend our results to a functional version of the periods (2), where one replaces $1 - x_1 \cdots x_n$ in the denominator by $1 - z\, x_1 \cdots x_n$, with z a complex parameter. Such functions have already been considered in [Riv00, BR01]. The relevant geometric objects are variations of mixed Hodge−Tate structures on $\mathbb{C} - \{0,1\}$, or mixed Tate motives over $\mathbb{A}^1_{\mathbb{Q}} - \{0,1\}$.

1.4. **Contents.** In §2 we recall some general facts about the categories in which the objects that we will be considering live, and in particular the categories $\mathsf{MT}(\mathbb{Z})$ and $\mathsf{MT}(\mathbb{Q})$ of mixed Tate motives over $\mathbb{Z}$ and $\mathbb{Q}$. In §3 we introduce the zeta motives and examine their Betti and de Rham realizations. In §4, which is more technical than the rest of the paper, we compute the full period matrix of the zeta motives, which allows us to define the odd zeta motives. In §5, we apply our results to proving vanishing results for the coefficients of some linear forms in zeta values.

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2. Mixed Tate motives and their period matrices

We recall the construction of the categories MHTS, $\mathsf{MT}(\mathbb{Q})$ and $\mathsf{MT}(\mathbb{Z})$, which sit as full subcategories of one another, as follows:

$$\mathsf{MT}(\mathbb{Z}) \hookrightarrow \mathsf{MT}(\mathbb{Q}) \hookrightarrow \mathsf{MHTS}\ .$$

2.1. Mixed Hodge−Tate structures and their period matrices.

Definition 2.1. A *mixed Hodge−Tate structure* is a triple $H = (H_{\mathrm{dR}}, H_{\mathrm{B}}, \alpha)$ consisting of:

- a finite-dimensional $\mathbb{Q}$-vector space H_{B}, together with a finite increasing filtration indexed by even integers: $\cdots \subset W_{2(n-1)}H_{\mathrm{B}} \subset W_{2n}H_{\mathrm{B}} \subset \cdots \subset H_{\mathrm{B}}$;
- a finite-dimensional $\mathbb{Q}$-vector space H_{dR}, together with a grading indexed by even integers: $H_{\mathrm{dR}} = \bigoplus_n (H_{\mathrm{dR}})_{2n}$;
- an isomorphism $\alpha : H_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\simeq} H_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$;

which satisfy the following conditions:

- for every integer n, the isomorphism α sends $(H_{\mathrm{dR}})_{2n} \otimes_{\mathbb{Q}} \mathbb{C}$ to $W_{2n}H_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$;
- for every integer n, it induces an isomorphism $\alpha_n : (H_{\mathrm{dR}})_{2n} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\simeq} (W_{2n}H_{\mathrm{B}}/W_{2(n-1)}H_{\mathrm{B}}) \otimes_{\mathbb{Q}} \mathbb{C}$, which sends $(H_{\mathrm{dR}})_{2n}$ to $(W_{2n}H_{\mathrm{B}}/W_{2(n-1)}H_{\mathrm{B}}) \otimes_{\mathbb{Q}} (2i\pi)^n\mathbb{Q}$.

We call H_{B} and H_{dR} respectively the *Betti realization* and the *de Rham realization* of the mixed Hodge−Tate structure, and α the *comparison isomorphism.* The filtration W on H_{B} is called the *weight filtration.* The grading on H_{dR} is called the *weight grading*, and the corresponding filtration $W_{2n}H_{\mathrm{dR}} := \bigoplus_{k\leqslant n}(H_{\mathrm{dR}})_{2k}$ the *weight filtration.*

Remark 2.2. More classically, a mixed Hodge−Tate structure is defined to be a mixed Hodge structure [Del71, Del74] whose weight-graded quotients are of Tate type, i.e. of type (p,p) for some integer p. One passes from that classical definition to Definition 2.1 by setting $H_{\mathrm{B}} := H$ and $H_{\mathrm{dR}} := \bigoplus_n W_{2n}H/W_{2(n-1)}H$. The isomorphism α is induced by the inverses of the isomorphisms

$$(W_{2n}H/W_{2(n-1)}H)\otimes_{\mathbb{Q}}\mathbb{C} \xleftarrow{\cong} W_{2n}H\otimes_{\mathbb{Q}}\mathbb{C}\cap F^nH\otimes_{\mathbb{Q}}\mathbb{C}$$

(multiplied by $(2i\pi)^n$) which express the fact that the weight-graded quotients are of Tate type.

It is convenient to view the comparison isomorphism $\alpha : H_{\mathrm{dR}}\otimes_{\mathbb{Q}}\mathbb{C} \xrightarrow{\simeq} H_{\mathrm{B}}\otimes_{\mathbb{Q}}\mathbb{C}$ as a pairing

$$H_{\mathrm{B}}^{\vee}\otimes_{\mathbb{Q}} H_{\mathrm{dR}} \longrightarrow \mathbb{C}\ ,\ \varphi\otimes v\mapsto \langle\varphi, v\rangle\ , \tag{7}$$

where $(\cdot)^{\vee}$ denotes the linear dual. The weight filtration on $H_{\mathrm{B}}^{\vee}$ is defined by

$$W_{-2n}H_{\mathrm{B}}^{\vee} := (H_{\mathrm{B}}/W_{2(n-1)}H_{\mathrm{B}})^{\vee}\ ,$$

so that we have

$$W_{-2n}H_{\mathrm{B}}^{\vee}/W_{-2(n+1)}H_{\mathrm{B}}^{\vee} \cong (W_{2n}H_{\mathrm{B}}/W_{2(n-1)}H_{\mathrm{B}})^{\vee}\ .$$

The pairing (7) is compatible with the weight filtrations in that we have $\langle\varphi, v\rangle = 0$ for $\varphi\in W_{-2m}H_{\mathrm{B}}^{\vee}$, $v\in W_{2n}H_{\mathrm{dR}}$ and $m<n$.

If we choose bases for the $\mathbb{Q}$-vector spaces H_{dR} and H_{B}, then the matrix of α in these bases, or equivalently the matrix of the pairing (7), is called a *period matrix* of the mixed Hodge−Tate structure. We will always make the following assumptions on the choice of bases:

- the basis of H_{B} is compatible with the weight filtration;
- the basis of H_{dR} is compatible with the weight grading;
- for every n, the matrix of the comparison isomorphism α_n in the corresponding basis is $(2i\pi)^n$ times the identity.

This implies that any period matrix is block upper-triangular with successive blocks of $(2i\pi)^n\,\mathrm{Id}$ on the diagonal. Conversely, any block upper-triangular matrix with successive blocks of $(2i\pi)^n\,\mathrm{Id}$ on the diagonal is a period matrix of a mixed Hodge−Tate structure.

Example 2.3. Any matrix of the form

$$\begin{pmatrix} 1 & * & * & * & * \\ 0 & 2i\pi & 0 & * & * \\ 0 & 0 & 2i\pi & * & * \\ 0 & 0 & 0 & (2i\pi)^2 & 0 \\ 0 & 0 & 0 & 0 & (2i\pi)^2 \end{pmatrix}$$

defines a mixed Hodge−Tate structure H such that $H_{\mathrm{dR}} = (H_{\mathrm{dR}})_0\oplus(H_{\mathrm{dR}})_2\oplus(H_{\mathrm{dR}})_4$ has graded dimension $(1,2,2)$.

2.2. **The category of mixed Hodge−Tate structures.** We denote by MHTS the category of mixed Hodge−Tate structures. It is a neutral tannakian category over $\mathbb{Q}$, which means in particular that it is an abelian $\mathbb{Q}$-linear category equipped with a $\mathbb{Q}$-linear tensor product $\otimes$. We note that an object $H\in\mathsf{MHTS}$ is endowed with a canonical weight filtration W by sub-objects, such that the morphisms in MHTS are strictly compatible with W. We have two natural fiber functors

$$\omega_{\mathrm{B}} : \mathsf{MHTS}\to\mathsf{Vect}_{\mathbb{Q}} \qquad\text{and}\qquad \omega_{\mathrm{dR}} : \mathsf{MHTS}\to\mathsf{Vect}_{\mathbb{Q}} \tag{8}$$

into the category of finite-dimensional vector spaces, which only remember the Betti realization H_{B} and the de Rham realization H_{dR} respectively. We note that the de Rham realization functor ω_{dR} factors through the category of finite-dimensional graded vector spaces. The comparison isomorphisms α gives an isomorphism between the complexifications of the two fiber functors:

$$\omega_{\mathrm{dR}}\otimes_{\mathbb{Q}}\mathbb{C} \xrightarrow{\simeq} \omega_{\mathrm{B}}\otimes_{\mathbb{Q}}\mathbb{C}\ . \tag{9}$$

For an integer n, we denote by $\mathbb{Q}(-n)$ the mixed Hodge−Tate structure whose period matrix is the 1×1 matrix $((2i\pi)^n)$. Its weight grading and filtration are concentrated in weight $2n$, hence we call it the *pure Tate structure* of weight $2n$. For H a mixed Hodge−Tate structure, the tensor product $H\otimes\mathbb{Q}(-n)$ is simply denoted by $H(-n)$ and called the n-th *Tate twist* of H. A period matrix of $H(-n)$ is obtained by multiplying a period matrix of H by $(2i\pi)^n$. The weight grading and filtration of $H(-n)$ are those of H, shifted by $2n$.

2.3. **Extensions between pure Tate structures.** The pure Tate structures $\mathbb{Q}(-n)$ are the only simple objects of the category MHTS. The extensions between them are easily described. Up to a Tate twist, it is enough to describe the extensions of $\mathbb{Q}(-n)$ by $\mathbb{Q}(0)$ for some integer n. The corresponding extension group is given by

$$\mathrm{Ext}^1_{\mathsf{MHTS}}(\mathbb{Q}(-n), \mathbb{Q}(0)) = \begin{cases} \mathbb{C}/(2i\pi)^n\mathbb{Q} & \text{if } n > 0; \\ 0 & \text{otherwise.} \end{cases}$$

More concretely, the extension corresponding to a number $z \in \mathbb{C}/(2i\pi)^n\mathbb{Q}$ has a period matrix

$$\begin{pmatrix} 1 & z \\ 0 & (2i\pi)^n \end{pmatrix}.$$

We note that the higher extension groups vanish: $\mathrm{Ext}^r_{\mathsf{MHTS}}(H, H') = 0$ for $r \geqslant 2$ and H, H' two mixed Hodge−Tate structures.

Example 2.4. For a complex number $a \in \mathbb{C} - \{0, 1\}$, the cohomology group $H^1(\mathbb{C}^*, \{1, a\})$ is an extension of $\mathbb{Q}(-1)$ by $\mathbb{Q}(0)$ corresponding to $z = \log(a) \in \mathbb{C}/(2i\pi)\mathbb{Q}$. It is called the *Kummer extension* of parameter a.

2.4. **Mixed Tate motives over $\mathbb{Q}$.** Let $\mathsf{MT}(\mathbb{Q})$ denote the category of mixed Tate motives over $\mathbb{Q}$, as defined in [Lev93]. It is a tannakian category. There is a faithful and exact functor

$$\mathsf{MT}(\mathbb{Q}) \to \mathsf{MHTS} \tag{10}$$

into the category MHTS of mixed Hodge−Tate structures, which is called the *Hodge realization functor* ([DG05, §2.13], see also [Hub00, Hub04]). Composing it with the fiber functors (8) gives the de Rham and Betti realization functors, still denoted by

$$\omega_{\mathrm{B}} : \mathsf{MT}(\mathbb{Q}) \to \mathsf{Vect}_{\mathbb{Q}} \qquad \text{and} \qquad \omega_{\mathrm{dR}} : \mathsf{MT}(\mathbb{Q}) \to \mathsf{Vect}_{\mathbb{Q}} \ , \tag{11}$$

and we still have a comparison isomorphism (9). We note that any object in $\mathsf{MT}(\mathbb{Q})$ is endowed with a canonical weight filtration W by sub-objects such that the morphisms in $\mathsf{MT}(\mathbb{Q})$ are strictly compatible with W. The realization morphisms are compatible with the weight filtrations.

Deciding whether a given mixed Hodge−Tate structure is in the essential image of the realization functor (10) is generally difficult. One can at least say that for every integer n, the object $\mathbb{Q}(-n)$ is the realization of a mixed Tate motive over $\mathbb{Q}$ denoted by $\mathbb{Q}(-n)$ as well, and called the *pure Tate motive* of weight $2n$. The extension groups between these objects are computed by the rational K-theory of $\mathbb{Q}$ [Lev93, §4] and hence given by

$$\mathrm{Ext}^1_{\mathsf{MT}(\mathbb{Q})}(\mathbb{Q}(-n), \mathbb{Q}(0)) = \begin{cases} \bigoplus_{p \text{ prime}} \mathbb{Q} & \text{if } n = 1; \\ \mathbb{Q} & \text{if } n \text{ is odd} \geqslant 3; \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

As in the category MHTS, the higher extension groups vanish in the category $\mathsf{MT}(\mathbb{Q})$. The morphisms

$$\mathrm{Ext}^1_{\mathsf{MT}(\mathbb{Q})}(\mathbb{Q}(-n), \mathbb{Q}(0)) \longrightarrow \mathrm{Ext}^1_{\mathsf{MHTS}}(\mathbb{Q}(-n), \mathbb{Q}(0)) \cong \mathbb{C}/(2i\pi)^n\mathbb{Q} \tag{13}$$

induced by (10) are easy to describe. For $n = 1$, the image of the direct summand indexed by a prime p is the line spanned by $\log(p)$. For $n \geqslant 3$ odd, the image is the line spanned by $\zeta(n)$. Thus, the morphism (13) is injective for every n. This implies the following theorem [DG05, Proposition 2.14].

Theorem 2.5. *The realization functor (10) is fully faithful.*

This theorem is very helpful, since it allows one to compute in the category $\mathsf{MT}(\mathbb{Q})$ with period matrices; in other words, a mixed Tate motive over $\mathbb{Q}$ is uniquely determined by its period matrix.

2.5. **Mixed Tate motives over $\mathbb{Z}$.** Let $\mathsf{MT}(\mathbb{Z})$ denote the category of mixed Tate motives over $\mathbb{Z}$, as defined in [DG05]. By definition, it is a full tannakian subcategory

$$\mathsf{MT}(\mathbb{Z}) \hookrightarrow \mathsf{MT}(\mathbb{Q})$$

of the category of mixed Tate motives over $\mathbb{Q}$, which contains the pure Tate motives $\mathbb{Q}(-n)$ for every integer n. It satisfies the following properties:
1. $\mathrm{Ext}^1_{\mathsf{MT}(\mathbb{Z})}(\mathbb{Q}(-1), \mathbb{Q}(0)) = 0$;
2. the natural morphism $\mathrm{Ext}^1_{\mathsf{MT}(\mathbb{Z})}(\mathbb{Q}(-n), \mathbb{Q}(0)) \to \mathrm{Ext}^1_{\mathsf{MT}(\mathbb{Q})}(\mathbb{Q}(-n), \mathbb{Q}(0))$ is an isomorphism for $n \neq 1$.

As in the categories MHTS and $\mathsf{MT}(\mathbb{Q})$, the higher extension groups vanish in the category $\mathsf{MT}(\mathbb{Z})$.

For n odd $\geqslant 3$, there is an essentially unique non-trivial extension of $\mathbb{Q}(-n)$ by $\mathbb{Q}(0)$ in the category $\mathsf{MT}(\mathbb{Q})$, which actually lives in $\mathsf{MT}(\mathbb{Z})$. A period matrix for such an extension is

$$\begin{pmatrix} 1 & \zeta(n) \\ 0 & (2i\pi)^n \end{pmatrix}.$$

Apart from the case $n = 3$ (see [Bro14, Corollary 11.3] or Proposition 4.11 below), we do not know of any *geometric* construction of these extensions.

3. Definition of the zeta motives $\mathcal{Z}_n$

We define the zeta motives $\mathcal{Z}_n$ and explain how to define elements of their Betti and de Rham realizations. In particular, we define the classes of the Eulerian differential forms, which are elements of the de Rham realization $\mathcal{Z}_{n,\mathrm{dR}}$ constructed out of the family of Eulerian polynomials. We also note that the zeta motives fit into an inductive system $\cdots \to \mathcal{Z}_{n-1} \to \mathcal{Z}_n \to \cdots$ which is compatible with the Eulerian differential forms.

3.1. **The definition.** Let $n \geqslant 1$ be an integer. In the affine n-space $X_n = \mathbb{A}^n_{\mathbb{Q}}$ we consider the hypersurfaces

$$A_n = \{x_1 \cdots x_n = 1\} \quad \text{and}$$
$$B_n = \bigcup_{1 \leqslant i \leqslant n} \{x_i = 0\} \cup \bigcup_{1 \leqslant i \leqslant n} \{x_i = 1\} \ .$$

The union $A_n \cup B_n$ is almost a normal crossing divisor inside X_n: around the point $P_n = (1, \ldots, 1)$, it looks like $z_1 \cdots z_n(z_1 + \cdots + z_n) = 0$ (set $x_i = \exp(z_i)$). Let

$$\pi_n : \widetilde{X}_n \to X_n$$

be the blow-up along P_n, and $E_n = \pi_n^{-1}(P_n)$ be the exceptional divisor. We denote respectively by $\widetilde{A}_n$ and $\widetilde{B}_n$ the strict transforms of A_n and B_n along π_n. The union $\widetilde{A}_n \cup \widetilde{B}_n \cup E_n$ is a simple normal crossing divisor inside $\widetilde{X}_n$.

There is an object $\mathcal{Z}_n \in \mathsf{MT}(\mathbb{Q})$, which we may denote by

$$\mathcal{Z}_n = H^n(\widetilde{X}_n - \widetilde{A}_n, (\widetilde{B}_n \cup E_n) - (\widetilde{B}_n \cup E_n) \cap \widetilde{A}_n) \ ,$$

such that its Betti and de Rham realizations (11) are $(? \in \{\mathrm{B}, \mathrm{dR}\})$

$$\mathcal{Z}_{n,?} = H^n_?(\widetilde{X}_n - \widetilde{A}_n, (\widetilde{B}_n \cup E_n) - (\widetilde{B}_n \cup E_n) \cap \widetilde{A}_n) \ .$$

We now give the precise definition of $\mathcal{Z}_n$, along the lines of [Gon02, Proposition 3.6]. Let us write $Y = \widetilde{X}_n - \widetilde{A}_n$ and $\partial Y = (\widetilde{B}_n \cup E_n) - (\widetilde{B}_n \cup E_n) \cap \widetilde{A}_n$, viewed as schemes defined over $\mathbb{Q}$. We have a decomposition into smooth irreducible components $\partial Y = \bigcup_i \partial_i Y$, where i runs in a set of cardinality $2n + 1$. For a set $I = \{i_1, \ldots, i_r\}$ of indices, we denote by $\partial_I Y = \partial_{i_1} Y \cap \cdots \cap \partial_{i_r} Y$ the corresponding intersection; it is either empty or a smooth subvariety of X of codimension r.

We thus get a complex

$$\cdots \to \bigsqcup_{|I|=3} \partial_I Y \to \bigsqcup_{|I|=2} \partial_I Y \to \bigsqcup_{|I|=1} \partial_I Y \to Y \to 0 \tag{14}$$

in Voevodsky's triangulated category $\mathsf{DM}(\mathbb{Q})$ of mixed motives over $\mathbb{Q}$, see [Voe00]. The differentials are the alternating sums of the natural closed immersions. One readily checks that the complex (14) lives in the triangulated Tate subcategory $\mathsf{DMT}(\mathbb{Q})$, which has a natural t-structure whose heart is $\mathsf{MT}(\mathbb{Q})$ [Lev93]. By definition, the object $\mathcal{Z}_n$ in $\mathsf{MT}(\mathbb{Q})$ is the n-th cohomology group of the complex (14) with respect to this t-structure.

Definition 3.1. For $n \geqslant 1$, we call $\mathcal{Z}_n \in \mathsf{MT}(\mathbb{Q})$ the *n-th zeta motive.*

Note that for $n = 1$, the blow-up map $\pi_1 : \widetilde{X}_1 \to X_1$ is an isomorphism and $\widetilde{A}_1 = \varnothing$, so that we get $\mathcal{Z}_1 = H^1(\mathbb{A}^1_{\mathbb{Q}}, \{0,1\})$.

Remark 3.2. We will prove in Proposition 4.12 that $\mathcal{Z}_n$ is actually an object of the full subcategory $\mathsf{MT}(\mathbb{Z}) \hookrightarrow \mathsf{MT}(\mathbb{Q})$. It would be possible, but a little technical, to prove directly from the definition by using the criterion [GM04, Proposition 4.3] on some compactification of $\widetilde{X}_n - \widetilde{A}_n$.

3.2. **Betti and de Rham realizations, 1.** We now give a first description of the Betti and de Rham realizations of the zeta motive $\mathcal{Z}_n$.

We let $C_\bullet$ denote the functor of singular chains with rational coefficients on topological spaces. By definition, the dual of the Betti realization $\mathcal{Z}_{n,\mathrm{B}}^\vee$ is the n-th homology group of the total complex of the double complex

$$
\begin{array}{ccccc}
\cdots\dashrightarrow \bigoplus_{|I|=2} C_0(\partial_I Y(\mathbb{C})) & \longrightarrow & \bigoplus_{|I|=1} C_0(\partial_I Y(\mathbb{C})) & \longrightarrow & C_0(Y(\mathbb{C})) \\
\vdots\uparrow & & \uparrow & & \uparrow \\
\cdots\dashrightarrow & & \bigoplus_{|I|=1} C_1(\partial_I Y(\mathbb{C})) & \longrightarrow & C_1(Y(\mathbb{C})) \\
 & & \vdots\uparrow & & \uparrow \\
 & & \cdots\dashrightarrow & & C_2(Y(\mathbb{C})) \\
 & & & & \vdots\uparrow
\end{array}
\tag{15}
$$

obtained by applying the functor $C_\bullet$ to the complex (14). One readily verifies that this complex is quasi-isomorphic to the quotient complex $C_\bullet(Y(\mathbb{C}))/C_\bullet(\partial Y(\mathbb{C}))$, classically used to define the relative homology groups $H_\bullet^{\mathrm{B}}(Y, \partial Y) = H_\bullet^{\mathrm{sing}}(Y(\mathbb{C}), \partial Y(\mathbb{C}))$.

We let $\Omega^\bullet_{\partial_I Y}$ denote the complex of sheaves of algebraic differential forms on the smooth variety $\partial_I Y$, extended by zero to Y. By definition, the de Rham realization $\mathcal{Z}_{n,\mathrm{dR}}$ is the hypercohomology of the total complex of the double complex of sheaves

$$
\begin{array}{ccccc}
\dashleftarrow\cdots \bigoplus_{|I|=2} \Omega^0_{\partial_I Y} & \longleftarrow & \bigoplus_{|I|=1} \Omega^0_{\partial_I Y} & \longleftarrow & \Omega^0_Y \\
\vdots\downarrow & & \downarrow & & \downarrow \\
\dashleftarrow\cdots & & \bigoplus_{|I|=1} \Omega^1_{\partial_I Y} & \longleftarrow & \Omega^1_Y \\
 & & \vdots\downarrow & & \downarrow \\
 & & \dashleftarrow\cdots & & \Omega^2_Y \\
 & & & & \vdots\downarrow
\end{array}
\tag{16}
$$

where the vertical arrows are the exterior derivatives and the horizontal arrows are the alternating sums of the natural restriction maps as in the complex (14).

The comparison morphism between the Betti and de Rham realizations of $\mathcal{Z}_n$ is induced, after complexification, by the morphism from the double complex (16) to the double complex (15) given by integration. Note that one first has to replace (15) by the double complex of sheaves of singular cochains.

3.3. **Betti and de Rham realizations, 2.** We now give a description of the Betti and de Rham realizations of $\mathcal{Z}_n$ that allow to work directly in the affine space X_n and do not require to work in the blow-up $\widetilde{X}_n$. The justification of the blow-up process goes as follows. Suppose that one wants to find a motive whose periods include all absolutely convergent integrals of the form

$$\int_{[0,1]^n} \frac{P(x_1,\ldots,x_n)}{(1-x_1\cdots x_n)^N}\, dx_1\cdots dx_n \tag{17}$$

where $P(x_1,\ldots,x_n)$ is a polynomial with rational coefficients, and $N \geqslant 0$ is an integer. On the Betti side, the blow-up process is required in order to have a class that represents the integration domain $[0,1]^n$; on the de Rham side, the blow-up process is required in order to only consider *absolutely convergent* integrals of the form (17). This is made precise by Propositions 3.3 and 3.5 below.

We start with the Betti realization. Let us write $\mathring{A}_n = A_n - P_n$ and note that this is not a closed subset, but only a locally closed subset, of X_n.

Proposition 3.3. *The blow-up morphism $\pi_n : \widetilde{X}_n \to X_n$ induces an isomorphism*

$$\mathcal{Z}_{n,\mathrm{B}}^\vee \xrightarrow{\cong} H_n^{\mathrm{sing}}(X_n(\mathbb{C}) - \mathring{A}_n(\mathbb{C}), B_n(\mathbb{C}) - B_n(\mathbb{C}) \cap \mathring{A}_n(\mathbb{C}))\ .$$

Proof. The blow-up morphism π_n is the contraction of the exceptional divisor E_n onto the point P_n. Thus, this is a consequence of the classical excision theorem in singular homology, see for instance [Hat02, Proposition 2.22]. $\square$

As a consequence of Proposition 3.3, we see that the unit n-square $\square_n = [0,1]^n \subset X_n(\mathbb{C}) - \mathring{A}_n(\mathbb{C})$ defines a class

$$[\square_n] \in \mathcal{Z}_{n,\mathrm{B}}^\vee\ .$$

When viewed in $\widetilde{X}_n(\mathbb{C}) - \widetilde{A}_n(\mathbb{C})$, it is the class of the strict transform $\widetilde{\square}_n$, which has the combinatorial structure of an n-cube truncated at one of its vertices.

We now turn to a description of the de Rham realization of $\mathcal{Z}_n$. Instead of giving a general description in terms of algebraic differential forms on $X_n - A_n$, we will only give a way of defining many classes in $\mathcal{Z}_{n,\mathrm{dR}}$, which will turn out to be enough.

Definition 3.4. An algebraic differential n-form on $X_n - A_n$ is said to be *integrable* if it can be written as a linear combination of forms of the type

$$\omega = \frac{(1-x_1)^{v_1-1}\cdots(1-x_n)^{v_n-1} f(x_1,\ldots,x_n)}{(1-x_1\cdots x_n)^N}\, dx_1\cdots dx_n \tag{18}$$

with $v_1,\ldots,v_n \geqslant 1$ and $N \geqslant 0$ integers such that $v_1+\cdots+v_n \geqslant N+1$, and $f(x_1,\ldots,x_n)$ a polynomial with rational coefficients.

The terminology is justified by the following proposition.

Proposition 3.5. *Let ω be an algebraic differential n-form on $X_n - A_n$. If ω is integrable, then $\pi_n^*(\omega)$ does not have a pole along E_n, and thus defines a class in $\mathcal{Z}_{n,\mathrm{dR}}$. In particular, the integral*

$$\int_{\widetilde{\square}_n} \pi_n^*(\omega) = \int_{\square_n} \omega$$

is absolutely convergent and is a period of $\mathcal{Z}_n$.

Proof. We write ω as in (18). We note that the only problem for absolute convergence is around the point $(1,\ldots,1)$. Let us thus make the change of variables $y_i = 1-x_i$ for $i=1,\ldots,n$, and $g(y_1,\ldots,y_n) = (-1)^n f(x_1,\ldots,x_n)$. We write $h(y_1,\ldots,y_n) = 1-(1-y_1)\cdots(1-y_n)$ so that we have

$$\omega = \frac{y_1^{v_1-1}\cdots y_n^{v_n-1} g(y_1,\ldots,y_n)}{h(y_1,\ldots,y_n)^N}\, dy_1\cdots dy_n \ .$$

There are n natural affine charts for the blow-up $\pi_n : \widetilde{X}_n \to X_n$ of the point $(0,\ldots,0)$, and by symmetry it is enough to work in the first one. We then have local coordinates $(z_1,\ldots,z_n)$ on $\widetilde{X}_n$, which are linked to the coordinates $(y_1,\ldots,y_n) = \pi_n(z_1,\ldots,z_n)$ by the formula

$$(y_1,\ldots,y_n) = (z_1, z_1z_2, \ldots, z_1z_n) \ .$$

The problem of convergence occurs in the neighborhood of the exceptional divisor E_n, which is defined by the equation $z_1=0$. Since $h(0,\ldots,0)=0$, we may write

$$h(z_1, z_1z_2,\ldots,z_1z_n) = z_1\, \widetilde{h}(z_1,\ldots,z_n)$$

with $\widetilde{h}(z_1,\ldots,z_n)$ a polynomial such that $\widetilde{h}(0,\ldots,0)=1$. The strict transform $\widetilde{A}_n$ of A_n is thus defined by the equation $\widetilde{h}(z_1,\ldots,z_n)=0$. We note that we have $dy_1\cdots dy_n = z_1^{n-1}dz_1\cdots dz_n$, so that we can write

$$\pi_n^*(\omega) = \frac{z_1^{v_1-1}(z_1z_2)^{v_2-1}\cdots(z_1z_n)^{v_n-1}g(z_1,z_1z_2,\ldots,z_1z_n)}{z_1^N\widetilde{h}(z_1,\ldots,z_n)^N}\, z_1^{n-1}dz_1\cdots dz_n = z_1^{v_1+\cdots+v_n-N-1}\Omega \ ,$$

where Ω has a pole along $\widetilde{A}_n$ but not along E_n. The claim follows. □

We make an abuse of notation and denote by

$$[\omega] \in \mathcal{Z}_{n,\mathrm{dR}}$$

the class of the pullback $\pi_n^*(\omega)$ for ω integrable, so that the comparison isomorphism reads

$$\langle [\square_n], [\omega]\rangle = \int_{\square_n} \omega \ .$$

We note the converse of Proposition 3.5, that we will not use.

Proposition 3.6. *Let ω be an algebraic differential n-form on $X_n - A_n$. If the integral $\int_{\square_n}\omega$ is absolutely convergent, then ω is integrable.*

Proof. In the coordinates $y_i = 1-x_i$, we write

$$\omega = \frac{P(y_1,\ldots,y_n)}{h(y_1,\ldots,y_n)^N}\, dy_1\cdots dy_n$$

with $P(y_1,\ldots,y_n)$ a polynomial with rational coefficients. If the integral $\int_{\square_n}\omega$ is absolutely convergent in the neighborhood of the point $(0,\cdots,0)$, then after the change of variables $\phi(z_1,\ldots,z_n) = (z_1,z_1z_2,\ldots,z_1z_n)$ we get an absolutely convergent integral in the nieghborhood of $z_1=0$. We write, as in the proof of Proposition 3.5:

$$\phi^*(\omega) = \frac{P(z_1,z_1z_2,\ldots,z_1z_n)}{z_1^{N-n+1}\widetilde{h}(z_1,\ldots,z_n)^N}\, dz_1\cdots dz_n \ .$$

Let us write

$$P(y_1,\ldots,y_n) = \sum_{\underline{a}} \lambda_{\underline{a}}\, y_1^{a_1-1}\cdots y_n^{a_n-1}$$

with $\lambda_{\underline{a}}\in\mathbb{Q}$ for every multi-index $\underline{a} = (a_1,\ldots,a_n)$. We then have

$$P(z_1,z_1z_2,\ldots,z_1z_n) = \sum_{\underline{a}} \lambda_{\underline{a}}\, z_1^{a_1+\cdots+a_n-n} z_2^{a_2-1}\cdots z_n^{a_n-1} \ .$$

Let v denote the smallest integer such that there exists a multi-index $\underline{a}$ with $|\underline{a}| := a_1+\cdots+a_n = v$. We then have an equivalence

$$P(z_1,z_1z_2,\ldots,z_1z_n) \sim_{z_1\to 0} z_1^{v-n}Q(z_2,\ldots,z_n)$$

where $Q(z_2,\ldots,z_n)=\sum_{|\underline{a}|=v}\lambda_{\underline{a}}\, z_2^{a_2-1}\cdots z_n^{a_n-1}$. We then have

$$\phi^*(\omega)\sim_{z_1\to 0} z_1^{v-N-1}dz_1\ \frac{Q(z_2,\ldots,z_n)}{(1+z_2+\cdots+z_n)^N}\,dz_2\cdots dz_n\ .$$

This gives an absolutely convergent integral in the neighborhood of $z_1=0$ if and only if $v\geqslant N+1$, which is exactly the integrability condition. □

3.4. **The Eulerian differential forms.** Recall that the family of *Eulerian polynomials* $E_r(x)$, $r\geqslant 0$, is defined by the equation

$$\frac{E_r(x)}{(1-x)^{r+1}}=\sum_{k\geqslant 0}(k+1)^r x^k\ . \tag{19}$$

If $r\geqslant 1$ this is equivalent to

$$\frac{E_r(x)}{(1-x)^{r+1}}=\frac{1}{x}\left(x\frac{d}{dx}\right)^r\frac{1}{1-x}\ .$$

For instance, we have $E_0(x)=E_1(x)=1$, $E_2(x)=1+x$, $E_3(x)=1+4x+x^2$. The Eulerian polynomials satisfy the recurrence relation

$$E_{r+1}(x)=x(1-x)E_r'(x)+(1+rx)E_r(x)\ . \tag{20}$$

For integers $n\geqslant 2$ and $d=2,\ldots,n$, we define a differential form

$$\omega_n^{(d)}=\frac{E_{n-d}(x_1\cdots x_n)}{(1-x_1\cdots x_n)^{n-d+1}}\,dx_1\cdots dx_n.$$

Note that $\omega_n^{(n)}=\dfrac{dx_1\cdots dx_n}{1-x_1\cdots x_n}$.

Lemma 3.7. *The form $\omega_n^{(d)}$ defines a class $[\omega_n^{(d)}]\in\mathcal{Z}_{n,\mathrm{dR}}$ and we have*

$$\langle[\square_n],[\omega_n^{(d)}]\rangle=\int_{\square_n}\omega_n^{(d)}=\zeta(d)\ . \tag{21}$$

Proof. The first statement follows from Proposition 3.5. The computation of the period is then straightforward using the definition (19) of Eulerian polynomials:

$$\int_{\square_n}\omega_n^{(d)}=\sum_{k\geqslant 0}(k+1)^{n-d}\int_{[0,1]^n}(x_1\cdots x_n)^k\,dx_1\cdots dx_n=\sum_{k\geqslant 0}(k+1)^{-d}=\zeta(d)\ .$$

□

For every $n\geqslant 0$, we define $\omega_n^{(0)}=dx_1\cdots dx_n$; we also have the class of $[\omega_n^{(0)}]\in\mathcal{Z}_{n,\mathrm{dR}}$, whose pairing with the class $\square_n$ is

$$\langle[\square_n],[\omega_n^{(0)}]\rangle=\int_{\square_n}\omega_n^{(0)}=1\ .$$

We call the differential forms $\omega_n^{(d)}$, for $d=0,2,\ldots,n$, the *Eulerian differential forms.*

3.5. **An inductive system.** For $n\geqslant 2$ there are natural morphisms

$$i_n:\mathcal{Z}_{n-1}\to\mathcal{Z}_n \tag{22}$$

in the category $\mathsf{MT}(\mathbb{Q})$, that we now define. We fix the identification $X_{n-1}=\{x_n=1\}\subset X_n$, which implies the equality $A_{n-1}=A_n\cap X_{n-1}$. Let us set

$$B_n'=\bigcup_{1\leqslant i\leqslant n}\{x_i=0\}\cup\bigcup_{1\leqslant i\leqslant n-1}\{x_i=1\}\ ,$$

so that we have $B_n=B_n'\cup X_{n-1}$, and $B_{n-1}=B_n'\cap X_{n-1}$.

In the blow-up $\widetilde{X}_n$, we thus get an embedding $\widetilde{X}_{n-1} \subset \widetilde{X}_n$ and identifications $\widetilde{A}_{n-1} = \widetilde{A}_n \cap \widetilde{X}_{n-1}$, $\widetilde{B}_{n-1} = \widetilde{B}'_n \cap \widetilde{X}_{n-1}$ and $E_{n-1} = E_n \cap \widetilde{X}_{n-1}$. Thus, the complex in $\mathsf{DM}(\mathbb{Q})$ that we have used to define $\mathcal{Z}_{n-1}$ is the subcomplex

$$\cdots \to \bigsqcup_{\substack{|I|=3 \\ \partial_I Y \subset \widetilde{X}_{n-1}}} \partial_I Y \to \bigsqcup_{\substack{|I|=2 \\ \partial_I Y \subset \widetilde{X}_{n-1}}} \partial_I Y \to \widetilde{X}_{n-1} \to 0 \to 0 \tag{23}$$

of the complex (14) that we have used to define $\mathcal{Z}_n$, shifted by 1. Taking the n-th cohomology groups with respect to the t-structure gives the morphism (22).

In Betti and de Rham realizations, the morphism (22) is also induced by the inclusion of double subcomplexes of (15) and (16).

Remark 3.8. There are signs in the differentials of the complexes (14), (15), (16), that we leave to the reader. This also induces signs on the different components of the inclusions of subcomplexes such as (23).

We define the ind-motive

$$\mathcal{Z} = \varinjlim_{n} \mathcal{Z}_n \ ,$$

viewed as an ind-object in the category $\mathsf{MT}(\mathbb{Q})$, and simply call it the *zeta motive*.

If the signs are chosen consistently (see Remark 3.8), then the map $i^\vee_{n,\mathrm{B}} : \mathcal{Z}^\vee_{n,\mathrm{B}} \to \mathcal{Z}^\vee_{n-1,\mathrm{B}}$ sends the class $[\square_{n-1}]$ to the class $[\square_n]$. This allows us to define a class

$$[\square] \in \mathcal{Z}^\vee_{\mathrm{B}} := \varprojlim_{n} \mathcal{Z}^\vee_{n,\mathrm{B}} \ .$$

Loosely speaking, if σ is a chain on $\widetilde{X}_n(\mathbb{C}) - \widetilde{A}_n(\mathbb{C})$ whose boundary is on $\widetilde{B}_n(\mathbb{C}) \cup E_n(\mathbb{C})$, then $i^\vee_{n,\mathrm{B}}([\sigma])$ is the class of "the component of the boundary of σ that lives on $\widetilde{X}_{n-1}(\mathbb{C})$". According to Proposition 3.3, one can also work with chains on $X_n(\mathbb{C}) - \mathring{A}_n(\mathbb{C})$.

The next proposition shows that the Eulerian differential forms $\omega_n^{(d)}$ are compatible with the inductive structure on the zeta motives.

Proposition 3.9. *For integers $n \geqslant 2$ and $d = 0, 2, \ldots, n-1$, the map $i_{n,\mathrm{dR}} : \mathcal{Z}_{n-1,\mathrm{dR}} \to \mathcal{Z}_{n,\mathrm{dR}}$ sends the class $[\omega_{n-1}^{(d)}]$ to the class $[\omega_n^{(d)}]$.*

Proof. Since all the differential forms that we are manipulating have no poles along the exceptional divisors E_{n-1} and E_n, it is safe to do the computations in the affine spaces X_{n-1} and X_n; we leave it to the reader to turn them into computations in $\widetilde{X}_{n-1}$ and $\widetilde{X}_n$ by working in local charts as in the proof of Proposition 3.5. Let us assume first that $d \in \{2, \ldots, n-1\}$. We put

$$\eta_{n-1}^{(d)} = \frac{x_n E_{n-1-d}(x_1 \cdots x_n)}{(1 - x_1 \cdots x_n)^{n-d}} \, dx_1 \cdots dx_{n-1} \ ,$$

viewed as a form on X_n. Then we have $(\eta_{n-1}^{(d)})_{|X_{n-1}} = \omega_{n-1}^{(d)}$ and $(\eta_{n-1}^{(d)})_{|B'_{n-1}} = 0$. A diagram chase in the double complex (16) shows that $i_{n,\mathrm{dR}}([\omega_{n-1}^{(d)}])$ is the class of

$$(-1)^{n-1} \, (d(\eta_{n-1}^{(d)}))$$

(the sign is here to be consistent with the Betti version, see Remark 3.8). We have

$$(-1)^{n-1} \, d(\eta_{n-1}^{(d)}) = \frac{\partial}{\partial x_n} \left(\frac{x_n E_{n-1-d}(x_1 \cdots x_n)}{(1 - x_1 \cdots x_n)^{n-d}} \right) dx_1 \cdots dx_n$$

and one easily sees that setting $x = x_1 \cdots x_n$ we have

$$\frac{\partial}{\partial x_n} \left(\frac{x_n E_{n-1-d}(x_1 \cdots x_n)}{(1 - x_1 \cdots x_n)^{n-d}} \right) = \frac{x(1-x)E'_{n-1-d}(x) + (1 + (n-1-d)x)E_{n-1-d}(x)}{(1-x)^{n-d+1}} \ .$$

Using the recurrence relation (20), one then concludes that

$$(-1)^{n-1} d(\eta_{n-1}^{(d)}) = \frac{E_{n-d}(x_1 \cdots x_n)}{(1 - x_1 \cdots x_n)^{n-d+1}} \, dx_1 \cdots dx_n = \omega_n^{(d)}.$$

For $d = 0$, this is the same computation with $\eta_n^{(0)} = x_n\, dx_1 \cdots dx_{n-1}$ and

$$(-1)^{n-1} d(\eta_{n-1}^{(0)}) = dx_1 \cdots dx_n = \omega_n^{(0)}.$$

□

Proposition 3.9 allows us to unambiguously define classes

$$[\omega^{(d)}] \in \mathcal{Z}_{\infty,\mathrm{dR}}$$

for $d = 0, 2, 3, \ldots$

Remark 3.10. The proof of Proposition 3.9 can be thought of as a cohomological version of the relation

$$\int_{\square_n} \omega_n^{(d)} = \int_{\square_{n-1}} \omega_{n-1}^{(d)},$$

which may be proved using Stokes' theorem and the recurrence relation (20).

Proposition 3.11. *For integers $n \geqslant 1$ and $d = 0, 2, \ldots, n$, the class of $[\omega_n^{(d)}]$ lives in the pure weight $2d$ component of $\mathcal{Z}_{n,\mathrm{dR}}$.*

Proof. For $d = 0$, Proposition 3.9 and the fact that the maps $i_{n,\mathrm{dR}}$ are compatible with the weight gradings implies that it is enough to do the proof for $n = 1$; this case is easy since $\mathcal{Z}_1 = \mathbb{Q}(0)$ only has weight 0. We now turn to the case $d = 2, \ldots, n$. Thanks to Proposition 3.9 and the fact that the maps $i_{n,\mathrm{dR}}$ are compatible with the weight gradings, it is enough to check it for $d = n$. We have to show that the class of $\pi_n^*(\omega_n^{(n)})$ is in $F^n\mathcal{Z}_{n,\mathrm{dR}}$. By the construction of the Hodge filtration [Del71], it is enough to prove that there is an embedding $\widetilde{X}_n - \widetilde{A}_n \subset Y_n$ such that

(1) Y_n is a smooth projective variety;
(2) the divisor at infinity $\partial Y_n = Y_n - (\widetilde{X}_n - \widetilde{A}_n)$ is a normal crossing divisor;
(3) the differential form $\omega_n^{(n)}$ has logarithmic poles along ∂Y_n.

One can choose $Y_n = \overline{\mathcal{M}}_{0,n+3}$. The embedding of $\widetilde{X}_n - \widetilde{A}_n$ is described in [Bro09, §2], and the divisor at infinity ∂Y_n is a union of irreducible components of the normal crossing divisor $\partial\overline{\mathcal{M}}_{0,n+3}$. The differential form $\omega_n^{(n)}$ is thus, after the change of variables $(t_1, \ldots, t_n) = (x_1 \cdots x_n, x_2 \cdots x_n, \ldots, x_{n-1}x_n, x_n)$, the form $\Omega(1, 0, \ldots, 0)$ of [GM04], which has logarithmic poles along $\partial\overline{\mathcal{M}}_{0,n+3}$. □

3.6. **A long exact sequence.** We now show that the morphism $i_n : \mathcal{Z}_{n-1} \to \mathcal{Z}_n$ fits into a long exact sequence. We first define objects of $\mathsf{MT}(\mathbb{Q})$:

$$\mathcal{Z}_n^k = H^k(\widetilde{X}_n - \widetilde{A}_n, (\widetilde{B}_n \cup E_n) - (\widetilde{B}_n \cup E_n) \cap \widetilde{A}_n) \quad \text{and} \quad {}'\mathcal{Z}_n^k = H^k(\widetilde{X}_n - \widetilde{A}_n, (\widetilde{B}'_n \cup E_n) - (\widetilde{B}'_n \cup E_n) \cap \widetilde{A}_n)\ ,$$

so that $\mathcal{Z}_n = \mathcal{Z}_n^n$. We leave it to the reader to fill in the technical definitions of these objects by mimicking that of $\mathcal{Z}_n$ from §3.1.

Proposition 3.12. *For $n \geqslant 2$, we have a long exact sequence in $\mathsf{MT}(\mathbb{Q})$:*

$$\cdots \to \mathcal{Z}_{n-1}^{k-1} \to \mathcal{Z}_n^k \to {}'\mathcal{Z}_n^k \to \mathcal{Z}_{n-1}^k \to \mathcal{Z}_n^{k+1} \to \cdots \tag{24}$$

Proof. The objects $\mathcal{Z}_{n-1}^\bullet$, $\mathcal{Z}_n^\bullet$ and ${}'\mathcal{Z}_n^\bullet$ are defined via objects in $\mathsf{DMT}(\mathbb{Q})$ that we denote by C_{n-1}, C_n and ${}'C_n$ respectively, C_n being the complex (14) and C_{n-1} the subcomplex (23). Now there is an obvious exact triangle

$$C_{n-1}[-1] \longrightarrow C_n \longrightarrow {}'C_n \xrightarrow{+1}\ ,$$

in $\mathsf{DMT}(\mathbb{Q})$, which gives the desired long exact sequence after taking the cohomology with respect to the t-structure. □

4. Computation of the zeta motives $\mathcal{Z}_n$

This section is the technical heart of this article, where we compute (Theorem 4.6) the full period matrix of the zeta motives $\mathcal{Z}_n$. The main difficulty is showing that the motives $\mathcal{T}_n$, introduced below, are semi-simple. For that we use the involution τ defined in the introduction and the computation of the extension groups in the category $\mathsf{MT}(\mathbb{Q})$. We conclude with the definition of the odd zeta motive and the computation of its period matrix.

4.1. **The Gysin long exact sequence.** Since the divisor A_n is smooth, it is natural to decompose the motives $\mathcal{Z}_n^k$ thanks to a Gysin long exact sequence. In the next Proposition, the definition of the objects $H^\bullet(X_n, B_n)$ and $H^\bullet(A_n, B_n \cap A_n)$ of $\mathsf{MT}(\mathbb{Q})$ is similar to that of $\mathcal{Z}_n$ from §3.1.

Proposition 4.1. *For $n \geqslant 1$, we have a long exact sequence in $\mathsf{MT}(\mathbb{Q})$:*

$$\cdots \to H^k(X_n, B_n) \to \mathcal{Z}_n^k \to H^{k-1}(A_n, B_n \cap A_n)(-1) \to H^{k+1}(X_n, B_n) \to \mathcal{Z}_n^{k+1} \to \cdots \tag{25}$$

Proof. Recall [Voe00, (3.5.4)] the existence of a Gysin exact triangle in the category $\mathsf{DM}(\mathbb{Q})$. For the pair $(\widetilde{X}_n, \widetilde{A}_n)$, it reads (with cohomological conventions)

$$\widetilde{X}_n \longrightarrow \widetilde{X}_n - \widetilde{A}_n \longrightarrow \widetilde{A}_n(-1)[-1] \xrightarrow{+1}$$

and is an exact triangle in the category $\mathsf{DMT}(\mathbb{Q})$. Applying this triangle to every pair $(\partial_I Y, \partial_I Y \cap \widetilde{A}_n)$ in the complex (14) and taking the cohomology with respect to the t-structure leads to a long exact sequence

$$\cdots \to H^k(\widetilde{X}_n, \widetilde{B}_n \cup E_n) \to H^k(\widetilde{X}_n - \widetilde{A}_n, (\widetilde{B}_n \cup E_n) - (\widetilde{B}_n \cup E_n) \cap \widetilde{A}_n) \to H^{k-1}(\widetilde{A}_n, (\widetilde{B}_n \cup E_n) \cap \widetilde{A}_n)(-1) \to \cdots$$

in $\mathsf{MT}(\mathbb{Q})$. One concludes with the fact that the natural morphisms

$$H^k(\widetilde{X}_n, \widetilde{B}_n \cup E_n) \to H^k(X_n, B_n) \quad \text{and} \quad H^{k-1}(\widetilde{A}_n, (\widetilde{B}_n \cup E_n) \cap \widetilde{A}_n) \to H^{k-1}(A_n, B_n \cap A_n)$$

are isomorphisms. This can be checked in the Betti realization, where it is a consequence of the excision theorem as in the proof of Proposition 3.3. □

4.2. **The motives $H^\bullet(X_n, B_n)$.** The computation of the motives $H^\bullet(X_n, B_n)$ appearing in the long exact sequence (25) is relatively easy.

Proposition 4.2. *(1) We have $H^k(X_n, B_n) = 0$ for $k \neq n$, and an isomorphism $H^n(X_n, B_n) \cong \mathbb{Q}(0)$.*

(2) A basis for the de Rham realization $H^n_{\mathrm{dR}}(X_n, B_n)$ is the class of the form $dx_1 \cdots dx_n$.

(3) A basis for the Betti realization $H_n^{\mathrm{B}}(X_n, B_n)$ is the class of the unit n-cube $\square_n = [0,1]^n$.

Proof. If (1) is proved then proving (2) and (3) amounts to showing that the classes of $dx_1 \cdots dx_n$ and of $\square_n$ are non-zero. This is clear since their pairing is $\int_{\square_n} dx_1 \cdots dx_n = 1 \neq 0$. Let us write $X = X_n$ and $\partial X = B_n = \bigcup_{i=1}^{2n} \partial_i X$ where $\partial_i X$ is either some $\{x_j = 0\}$ or some $\{x_j = 1\}$. Then $H^\bullet(X_n, B_n)$ is defined from the complex of varieties

$$\cdots \to \bigsqcup_{|I|=3} \partial_I X \to \bigsqcup_{|I|=2} \partial_I X \to \bigsqcup_{|I|=1} \partial_I X \to X \to 0$$

similar to (14). There is a natural spectral sequence

$$E_1^{p,q} = \bigoplus_{|I|=p} H^q(\partial_I X) \quad \Rightarrow \quad H^{p+q}(X, \partial X)$$

in $\mathsf{MT}(\mathbb{Q})$ where the differential on the E_1 page in the alternating sum of the natural restrictions. Since all the varieties $\partial_I X$ are affine spaces, we have $H^q(\partial_I X) = 0$ for $q \neq 0$ and $H^0(\partial_I X) \cong \mathbb{Q}(0)$. Thus, the only non-zero row is $q = 0$ and the spectral sequence degenerates at E_2. In the Betti realization, the row $q = 0$ is the complex computing the cellular cohomology of the unit n-cube $[0,1]^n$, shifted by n, and claim (1) follows. □

4.3. **The motives $H^\bullet(A_n, B_n \cap A_n)$.** For $n \geqslant 1$, we realize the n-torus as $T_n = \{x_1 \cdots x_{n+1} = 1\}$, and we have subtori $T_i^{n-1} = \{x_i = 1\} \subset T^n$ for $i = 1, \ldots, n+1$. We define

$$\mathcal{T}_n^k = H^k(T^n, \textstyle\bigcup_{1 \leqslant i \leqslant n+1} T_i^{n-1}) \quad \text{and} \quad {}'\mathcal{T}_n^k = H^k(T^n, \textstyle\bigcup_{1 \leqslant i \leqslant n} T_i^{n-1}) \ ,$$

which are objects in $\mathsf{MT}(\mathbb{Q})$ (whose definition is similar to that of $\mathcal{Z}_n$ from §3.1) and write $\mathcal{T}_n = \mathcal{T}_n^n$, ${}'\mathcal{T}_n = {}'\mathcal{T}_n^n$. We then have

$$H^{k-1}(A_n, B_n \cap A_n) \cong \mathcal{T}_{n-1}^{k-1} \ .$$

By mimicking the proof of Proposition 3.12, one produces a long exact sequence in $\mathsf{MT}(\mathbb{Q})$:

$$\cdots \to \mathcal{T}_{n-1}^{k-1} \to \mathcal{T}_n^k \to {}'\mathcal{T}_n^k \to \mathcal{T}_{n-1}^k \to \mathcal{T}_n^{k+1} \to \cdots \tag{26}$$

Proposition 4.3. *(1) We have ${}'\mathcal{T}_n^k = 0$ for $k \neq n$, and an isomorphism ${}'\mathcal{T}_n \cong H^n(T^n) \cong \mathbb{Q}(-n)$.*

(2) *We have* $\mathcal{T}_n^k = 0$ *for* $k \neq n$*, and short exact sequences in* $\mathsf{MT}(\mathbb{Q})$*:*

$$0 \to \mathcal{T}_{n-1} \to \mathcal{T}_n \to H^n(T^n) \to 0 \ . \tag{27}$$

Proof. If (1) is proved then (2) follows from the long exact sequence (26). We choose coordinates $(x_1, \ldots, x_n)$ on T^n. Let us write, for $I \subset \{1, \ldots, n\}$, $T_I = \{\forall i \in I\,,\, x_i = 1\}$. It is a subtorus of $T^n = T_\varnothing$ of codimension the cardinality of I. By mimicking the proof of Proposition 4.2, one produces a spectral sequence

$$E_1^{p,q} = \bigoplus_{|I|=p} H^q(T_I) \ \ \Rightarrow \ \ {}'\mathcal{T}_n^{p+q}$$

in $\mathsf{MT}(\mathbb{Q})$, where the differential on the E_1 page in the alternating sum of the natural restrictions. We note that the row $q = n$ only contains

$$E_1^{0,n} = H^n(T^n) \cong H^1(T^1)^{\otimes n} \cong \mathbb{Q}(-1)^{\otimes n} = \mathbb{Q}(-n) \ .$$

Thus, we are done if we prove that the rows $q = 0, \ldots, n-1$ are all exact. We work in de Rham realization for simplicity. For $I \subset \{1, \ldots, n\}$, we put

$$\Lambda^\bullet(I) = \Lambda^\bullet(e_1, \ldots, e_n)/(e_i \,,\, i \in I) \ .$$

Then we have natural identifications $H^\bullet(T_I) \cong \Lambda^\bullet(I)$, and thus

$$E_1^{p,q} \cong \bigoplus_{|I|=p} \Lambda^q(I) \ ,$$

where the differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is the alternating sum of the natural quotient maps

$$\Lambda^q(I) \twoheadrightarrow \Lambda^q(I \cup \{i\})$$

for $i \notin I$. We have natural splittings $\Lambda^\bullet(I \cup \{i\}) \hookrightarrow \Lambda^\bullet(I)$, which give (with proper signs) a map $h : E_1^{p+1,q} \to E_1^{p,q}$. One then easily shows that we have $dh + hd = (n-q)\,\mathrm{id}$, thus h defines a contracting homotopy for the complex $E_1^{\bullet,q}$ for $q = 0, \ldots, n-1$, and we are done. □

We note that we have $\mathcal{T}_0 = H^0(\mathrm{pt}, \mathrm{pt}) = 0$, so that Proposition 4.3 implies that we have

$$\mathrm{gr}_k^W \mathcal{T}_n = \begin{cases} \mathbb{Q}(-k) & \text{if } k \in \{1, \ldots, n\}; \\ 0 & \text{otherwise.} \end{cases}$$

In the next proposition, we will prove that the weight filtration of $\mathcal{T}_n$ actually splits in $\mathsf{MT}(\mathbb{Q})$. For that we introduce the involution τ which acts on the tori T^n by

$$\tau : (x_1, \ldots, x_{n+1}) \mapsto (x_1^{-1}, \ldots, x_{n+1}^{-1}) \ .$$

This induces an involution, still denoted by τ, on the objects $\mathcal{T}_n^k$ and ${}'\mathcal{T}_n^k$ of $\mathsf{MT}(\mathbb{Q})$, such that all the maps in the long exact sequence (26) commute with τ.

Proposition 4.4. (1) *The short exact sequences (27) split in* $\mathsf{MT}(\mathbb{Q})$*, hence we have isomorphisms:*

$$\mathcal{T}_n \cong \mathbb{Q}(-1) \oplus \mathbb{Q}(-2) \oplus \cdots \oplus \mathbb{Q}(-n) \ .$$

In other words, a period matrix for $\mathcal{T}_n$ *is the diagonal matrix* $\mathrm{Diag}(2i\pi, (2i\pi)^2, \ldots, (2i\pi)^n)$*.*

(2) *The involution* τ *acts on the direct summand* $\mathbb{Q}(-k)$ *of* $\mathcal{T}_n$ *by multiplication by* $(-1)^k$*.*

Proof. We prove the proposition by induction on n, the case $n = 0$ being trivial. We first note that τ acts on $H^1(T^1)$ by multiplication by -1. It is enough to prove it in de Rham realization, where it follows from $\tau.\,\mathrm{dlog}(x_1) = -\mathrm{dlog}(x_1)$. Thus, τ acts on $\mathrm{gr}_n^W \mathcal{T}_n \cong H^n(T^n) = H^1(T^1)^{\otimes n}$ by multiplication by $(-1)^n$, and we are left with proving (1).

Thanks to the induction hypothesis, there exists a basis $(w_1, \ldots, w_{n-1})$ (resp. $(\psi_1, \ldots, \psi_{n-1})$) of $\mathcal{T}_{n-1,\mathrm{dR}}$ (resp. $\mathcal{T}_{n-1,\mathrm{B}}^\vee$) which is compatible with the weight grading (resp. filtration) and satisfies $\tau.w_k = (-1)^k w_k$ (resp. $\tau.\psi_k = (-1)^k \psi_k$) for every k. We assume for simplicity that we have $\langle \psi_k, w_k \rangle = (2i\pi)^k$ for every k. Thanks to the short exact sequence (27), we can complete it to get a basis $(w_1, \ldots, w_{n-1}, w_n)$ (resp. $(\psi_1, \ldots, \psi_{n-1}, \psi_n)$) of $\mathcal{T}_{n,\mathrm{dR}}$ (resp. $\mathcal{T}_{n,\mathrm{B}}^\vee$) such that the following conditions are verified:

(a) $(w_1, \ldots, w_n)$ (resp. $(\psi_1, \ldots, \psi_n)$) is compatible with the weight grading (resp. filtration);

(b) $\langle \psi_k, w_k \rangle = (2i\pi)^k$ for every $k = 1, \ldots, n$;

(c) $\tau.w_k = (-1)^k w_k$ (resp. $\tau.\psi_k = (-1)^k \psi_k$) for every $k = 1, \dots, n$;
For $k = 1, \dots, n-1$ we put $p_k := \langle \psi_k, w_n \rangle$, so that the period matrix of $\mathcal{T}_n$ in these bases is

$$\begin{pmatrix} 2i\pi & & 0 & & p_1 \\ & (2i\pi)^2 & & & p_2 \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ & 0 & & (2i\pi)^{n-1} & p_{n-1} \\ & & & & (2i\pi)^n \end{pmatrix}.$$

We compute

$$p_k = \langle \psi_k, w_n \rangle = (-1)^k \langle \tau.\psi_k, w_n \rangle = (-1)^k \langle \psi_k, \tau.w_n \rangle = (-1)^{k+n} \langle \psi_k, w_n \rangle = (-1)^{k+n} p_k$$

where we have used the fact the pairing $\langle \cdot, \cdot \rangle$ is compatible with τ. This implies that $p_{n-2i+1} = 0$ for every $i > 0$.

Now the period matrix for $\mathcal{T}_n / W_{n-3}\mathcal{T}_n$ is the matrix

$$\begin{pmatrix} (2i\pi)^{n-2} & 0 & p_{n-2} \\ 0 & (2i\pi)^{n-1} & 0 \\ 0 & 0 & (2i\pi)^n \end{pmatrix}$$

Thus, $\mathcal{T}_n / W_{n-3}\mathcal{T}_n$ is the direct sum of $\mathbb{Q}(-(n-1))$ and an extension of $\mathbb{Q}(-n)$ by $\mathbb{Q}(-(n-2))$. By (12), all such extensions are trivial, hence $p_{n-2} = \lambda(2i\pi)^n$ with λ a rational number. If we replace ψ_{n-2} by $\psi_{n-2} - \lambda\psi_n$, we can thus assume that we have $p_{n-2} = 0$. Note that conditions (a), (b), (c) are still verified. By looking at $\mathcal{T}_n / W_{n-5}\mathcal{T}_n$ and using the fact that all extensions of $\mathbb{Q}(-n)$ by $\mathbb{Q}(-(n-4))$ are trivial, we can assume that we have $p_{n-4} = 0$. By induction, we can assume that we have $p_{n-2i} = 0$ for every $i > 0$, hence the period matrix of $\mathcal{T}_n$ is the diagonal matrix $\mathrm{Diag}(2i\pi, (2i\pi)^2, \dots, (2i\pi)^n)$ and we are done. □

4.4. **The structure of the zeta motives.** We can now determine the structure of the zeta motives $\mathcal{Z}_n$, for $n \geqslant 1$.

Theorem 4.5. *(1) We have a short exact sequence in* $\mathsf{MT}(\mathbb{Q})$*:*

$$0 \to \mathbb{Q}(0) \to \mathcal{Z}_n \to \mathcal{T}_{n-1}(-1) \to 0 \;, \tag{28}$$

with $\mathcal{T}_{n-1}(-1) \cong \mathbb{Q}(-2) \oplus \cdots \oplus \mathbb{Q}(-n)$.

(2) We have a short exact sequence in $\mathsf{MT}(\mathbb{Q})$*:*

$$0 \to \mathcal{Z}_{n-1} \xrightarrow{i_n} \mathcal{Z}_n \to \mathbb{Q}(-n) \to 0 \;. \tag{29}$$

(3) These short exact sequences fit into a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbb{Q}(0) & \to & \mathcal{Z}_{n-1} & \to & \mathcal{T}_{n-2}(-1) & \to & 0 \\ & & \downarrow{=} & & \downarrow{i_{n-1}} & & \downarrow & & \\ 0 & \to & \mathbb{Q}(0) & \to & \mathcal{Z}_n & \to & \mathcal{T}_{n-1}(-1) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & 0 & \to & \mathbb{Q}(-n) & \xrightarrow{=} & \mathbb{Q}(-n) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{30}$$

where all rows and columns are exact.

Proof. Assertion (1) follows from Propositions 4.1, 4.2 and 4.4. The commutativity of (30) follows from the compatibility of the long exact sequences (24) and (26). A diagram chase implies that (29) is exact. ☐

Theorem 4.6. *(1) The classes*

$$v_d := [\omega_n^{(d)}] \qquad (d = 0, 2, \ldots, n)$$

of the Eulerian differential forms provides a basis $(v_0, v_2, \ldots, v_n)$ of the de Rham realization $\mathcal{Z}_{n,\mathrm{dR}}$ which is compatible with the weight grading.

(2) There exists a unique basis $(\varphi_0, \varphi_2, \ldots, \varphi_n)$ for the dual of the Betti realization $\mathcal{Z}_{n,\mathrm{B}}^\vee$ which is compatible with the weight filtration and such that the period matrix for $\mathcal{Z}_n$ with bases $(v_0, v_2, \ldots, v_n)$ and $(\varphi_0, \varphi_2, \ldots, \varphi_n)$ is

$$\begin{pmatrix} 1 & \zeta(2) & \zeta(3) & \cdots & \cdots & \zeta(n-1) & \zeta(n) \\ & (2i\pi)^2 & & & & & \\ & & (2i\pi)^3 & & & 0 & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & 0 & & & & (2i\pi)^{n-1} & \\ & & & & & & (2i\pi)^n \end{pmatrix}. \tag{31}$$

Proof. (1) Proposition 3.11 implies that v_d is in the pure weight $2d$ component of $\mathcal{Z}_{n,\mathrm{dR}}$. Thus, it is enough to show that it is non-zero, which is a consequence of the equality $\langle [\square_n], v_d \rangle = \zeta(d) \neq 0$.

(2) We put $\varphi_0 = [\square_n]$. Let $(\psi_1, \ldots, \psi_{n-1})$ be a basis of $\mathcal{T}_{n-1,\mathrm{B}}^\vee$ for which the period matrix is diagonal, as in Proposition 4.4. Let p denote the morphism $\mathcal{Z}_n \to \mathcal{T}_{n-1}(-1)$, and let us consider the transpose of its Betti realization $p_\mathrm{B}^\vee : \mathcal{T}_{n-1,B}^\vee \to \mathcal{Z}_{n,\mathrm{B}}^\vee$. Then we can put $\varphi_d = p_\mathrm{B}^\vee(\psi_{d-1})$ for $d = 2, \ldots, n$. The fact that this gives a basis of $\mathcal{Z}_{n,B}^\vee$ is a consequence of the short exact sequence (28). The fact that the period matrix is as required follows from Lemma 3.7 and Proposition 4.4. The uniqueness statement is obvious.

☐

We have already noted that the classes v_d are compatible with the inductive system of the zeta motives. By the uniqueness statement in Theorem 4.6, this is also the case for the classes σ_d, and the zeta motive $\mathcal{Z}$ has an infinite period matrix

$$\begin{pmatrix} 1 & \zeta(2) & \zeta(3) & \zeta(4) & \cdots & \cdots & \cdots \\ & (2i\pi)^2 & & & & & \\ & & (2i\pi)^3 & & & 0 & \\ & & & (2i\pi)^4 & & & \\ & & & & \ddots & & \\ & 0 & & & & \ddots & \\ & & & & & & \ddots \end{pmatrix}.$$

4.5. **Classes in the Betti realization.** We now explain how to work with the classes $(\varphi_2, \ldots, \varphi_n)$ in the dual of the Betti realization $\mathcal{Z}_{n,\mathrm{B}}^\vee$. The subtlety here is that these classes can be represented by relative cycles which are invariant, up to a sign, by τ, but that τ does not act on the motive $\mathcal{Z}_n$. We thus introduce a motive $\mathcal{Z}_n^*$ on which τ acts, such that the classes $(\varphi_2, \ldots, \varphi_n)$ are images of classes in $\mathcal{Z}_{n,\mathrm{B}}^{*,\vee}$.

We put

$$X_n^* = X_n - \bigcup_{1 \leqslant i \leqslant n} \{x_i = 0\} = (\mathbb{A}^1_\mathbb{Q} - \{0\})^n \quad \text{and} \quad B_n^* = X_n^* \cap \bigcup_{1 \leqslant i \leqslant n} \{x_i = 1\} .$$

We then define, with the obvious notations:

$$\mathcal{Z}_n^* = H^n(\widetilde{X}_n^* - \widetilde{A}_n, (\widetilde{B}_n^* \cup E_n) - (\widetilde{B}_n^* \cup E_n) \cap \widetilde{A}_n) ,$$

viewed as an object of $\mathsf{MT}(\mathbb{Q})$. Since A_n does not meet the coordinate hyperplanes $\{x_i = 0\}$, the morphism $p : \mathcal{Z}_n \to \mathcal{T}_{n-1}(-1)$ factors through $\mathcal{Z}_n^*$:

$$\mathcal{Z}_n \longrightarrow \mathcal{Z}_n^* \xrightarrow{p^*} \mathcal{T}_{n-1}(-1)\ . \tag{32}$$

Proposition 4.7. *The morphism $\mathcal{Z}_n \to \mathcal{Z}_n^*$ fits into a short exact sequence*

$$0 \to \mathbb{Q}(0) \to \mathcal{Z}_n \oplus \mathbb{Q}(-n) \to \mathcal{Z}_n^* \to 0$$

in $\mathsf{MT}(\mathbb{Q})$.

Proof. By mimicking the proof of Theorem 4.5, we see that there is a short exact sequence

$$0 \to \mathbb{Q}(-n) \to \mathcal{Z}_n^* \to \mathcal{T}_{n-1}(-1) \to 0\ .$$

We then have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Q}(0) & \longrightarrow & \mathcal{Z}_n & \longrightarrow & \mathcal{T}_{n-1}(-1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & \mathbb{Q}(-n) & \longrightarrow & \mathcal{Z}_n^* & \longrightarrow & \mathcal{T}_{n-1}(-1) & \longrightarrow & 0 \end{array}$$

which finishes the proof. □

Note that this implies that $\mathcal{Z}_n^*$ is isomorphic to $\mathbb{Q}(-1) \oplus \cdots \oplus \mathbb{Q}(-(n-1)) \oplus \mathbb{Q}(-n)^{\oplus 2}$.

In the proof of Theorem 4.6, we defined the classes $\varphi_k \in \mathcal{Z}_{n,\mathrm{B}}^\vee$, for $k = 2, \ldots, n$, in the image of the morphism $p_{\mathrm{B}}^\vee : \mathcal{T}_{n-1,\mathrm{B}}^\vee \to \mathcal{Z}_{n,\mathrm{B}}^\vee$. In view of the factorization (32), we have classes

$$\varphi_k^* \in \mathcal{Z}_{n,\mathrm{B}}^{*,\vee} \qquad (k = 2, \ldots, n)\ .$$

The reason for introducing $\mathcal{Z}_n^*$ is that it is naturally endowed with an involution τ, induced by $(x_1, \ldots, x_n) \mapsto (x_1^{-1}, \ldots, x_n^{-1})$. This is not the case for $\mathcal{Z}_n$.

Lemma 4.8. *We have the equality $\tau.\varphi_k^* = (-1)^{k-1}\varphi_k^*$ in $\mathcal{Z}_{n,\mathrm{B}}^{*\vee}$.*

Proof. This is obvious from the proof of Theorem 4.6 since the morphism $p_{\mathrm{B}}^\vee : \mathcal{T}_{n-1,\mathrm{B}}^\vee \to \mathcal{Z}_{n,\mathrm{B}}^{*,\vee}$ is compatible with the involutions τ. □

Note that the morphism $p_{\mathrm{B}}^{*,\vee} : \mathcal{T}_{n-1,\mathrm{B}}^\vee \to \mathcal{Z}_{n,\mathrm{B}}^\vee$ can be described explicitly in the following way. Let $f : T \to A_n(\mathbb{C})$ be the boundary of a tubular neighborhood of $A_n(\mathbb{C})$ in $X_n^*(\mathbb{C})$, which means that locally T looks like the product $S^1 \times A_n(\mathbb{C})$. If α is the representative of a $(n-1)$-cycle on $A_n(\mathbb{C})$ with boundary on $A_n(\mathbb{C}) \cap B_n(\mathbb{C})$, then $p_{\mathrm{B}}^{*,\vee}([\alpha])$ is the class of $f^{-1}(\alpha)$.

Lemma 4.9. *The class $\varphi_n \in \mathcal{Z}_{n,\mathrm{B}}^\vee$ can be represented by the cycle $(S^1)^n \to X_n(\mathbb{C}) - A_n(\mathbb{C})$ given by the equations*

$$|x_1| = \rho_1,\ \ldots,\ |x_{n-1}| = \rho_{n-1},\ \left| x_n - \frac{1}{x_1 \cdots x_{n-1}} \right| = \rho_n\ , \tag{33}$$

for any choice of $\rho_1, \ldots, \rho_{n-1}, \rho_n > 0$.

Proof. The homology class of these cycles obviously does not depend on the parameters $\rho_1, \ldots, \rho_n$, so we can assume that we have $\rho_1 \cdots \rho_n \neq 1$. This way, the cycle (33) lives in $X_n^*(\mathbb{C}) - A_n(\mathbb{C})$. Note that the domain T defined by the equation $\left| x_n - \frac{1}{x_1 \cdots x_{n-1}} \right| = \rho_n$ is the boundary of a tubular neighborhood of $A_n(\mathbb{C})$, the map $f : T \to A_n(\mathbb{C})$ being given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, \frac{1}{x_1 \cdots x_{n-1}})$. With the notations of the proof of Theorem 4.6, the homology class $\psi_{n-1} \in \mathcal{T}_{n-1,\mathrm{B}}^\vee$ can obviously be represented by the cycle $|x_1| = \rho_1, \ldots, |x_{n-1}| = \rho_{n-1}$, hence the result. □

4.6. **The odd zeta motive.** Let us write $\mathcal{T}_{n-1} = \mathcal{T}^+_{n-1} \oplus \mathcal{T}^-_{n-1}$ for the decomposition into direct summands on which the involution τ acts positively or negatively, and let us write $p : \mathcal{Z}_n \to \mathcal{T}_{n-1}(-1)$ for the surjection appearing in the short exact sequence (28).

Definition 4.10. The n-th *odd zeta motive* $\mathcal{Z}^{\mathrm{odd}}_n$ is the object of $\mathsf{MT}(\mathbb{Q})$ defined by

$$\mathcal{Z}^{\mathrm{odd}}_n := p^{-1}(\mathcal{T}^+_{n-1}(-1)) \ .$$

We obviously have a short exact sequence

$$0 \to \mathbb{Q}(0) \to \mathcal{Z}^{\mathrm{odd}}_n \to \mathcal{T}^+_{n-1}(-1) \to 0 \tag{34}$$

with

$$\mathcal{T}^+_{n-1}(-1) \cong \bigoplus_{3\leqslant 2k+1\leqslant n} \mathbb{Q}(-(2k+1)) \ .$$

We note that there are morphisms

$$i^{\mathrm{odd}}_n : \mathcal{Z}^{\mathrm{odd}}_{n-1} \to \mathcal{Z}^{\mathrm{odd}}_n$$

such that i^{odd}_{2n} is an isomorphism for every integer n. The limit

$$\mathcal{Z}^{\mathrm{odd}} := \varinjlim_{n} \mathcal{Z}^{\mathrm{odd}}_n$$

is an ind-object in $\mathsf{MT}(\mathbb{Q})$ that we simply call the *odd zeta motive.*

Proposition 4.11. *(1) We have a direct sum decomposition*

$$\mathcal{Z}_n \cong \mathcal{Z}^{\mathrm{odd}}_n \oplus \bigoplus_{2\leqslant 2k\leqslant n} \mathbb{Q}(-2k) \ . \tag{35}$$

(2) A period matrix for $\mathcal{Z}^{\mathrm{odd}}_{2n+1} = \mathcal{Z}^{\mathrm{odd}}_{2n+2}$ is

$$\begin{pmatrix} 1 & \zeta(3) & \zeta(5) & \cdots & \cdots & \zeta(2n-1) & \zeta(2n+1) \\ & (2i\pi)^3 & & & & & \\ & & (2i\pi)^5 & & & 0 & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & 0 & & & & (2i\pi)^{2n-1} & \\ & & & & & & (2i\pi)^{2n+1} \end{pmatrix} . \tag{36}$$

Proposition 4.11 implies that the odd zeta motive $\mathcal{Z}^{\mathrm{odd}}$ has an infinite period matrix (6).

Proof. A basis for $\mathcal{Z}^{\mathrm{odd}}_{n,\mathrm{dR}}$ is given by v_0 and the v_{2k+1}, for $3 \leqslant 2k+1 \leqslant n$, and a basis for $\mathcal{Z}^{\mathrm{odd},\vee}_{n,\mathrm{B}}$ is given by φ_0 and the φ_{2k+1}, for $3 \leqslant 2k+1 \leqslant n$. This gives the desired shape for the period matrix (36). Now, Euler's solution to the Basel problem implies that we have $\zeta(2k) = \lambda_{2k}(2i\pi)^{2k}$ for every integer $k \geqslant 1$, with $\lambda_{2k} = -\frac{B_{2k}}{2(2k)!} \in \mathbb{Q}$. Thus, we may replace the basis $(\varphi_0, \varphi_2, \ldots, \varphi_n)$ of Theorem 4.6 by the basis $(\varphi'_0, \varphi_2, \ldots, \varphi_n)$ with

$$\varphi'_0 = \varphi_0 - \sum_{2\leqslant 2k\leqslant n} \lambda_{2k}\, \varphi_{2k}$$

to get a period matrix similar to (31) where the even zeta values $\zeta(2k)$ in the first row are replaced by 0. This implies the direct sum decomposition (35). ☐

We finish by proving that all the objects in $\mathsf{MT}(\mathbb{Q})$ considered earlier actually live in the full subcategory $\mathsf{MT}(\mathbb{Z})$.

Proposition 4.12. *The zeta motives $\mathcal{Z}_n$ and the odd zeta motives $\mathcal{Z}^{\mathrm{odd}}_n$ are objects of the category* $\mathsf{MT}(\mathbb{Z})$.

Proof. Thanks to the direct sum decomposition (35), it is enough to prove it for the odd zeta motives. Let us recall the definition [DG05, Définition 1.4] of the category $\mathsf{MT}(\mathbb{Z})$. According to the Tannakian formalism, the de Rham realization functor $\mathsf{MT}(\mathbb{Q}) \to \mathsf{grVect}_{\mathbb{Q}}$ induces an equivalence of categories

$$\mathsf{MT}(\mathbb{Q}) \cong \mathsf{grRep}(\mathfrak{g}^{\mathrm{dR}})$$

between $\mathsf{MT}(\mathbb{Q})$ and the category of graded finite-dimensional representations of a graded Lie algebra $\mathfrak{g}^{\mathrm{dR}}$. The degree in $\mathfrak{g}^{\mathrm{dR}}$ is half the weight. This Lie algebra is non-positively graded. The category $\mathsf{MT}(\mathbb{Z})$ is defined as the full subcategory of $\mathsf{MT}(\mathbb{Q})$ consisting on objects H such that the degree -1 component $\mathfrak{g}^{\mathrm{dR}}_{-1}$ acts trivially on H_{dR}. This is trivially the case for $\mathcal{Z}^{\mathrm{odd}}_n$, which is concentrated in weights 0 and $2(2k+1)$ with $2k+1 \geqslant 3$ by the short exact sequence (34). ☐

Remark 4.13. A tannakian interpretation of the odd zeta motive goes as follows. Let $\mathfrak{g}^\vee$ be the graded dual of the fundamental Lie algebra $\mathfrak{g}$ of the Tannakian category $\mathsf{MT}(\mathbb{Z})$. It is an ind-object in $\mathsf{MT}(\mathbb{Z})$, independent of the choice of a fiber functor [Del89, Définition 6.1]. Then one has a short exact sequence

$$0 \to \mathbb{Q}(0) \to \mathfrak{g}^\vee \to \mathfrak{u}^\vee \to 0 \ ,$$

where $\mathfrak{u}$ is the pro-unipotent radical of $\mathfrak{g}$. One views $\mathcal{Z}^{\mathrm{odd}}$ inside the exact subsequence

$$0 \to \mathbb{Q}(0) \to \mathcal{Z}^{\mathrm{odd}} \to (\mathfrak{u}^{\mathrm{ab}})^\vee \to 0 \ ,$$

where $(\mathfrak{u}^{\mathrm{ab}})^\vee \cong \bigoplus_{k\geqslant 1} \mathbb{Q}(-(2k+1))$ is the dual of the abelianization of $\mathfrak{u}$.

5. Linear forms in odd/even zeta values

We apply our results from the previous section to prove Theorems 1.1 and 1.2 from the Introduction.

5.1. **Vanishing of coefficients.**

Theorem 5.1. *For ω an integrable algebraic differential form on $X_n - A_n$, we have*

$$\int_{[0,1]^n} \omega = a_0(\omega) + a_2(\omega)\zeta(2) + \cdots + a_n(\omega)\zeta(n) \tag{37}$$

with $a_k(\omega)$ a rational number for every k, given for $k = 2, \ldots, n$ by the formula

$$a_k(\omega) = \frac{1}{(2i\pi)^k} \langle \varphi_k, [\omega] \rangle \ . \tag{38}$$

Proof. According to Proposition 3.5, the class $[\omega]$ defines an element in $\mathcal{Z}_{n,\mathrm{dR}}$, hence we may write

$$[\omega] = a_0(\omega)v_0 + a_2(\omega)v_2 + \cdots + a_n(\omega)v_n$$

with $a_k(\omega) \in \mathbb{Q}$ for every k. Pairing with the class $\varphi_0 = [\square_n]$ gives the equality (37), and pairing with the class φ_k, $k = 2, \ldots, n$, gives the equality (38). ☐

If we represent the class φ_k by a relative cycle σ_k, then (38) becomes

$$a_k(\omega) = \frac{1}{(2i\pi)^k} \int_{\sigma_k} \omega \ .$$

In view of Lemma 4.9, we see that the case $n = k = 2$ of Theorem 5.1 is Rhin and Viola's contour integral for $\zeta(2)$ [RV96, Lemma 2.6].

Theorem 5.2. *For ω an integrable algebraic differential form on $X_n - A_n$, we have:*

1. *if $\tau.\omega = \omega$ then $a_k(\omega) = 0$ for $k \neq 0$ even;*
2. *if $\tau.\omega = -\omega$ then $a_k(\omega) = 0$ for k odd.*

Proof. Let us assume that we have $\tau.\omega = \omega$, and let us write $[\omega]^*$ for the class of ω in $\mathcal{Z}^*_{n,\mathrm{dR}}$. We have $\tau.[\omega]^* = [\omega]^*$. The pairing between $\mathcal{Z}^*_{n,\mathrm{dR}}$ and $\mathcal{Z}^{*,\vee}_{n,\mathrm{B}}$ is compatible with τ, hence we have

$$(2i\pi)^k\, a_k(\omega) = \langle [\varphi_k^*, [\omega]^*] \rangle = \langle \varphi_k^*, \tau.[\omega]^* \rangle = \langle \tau.\varphi_k^*, [\omega]^* \rangle = (-1)^{k-1} \langle \varphi_k^*, [\omega]^* \rangle = (-1)^{k-1} (2i\pi)^k\, a_k(\omega) \ ,$$

where we have used Lemma 4.8. This implies that $a_k(\omega) = 0$ if $k \neq 0$ is even. The second case is similar. ☐

Let us write an integrable form as

$$\omega = \frac{P(x_1,\ldots,x_n)}{(1-x_1\cdots x_n)^N}\,dx_1\cdots dx_n \tag{39}$$

with $P(x_1,\ldots,x_n)$ a polynomial with rational coefficients and $N\geqslant 0$ an integer. Then we have

$$\tau.\omega = \pm\omega \quad\Leftrightarrow\quad P(x_1,\ldots,x_n) = \pm(-1)^{N+n}(x_1\cdots x_n)^{N-2}P(x_1^{-1},\ldots,x_n^{-1})\ . \tag{40}$$

Remark 5.3. We note that apart from the integral formula (38) for the coefficients, there is a direct proof of Theorems 5.1 and 5.2 which goes as follows. Expanding $(1-x_1\cdots x_n)^{-N}$ as a series allows one to rewrite the integral of any integrable algebraic differential form (39) as the sum of a convergent series

$$\sum_{k\geqslant 0} R(k)\ ,$$

where $R(k)$ is a rational function with rational coefficients and poles in $\{-1,-2,\ldots\}$. By decomposing $R(k)$ into partial fractions, one easily sees that the sum evaluates to a linear combination $a_0+a_2\zeta(2)+\cdots+a_n\zeta(n)$ with $a_k\in\mathbb{Q}$ for every k. The vanishing of the even/odd coefficients comes from the comparison of $R(-k-N)$ and $R(k)$ as in [BR01], see also [Zud04, §8]. One can show that the coefficients of the linear forms that we get in this way are the same as the ones defined by Theorem 5.1. This was suggested by Don Zagier.

5.2. **The Ball−Rivoal integrals.** We apply Theorems 5.1 and 5.2 to a special family of integrals.

Corollary 5.4. *Let $u_1,\ldots,u_n,v_1,\ldots,v_n\geqslant 1$ and $N\geqslant 0$ be integers such that $v_1+\cdots+v_n\geqslant N+1$. Then the integral*

$$\int_{[0,1]^n}\frac{x_1^{u_1-1}\cdots x_d^{u_n-1}(1-x_1)^{v_1-1}\cdots(1-x_n)^{v_n-1}}{(1-x_1\cdots x_n)^N}\,dx_1\cdots dx_n \tag{41}$$

is absolutely convergent and evaluates to a linear combination

$$a_0+a_2\zeta(2)+a_3\zeta(3)+\cdots+a_n\zeta(n).$$

If furthermore we have $2u_i+v_i=N+1$ for every i, then we get:

(1) if $(n+1)(N+1)$ is odd then $a_k=0$ for $k\neq 0$ even;
(2) if $(n+1)(N+1)$ is even then $a_k=0$ for k odd.

Proof. This is a direction application of Theorem 5.2. The polynomial $P(x_1,\ldots,x_n)=x_1^{u_1-1}\cdots x_d^{u_n-1}(1-x_1)^{v_1-1}\cdots(1-x_n)^{v_n-1}$ satisfies

$$P(x_1,\ldots,x_n) = (-1)^{n+v_1+\cdots+v_n}x_1^{2u_1+v_1-3}\cdots x_n^{2u_n+v_n-3}P(x_1^{-1},\ldots,x_n^{-1})\ .$$

Let us assume that we have $2u_i+v_i=N+1$ for every i, then $v_1+\cdots+v_n\equiv n(N+1)\mod 2$ and we get

$$P(x_1,\ldots,x_n) = -(-1)^{(n+1)(N+1)}(-1)^{N+n}(x_1\cdots x_n)^{N-2}P(x_1^{-1},\ldots,x_n^{-1})\ ,$$

hence the result, in view of (40). □

Corollary 5.4 applies in particular to the special case

$$N=(2r+1)m+2,\ u_i=rm+1,\ v_i=m+1$$

for some integer parameters $r,m\geqslant 0$ satisfying $n(m+1)\geqslant(2r+1)m+3$. We then recover the integrals considered by Ball and Rivoal [BR01, Lemme 2]. The vanishing of the coefficients is [BR01, Lemme 1]. The notations (a,n,r) in [BR01] correspond to our notations $(n-1,m,r)$.

The integrals (41) can be expressed as generalized hypergeometric series

$$\left(\prod_{i=1}^n\frac{(u_i-1)!(v_i-1)!}{(u_i+v_i-1)!}\right){}_{n+1}F_n\left(\begin{matrix}u_1,\ldots,u_n,N\\ u_1+v_1,\ldots,u_n+v_n\end{matrix};1\right) = \frac{\prod_{i=1}^n(v_i-1)!}{(N-1)!}\sum_{k\geqslant 0}\frac{(k)_{u_1}\cdots(k)_{u_n}(k+1)_{N-1}}{(k)_{u_1+v_1}\cdots(k)_{u_n+v_n}}\ .$$

If $2u_i+v_i=N+1$ then the corresponding generalized hypergeometric series is said to be *well-poised.*

5.3. **Weight drop.** In the context of Theorem 5.1, we say that the integral $\int_{[0,1]^n}\omega$ has *weight drop* if the highest weight coefficient $a_n(\omega)$ vanishes. This amounts to saying that the class $[\omega]$ actually lives in the step $W_{2(n-1)}\mathcal{Z}_{n,\mathrm{dR}}$ of the weight filtration, hence the terminology. We give a sufficient condition for this phenomenon to happen.

Lemma 5.5. *Let $u, v \geqslant 1$ and $N \geqslant 0$ be integers such that $u+v \leqslant N$. Then there exists a polynomial $P(t)$ with rational coefficients such that*

$$\int_0^1 \frac{x^{u-1}(1-x)^{v-1}}{(1-tx)^N}\,dx = \frac{P(t)}{(1-t)^{N-v}}$$

for every $0 \leqslant t < 1$.

Proof. We can write

$$x^{u-1}(1-x)^{v-1} = \sum_{k=0}^{u+v-2} a_k(t)(1-tx)^k$$

with $a_k(t)$ a Laurent polynomial with rational coefficients for every k. We then have

$$\frac{x^{u-1}(1-x)^{v-1}}{(1-tx)^N} = \sum_{k=0}^{u+v-2} \frac{a_k(t)}{(1-tx)^{N-k}}$$

and all the powers of $(1-tx)$ appearing in the denominators are $\geqslant N-(u+v-2) \geqslant N-u-v+2 \geqslant 2$. Thus, we may integrate and get

$$\int_0^1 \frac{x^{u-1}(1-x)^{v-1}}{(1-tx)^N}dx = \frac{Q(t)}{(1-t)^{N-1}}$$

with $Q(t)$ a Laurent polynomial with rational coefficients. The left-hand side has a limit when t tends to 0, so $Q(t)$ has to be a polynomial. To conclude, it is enough to show that

$$(1-t)^{N-v}\int_0^1 \frac{x^{u-1}(1-x)^{v-1}}{(1-tx)^N}\,dx$$

is bounded when t approaches 1. We make the change of variables $s = 1-t$, $y = 1-x$, and consider integrals

$$s^{N-v}\int_0^1 \frac{(1-y)^{u-1}y^{v-1}}{(y+s-ys)^N}\,dy$$

with s approaching 0. Since $(1-y)^{u-1} \leqslant 1$ and $y+s-ys \geqslant \frac{1}{2}(y+s)$, it is enough to prove that the quantities

$$s^{N-v}\int_0^1 \frac{y^{v-1}}{(y+s)^N}\,dy$$

are bounded when s approaches 0. This equals

$$s^{N-v}\int_0^1 \left(\frac{y}{y+s}\right)^{v-1}\frac{1}{(y+s)^{N-v+1}}\,dy \leqslant s^{N-v}\int_0^1 \frac{1}{(y+s)^{N-v+1}}\,dy = \frac{1}{N-v}\left(1-\left(\frac{s}{1+s}\right)^{N-v}\right)$$

and we are done. ∎

Proposition 5.6. *Let $u_1, \ldots, u_n, v_1, \ldots, v_n \geqslant 1$ and $N \geqslant 0$ be integers such that $v_1+\cdots+v_n \geqslant N+1$. Let us assume that there exists $i \in \{1, \ldots, N\}$ such that*

$$u_i + v_i \leqslant N\ .$$

Then the integral

$$\int_{[0,1]^n} \frac{x_1^{u_1-1}\cdots x_n^{u_n-1}(1-x_1)^{v_1-1}\cdots(1-x_n)^{v_n-1}}{(1-x_1\cdots x_n)^N}\,dx_1\cdots dx_n$$

is absolutely convergent and evaluates to a linear combination

$$a_0 + a_2\zeta(2) + a_3\zeta(3) + \cdots + a_{n-1}\zeta(n-1)$$

with $a_i \in \mathbb{Q}$ for every i.

Proof. By symmetry, we can assume that $u_n + v_n \leqslant N$. Therefore, applying Lemma 5.5 to the variables $x = x_n$ and $t = x_1 \cdots x_{n-1}$ in the integral leads to the $(n-1)$-dimensional integral

$$\int_{[0,1]^{n-1}} \frac{x_1^{u_1-1} \cdots x_{n-1}^{u_{n-1}-1}(1-x_1)^{v_1-1} \cdots (1-x_{n-1})^{v_{n-1}-1} P(x_1 \cdots x_{n-1})}{(1-x_1 \cdots x_{n-1})^{N-v_n}} \, dx_1 \cdots dx_{n-1} \ .$$

Since $v_1 + \cdots + v_{n-1} \geqslant N - v_n + 1$, one can then conclude thanks to Theorem 5.2. □

Note that Proposition 5.6 applies in particular if for every i, $2u_i + v_i = N + 1$. This gives in particular a geometric interpretation of the weight drop in the Ball−Rivoal integrals [Riv00, BR01]. Note that a careful analysis of the degree of the polynomial $P(t)$ in Lemma 5.5 can lead to sufficient conditions for the vanishing of more highest weight coefficients.

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