A SOLUTION OF GROMOV'S HÖLDER EQUIVALENCE PROBLEM FOR THE HEISENBERG GROUP

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ABSTRACT. We show that a map with Hölder exponent bigger than 1/2 from a quasi-convex metric space with vanishing first Lipschitz homology into the Sub-Riemannian Heisenberg group factors through a tree. In particular, if the domain contains a disk, such a map can't be injective. This gives an answer to a question of Gromov for the simplest nontrivial case. The same tools allow to improve on a result of Borisov and it is shown that an isometric immersion of class $C^{1,\alpha}$ of a Riemannian surface with positive Gauss curvature into $\mathbb{R}^3$ has bounded extrinsic curvature if $\alpha > 1/2$.

1. Introduction

Apart from $\mathbb{R}^3$, the Heisenberg group $\mathbb{H}$ is the only simply-connected nilpotent Lie group of dimension three. Equipped with the Carnot-Carathéodory distance, $(\mathbb{H}, d_{\mathrm{cc}})$ is a metric space for which there is a homeomorphism $\mathbb{R}^3 \to \mathbb{H}$ that is locally of class $C^{\frac{1}{2}}$. In [10, §2.1] Gromov showed that any topological surface in $\mathbb{H}$ has Hausdorff-dimension at least three. In particular there can't be an embedding $\mathrm{B}^2(0,1) \hookrightarrow \mathbb{H}$ of class C^α for $\alpha > \frac{2}{3}$, and the question remained open whether the same is true for $\alpha > \frac{1}{2}$. It was observed by Wenger and Young [17] that Lipschitz maps $\varphi : X \to \mathbb{H}$ defined on a nice enough metric space X factor through trees. This was extended to maps in C^α for $\alpha > \frac{2}{3}$ in [21]. In these notes we give an optimal characterization for Hölder maps into the sub-Riemannian Heisenberg group.

Theorem 1.1. *Let X be a quasi-convex metric space with $H_1^{\mathrm{Lip}}(X) = 0$ and $\varphi : X \to \mathbb{H}$ be a map of class C^α for some $\alpha > \frac{1}{2}$. Then φ factors through a tree. In particular, if $\dim(X) > 1$, then φ can't be an embedding.*

In Theorem 4.1 we give some more details on the properties of the tree and the maps that arise. As a particular consequence, any map $\varphi : S^n \to \mathbb{H}$ of Hölder regularity $\alpha > \frac{1}{2}$ has an extension to a map on $\mathrm{B}^{n+1}(0,1)$ of the same regularity in case $n \geq 2$, whereas for $n = 1$ this is in general false, see Corollary 4.2. Along the way, Theorem 3.4 is an improvement of [21, Theorem 1.2] and gives a necessary and sufficient condition for Hölder maps into the plane to factor through a tree.

Related to this, at least to some extent, is a problem about isometric immersions of surfaces. Given a Riemannian surface (M, g) with a metric g of class C^2, there is a big difference in the behavior of isometric immersions $\varphi : M \to \mathbb{R}^3$ of class C^1 and C^2 due to curvature. The Nash-Kuiper Theorem implies that any short embedding (or immersion) into $\mathbb{R}^3$ can be approximated by isometric embeddings (or immersions) of class C^1. Extending the h-principle to Hölder classes it is shown

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in [7] that the same holds for approximations in $C^{1,\alpha}$ for $\alpha < \frac{1}{5}$. On the other side, if M has positive Gauss curvature, a result of Borisov [3, 4] states that any isometric immersion of class $C^{1,\alpha}$ are of bounded extrinsic curvature for $\alpha > \frac{2}{3}$ and applying results of Pogorelov, rigidity phenomena arise in this range of regularity. Following closely the simplified proof presented in [6] we can lower the threshold for rigidity to $\alpha > \frac{1}{2}$.

Theorem 1.2. *Let (M^2, g) be a Riemannian surface without boundary and g be a metric of class C^2 with positive Gauss curvature. If $\varphi : M \to \mathbb{R}^3$ is an isometric immersion of class $C^{1,\alpha}$ for some $\alpha > \frac{1}{2}$, then $\varphi(M)$ is of locally bounded extrinsic curvature.*

From the work of Pogorelov it follows that M can be covered by open sets U for which $\varphi(U)$ is contained in the boundary of a convex set, [15, Theorem 8, pp. 650]. If further $M \approx S^2$ and φ is an embedding, then $\varphi(M)$ is the boundary of a convex set, [15, Theorem 9, pp. 650], and hence determined up to isometries of $\mathbb{R}^3$, [15, Theorem 1, pp. 167].

The crucial ingredient of both proofs is the following observation which may be interesting in its own right.

Proposition 1.3. *Set $Q := [0,1]^2 \subset \mathbb{R}^2$ and let $\Gamma = (\Gamma^1, \Gamma^2) : Q \to \mathbb{R}^2$ be a map such that Γ^i is of class C^{α_i} for $i = 1, 2$ and $\alpha_1 + \alpha_2 > 1$. If for any subsquare $R \subset Q$,*

$$\int_{\mathbb{R}^2} \deg(\Gamma, R, q)\, d\mathscr{L}^2(q) \geq 0\,,$$

then

$$\deg(\Gamma, Q, q) \geq 0\,,$$

whenever $q \in \mathbb{R}^2 \setminus \Gamma(\partial Q)$.

The same is of course true with opposite signs as well and there is also a very similar statement with strict inequalities in Corollary 3.3. The reason this condition pops up for maps into the Heisenberg group has to do with the path lifting property. If $\varphi : Q \to \mathbb{H}$ is of class C^{α} for $\alpha > \frac{1}{2}$ and $\Gamma : Q \to \mathbb{R}^2$ is the horizontal projection, then for all squares $R \subset Q$,

$$\int_{\mathbb{R}^2} \deg(\Gamma, R, q)\, d\mathscr{L}^2(q) = \frac{1}{2}\left(\int_{\partial R} \Gamma_x\, d\Gamma_y - \Gamma_y\, d\Gamma_x\right) = 0\,.$$

On the other hand, if g is a Riemannian metric on $Q \subset \mathbb{R}^2$ of class C^2 with Gauss curvature $\kappa \geq 0$ and $\varphi : Q \to \mathbb{R}^3$ is a $C^{1,\alpha}$-isometric immersion for some $\alpha > \frac{1}{2}$, then

$$\int_{S^2} \deg(\Gamma, R, q)\, d\mathscr{H}^2(q) = \int_R \kappa(p)\, d\mathscr{H}^2_g(p) \geq 0\,,$$

where $\Gamma : Q \to S^2$ is the associated Gauss map, compare with the proof of Proposition 5.2. In case $\alpha > \frac{2}{3}$, Proposition 1.3 follows immediately from Lemma 2.1 because the following change of variables formula holds for Lipschitz test functions f,

$$\int f(q) \deg(\Gamma, Q, q) = \lim_{k\to\infty} \sum_{b_R \in R \in \mathscr{P}_k(Q)} (f \circ \Gamma)(b_R) \int \deg(\Gamma, R, q)\,. \tag{1.1}$$

In the general case, the idea for the proof of Proposition 1.3 is to consider Γ as a map on S^2 by gluing together two copies of Q along the equator. For a Lipschitz test function $f : S^2 \to \mathbb{R}$, the map $\Gamma \times f : S^2 \to \mathbb{R}^3$ contains information about Γ by means of its winding number function $q \mapsto \mathrm{w}(\Gamma \times f, q)$. It is shown in Proposition 2.4 that $\mathrm{w}(\Gamma \times f, \cdot)$ is in $L^1(\mathbb{R}^3)$. A simple homological calculation is then used in Lemma 3.1 to investigate $\int_{\mathbb{R}^3} \mathrm{w}(\Gamma \times f, q)\, dq$ with respect to perturbations of f. The reason this seems to work for $\alpha > \frac{1}{2}$ but in (1.1) it does not, is that therein we sum over $f \circ \Gamma$ and even if f is Lipschitz, this composition is in general only as regular as Γ.

The bulk of the work is contained in the technical part, Section 2, where the theory of currents is used to show that under the right conditions these winding number functions are integrable. Although more general than we need, Theorem 2.2 gives meaning to the push-forward $\varphi_\# T$ if the coordinate functions of the map $\varphi : X \to \mathbb{R}^n$ are Hölder continuous with varying regularity and $T \in \mathbf{N}_{m,c}(X)$ is a normal metric current with compact support. The benefit of allowing the coordinate functions to have varying regularity is precisely that we can work with maps like $\Gamma \times f$ mentioned above.

2. Mapping degree and currents

Before we proceed we recall some definitions that are used in the main theorems. Let (X, d_X) be a metric space. (X, d) is *C-quasi-convex* if for any two points $x, x' \in X$ there is a curve $\gamma : [0,1] \to X$ connecting x and x' of length $\ell(\gamma) \leq C d_X(x, x')$. With $H_k^{\mathrm{Lip}}(X)$ we denote the kth singular Lipschitz homology group of X. A metric space (T, d_T) is called a *tree* if it is uniquely arc-connected. This means that for any two points $p, p' \in T$ there is an injective curve $\gamma : [0,1] \to T$ connecting x with x' and any other such curve is a reparametrization of γ. If $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a homeomorphism (with $\sigma(0) = 0$), then $\varphi : (X, d_X) \to (Y, d_Y)$ is *σ-continuous* in case $d_Y(\varphi(x), \varphi(x')) \leq \sigma(d_X(x, x'))$ holds for all $x.x' \in X$. The *σ-variation* of a curve $\gamma : [a,b] \to (X, d_X)$ is given by

$$V_\sigma(\gamma) := \sup \sum_{i=1}^{n-1} \sigma^{-1}(d_X(\gamma(t_i), \gamma(t_{i+1})))\,,$$

where the supremum is taken over all finite sequences $a \leq t_1 \leq \dots t_n \leq b$. If γ is σ-continuous, then one can check that $V_\sigma(\gamma) \leq |b - a|$.

A map $\varphi : X \to Y$ between metric spaces is called *Hölder continuous of regularity* $\alpha > 0$ if there is a constant $H \geq 0$ such that for all $x, x' \in X$,

$$d_Y(\varphi(x), \varphi(x')) \leq H d_X(x, x')^\alpha\,.$$

The infimum over all such H is denoted by $\mathrm{H}^\alpha(\varphi)$ and $\mathrm{H}^\alpha(X, Y)$ is the set of all Hölder continuous maps of regularity α from X to Y. For $Y = \mathbb{R}$ we abbreviate $\mathrm{H}^\alpha(X) := \mathrm{H}^\alpha(X, \mathbb{R})$. We write $\varphi_k \xrightarrow{\alpha} \varphi$ if a sequence φ_k in $\mathrm{H}^\alpha(X,Y)$ converges uniformly $\lim_{k \to \infty} \sup_{x \in X} d_Y(\varphi_k(x), \varphi(x)) = 0$ and $\sup_k \mathrm{H}^\alpha(\varphi_k) < \infty$. The limit satisfies $\mathrm{H}^\alpha(\varphi) \leq \liminf_k \mathrm{H}^\alpha(\varphi_n)$ and hence $\varphi \in \mathrm{H}^\alpha(X,Y)$.

Given two functions $f \in \mathrm{H}^\alpha([a,b])$ and $g \in \mathrm{H}^\beta([a,b])$ with $\alpha + \beta > 1$, it follows from a result of Young [18] that the Riemann-Stieltjes integral $\int_a^b f\, dg$ is well defined and satisfies

$$\left| \int_a^b f\, dg - f(c)(g(b) - g(a)) \right| \leq C\, \mathrm{H}^\alpha(f)\, \mathrm{H}^\beta(g) |b - a|^{\alpha+\beta}\,, \tag{2.1}$$

for any $c \in [a,b]$ and some constant $C = C(\alpha,\beta)$. Moreover, if $f_k \xrightarrow{\alpha} f$ and $g_k \xrightarrow{\beta} g$, then

$$\lim_{k\to\infty} \int_a^b f_k\, dg_k = \int_a^b f\, dg\,. \tag{2.2}$$

The Riemann-Stieltjes integral over Hölder functions can be generalized to higher dimensions. For a rectangle $Q \subset \mathbb{R}^2$ we denote by $\mathscr{P}_k(Q)$ the partition of Q into 4^k similar rectangles. Given functions $f, g^1, g^2 : Q \to \mathbb{R}$ we define the approximate functionals

$$I_{Q,k}(f,g^1,g^2) := \sum_{R\in\mathscr{P}_k(Q)} f(b_R) \int_{\partial R} g^1\, dg^2\,,$$

where b_R is the barycenter of R (any other point in R would work as well for our purposes) and assuming the integrals inside the sum make sense. They are to be understood as Riemann-Stieltjes integrals running counterclockwise around the boundary of the indicated rectangle. In particular, if g_1 and g_2 have a large enough Hölder exponent, then $I_{Q,k}(f,g^1,g^2)$ is well defined for all k by the result of Young mentioned above. The following lemma is the two-dimensional case of [20, Theorem 3.2].

Lemma 2.1. *Let $\alpha,\beta_1,\beta_2 \in (0,1]$ with $\alpha+\beta_1+\beta_2 > 2$. Then the limit functional*

$$I_Q : \mathrm{H}^\alpha(Q)\times \mathrm{H}^{\beta_1}(Q)\times \mathrm{H}^{\beta_2}(Q) \to \mathbb{R}\,,$$
$$I_Q(f,g^1,g^2) := \lim_{k\to\infty} I_{Q,k}(f,g^1,g^2)\,,$$

is well defined. Further, I_Q satisfies and is uniquely determined by the following properties:

1. *I_Q is linear in each argument,*
2. *$I_Q(f,g^1,g^2) = \int_Q f \det D(g^1,g^2)\, d\mathscr{L}^2 = \int_Q f\, g^1 \wedge g^2$ if all three functions are Lipschitz,*
3. *$I_Q(f_k,g^1_k,g^2_k) \to I_Q(f,g^1,g^2)$ if $f_k \xrightarrow{\alpha} f$ and $g^i_k \xrightarrow{\beta_i} g^i$ for $i=1,2$.*

The proof of this lemma is rather straightforward and uses the estimate (2.1) in order to show that $(I_{Q,k}(f,g^1,g^2))_k$ converges geometrically to some limit. The continuity property of I_Q is then a consequence of the continuity of the one dimensional Riemann-Stieltjes integrals (2.2). Iteratively it is possible to define similar functionals over higher dimensional boxes, but we won't need this here.

The functional I_Q is closely related to those that appear in the theory of currents in metric spaces. Such a theory was introduced by Ambrosio and Kirchheim [1] extending the classical theory as described in the monograph of Federer [9]. Since in combination with Hölder maps some currents with infinite mass emerge naturally, we also refer to the theory of Lang [12] which does not rely on the finite mass assumption in its development. Following [12], a current $T \in \mathscr{D}_m(X)$ with compact support in a metric space X is a multilinear functional $T : \mathrm{Lip}(X)^{m+1} \to \mathbb{R}$ that satisfies:

1. $T(f,g^1,\dots,g^m) = 0$ whenever some g^i is constant in the neighborhood of $\mathrm{spt}(f)$;
2. There is a compact set $K \subset X$ such that $T(f,g^1,\dots,g^m) = 0$ whenever $\mathrm{spt}(f)$ and K are disjoint;

(3) $\lim_{k\to\infty} T(f_k, g_k^1, \ldots, g_k^m) = T(f, g^1, \ldots, g^m)$ if $f_k \xrightarrow{1} f$ and $g_k^i \xrightarrow{1} g^i$ for all i. That is, the functions converge uniformly with bounded Lipschitz constants.

For $w \in L^1(\mathbb{R}^n)$ with compact support we write $[\![w]\!]$ for the current in $\mathscr{D}_n(\mathbb{R}^n)$ defined by

$$[\![w]\!](f, g^1, \ldots, g^n) := [\![w]\!](f\, g^1 \wedge \cdots \wedge g^n) := \int_{\mathbb{R}^n} w\, f \det D(g^1, \ldots, g^n)\, d\mathscr{L}^n \,.$$

Similarly, for a compact oriented Lipschitz manifold M^m, or a bounded measurable subset of $\mathbb{R}^m$, we write $[\![M]\!]$ for the m-dimensional current induced by integrating m-forms over M. In this sense, the functional I_Q of Lemma 2.1 is the continuous extension of $[\![Q]\!]$ to Hölder test functions with respect to an appropriate topology. If $M \subset \mathbb{R}^n$ is a closed submanifold of codimension one, there is some connection between the winding number function $q \mapsto \mathrm{w}(\varphi, q)$ of $\varphi : M \to \mathbb{R}^n$ and the push-forward $\varphi_\#[\![M]\!]$ as we will see in Proposition 2.4. Most properties of the mapping degree and the winding number we need can be found for example in [14]. Here is a short summary. Let $\varphi : U \to \mathbb{R}^n$ be a smooth map defined on a bounded open set $U \subset \mathbb{R}^n$. If q is a regular value, the *local degree of* φ *at* q is defined by

$$\deg(\varphi, U, q) := \sum_{p \in \varphi^{-1}(q) \cap U} \operatorname{sign}(\det(D\varphi_p))\,.$$

By Sard's Theorem, the regular values form a set of full Lebesgue measure. Moreover, if φ has a smooth extension to a neighborhood of $\bar{U}$, $\deg(\varphi, U, \cdot)$ is locally constant and homotopy invariant on regular values outside $\varphi(\partial U)$, [14, Propositions IV.1.2, IV.1.4]. This allows to define the local degree for continuous maps $\varphi : \bar{U} \to \mathbb{R}^n$ and points $q \in \mathbb{R}^n \setminus \varphi(\partial U)$ by approximation, [14, Proposition IV.2.2]. We will make use of the following properties of the local degree:

(Homotopy invariance): Let $H : [0,1] \times \bar{U} \to \mathbb{R}^n$ and $\gamma : [0,1] \to \mathbb{R}^n$ be continuous maps. If $\gamma(t) \notin H_t(\partial U)$ for all t, then $\deg(H_t, U, \gamma(t))$ doesn't depend on t, [14, Proposition IV.2.4].

(Local invariance): The map $q \mapsto \deg(\varphi, U, q)$ is locally constant on $\mathbb{R}^n \setminus \varphi(\partial U)$. This follows from homotopy invariance or by smooth approximation.

(Additivity property): If $U_1, U_2 \subset U$ are two disjoint open sets and $q \notin \varphi(\bar{U} \setminus U_1 \cup U_2)$, then

$$\deg(\varphi, U, q) = \deg(\varphi, U_1, q) + \deg(\varphi, U_2, q)\ .$$

This is immediate by smooth approximation.

(Winding number): Let $\varphi : \partial U \to \mathbb{R}^{n+1}$ be a continuous map and $q \in \mathbb{R}^n \setminus \varphi(X)$. Then the local degree $\deg(\bar{\varphi}, U, q)$ doesn't depend on the continuous extension $\bar{\varphi} : \bar{U} \to \mathbb{R}^n$ of φ, [14, Proposition IV.4.1]. This number is called the *winding number of* φ *at* q and denoted by $\mathrm{w}(\varphi, q)$.

(Homological degree): The *homological degree* $\deg(f)$ of a map $f : S^n \to S^n$ is the integer that satisfies $f_*(g) = \deg(f) g$ for all $g \in H_n(S^n) \approx \mathbb{Z}$. If $\varphi : S^n \to \mathbb{R}^{n+1}$ is continuous and $q \in \mathbb{R}^{n+1} \setminus \varphi(S^n)$, then

$$\mathrm{w}(\varphi, q) = \deg(\varphi_q)\,,$$

where $\varphi_q : S^n \to S^n$ is defined by $\varphi_q(p) := \frac{\varphi(p) - q}{|\varphi(p) - q|}$. This follows for example from [14, Proposition IV.4.6] and [2, Corollary IV.7.5].

As defined in [1, Proposition 2.7] or [12, Definition 4.1], the *mass* of a current $T \in \mathscr{D}_m(X)$ with compact support in a metric space X can be defined by

$$\mathbf{M}(T) := \sup \sum_{\lambda\in\Lambda} T(f_\lambda, g_\lambda^1, \dots, g_\lambda^m)\,,$$

where the supremum ranges over all finite collections $\{(f_\lambda, g_\lambda^1, \dots, g_\lambda^m)\}_{\lambda\in\Lambda}$, of $\mathrm{Lip}(X)^{m+1}$ that satisfy $\sum_{\lambda\in\Lambda} |f_\lambda| \leq 1$ and $\mathrm{Lip}(g_\lambda^i) \leq 1$ for all i and λ. In case $X = \mathbb{R}^n$ we can estimate the mass of T by

$$\mathbf{M}(T) \leq \sum_{\lambda\in\Lambda(n,m)} \sup_{|f_\lambda|\leq 1} T\left(f_\lambda, \pi^{\lambda(i)}, \dots, \pi^{\lambda(m)}\right)\,, \tag{2.3}$$

where $\Lambda(n,m)$ is the collection of all strictly increasing functions from $\{1,\dots,m\}$ to $\{1,\dots,n\}$, $\pi^i : \mathbb{R}^n \to \mathbb{R}$ denotes the ith coordinate function $\pi^i(p) = p_i$ and each $f_\lambda \in \mathrm{Lip}(\mathbb{R}^n)$ satisfies $\|f_\lambda\|_\infty \leq 1$. This estimate follows from a smoothing argument for Lipschitz function and the chain rule for currents, compare with [12, Theorem 5.5]. For $T \in \mathbf{N}_{m,c}(\mathbb{R}^n)$ (this means that $T \in \mathscr{D}_m(\mathbb{R}^n)$ has compact support and $\mathbf{N}(T) := \mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$) define

$$\mathbf{F}(T) := \inf\{\mathbf{M}(T - \partial S) + \mathbf{M}(S) : S \in \mathbf{N}_{m+1,c}(\mathbb{R}^n)\}\,.$$

With $\mathbf{F}_m(\mathbb{R}^n)$ we denote those metric currents in $\mathscr{D}_m(\mathbb{R}^n)$ with compact support that are in the closure of $\mathbf{N}_{m,c}(\mathbb{R}^n)$ with respect to $\mathbf{F}$, compare with [9, §4.1.12]. Note that the metric definition of mass may differ from the classical one in [9], but it is stated in [12, Theorem 5.5] that these two norms are comparable and that the space $\mathbf{F}_m(\mathbb{R}^n)$ agrees with the classical definition of flat chains. Similarly, for $T \in \mathbf{I}_{m,c}(\mathbb{R}^n)$ (this means that $T \in \mathscr{D}_m(\mathbb{R}^n)$ is an integral current with compact support) we define

$$\mathscr{F}(T) := \inf\{\mathbf{M}(T - \partial S) + \mathbf{M}(S) : S \in \mathbf{I}_{m+1,c}(\mathbb{R}^n)\}\,.$$

With $\mathscr{F}_m(\mathbb{R}^n)$ we denote those metric currents in $\mathscr{D}_m(\mathbb{R}^n)$ with compact support in the closure of $\mathbf{I}_{m,c}(\mathbb{R}^n)$ with respect to $\mathscr{F}$, compare with [9, §4.1.24]. It is easy to check that both $\mathbf{F}$ and $\mathscr{F}$ define norms. The following theorem is a generalization of [19, Proposition 4.4].

Theorem 2.2. *Let (X,d) be a metric space, $T \in \mathbf{N}_{m,c}(X)$ (or $T \in \mathbf{I}_{m,c}(X)$) for some integer $m \geq 1$ and let $\varphi = (\varphi^1, \dots, \varphi^n) : X \to \mathbb{R}^n$ be a map with coordinate functions $\varphi^i \in \mathrm{H}^{\alpha_i}(X)$ for $\alpha_i \in (0,1]$ and $i = 1,\dots,n$. If*

$$\sigma(m) := \min_{\lambda\in\Lambda(n,m+1)} \alpha_{\lambda(1)} + \cdots + \alpha_{\lambda(m+1)} > m\,,$$

then $\varphi_\# T$ is a well defined element of $\mathbf{F}_m(\mathbb{R}^n)$ (or $\mathscr{F}_m(\mathbb{R}^n)$) in the sense that $\varphi_{k\#}T$ converges with respect to $\mathbf{F}$ (or $\mathscr{F}$) to $\varphi_\# T$ whenever $\varphi_k \in \mathrm{Lip}(X,\mathbb{R}^n)$ and $\varphi_k^i \xrightarrow{\alpha_i} \varphi^i$ for all i. Moreover, there is a constant $C = C(m,n,\sigma(m))$ with the following property: If $\varphi_1, \varphi_2 : X \to \mathbb{R}^n$, $\delta \in [0,1]$ and $H \geq 1$ satisfy for all i,

(A) $\max\{\mathrm{H}^{\alpha_i}(\varphi_1^i), \mathrm{H}^{\alpha_i}(\varphi_2^i)\} \leq H$,
(B) $\|\varphi_1^i - \varphi_2^i\|_\infty \leq H\delta^{\alpha_i}$,

then there are $R \in \mathbf{F}_m(\mathbb{R}^n) \cap \mathbf{M}_m(\mathbb{R}^n)$ and $S \in \mathbf{F}_{m+1}(\mathbb{R}^n) \cap \mathbf{M}_{m+1}(\mathbb{R}^n)$ (respectively $R \in \mathscr{R}_{m,c}(\mathbb{R}^n)$ and $S \in \mathscr{R}_{m+1,c}(\mathbb{R}^n)$) such that

(1) $\varphi_{1\#}T - \varphi_{2\#}T = R + \partial S$;
(2) $\mathbf{M}(R) + \mathbf{M}(S) \leq CH^{m+1}\,\mathbf{N}(T)\delta^{\sigma(m)-m}$;
(3) $\mathrm{spt}(R) \cup \mathrm{spt}(S) \subset \mathrm{B}(\varphi_1(\mathrm{spt}(T)), 10\sqrt{n}H\delta^\alpha)$, *where* $\alpha := \min_i \alpha_i$;

(4) *if* $\partial T = 0$, *then* $R = 0$.

Proof. We formulate the proof for $T \in \mathbf{N}_{m,c}(X)$. Replacing $\mathbf{F}$ with $\mathscr{F}$ and normal currents with integral currents wherever they appear, the proof for $T \in \mathbf{I}_{m,c}(X)$ is the same if not indicated otherwise. Because T has compact support we can assume that X is compact by restricting the functions to the support of T. Let $0 \leq a < b \leq 1$. As in [16, Theorem 5.2], which is an adaption of the cone construction in [1, Proposition 10.2], the functional $[\![a,b]\!] \times T$ on $\mathrm{Lip}([0,1] \times X)^{m+2}$ defined by

$$([\![a,b]\!] \times T)(f, g^1, \dots, g^{m+1}) \\ := \sum_{i=1}^{m+1} (-1)^{i+1} \int_a^b T\left(f_t \partial_t g_t^i, g_t^1, \dots, g_t^{i-1}, g_t^{i+1}, \dots, g_t^{m+1}\right) dt$$

is an element of $\mathbf{N}_{m+1}([0,1] \times X)$. For convenience sake we put the ℓ_1 metric on the product $[0,1] \times X$. This construction of a product with an interval has similar properties to the classical one in [9, §4.1.8]. For example,

$$\partial([\![a,b]\!] \times T) = ([\![b]\!] \times T) - ([\![a]\!] \times T) - ([\![a,b]\!] \times \partial T), \tag{2.4}$$

where for any $t \in [0,1]$ the current $[\![t]\!] \times T$ in $\mathbf{N}_m([0,1] \times X)$ is given by

$$([\![t]\!] \times T)(f, g^1, \dots, g^m) := T(f_t, g_t^1, \dots, g_t^m)\,.$$

From the definition of mass and of $[\![a,b]\!] \times T$ it is clear that

$$\mathbf{M}([\![a,b]\!] \times T) \leq (m+1)(b-a)\,\mathbf{M}(T)\,. \tag{2.5}$$

Set $H := \max_i \mathrm{H}^{\alpha_i}(\varphi_i)$ and define $\tilde{\varphi} : [0,1] \times X \to \mathbb{R}^m$ coordinate-wise by

$$\tilde{\varphi}_t^i(x) := \inf_{y \in X} \varphi^i(y) + H t^{\alpha_i - 1} d(x,y)\,,$$

This construction to approximate Hölder functions with Lipschitz functions is described in the appendix of [11] written by Semmes. The following properties are direct. For a proof see [11, Theorem B.6.16] or [20, Lemma 2.2]. For all i and $t \in (0,1]$:

$$\tilde{\varphi}_t^i(x) = \inf\{\varphi^i(y) + H t^{\alpha_i - 1} d(x,y) : y \in \mathrm{B}(x,t)\}\,; \tag{2.6}$$

$$\mathrm{Lip}(\tilde{\varphi}_t^i) \leq H t^{\alpha_i - 1}\,; \tag{2.7}$$

$$\|\tilde{\varphi}_t^i - \varphi^i\|_\infty \leq H t^{\alpha_i}\,. \tag{2.8}$$

Using (2.6), we see that for any fixed $x \in X$ and i, the function $t \mapsto \tilde{\varphi}_t^i(x)$ is $H s_2 s_1^{\alpha_i - 2}$-Lipschitz on $[s_1, s_2]$ in case $0 < s_1 < s_2 \leq 1$. Together with (2.7), and the fact that we have chosen the ℓ_1-metric on $[0,1] \times X$, each function $\tilde{\varphi}^i$ is

$2Hs^{\alpha_i-1}$-Lipschitz on $[s,2s]\times X$ for all $s\in(0,\frac{1}{2}]$. Hence with (2.3) and (2.5),

$$\begin{aligned}
&\mathbf{M}\left(\tilde\varphi_\#([\![s,2s]\!]\times T)\right)\\
&\quad\leq \sum_{\lambda\in\Lambda(n,m+1)} \sup_{|f_\lambda|\leq 1} \tilde\varphi_\#([\![s,2s]\!]\times T)\left(f_\lambda,\pi^{\lambda(1)},\dots,\pi^{\lambda(m+1)}\right)\\
&\quad= \sum_{\lambda\in\Lambda(n,m+1)} \sup_{|f_\lambda|\leq 1} ([\![s,2s]\!]\times T)\left(f_\lambda\circ\tilde\varphi,\tilde\varphi^{\lambda(1)},\dots,\tilde\varphi^{\lambda(m+1)}\right)\\
&\quad\leq \sum_{\lambda\in\Lambda(n,m+1)} \mathbf{M}\left([\![s,2s]\!]\times T\right)\prod_{i=1}^{m+1}\operatorname{Lip}\left(\tilde\varphi^{\lambda(i)}|_{[s,2s]\times X}\right)\\
&\quad\leq \sum_{\lambda\in\Lambda(n,m+1)} (m+1)s\,\mathbf{M}(T)(2H)^{m+1}s^{\alpha_{\lambda(1)}+\dots+\alpha_{\lambda(m+1)}-(m+1)}\\
&\quad\leq \binom{n}{m+1}(m+1)(2H)^{m+1}\,\mathbf{M}(T)s^{\sigma(m)-m}\,. \qquad (2.9)
\end{aligned}$$

The hypothesis $\sigma(m)>m$ now implies that $(\tilde\varphi_\#([\![2^{-k},1]\!]\times T))_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbf{M}_{m+1}(\mathbb{R}^m)$ equipped with the norm $\mathbf{M}$. This sequence converges to a current $\tilde\varphi_\#([\![0,1]\!]\times T)\in\mathbf{M}_{m+1}(\mathbb{R}^n)$ because $(\mathbf{M}_{m+1}(\mathbb{R}^n),\mathbf{M})$ is a Banach space by [12, Proposition 4.2]. As a result, $\tilde\varphi_\#([\![0,1]\!]\times T)\in\mathbf{F}_{m+1}(\mathbb{R}^n)\cap\mathbf{M}_{m+1}(\mathbb{R}^n)$. In case $T\in\mathbf{I}_{m,c}(X)$, $\tilde\varphi_\#([\![0,1]\!]\times T)\in\mathcal{R}_{m+1,c}(\mathbb{R}^n)$ by [9, §4.1.17] and [9, §4.1.24]. The same reasoning applies to the boundary ∂T. Note that $\sigma(m)>m$ implies that

$$\begin{aligned}
\sigma(m-1)-(m-1) &= \min_{\lambda\in\Lambda(n,m)} \alpha_{\lambda(1)}+\dots+\alpha_{\lambda(m)}-(m-1)\\
&\geq \sigma(m)-m>0\,, \qquad (2.10)
\end{aligned}$$

which is the appropriate hypothesis in order to obtain similar estimates for ∂T in place of T. We abbreviate $\tilde\varphi_k:=\tilde\varphi_{2^{-k}}$. From (2.4) it follows that

$$\begin{aligned}
\tilde\varphi_{k\#}T &= \tilde\varphi_\#([\![2^{-k}]\!]\times T)\\
&= \tilde\varphi_\#\left([\![1]\!]\times T-([\![2^{-k},1]\!]\times\partial T)-\partial([\![2^{-k},1]\!]\times T)\right)\\
&= \tilde\varphi_\#([\![1]\!]\times T)-\tilde\varphi_\#\left([\![2^{-k},1]\!]\times\partial T\right)-\partial\tilde\varphi_\#\left([\![2^{-k},1]\!]\times T\right)\\
&\to \tilde\varphi_\#([\![1]\!]\times T)-\tilde\varphi_\#([\![0,1]\!]\times\partial T)-\partial\tilde\varphi_\#([\![0,1]\!]\times T)\,,
\end{aligned}$$

where convergence is with respect to $\mathbf{F}$. The limit of the sequence $\tilde\varphi_{k\#}T$, we denote it by $\tilde\varphi_\#T$, is hence a current in $\mathbf{F}_m(\mathbb{R}^n)$. From (2.9) and (2.10) it follows that there is a constant C_1 depending on m, n and $\sigma(m)$ such that for $R_{\varphi,k}:=\tilde\varphi_\#\left([\![0,2^{-k}]\!]\times\partial T\right)$ and $S_{\varphi,k}:=\tilde\varphi_\#\left([\![0,2^{-k}]\!]\times T\right)$ we have $\partial R_{\varphi,k}=0$ in case $\partial T=0$,

$$\tilde\varphi_{k\#}T-\tilde\varphi_\#T=R_{\varphi,k}+\partial S_{\varphi,k}\,, \qquad (2.11)$$

$$\mathbf{M}(R_{\varphi,k})+\mathbf{M}(S_{\varphi,k})\leq C_1\max\{1,H\}^{m+1}\,\mathbf{N}(T)2^{k(m-\sigma(m))}\,, \qquad (2.12)$$

$$\operatorname{spt}(R_{\varphi,k})\cup\operatorname{spt}(S_{\varphi,k})\subset \mathrm{B}(\varphi(\operatorname{spt}(T)),\sqrt{n}H2^{-k\alpha})\,. \qquad (2.13)$$

In order for $\tilde\varphi_\#$ to be useful it should agree with the usual definition of the push-forward $\varphi_\#T$ if φ is Lipschitz. In this case we can choose some L for which $L\geq \mathrm{H}^{\alpha_i}(\varphi^i)$ for all i. This is possible because $\operatorname{diam}(X)<\infty$ implies that $\operatorname{Lip}(X)\subset\mathrm{H}^\alpha(X)$ for any α. Let t_0 be such that $Lt^{\alpha_i-1}\geq\operatorname{Lip}(\varphi^i)$ for all $t\in(0,t_0]$ and all i

(if $\alpha_i = 1$, then $Lt^{\alpha_i-1} = L \geq \mathrm{Lip}(\varphi^i)$). For such a $t \in (0, t_0]$, $x, y \in X$ and i, there holds

$$\varphi^i(y) + Lt^{\alpha_i-1}d(x,y) \geq \varphi^i(y) + \mathrm{Lip}(\varphi^i)d(x,y) \geq \varphi^i(x)\,.$$

Hence, if $\varphi \in \mathrm{Lip}(X, \mathbb{R}^n)$, then $\tilde{\varphi}_t = \varphi$ if t is small enough, and therefore

$$\tilde{\varphi}_{\#}T = \varphi_{\#}T\,. \tag{2.14}$$

In the next step we show that the definition of $\tilde{\varphi}_{\#}T$ doesn't depend on the particular approximating sequence $\tilde{\varphi}_{2^{-k}}$ of φ.

Consider to maps $\psi_0, \psi_1 \in \mathrm{Lip}(X, \mathbb{R}^n)$ and a constant $L > 0$ with $\|\psi_0^i - \psi_1^i\|_\infty \leq L\epsilon^{\alpha_i}$ and $\mathrm{Lip}(\psi_j^i) \leq L\epsilon^{\alpha_i-1}$ for all $i = 1, \ldots, n$ and $j = 0, 1$. We want to estimate $\mathbf{F}(\psi_{1\#}T - \psi_{0\#}T)$. Let $\psi : [0,1] \times X \to \mathbb{R}^n$ be the linear homotopy

$$\psi(t,x) := t\psi_1(x) + (1-t)\psi_0(x)\,.$$

For all i and $t \in (0,1)$ there holds $\|\partial_t\psi_t^i\|_\infty \leq L\epsilon^{\alpha_i}$ and $\mathrm{Lip}(\psi_t^i) \leq L\epsilon^{\alpha_i-1}$. For each $\lambda \in \Lambda(n, m+1)$ set

$$\hat{\psi}_t^{\lambda,i} := \left(\psi_t^{\lambda(1)}, \ldots, \psi_t^{\lambda(i-1)}, \psi_t^{\lambda(i+1)}, \ldots, \psi_t^{\lambda(m+1)}\right)\,.$$

Similar to (2.9),

$$\begin{aligned}
\mathbf{M}\left(\psi_{\#}([\![0,1]\!] \times T)\right) &\leq \sum_{\lambda\in\Lambda(n,m+1)} \sup_{|f_\lambda|\leq 1} \psi_{\#}([\![0,1]\!] \times T)\left(f_\lambda, \pi^{\lambda(1)}, \ldots, \pi^{\lambda(m+1)}\right) \\
&= \sum_{\lambda\in\Lambda(n,m+1)} \sup_{|f_\lambda|\leq 1} \sum_{i=1}^{m+1} (-1)^{i+1} \int_0^1 T\left(f_{\lambda,t}\partial_t\psi_t^{\lambda(i)}, \hat{\psi}_t^{\lambda,i}\right) dt \\
&\leq \sum_{\lambda\in\Lambda(n,m+1)} \sum_{i=1}^{m+1} \mathbf{M}(T) \left\|\partial_t\psi_t^{\lambda(i)}\right\|_\infty \prod_{j\neq i} \mathrm{Lip}\left(\psi_t^{\lambda(j)}\right) \\
&\leq \binom{n}{m+1}(m+1)L^{m+1}\,\mathbf{M}(T)\epsilon^{\sigma(m)-m}\,.
\end{aligned}$$

The estimate (2.10) allows to obtain similar bounds for the boundary ∂T and we conclude that there is a constant C_2 depending on m and n such that for the currents $R_{\psi_0,\psi_1} := \psi_{\#}([\![0,1]\!] \times \partial T) \in \mathbb{N}_{m,c}(\mathbb{R}^n)$ and $S_{\psi_0,\psi_1} := \psi_{\#}([\![0,1]\!] \times T) \in \mathbb{N}_{m+1,c}(\mathbb{R}^n)$ we have $R_{\psi_0,\psi_1} = 0$ if $\partial T = 0$ and further

$$\psi_{1\#}T - \psi_{0\#}T = R_{\psi_0,\psi_1} + \partial S_{\psi_0,\psi_1}\,, \tag{2.15}$$

$$\mathbf{M}(R_{\psi_0,\psi_1}) + \mathbf{M}(S_{\psi_0,\psi_1}) \leq C_2 \max\{1, L\}^{m+1}\,\mathbf{N}(T)\epsilon^{\sigma(m)-m}\,, \tag{2.16}$$

$$\mathrm{spt}(R_{\psi_0,\psi_1}) \cup \mathrm{spt}(S_{\psi_0,\psi_1}) \subset \mathrm{B}(\psi_1(\mathrm{spt}(T)), \sqrt{n}L\epsilon^{\alpha})\,. \tag{2.17}$$

Assume that $\gamma : X \to \mathbb{R}^n$ satisfies $G := \max_i \mathrm{H}^{\alpha_i}(\gamma^i) < \infty$ and there are constants $\delta \in (0,1]$ and $\bar{H} \geq \max\{1, H, G\}$ such that $\|\gamma^i - \varphi^i\|_\infty \leq \bar{H}\delta^{\alpha_i}$ for all i. Let $k \geq 0$ be the unique integer such that $2^{-k-1} \leq \delta \leq 2^{-k}$. Then for all i we obtain from (2.8),

$$\begin{aligned}
\|\tilde{\gamma}_k^i - \tilde{\varphi}_k^i\|_\infty &\leq \|\tilde{\gamma}_k^i - \gamma^i\|_\infty + \|\gamma^i - \varphi^i\|_\infty + \|\tilde{\varphi}_k^i - \tilde{\varphi}^i\|_\infty \\
&\leq G2^{-k\alpha_i} + \bar{H}\delta^{\alpha_i} + H2^{-k\alpha_i} \\
&\leq 3\bar{H}2^{-k\alpha_i}\,. \qquad (2.18)
\end{aligned}$$

Also, (2.7) implies for all i,

$$\max\left\{\operatorname{Lip}(\tilde{\varphi}^i_k), \operatorname{Lip}(\tilde{\gamma}^i_k)\right\} \leq \bar{H}\delta^{\alpha_i - 1} \leq \bar{H}2^{(k+1)(1-\alpha_i)}$$
$$\leq 2\bar{H}2^{-k(\alpha_i-1)}\,. \tag{2.19}$$

With (2.12), (2.16), (2.18) and (2.19) we obtain $R \in \mathbf{F}_m(\mathbb{R}^n) \cap \mathbf{M}_m(\mathbb{R}^n)$ and $S \in \mathbf{F}_{m+1}(\mathbb{R}^n) \cap \mathbf{M}_{m+1}(\mathbb{R}^n)$ (respectively, $R \in \mathscr{R}_{m,c}(\mathbb{R}^n)$ and $S \in \mathscr{R}_{m+1,c}(\mathbb{R}^n)$ if $T \in \mathbf{I}_{m,c}(X)$) with $\tilde{\gamma}_\# T - \tilde{\varphi}_\# T = R + \partial S$ and

$$\begin{aligned}\mathbf{M}(R) + \mathbf{M}(S) &\leq \mathbf{M}(R_{\gamma,k} + R_{\varphi_k,\gamma_k} - R_{\varphi,k}) + \mathbf{M}(S_{\gamma,k} + S_{\varphi_k,\gamma_k} - S_{\varphi,k})\\ &\leq (2C_1\bar{H}^{m+1} + C_2(3\bar{H})^{m+1})\,\mathbf{N}(T)2^{k(m-\sigma(m))}\\ &\leq C_3\bar{H}^{m+1}\,\mathbf{N}(T)\delta^{\sigma(m)-m}\,.\end{aligned} \tag{2.20}$$

for some constant C_3 depending on m, n and $\sigma(m)$. It is also clear that $R = 0$ in case $\partial T = 0$. For any point $x \in \operatorname{spt}(R) \cup \operatorname{spt}(S)$ it follows from (2.13), (2.17) and (2.18) that

$$\operatorname{dist}(x, \varphi(\operatorname{spt}(T))) \leq 5\sqrt{n}\bar{H}2^{-k\alpha} \leq 10\sqrt{n}\bar{H}\delta^{\alpha}\,.$$

Let $\varphi_k \in \operatorname{Lip}(X,\mathbb{R}^n)$ be a sequence such that $\varphi^i_k \xrightarrow{\alpha_i} \varphi^i$ for all i. With (2.14) and (2.20) we conclude that $\varphi_{k\#}T$ converges to $\tilde{\varphi}_\# T$ with respect to $\mathbf{F}$. Hence $\tilde{\varphi}_\# T$ doesn't depend on the particular approximating sequence used in its definition. This justifies to use the notation $\varphi_\# T$ for $\tilde{\varphi}_\# T$ and finishes the theorem. □

The following is a direct consequence Theorem 2.2.

Corollary 2.3. *Assume that $T \in \mathbf{N}_{n-1,c}(X)$ satisfies $\partial T = 0$ and let $\varphi = (\varphi^1,\ldots,\varphi^n) : X \to \mathbb{R}^n$ be a map with $\varphi^i \in \mathrm{H}^{\alpha_i}(X)$ and $\sum_i \alpha_i > n-1$. Then there is a unique $w_\varphi \in L^1_c(\mathbb{R}^n)$ such that $\partial[\![w_\varphi]\!] = \varphi_\# T$. If $T \in \mathbf{I}_{n-1,c}(X)$, then $w_\varphi \in L^1_c(\mathbb{R}^n,\mathbb{Z})$. Moreover, if $\varphi_k : X \to \mathbb{R}^n$ is a sequence with $\varphi^i_k \xrightarrow{\alpha_i} \varphi$ for all i, then*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |w_{\varphi_k} - w_\varphi|\, d\mathscr{L}^n = 0\,.$$

Proof. If we set $\varphi_1 = \varphi$ and $\varphi_2 = 0$ in Theorem 2.2, we obtain a current $S \in \mathbf{F}_n(\mathbb{R}^n) \cap \mathbf{M}_n(\mathbb{R}^n)$, respectively $S \in \mathscr{R}_{n,c}(\mathbb{R}^n)$ if $T \in \mathbf{I}_{n-1,c}(X)$, with $\partial S = \varphi_\# T$. By [9, §4.1.18], S can be represented by an integrable function $w_\varphi \in L^1_c(\mathbb{R}^n)$. This is obvious in case $S \in \mathscr{R}_{n,c}(\mathbb{R}^n)$. The constancy theorem [9, §4.1.7] implies that this filling is unique among all classical currents. Let φ_k be a sequence as in the statement. By the constancy theorem, $[\![w_{\varphi_k} - w_\varphi]\!]$ is the unique filling with compact support of $\varphi_{k\#}T - \varphi_\# T$. Hence the mass estimate of Theorem 2.2 applies to $[\![w_{\varphi_k} - w_\varphi]\!]$ and therefore

$$\int_{\mathbb{R}^n} |w_{\varphi_k} - w_\varphi|\, d\mathscr{L}^n = \mathbf{M}([\![w_{\varphi_k} - w_\varphi]\!]) \to 0\,,$$

for $k \to \infty$. □

The *perimeter* of a $\mathscr{L}^n$-measurable set $B \subset \mathbb{R}^n$ is equal to

$$P(B) := \sup\left\{\int_B \operatorname{div}(\psi)\, d\mathscr{L}^n \,:\, \psi \in C^1_c(\mathbb{R}^n,\mathbb{R}^n),\ \|\psi\|_\infty \leq 1\right\}.$$

By [12, Theorem 7.2] we get the identity

$$\mathbf{M}(\partial[\![B]\!]) = P(B)\,.$$

For example, if the topological boundary of $B \subset \mathbb{R}^n$ satisfies

$$\mathscr{H}^{n-1}(\partial B) < \infty\,,$$

then B has finite perimeter. See for example [9, Theorem 4.5.11] for this and related properties of the perimeter. As an application of Theorem 2.2 we can relate currents induced by Hölder maps with the winding number function and obtain an integrability condition on the latter. The following is a generalization of [19, Proposition 4.6].

Proposition 2.4. *Let $U \subset \mathbb{R}^n$ be a bounded open set with finite perimeter. Let $\varphi = (\varphi^1, \dots, \varphi^n) : \partial U \to \mathbb{R}^n$ be a map such that $\varphi^i \in \mathrm{H}^{\alpha_i}(\partial U)$ and $\sum_i \alpha_i > n-1$. Then $\varphi_\#(\partial[\![U]\!])$ has a unique filling that is given by $[\![w_\varphi]\!]$ for some $w_\varphi \in L^1_c(\mathbb{R}^n, \mathbb{Z})$ and moreover $w_\varphi = \mathrm{w}(\varphi, \cdot)$ almost everywhere on $\mathbb{R}^n \setminus \varphi(\partial U)$. If $\mathscr{H}^{n-1}(\partial U) < \infty$, then $\mathscr{L}^n(\varphi(\partial U)) = 0$ and $w_\varphi = \mathrm{w}(\varphi, \cdot)$ almost everywhere. This means that*

$$\mathrm{w}(\varphi, \cdot) \in L^1_c(\mathbb{R}^n, \mathbb{Z})\,, \quad \text{and} \quad \varphi_\#(\partial[\![U]\!]) = [\![\mathrm{w}(\varphi, \cdot)]\!]\,. \tag{2.21}$$

If $\varphi_k : \partial U \to \mathbb{R}^n$ is a sequence that satisfies $\varphi_k^i \xrightarrow{\alpha_i} \varphi^i$ for all i, then $\mathbf{M}([\![w_{\varphi_k}]\!] - [\![w_\varphi]\!]) \to 0$, respectively if $\mathscr{H}^{n-1}(\partial U) < \infty$, then

$$\int_{\mathbb{R}^n} |\,\mathrm{w}(\varphi_k, q) - \mathrm{w}(\varphi, q)|\, d\mathscr{L}^n(q) \to 0\,. \tag{2.22}$$

Proof. First note that (2.21) follows from the general case if we can show that $\mathscr{L}^n(\mathrm{im}(\varphi)) = 0$ whenever $\mathscr{H}^{n-1}(\partial U) < \infty$. The finiteness of $\mathscr{H}^{n-1}(\partial U)$ implies that there is some $C > 0$ such that for all $\delta > 0$ there is a finite covering of ∂U by balls $\mathrm{B}^n(x_1, r_1), \dots, \mathrm{B}^n(x_k, r_k)$ with $r_i \leq \delta$ and $\sum_i r_i^{n-1} \leq C$. Each set $\varphi(\mathrm{B}(x_i, r_i) \cap \partial U)$ is contained in the box

$$\varphi(x_i) + \left([-Hr_i^{\alpha_1}, Hr_i^{\alpha_1}] \times \dots \times [-Hr_i^{\alpha_n}, Hr_i^{\alpha_n}]\right),$$

where $H := \max_i \mathrm{H}^{\alpha_i}(\varphi^i)$. Hence

$$\begin{aligned}\mathscr{L}^n(\varphi(\partial U)) &\leq \sum_i (2Hr_i^{\alpha_1}) \dots (2Hr_i^{\alpha_n}) \leq (2H)^n \sum_i r_i^{\alpha_1 + \dots + \alpha_n} \\ &\leq (2H)^n \delta^{\alpha_1 + \dots + \alpha_n - (n-1)} \sum_i r_i^{n-1} \\ &\leq (2H)^n C \delta^{\alpha_1 + \dots + \alpha_n - (n-1)}\,,\end{aligned}$$

and this converges to zero provided δ does. It is interesting to note that this estimate is very similar to the one obtained in (2.9) within the proof of Theorem 2.2.

The current $[\![U]\!]$ is an integer rectifiable current with finite boundary mass $\mathbf{M}(\partial[\![U]\!]) = P(U) < \infty$. The boundary rectifiability theorem, see for example [12, Theorem 8.7] or [9, Theorem 4.2.16], implies that $\partial[\![U]\!]$ is an element of $\mathbf{I}_{n-1}(\mathbb{R}^n)$. As a consequence of Corollary 2.3, there is a unique $w_\varphi \in L^1_c(\mathbb{R}^n, \mathbb{Z})$ with $\partial[\![w_\varphi]\!] = \varphi_\#(\partial[\![U]\!])$.

If $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map, then $\psi_\#[\![U]\!] = [\![\deg(\psi, U, \cdot)]\!]$ and hence $\deg(\psi, U, \cdot) = w_\psi$. This is a consequence of the change of variables formula [9, Theorem 3.2.3(2)] and the definition of the local degree for smooth maps. If f :

$\mathbb{R}^n \to \mathbb{R}$ is smooth (or Lipschitz), then

$$
\begin{aligned}
\psi_\#[\![U]\!](f\, d\pi^1 \wedge \cdots \wedge d\pi^n) &= \int_U f \circ \psi \det(D\psi_p)\, d\mathscr{L}^n(p) \\
&= \int_{\mathbb{R}^n} f(q) \sum_{p \in \psi^{-1}(q) \cap U} \operatorname{sign}(\det(D\psi_p))\, d\mathscr{L}^n(q) \\
&= \int_{\mathbb{R}^n} f(q) \deg(\psi, U, q)\; d\mathscr{L}^n(q) \\
&= [\![\deg(\psi, U, \cdot)]\!](f\, d\pi^1 \wedge \cdots \wedge d\pi^n)\,. \qquad (2.23)
\end{aligned}
$$

Using the McShane-Whitney extension theorem we can extend φ to a map $\bar\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with $\max_i \mathrm{H}^{\alpha_i}(\bar\varphi^i) < \infty$. Smoothing each function $\bar\varphi^i$ appropriately by convolution with a mollifier, we can construct an approximating sequence $\varphi_k \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $\sup_{i,k} \mathrm{H}^{\alpha_i}(\varphi_k^i) < \infty$ and $\lim_{k\to\infty} \|\varphi_k - \bar\varphi\|_\infty = 0$. From Corollary 2.3 we obtain

$$
\lim_{k\to\infty} \mathbf{M}([\![\deg(\varphi_k, U, \cdot)]\!] - [\![w_\varphi]\!]) = 0\,. \qquad (2.24)
$$

Let $\mathrm{B}(q,r) \subset \mathbb{R}^n \setminus \varphi(\partial U)$. Since $\varphi_k|_{\partial U}$ converges uniformly to φ, there is an integer $k_0 \geq 1$ such that $\mathrm{B}(q,r) \subset \mathbb{R}^n \setminus \varphi_k(\partial U)$ and $\deg(\varphi_k, U, q') = \mathrm{w}(\varphi, q)$ for all $q' \in \mathrm{B}(q,r)$ and all $k \geq k_1$. This is a consequence of the homotopy invariance of the local degree. Hence, for all $k \geq k_0$,

$$
[\![\deg(\varphi_k, U, \cdot)]\!] \lfloor \mathrm{B}(q,r) = \mathrm{w}(\varphi, q) [\![\mathrm{B}(q,r)]\!]\,. \qquad (2.25)
$$

Since $\operatorname{spt}(\varphi_\#(\partial [\![U]\!])) \subset \varphi(\partial U)$ (this is true for Lipschitz maps and by approximation also for φ), the constancy theorem implies that there is an integer $m \in \mathbb{Z}$ such that $[\![w_\varphi]\!] \lfloor \mathrm{B}(q,r) = m[\![\mathrm{B}(q,r)]\!]$. The convergence of mass (2.24) and (2.25) force that $m = \mathrm{w}(\varphi, q)$. Since the winding number is locally constant and the above holds for all balls $\mathrm{B}(q,r) \subset \mathbb{R}^n \setminus \varphi(\partial U)$, we conclude that $[\![w_\varphi]\!] \lfloor \varphi(\partial U)^c = [\![\mathrm{w}(\varphi, q)]\!] \lfloor \varphi(\partial U)^c$. The last statement (2.22) follows directly from Corollary 2.3. □

The results of this section are more general than what is needed in the progress. Collecting the tools so far, we combine the push-forwards of currents with the Riemann-Stieltjes integrals. Fixing some notation first, set $B_+ := \{(x,y,z) \in S^2 : z \geq 0\}$ and $B_- := \{(x,y,z) \in S^2 : z \geq 0\}$ to be the northern and southern hemispheres with intersection $B_+ \cap B_- = S^1$. Set $Q := [0,1]^2$ and let $\psi_+ : Q \to B_+$ and $\psi_- : Q \to B_-$ be two bi-Lipschitz maps with $\psi_{+\#}[\![Q]\!] + \psi_{-\#}[\![Q]\!] = [\![S^2]\!]$.

Corollary 2.5. *If $\varphi : [0,1] \to \mathbb{R}^n$ with $\varphi^i \in \mathrm{H}^{\alpha_i}([0,1])$ for $i = 1,2$ and $\alpha_1 + \alpha_2 > 1$, then*

$$
\varphi_\#[\![0,1]\!](\pi^1\, d\pi^2) = \int_0^1 \varphi^1\, d\varphi^2\,.
$$

If $\varphi : S^1 \to \mathbb{R}^2$ with $\varphi^i \in \mathrm{H}^{\alpha_i}([0,1])$ for $i = 1,2$ and $\alpha_1 + \alpha_2 > 1$, then

$$
\varphi_\#[\![S^1]\!](\pi^1\, d\pi^2) = \int_{S^1} \varphi^1\, d\varphi^2 = \int_{\mathbb{R}^2} \mathrm{w}(\varphi, q)\, d\mathscr{L}^2(q)\,.
$$

If $\varphi : Q = [0,1]^2 \to \mathbb{R}^3$ with $\varphi^i \in \mathrm{H}^{\alpha_i}(Q)$ for $i = 1,2,3$ and $\alpha_1 + \alpha_2 + \alpha_3 > 2$, then

$$
\varphi_\#[\![Q]\!](\pi^1\, d\pi^2 \wedge d\pi^3) = I_Q(\varphi^1, \varphi^2, \varphi^2)\,.
$$

If $\varphi : S^2 \to \mathbb{R}^3$ *with* $\varphi^i \in \mathrm{H}^{\alpha_i}(S^2)$ *for* $i = 1, 2, 3$ *and* $\alpha_1 + \alpha_2 + \alpha_3 > 2$*, then*

$$\begin{aligned}\varphi_{\#}[\![S^2]\!](\pi^1\, d\pi^2 \wedge d\pi^3) &= \sum_{i=+,-} I_Q(\varphi^1 \circ \psi_i, \varphi^2 \circ \psi_i, \varphi^3 \circ \psi_i) \\ &= \int_{\mathbb{R}^3} \mathrm{w}(\varphi, q)\, d\mathscr{L}^3(q)\,.\end{aligned}$$

Proof. These statements are true for Lipschitz maps and with appropriate approximations they follow from (2.2), Lemma 2.1, Theorem 2.2 and Proposition 2.4. Here is a proof of the last identity, the others are similar. If $\varphi_k \in \mathrm{Lip}(S^2, \mathbb{R}^3)$ is a sequence with $\varphi_k^i \xrightarrow{\alpha_i} \varphi^i$ for all i, then

$$\begin{aligned}\int_{\mathbb{R}^3} \mathrm{w}(\varphi_k, q)\, d\mathscr{L}^3(q) &= [\![\mathrm{w}(\varphi_k, \cdot)]\!](d\pi^1 \wedge d\pi^2 \wedge d\pi^3) \\ &= \partial[\![\mathrm{w}(\varphi_k, \cdot)]\!](\pi^1\, d\pi^2 \wedge d\pi^3) \\ &= \varphi_{k\#}[\![S^2]\!](\pi^1\, d\pi^2 \wedge d\pi^3) \\ &= \sum_{i=+,-} (\varphi_k \circ \psi_i)_{\#}[\![Q]\!](\pi^1\, d\pi^2 \wedge d\pi^3) \\ &= \sum_{i=+,-} [\![Q]\!](\varphi_k^1 \circ \psi_i\, d(\varphi_k^2 \circ \psi_i) \wedge d(\varphi_k^3 \circ \psi_i)) \\ &= \sum_{i=+,-} I_Q(\varphi_k^1 \circ \psi_i, \varphi_k^2 \circ \psi_i, \varphi_k^3 \circ \psi_i)\,.\end{aligned}$$

Taking the limit, the result follows from Lemma 2.1 and Proposition 2.4. □

3. Winding number testing

The lemma below examines the behavior of the integrated winding number function with respect to some well chosen test functions.

Lemma 3.1. *Let* $\gamma : S^1 \to \mathbb{R}^2$ *be a closed curve and* $\bar\gamma : S^2 \to \mathbb{R}^2$ *be an extension with* $\bar\gamma^i \in \mathrm{H}^{\alpha_i}(S^2)$ *for* $i = 1, 2$ *and* $\alpha_1 + \alpha_2 > 1$*. If there is a point* $c \in \mathbb{R}^2 \setminus \gamma(S^1)$ *with*

$$\mathrm{w}(\gamma, c) < 0\,,$$

then there are Lipschitz functions $f_1, f_2 : S^2 \to \mathbb{R}$ *with* $f_1 = f_2$ *on* B_- *and* $f_1 \geq f_2$ *on* B_+ *such that*

$$\int_{\mathbb{R}^3} \mathrm{w}(\bar\gamma \times f_1, q)\, d\mathscr{L}^3(q) < \int_{\mathbb{R}^3} \mathrm{w}(\bar\gamma \times f_2, q)\, d\mathscr{L}^3(q)\,.$$

Proof. Fix some $r > 0$ with $\mathrm{B}(c, r) \subset \mathbb{R}^2 \setminus \gamma(S^1)$ and let $g : \mathbb{R}^2 \to \mathbb{R}$ be the function

$$g(q) := \max\{r - |q - c|, 0\}\,.$$

This function has support in $\mathrm{B}(c, r)$ and parametrizes a cone over this disk. Let $f : S^2 \to \mathbb{R}$ be the function given by

$$f(p) := \begin{cases} +g(\bar\gamma(p)) & \text{if } p \in B_+\,, \\ -g(\bar\gamma(p)) & \text{if } p \in B_-\,. \end{cases}$$

This function is well defined since $g(q) = 0$ if $q \in \bar\gamma(B_+ \cap B_-)$. Consider the product map $\varphi := \bar\gamma \times f : S^2 \to \mathbb{R}^3$. By construction φ is continuous, $(c, 0) \notin \varphi(S^2)$ and $\varphi(B_\pm) \subset \mathbb{R}^2 \times \mathbb{R}_\pm$. Accordingly, the associated map $\varphi_c : S^2 \to S^2$ given by

$\varphi_c(p) := \frac{\varphi(p)-(c,0)}{|\varphi(p)-(c,0)|}$ satisfies $g(B_+) \subset B_+$ and $g(B_-) \subset B_-$. φ_c restricts to a map on the equator $\varphi_c|_{S^1} : S^1 \to S^1$ and the homological degree on the circle satisfies

$$\deg(\varphi_c|_{S^1}) = \mathrm{w}(\gamma, c) < 0 \,. \tag{3.1}$$

We want to show that

$$\deg(\varphi_c) = \deg(\varphi_c|_{S^1}) \,, \tag{3.2}$$

which then implies with (3.1) that $\mathrm{w}(\varphi,(c,0)) < 0$. The observations below are for example contained in the proof of [2, Theorem IV.6.6] and follow directly from the Eilenberg-Steenrod axioms for homology. From the exactness axiom and the fact that B_+ and B_- are contractible it follows that $\partial_* : H_2(B_-, S^1) \to H_1(S^1)$ and $j_* : H_2(S^2) \to H_2(S^2, B_+)$ are isomorphisms. Let U be an open ball around the north pole with $\bar{U} \subset \mathrm{int}(B_+)$. There is a deformation retraction $r : (S^2 \setminus U, B_+ \setminus U) \to (B_-, S^1)$ and by use of the homotopy and excision axioms we get isomorphisms,

$$H_1(B_-, S^1) \xrightarrow{l_*} H_1(S^2 \setminus U, B_+ \setminus U) \xrightarrow{k_*} H_1(S^2, B_+) \,,$$

where k and l are the inclusions. Since $\partial_* \circ \varphi_{c*} = \varphi_{c*} \circ \partial_*$, $k \circ l \circ \varphi_c = \varphi_c \circ k \circ l$ and $j \circ \varphi_c = \varphi_c \circ j$ we obtain an isomorphism $\psi : H_2(S^2) \to H_1(S^1)$ with $\psi \circ \varphi_{c*} = \varphi_{c*} \circ \psi$. This immediately implies (3.2) and together with (3.1) that $\mathrm{w}(\varphi,(c,0)) < 0$. Due to the homotopy invariance of the winding number we can approximate f by a Lipschitz function f_1 such that $f_1 \leq 0$ on B_-, $f_1 \geq 0$ on B_+, $(c,0) \notin (\bar{\gamma} \times f_1)(S^2)$ and $\mathrm{w}(\bar{\gamma} \times f_1, (c,0)) = \mathrm{w}(\varphi,(c,0)) < 0$. Summing up the Hölder exponents of the coordinate functions of $\bar{\gamma} \times f_1$ we obtain $\alpha_1 + \alpha_2 + 1 > 2$. As a consequence of Proposition 2.4, $\mathrm{w}(\bar{\gamma} \times f_1, \cdot) \in L^1_c(\mathbb{R}^3)$ and $(\bar{\gamma} \times f_1)_\# [\![S^2]\!] = \partial [\![\mathrm{w}(\bar{\gamma} \times f_1, \cdot)]\!]$. Next we show that there is a small perturbation $f_2 \in \mathrm{Lip}(S^2)$ of f_1 as in the statement of the lemma.

Since $\mathrm{w}(\bar{\gamma} \times f_1, \cdot)$ has compact support and $(c,0) \notin (\bar{\gamma} \times f_1)(S^2)$, there are $\rho, h > 0$ such that both cylinders $\mathrm{B}(c,\rho) \times [-2\rho, 2\rho]$ and $\mathrm{B}(c,\rho) \times [h-2\rho, h+2\rho]$ are contained in $\mathbb{R}^3 \setminus (\bar{\gamma} \times f_1)(S^2)$ and $\mathrm{w}(\bar{\gamma} \times f_1, (c,h)) = 0$. Let $\eta : \mathbb{R}^3 \to \mathbb{R}^3$ be an orientation preserving bi-Lipschitz map with $\eta(q) = q$ if $q \notin \mathrm{B}(c,\rho) \times [-2\rho, h+2\rho]$ and

$$\eta(q) = (q_1, q_2, q_3 + |(q_1,q_2) - c| - \rho) \,,$$

if $q \in \mathrm{B}(c,\rho) \times [0,h]$. The multiplication formula for the local degree [14, Proposition IV.6.1], or a standard fact about currents as for example stated in [12, Lemma 3.7], shows that

$$\mathrm{w}(\eta \circ (\bar{\gamma} \times f_1), \eta(q)) = \mathrm{w}(\bar{\gamma} \times f_1, q) \,, \tag{3.3}$$

whenever $q \notin (\bar{\gamma} \times f_1)(S^2)$. By construction, $\eta \circ (\bar{\gamma} \times f_1) = \bar{\gamma} \times f_2$ for some $f_2 \in \mathrm{Lip}(S^2)$ with $f_2 \leq f_1$ on B_+ and $f_2 = f_1$ on B_-. There holds $\det D\eta = 1$ almost everywhere on $A := \mathbb{R}^3 \setminus \{(q,t) \in \mathrm{B}(c,\rho) \times \mathbb{R} : t \in [-2\rho, 0] \cup [h, h+2\rho]\}$ and hence with the change of variables formula and (3.3),

$$\int_A \mathrm{w}(\bar{\gamma} \times f_1, q) \, d\mathscr{L}^3(q) = \int_{\eta(A)} \mathrm{w}(\bar{\gamma} \times f_2, q) \, d\mathscr{L}^3(q) \,. \tag{3.4}$$

The homotopy invariance of the local degree implies that $\mathrm{w}(\bar{\gamma} \times f_2, q) = \mathrm{w}(\bar{\gamma} \times f_1, q) = \mathrm{w}(\bar{\gamma} \times f_1, (c,0))$ for $q \in B := \mathrm{B}(c,\rho) \times [-2\rho, 0]$ and $\mathrm{w}(\bar{\gamma} \times f_2, q) = \mathrm{w}(\bar{\gamma} \times f_1, q) = 0$ for $q \in \mathrm{B}(c,\rho) \times [h-2\rho, h+2\rho]$. By construction, the set

$$B \setminus \eta(B) = \{q \in \mathbb{R}^3 : (q_1,q_2) \in \mathrm{B}(c,\rho), q_3 \in (|(q_1,q_2) - c| - \rho, 0]\}$$

has positive measure and with (3.4) we conclude,

$$\begin{aligned}\int_{\mathbb{R}^3} \mathrm{w}(\bar\gamma \times f_2, q)\, dq &= \int_A \mathrm{w}(\bar\gamma \times f_1, q)\, dq + \int_{\mathbb{R}^3 \setminus \eta(A)} \mathrm{w}(\bar\gamma \times f_2, q)\, dq \\ &= \int_{\mathbb{R}^3 \setminus B} \mathrm{w}(\bar\gamma \times f_1, q)\, dq + \int_{\eta(B)} \mathrm{w}(\bar\gamma \times f_1, q)\, dq \\ &= \int_{\mathbb{R}^3} \mathrm{w}(\bar\gamma \times f_1, q)\, dq - \mathscr{L}^3(B \setminus \eta(B))\, \mathrm{w}(\bar\gamma \times f_1, (c,0)) \\ &> \int_{\mathbb{R}^3} \mathrm{w}(\bar\gamma \times f_1, q)\, dq\,.\end{aligned}$$

This proves the lemma. □

Next we combine this lemma with the two dimensional analogue of the Riemann-Stieltjes integral described in Lemma 2.1 to obtain Proposition 1.3 stated in the introduction.

Proposition 3.2. *Set $Q := [0,1]^2 \subset \mathbb{R}^2$ and let $\Gamma : Q \to \mathbb{R}^2$ be a map with $\Gamma^i \in \mathrm{H}^{\alpha_i}(Q)$ for $i = 1,2$ and $\alpha_1 + \alpha_2 > 1$. If for any dyadic square $R \subset Q$ there holds*

$$\int_{\mathbb{R}^2} \deg(\Gamma, \mathrm{int}(R), q)\, d\mathscr{L}^2(q) \geq 0\,,$$

then

$$\deg(\Gamma, \mathrm{int}(Q), q) \geq 0\,,$$

whenever $q \in \mathbb{R}^2 \setminus \Gamma(\partial Q)$.

Proof. With γ we denote the restriction of Γ to ∂Q. Consider bi-Lipschitz maps $\psi_+ : Q \to B_+$ and $\psi_- : Q \to B_-$ as in Corollary 2.5. Let $\bar\gamma : S^2 \to \mathbb{R}^2$ be equal to $\Gamma \circ \psi_+^{-1}$ on B_+ and extend each $\bar\gamma^i$ arbitrarily to a function in $\mathrm{H}^{\alpha_i}(S^2)$ for $i = 1, 2$. For any $f \in \mathrm{Lip}(S^2)$, it follows from Corollary 2.5 that

$$\sum_{i=+,-} I_Q(f \circ \psi_i, \bar\gamma^1 \circ \psi_i, \bar\gamma^2 \circ \psi_i) = \int_{\mathbb{R}^3} \mathrm{w}(\bar\gamma \times f, q)\, d\mathscr{L}^3(q)\,, \tag{3.5}$$

and also for any square $R \subset Q$, there holds

$$\int_{\partial R} \bar\gamma^1 \circ \psi_+\, d(\bar\gamma^2 \circ \psi_+) = \Gamma_\#(\partial[\![R]\!])(\pi^1\, d\pi^2) = \int_{\mathbb{R}^2} \mathrm{w}(\Gamma|_{\partial R}, q) \geq 0\,. \tag{3.6}$$

Let $f_1, f_2 \in \mathrm{Lip}(S^2)$ be Lipschitz functions with $f_1 \geq f_2$ on B_+ and $f_1 = f_2$ on B_-. Combining (3.5) and (3.6) with the definition of I_Q we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathrm{w}(\bar\gamma \times f_1, q) - \mathrm{w}(\bar\gamma \times f_2, q)\, d\mathscr{L}^3(q) \\ &\quad= I_Q(f_1 \circ \psi_+, \bar\gamma^1 \circ \psi_+, \bar\gamma^2 \circ \psi_+) - I_Q(f_2 \circ \psi_+, \bar\gamma^1 \circ \psi_+, \bar\gamma^2 \circ \psi_+) \\ &\quad= \lim_{k\to\infty} \sum_{R \in \mathscr{P}_k(Q)} \big[(f_1 \circ \psi_+)(b_R) - (f_2 \circ \psi_+)(b_R)\big] \int_{\partial R} \bar\gamma^1 \circ \psi_+\, d(\bar\gamma^2 \circ \psi_+) \\ &\quad\geq 0\,.\end{aligned}$$

With Lemma 3.1 we conclude that $\mathrm{w}(\bar\gamma|_{S^1}, q) \geq 0$ for all $q \in \mathbb{R}^2 \setminus \bar\gamma(S^1)$. Because $\psi_+|_{\partial Q}$ is a counterclockwise bi-Lipschitz parameterization of S^1, $\mathrm{w}(\bar\gamma|_{S^1}, q) = \mathrm{w}(\Gamma|_{\partial Q}, q)$ and the proposition follows. □

There is a simple consequence for strictly positive degrees which is motivated by the proof of [6, Theorem 3].

Corollary 3.3. *Let $U \subset \mathbb{R}^2$ be a bounded open set and $\varphi : \bar{U} \to \mathbb{R}^2$ be a map with $\varphi^i \in \mathrm{H}^{\alpha_i}(Q)$ and $\alpha_1 + \alpha_2 > 1$. If for all squares $R \subset U$,*

$$\int_{\mathbb{R}^2} \deg(\varphi, \mathrm{int}(R), q)\, d\mathscr{L}^2(q) \geq 0\,, \tag{3.7}$$

then $\deg(\varphi, U, q) \geq 0$ whenever $q \in \mathbb{R}^2 \setminus \varphi(\partial U)$. If strict inequality holds in (3.7) for all non-degenerate squares $R \subset U$, then $\deg(\varphi, U, q) \geq 1$ whenever $q \in \varphi(U) \setminus \varphi(\partial U)$.

Proof. Because of the regularity of φ, it follows from Proposition 2.4 that for all squares $Q \subset U$ and $q \in \mathbb{R}^2 \setminus \varphi(\partial Q)$,

$$\deg(\varphi, \mathrm{int}(Q), q) \geq 0\,. \tag{3.8}$$

If $c \notin \varphi(\bar{U})$, then obviously $\deg(\varphi, U, c) = 0$. So, fix some ball $\mathrm{B}^2(c, r) \subset \varphi(U) \setminus \varphi(\partial U)$ and set $V := \varphi^{-1}(\mathrm{U}^2(c, r))$. V is an open set that satisfies $\varphi(V) \subset \mathrm{U}^2(c, r)$ and $\varphi(\partial V) \subset \partial\, \mathrm{B}^2(c, r)$. From the additivity property and local invariance of the degree it follows that for all $q \in \mathbb{R}^2 \setminus \mathrm{U}^2(c, r)$,

$$\deg(\varphi, V, q) = \deg(\varphi, U, c)\,. \tag{3.9}$$

As an open set, $V = \bigcup_{i \in \mathbb{N}} Q_i$ for a countable collection of almost disjoint closed squares Q_i. This means that $Q_i \cap Q_j \subset \partial Q_i \cap \partial Q_j$ for different i and j. We can assume that any compact subset of V intersects only finitely many Q_i. This is for example the case for a Whitney decomposition of V. Because of Proposition 2.4, $\mathscr{H}^2(\varphi(\partial Q_i)) = 0$ for all i. As a countable union $A := \bigcup_i \partial Q_i$ satisfies $\mathscr{H}^2(\varphi(A)) = 0$ too. Hence for almost every $q \in \mathrm{U}^2(c, r)$, the preimage $\varphi^{-1}(q)$ is contained in the interior of finitely many Q_i and by the additivity property of the degree and (3.8),

$$\deg(\varphi, V, q) = \sum_{i \in \mathbb{N}} \deg(\varphi, \mathrm{int}(Q_i), q) \geq \deg(\varphi, \mathrm{int}(Q_1), q)\,.$$

With (3.9) we conclude,

$$\begin{aligned} \mathscr{L}^2(\mathrm{B}^2(c, r)) \deg(\varphi, U, c) &= \int_{\mathbb{R}^2} \deg(\varphi, V, q)\, d\mathscr{L}^2(q) \\ &\geq \int_{\mathbb{R}^2} \deg(\varphi, \mathrm{int}(Q_1), q)\, d\mathscr{L}^2(q) \\ &\geq 0\,. \end{aligned}$$

This shows that $\deg(\varphi, U, c) \geq 0$ and in case of strict inequality (and the fact that the degree is an integer) that $\deg(\varphi, U, c) \geq 1$. □

The analogous statements for non-positive or negative degrees are true as well. They follows with the same proofs or by a change of orientations, for example by switching φ^1 and φ^2. With the proposition above, we have the following strengthening of [21, Theorem 1.2].

Theorem 3.4. *Let (X, d) be a C-quasi-convex metric space with $H_1^{\mathrm{Lip}}(X) = 0$ and $\varphi : X \to \mathbb{R}^2$ be a map with $\varphi^i \in \mathrm{H}^{\alpha_i}(X)$ and $\alpha_1 + \alpha_2 > 1$. If for any Lipschitz curve $\gamma : S^1 \to X$,*

$$\int_{\mathbb{R}^2} \mathrm{w}(\varphi \circ \gamma, q)\, d\mathscr{L}^2(q) = 0\,, \tag{3.10}$$

then there is a tree (T, d_T) *and maps* $\psi : X \to T$, $\bar\varphi : T \to \varphi(X)$ *with* $\varphi = \bar\varphi \circ \psi$ *and*

$$|\varphi(x) - \varphi(x')| \leq d_T(\psi(x), \psi(x')) \leq \sigma(Cd(x, x'))\,,$$

where $\sigma(t) := (\mathrm{H}^{\alpha_1}(\varphi^1)^2 t^{2\alpha_1} + \mathrm{H}^{\alpha_2}(\varphi^2)^2 t^{2\alpha_2})^{\frac{1}{2}}$. *Moreover, the tree* (T, d_T) *satisfies:*

(1) d_T *is monotone on arcs in the sense that* $d_T(p, p') \leq d_T(q, q')$ *whenever* p *and* p' *are contained in the arc* $[q, q']$ *connecting* q *with* q'.
(2) $\dim(T, d_T) \leq 1$.
(3) *For any* $p \in T$ *there is a contraction* $\pi_p : T \times \mathbb{R}_{\geq 0} \to T$ *with* $\pi_p(q, t) \in [p, q]$, $\pi_p(q, 0) = p$, $\pi_p(q, t) = q$ *for* $t \geq V_\sigma([p, q])$ *and*
$$d_T(\pi_p(q, t), \pi_p(q', t')) \leq d_T(q, q') + \sigma(|t - t'|)\,.$$

On the other hand, if φ *factors through a tree,* (3.10) *holds.*

Proof. Note that $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a homeomorphism and the map $\varphi : X \to \mathbb{R}^2$ is σ-continuous by assumption. The conclusions follows at once from [21, Theorem 1.1] if we can show that φ has Property (T) as stated therein. Statement (3) is slightly altered and we replaced $C\,\mathrm{dist}_X(\psi^{-1}(p), \psi^{-1}(q))$ with the intrinsic $V_\sigma([p, q])$. This change is valid by the definition of π_p given in [21, pp. 84].

Since X is quasi-convex, any curve in X can be uniformly approximated by Lipschitz curves and in order to show Property (T) it is therefore enough to show it for Lipschitz curves in X only. Consider first a closed Lipschitz curve $\gamma : S^1 \to X$ for which there is a Lipschitz extension $\Gamma : \mathrm{B}^2(0, 1) \to X$. By assumption there holds

$$\int_{\mathbb{R}^2} \mathrm{w}(\varphi \circ \Gamma|_{\partial B}, q)\, d\mathscr{L}^2(q) = 0\,,$$

for any set $B \subset \mathrm{B}^2(0, 1)$ bi-Lipschitz equivalent to $\mathrm{B}^2(0, 1)$. Proposition 3.2 implies that $\mathrm{w}(\varphi \circ \gamma, \cdot) \geq 0$ almost everywhere. Changing the orientation of φ, for example by switching φ^1 and φ^2, we similarly obtain $\mathrm{w}(\varphi \circ \gamma, \cdot) \leq 0$ almost everywhere. From Proposition 2.4 it follows that

$$(\varphi \circ \gamma)_\# [\![S^1]\!] = \partial [\![\mathrm{w}(\varphi \circ \gamma, \cdot)]\!] = 0\,, \tag{3.11}$$

for any closed Lipschitz curve $\gamma : S^1 \to X$ for which there is a filling Γ as above. If $\gamma : S^1 \to X$ is an arbitrary Lipschitz curve, the assumption $H_1^{\mathrm{Lip}}(X) = 0$ implies that there are finitely many Lipschitz maps $\Gamma_i : \mathrm{B}^2(0, 1) \to X$ with $\partial \sum_i \Gamma_{i\#} [\![\mathrm{B}^2(0, 1)]\!] = \sum_i \Gamma_{i\#} [\![S^1]\!] = \gamma_\# [\![S^1]\!]$, see [21, Equation (2.2)] and the discussion leading to this for some details. Then (3.11) holds for any restriction $\gamma_i = \Gamma_i|_{S^1}$ and therefore also for γ.

Now fix two points $x_1, x_2 \in X$ with $\varphi(x_1) \neq \varphi(x_2)$ and let $\gamma : [0, 1] \to X$ be a Lipschitz curve connecting x_1 with x_2. With Theorem 2.2 we see that $R := (\varphi \circ \gamma)_\# [\![0, 1]\!]$ is a metric current in $\mathscr{F}_1(\mathbb{R}^2)$ with $\partial R = [\![\varphi(x_2)]\!] - [\![\varphi(x_1)]\!] \neq 0$. Hence $R \neq 0$ and since a nontrivial 1-dimensional metric current, or a nontrivial 1-dimensional flat chain, can't be supported on finitely many points, there is a point $q \in \mathrm{spt}(R) \setminus \{\varphi(x_1), \varphi(x_2)\}$. If $\eta : [0, 1] \to X$ is another Lipschitz curve connecting x_1 and x_2 and $S := (\varphi \circ \eta)_\# [\![0, 1]\!]$ is the induced current, it follows from (3.11) that $R = S$ and hence $p \in \mathrm{spt}(S) \subset \mathrm{im}(\varphi \circ \eta)$. Since η was arbitrary, φ has Property (T) and the theorem now follows from [21, Theorem 1.1].

For the other implication, if φ is a map that factors through a tree, then it has Property (T) and hence φ factors through a tree as in [21, Theorem 1.1], respectively as in the statement of this theorem. Using the contraction of T, for any map $\gamma \in \mathrm{Lip}(S^1, X)$ there is a continuous extension $\Gamma : \mathrm{B}^2(0,1) \to T$ of $\psi \circ \gamma$ with $\mathrm{im}(\Gamma) \subset \mathrm{im}(\psi \circ \gamma)$, compare with [21, Corollary 3.12]. Hence $\mathrm{im}(\bar\varphi \circ \Gamma) \subset \mathrm{im}(\varphi \circ \gamma)$ and therefore $\mathrm{w}(\varphi \circ \gamma, \cdot) = 0$ almost everywhere (note that $\mathscr{L}^2(\mathrm{im}(\varphi \circ \gamma)) = 0$ by Proposition 2.4). □

4. Maps into the Heisenberg group

The first Heisenberg group with the Carnot-Carathéodory metric $(\mathbb{H}, d_{\mathrm{cc}})$ is bi-Lipschitz equivalent to $\mathbb{R}^3$ equipped with the *Korányi metric*,

$$d_K(p,q) := \left[(|q_x - p_x|^2 + |q_y - p_y|^2)^2 + 16|q_z - p_z - \tfrac{1}{2}(p_x q_y - p_y q_x)|^2\right]^{\frac{1}{4}} ,$$

see for example [5, §2.2.1]. Since the statements about the Heisenberg group we consider do not depend on a change to a bi-Lipschitz equivalent metric, we work with $(\mathbb{R}^3, d_{\mathrm{K}})$ instead of $(\mathbb{H}, d_{\mathrm{cc}})$. It is rather direct to check that for any bounded subset $B \subset \mathbb{R}^3$ there is a constant $C > 0$ such that for all $p, q \in B$,

$$C^{-1} d_{\mathrm{E}}(p,q) \le d_{\mathrm{K}}(p,q) \le C d_{\mathrm{E}}(p,q)^{\frac{1}{2}} ,$$

where d_{E} denotes the Euclidean metric on $\mathbb{R}^3$. The following well known path lifting property is for example stated in [13, Lemma 3.2]. Given $\gamma : [a,b] \to (\mathbb{R}^3, d_{\mathrm{K}})$ of Hölder regularity $\alpha > 1/2$, we can recover the vertical components from its horizontal projections in the sense that

$$\gamma_z(b) - \gamma_z(a) = \frac{1}{2}\left(\int_a^b \gamma_x \, d\gamma_y - \int_a^b \gamma_y \, d\gamma_x\right) . \tag{4.1}$$

This is a consequence of the estimate (2.1) for the Riemann-Stieltjes integral and the specific form of the Korányi metric. If $\varphi : Q = [0,1]^2 \to (\mathbb{R}^3, d_K)$ is a map of Hölder regularity $\alpha > \frac{1}{2}$, then the horizontal projection $\varphi_h := \varphi_x \times \varphi_y : Q \to \mathbb{R}^2$ is of Hölder regularity α too. The path lifting property (4.1) and Corollary 2.5 imply that for any square $R \subset Q$ there holds,

$$\int_{\mathbb{R}^2} \mathrm{w}(\varphi_h|_{\partial R}, q) \, d\mathscr{L}^2(q) = \varphi_{h\#}(\partial [\![R]\!])(x\,dy) = \int_{\partial R} \varphi_x \, d\varphi_y = 0 .$$

From Proposition 3.2 it follows that the winding number $\mathrm{w}(\varphi_h|_{\partial Q}, \cdot)$ vanishes on $\mathbb{R}^2 \setminus \varphi_h(\partial Q)$. Then [13, Lemma 3.3] concludes that φ can't be an embedding. This also follows from the factorization through trees statement below, but since the proof of [13, Lemma 3.3] is simpler and also implies Gromov's non-embedding conjecture it is certainly interesting to note.

Theorem 4.1. *Let (X,d) be a C-quasi-convex metric space with $H_1^{\mathrm{Lip}}(X) = 0$ and $\varphi : X \to (\mathbb{H}, d_{cc})$ be a map of Hölder regularity $\alpha > \frac{1}{2}$. Then there is a tree (T, d_T) and maps $\psi : X \to T$, $\bar\varphi : T \to \varphi(X)$ with $\varphi = \bar\varphi \circ \psi$ and*

$$|\varphi(x) - \varphi(x')| \le d_T(\psi(x), \psi(x')) \le \sigma(C d(x,x')) ,$$

for $\sigma(t) := \mathrm{H}^\alpha(\varphi) t^\alpha$. Moreover, the tree (T, d_T) satisfies:

1. *d_T is monotone on arcs in the sense that $d_T(p,p') \le d_T(q,q')$ whenever p and p' are contained in the arc $[q,q']$ connecting q with q'.*
2. $\dim(T, d_T) \le 1$.
3. *For any $p \in T$ there is a contraction $\pi_p : T \times \mathbb{R}_{\ge 0} \to T$ with $\pi_p(q,t) \in [p,q]$, $\pi_p(q,0) = p$, $\pi_p(q,t) = q$ for $t \ge V_\sigma([p,q])$ and*
$$d_T(\pi_p(q,t), \pi_p(q',t')) \le d_T(q,q') + \mathrm{H}^\alpha(\varphi)|t - t'|^\alpha .$$

Proof. As in Theorem 3.4 we replaced $C \operatorname{dist}_X(\psi^{-1}(p), \psi^{-1}(q))$ in (3) with $V_\sigma([p,q])$. Treating φ as a map into $(\mathbb{R}^3, d_{\mathrm{K}})$, the horizontal projection $\varphi_h := \varphi_x \times \varphi_y : X \to \mathbb{R}^2$ is Hölder continuous with regularity α. The path lifting property (4.1) and Corollary 2.5 imply that for any Lipschitz curve $\gamma : S^1 \to X$ there holds

$$\int_{\mathbb{R}^2} \mathrm{w}(\varphi_h \circ \gamma, q)\, d\mathscr{L}^2(q) = (\varphi_h \circ \gamma)_\# [\![S^1]\!](x\, dy) = \int_{S^1} (\varphi_x \circ \gamma)\, d(\varphi_y \circ \gamma) = 0\,.$$

It follows from Theorem 3.4 that there is a tree (T, d_1) and maps $f \in \mathrm{H}^\alpha(X,T)$ and $g \in \mathrm{Lip}(T, \mathbb{R}^2)$ with $\varphi_h = g \circ f$, and say with $T = \mathrm{im}(f)$. g can be lifted to a map $\bar g : T \to \mathbb{R}^3$ such that $\varphi = \bar g \circ f$. To see this, fix a point $p = f(x) \in T$ and set $\bar g(p) := \varphi(x)$. The contraction property of T implies that for any $q \in T$ there is a curve $\gamma \in \mathrm{H}^\alpha([0,1], T)$ connecting p with q. Define

$$\bar g_z(q) := \bar g_z(p) + \frac{1}{2}\left(\int_0^1 (g_x \circ \gamma)\, d(g_y \circ \gamma) - \int_0^1 (g_y \circ \gamma)\, d(g_x \circ \gamma) \right)\,.$$

This definition is independent of the particular choice for γ. For $\gamma \in \mathrm{H}^\alpha(S^1, T)$, the contraction property of T gives a continuous extension $\Gamma : \mathrm{B}^2(0,1) \to T$ with $\mathrm{im}(\Gamma) \subset \mathrm{im}(\gamma)$, see [21, Corollary 3.12]. Hence $\mathrm{w}(g \circ \gamma, \cdot) = 0$ almost everywhere. Together with Corollary 2.5 this implies

$$\begin{aligned} 0 &= \frac{1}{2}(g \circ \gamma)_\# [\![S^1]\!](x\, dy - y\, dx) \\ &= \frac{1}{2}\left(\int_{S^1} (g_x \circ \gamma)\, d(g_y \circ \gamma) - \int_{S^1} (g_y \circ \gamma)\, d(g_x \circ \gamma) \right)\,. \end{aligned}$$

Since X is Lipschitz path-connected we can find for any $q \in T$ a Lipschitz curve $\gamma \in \mathrm{Lip}([0,1], X)$ for which $f \circ \gamma \in \mathrm{H}^\alpha([0,1], T)$ connects p with q. The path lifting property for $\varphi \circ \gamma$ then implies that $\varphi = \bar g \circ f$. It may be possible that $\bar g$ is not continous. For this reason define a new metric d_2 on T by

$$d_2(p,q) := \max\{d_K(\bar g(p), \bar g(q)), d_1(p,q)\}\,.$$

This is clearly a metric on T with respect to which $\bar g : (T, d_2) \to (\mathbb{R}^3, d_K)$ is Lipschitz. For all $x, y \in X$,

$$\begin{aligned} d_2(f(x), f(y)) &\leq \max\{d_K(\varphi(x), \varphi(y)), \mathrm{H}^\alpha(f) d(x,y)^\alpha\} \\ &\leq \max\{\mathrm{H}^\alpha(\varphi), \mathrm{H}^\alpha(f)\} d(x,y)^\alpha\,. \end{aligned}$$

It remains to check that (T, d_2) is a tree. Because f is continuous and X is path-connected, so is (T, d_2). In particular, (T, d_2) is arc-connected. Because $\mathrm{id}_T : (T, d_2) \to (T, d_1)$ is continuous, any arc in (T, d_2) is also an arc in (T, d_1). Hence (T, d_2) is uniquely arc-connected because (T, d_1) is. This shows that φ factors through a tree and hence has Property (T). Since this is a purely topological property, the factorization for the map $\varphi : X \to \mathbb{H}$ now follows from [21, Theorem 1.1] (the resulting tree may be different from (T, d_2)). □

The contraction property of the tree has immediate consequences for the homotopy groups of $(\mathbb{H}, d_{\mathrm{cc}})$ due to the continuous extensions of [21, Corollary 3.12].

Corollary 4.2. *There is a constant $C > 0$ with the following property. For any $n \geq 2$, $\alpha > \frac{1}{2}$ and $\gamma \in \mathrm{H}^\alpha(S^n, \mathbb{H})$, there is an extension $\Gamma : \mathrm{B}^{n+1}(0,1)$ with $\mathrm{H}^\alpha(\Gamma) \leq C\, \mathrm{H}^\alpha(\gamma)$. In contrast, for $\gamma \in \mathrm{H}^\alpha(S^1, \mathbb{H})$ there is in general no extension $\Gamma \in \mathrm{H}^\alpha(\mathrm{B}^2(0,1), \mathbb{H})$ (for example if γ is injective or more generally if $\gamma_\# [\![S^1]\!] \neq 0$).*

5. Isometric immersions

In the discussion hereafter we follow closely the arguments of [6], where a simplified proof of a result of Borisov [3, 4] is given. First we recall a definition from [15, pp. 572].

Definition 5.1. *Let M^2 be a surface without boundary and $\varphi : M \to \mathbb{R}^3$ be an immersion of class C^1. We say that $\varphi(M)$ is of* bounded extrinsic curvature *if there is a constant $C > 0$ such that for any finite collection $A_1, \dots, A_n$ of disjoint compact subsets of M there holds*

$$\sum_{i=1}^{n} \mathscr{H}^2(N(A_i)) \leq C\,,$$

where $N : U \to S^2$ is the Gauss map $N := \frac{\partial_1 u \times \partial_2 u}{|\partial_1 u \times \partial_2 u|}$.

We should clarify that in [15] the sets A_i are disjoint closed sets in $\mathbb{R}^3$, but since we are interested in a local statement the two definitions are the same. We prove the following result which combines Proposition 3.2 with the arguments in [6].

Proposition 5.2. *Let g be a Riemannian metric of class C^2 defined on a bounded open set $U \subset \mathbb{R}^2$ and let $\varphi : U \to \mathbb{R}^3$ be an isometric immersion of class $C^{1,\alpha}$ for some $\alpha > \frac{1}{2}$. Assume that $N(U) \subset B_+$. If g has positive (or negative) Gauss curvature κ, then $\varphi(U)$ has extrinsic curvature bounded by $\int_U |\kappa(p)|\, d\mathscr{H}^2_g(p)$.*

Proof. Let ψ be an area and orientation-preserving diffeomorphism from a neighborhood of B_+ onto a subset of $\mathbb{R}^2$. The projection ψ and the hypothesis $N(U) \subset B_+$ are considered only because we have introduced the local degree for maps into the plane and not for maps into surfaces. Also Proposition 3.2 is formulated this way. By post-composing a Gauss map with ψ, we will treat it as a map into $\mathbb{R}^2$ where convenient and won't mention ψ explicitly. Fix some open Lipschitz domain $\Omega \Subset U$ (this means that $\partial\Omega$ can be parametrized by finitely many closed Lipschitz curves) and let V be an open set with $\Omega \Subset V \Subset U$. We want to show that

$$\int_\Omega \kappa(p)\, d\mathscr{H}^2_g(p) = \int_{S^2} \deg(N, \Omega, q)\, d\mathscr{H}^2(q)\,. \tag{5.1}$$

As it will turn out, the reason we can do this for $\alpha \in (\frac{1}{2}, \frac{2}{3}]$ is that we do not integrate over a test function f as in [6]. It follows from [6, Proposition 1] that there is a sequence of smooth immersions $\varphi_n : V \to \mathbb{R}^3$ with corresponding Gauss maps N_n such that $\operatorname{im}(\varphi_n) \subset \operatorname{dom}(\psi)$, $N_n|_V \xrightarrow{\alpha} N|_V$ and

$$\lim_{n\to\infty} \|g_n - g\|_{C^1(V)} = 0\,, \tag{5.2}$$

where $g_n = \varphi_n^{\#} e$ for the standard inner product e on $\mathbb{R}^3$. With κ_n we denote the Gauss curvature induced by g_n. An oriented orhonormal frame (X_1, X_2) on V is defined by

$$X_1 := \frac{\partial_1}{|\partial_1|_g} \quad \text{and} \quad X_2 := \frac{\partial_2 - X_1 g(\partial_2, X_1)}{|\partial_2 - X_1 g(\partial_2, X_1)|_g}\,.$$

This frame is of class C^2 because g is, and so are the dual 1-forms $\omega_1, \omega_2 \in C^2(U, \Lambda^1\mathbb{R}^2)$ (they satisfy $\omega_i(X_j) = \delta_{ij}$). There is a 1-form $\omega_{12} \in C^1(U, \Lambda^1\mathbb{R}^2)$ with defining identities $\omega_{12}(X_1) = d\omega_1(X_1, X_2)$ and $\omega_{12}(X_2) = d\omega_2(X_1, X_2)$. As for example stated in [8, Proposition 2, pp. 92],

$$d\omega_{12}(p) = -\kappa(p)(\omega_1 \wedge \omega_2)(p)\,. \tag{5.3}$$

Similarly we can define frames (X_1^n, X_2^n) and corresponding forms $\omega_1^n, \omega_2^n, \omega_{12}^n \in C^\infty(V, \Lambda^1\mathbb{R}^2)$ with respect to g_n for all n. Since these constructions depend smoothly on the metric, (5.2) implies that $\lim_{n\to\infty} \|\omega_i^n - \omega_i\|_{C^1(V)} = 0$ for $i = 1, 2$, and further

$$\lim_{n\to\infty} \|\omega_{12}^n - \omega_{12}\|_{C^0(V)} = 0\,.$$

With (5.3) this leads to,

$$\begin{aligned}\int_\Omega \kappa(p)\, d\mathscr{H}_g^2(p) &= \int_\Omega \kappa\,\omega_1 \wedge \omega_2 = [\![\Omega]\!](\kappa\,\omega_1 \wedge \omega_2)\\ &= -\partial[\![\Omega]\!](\omega_{12}) = -\lim_{n\to\infty} \partial[\![\Omega]\!](\omega_{12}^n)\\ &= \lim_{n\to\infty} \int_\Omega \kappa_n(p)\, d\mathscr{H}_{g_n}^2(p)\,. \end{aligned} \tag{5.4}$$

For the convergence of the boundary terms note that $\partial[\![\Omega]\!] = \sum_i \gamma_{i\#}[\![0,1]\!]$ for finitely many closed Lipschitz curves $\gamma_i : [0,1] \to \partial\Omega$. If $\omega_{12} = a\,dx + b\,dy$, then

$$\gamma_{i\#}[\![0,1]\!](a\,dx_1 + b\,dx_2) = \int_0^1 a(\gamma_i(t))\gamma_{i,x}'(t) + b(\gamma(t))\gamma_{i,y}'(t)\,dt\,,$$

and the convergence is obvious since ω_{12}^n converges uniformly to ω_{12}. Let σ be the volume form of S^2. By construction, $\omega_1^n \wedge \omega_2^n$ is a volume form of (U, g_n) and because N_n is smooth, the classical formula $N_n^{\#}\sigma = \kappa_n\,\omega_1^n \wedge \omega_2^n$ is valid. Relating the push-forward of currents with the degree as in (2.23),

$$\int_\Omega \kappa_n(p)\, d\mathscr{H}_{g_n}^2(p) = N_{n\#}[\![\Omega]\!](\sigma) = \int_{S^2} \deg(N_n, \Omega, q)\, \mathscr{H}^2(q)\,. \tag{5.5}$$

Since $\alpha > \frac{1}{2}$, it follows from Proposition 2.4, that $\deg(N_n, \Omega, \cdot)$ converges in L^1 to $\deg(N, \Omega, \cdot)$. From (5.4) and (5.5) we obtain (5.1):

$$\begin{aligned}\int_\Omega \kappa(p)\, d\mathscr{H}_g^2(p) &= \lim_{n\to\infty} \int_\Omega \kappa_n(p)\, d\mathscr{H}_{g_n}^2(p)\\ &= \lim_{n\to\infty} \int_{S^2} \deg(N_n, \Omega, q)\, d\mathscr{H}^2(q)\\ &= \int_{S^2} \deg(N, \Omega, q)\, d\mathscr{H}^2(q)\,.\end{aligned}$$

Note that we could show this identity more generally for open domains $\Omega \Subset U$ with $\mathscr{H}^1(\Omega) < \infty$. This equation in particular holds for open squares $R \Subset U$. Since $\kappa > 0$ it follows from Corollary 3.3 that for all open sets $W \Subset U$ and $q \in N(W) \setminus N(\partial W)$,

$$\deg(N, W, q) \geq 1\,. \tag{5.6}$$

Now let $A_1, \ldots, A_n \subset U$ be a finite collection of disjoint compact sets. Thickening them if necessary, we find a cover $\Omega_1, \ldots, \Omega_n$ of disjoint open Lipschitz domains with compact closure in U. With (5.6) and (5.1) we conclude

$$\sum_i \mathscr{H}^2(N(A_i)) \leq \sum_i \mathscr{H}^2(N(\Omega_i) \setminus N(\partial\Omega_i)) \leq \sum_i \int_{S^2} \deg(N, \Omega_i, \cdot) \leq \int_U \kappa\,.$$

The proof in case $\kappa < 0$ is analogous. □

Theorem 1.2 stated in the introduction is an immediate consequence.

References

[1] L. Ambrosio and B. Kirchheim, Currents in metric spaces, Acta Math. 185 (2000), 1–80.
[2] G. E. Bredon, Topology and geometry, Graduate Texts in Mathematics, vol. 139, Springer, 1993.
[3] Y. F. Borisov, On the connection bewteen the spatial form of smooth surfaces and their intrinsic geometry, Vestnik Leningrad. Univ. 14 (1959), 20–26.
[4] Y. F. Borisov, On the question of parallel displacement on a smooth surface and the connection of space forms of smooth surfaces with their intrinsic geometries, Vestnik Leningrad. Univ. 15 (1960), 127–129.
[5] L. Capogna, D. Danielli, S. Pauls and J. Tyson, An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem, vol. 259 of Prog. Math., Birkhäuser, 2007.
[6] S. Conti, C. De Lellis and L. Székelyhidi Jr., h-principle and rigidity for $C^{1,\alpha}$ isometric embeddings, The Abel Symposium 2010,, pp. 83–116, Springer, 2012.
[7] C. De Lellis, D Inauen and L. Székelyhidi Jr., A Nash-Kuiper theorem for $C^{1,\frac{1}{5}-\delta}$ immersions of surfaces in 3 dimensions, arXiv:1510.01934, 2015.
[8] M. P. do Carmo, Differential forms and applications, Springer, 1994.
[9] H. Federer, Geometric measure theory, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer, 1969.
[10] M. Gromov, Carnot-Carathéodory spaces seen from within, in Sub-Riemannian geometry, vol. 144 of Prog. Math., pp. 79–323, Birkhäuser, 1996.
[11] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces (with appendices by M. Katz, P. Pansu, and S. Semmes), vol. 152 of Prog. Math., Birkhäuser, 1999.
[12] U. Lang, Local currents in metric spaces, J. Geom. Anal. 21 (2011), 683–742.
[13] E. Le Donne and R. Züst, Some properties of Hölder surfaces in the Heisenberg group, Illinois J. Math. 57 (2013), 229–249.
[14] E. Outerelo and J. M. Ruiz, Mapping degree theory, Graduate Studies in Mathematics, vol. 108, American Mathematical Society, 2009.
[15] A. V. Pogorelov, Extrinsic geometry of convex surfaces, American Mathematical Society, 1973, Translated from the Russian by Israel Program for Scientific Translations, Translations of Mathematical Monographs vol. 35.
[16] S. Wenger, Isoperimetric inequalities of Euclidean type and applications, phd thesis, ETH Zurich, 2004.
[17] S. Wenger and R. Young, Lipschitz homotopy groups of the Heisenberg groups, Geom. Funct. Anal. 24 (2014), 387–402.
[18] L. C. Young, An inequality of the Hölder type, connected with Stieltjes integration, Acta Math. 67 (1936), 251–282.
[19] R. Züst, Currents in snowflaked metric spaces, phd thesis, ETH Zurich, 2011.
[20] R. Züst, Integration of Hölder forms and currents in snowflake spaces, Calc. Var. and PDE 40 (2011), 99–124.
[21] R. Züst, Some results on maps that factor through a tree, Anal. Geom. Metr. Spaces 3 (2015), 73–92.

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